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PROPERTIES OF FINITE DUAL FUSION FRAMES

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ABSTRACT. A new notion of dual fusion frame has been recently introduced by the authors. In this article that notion is further motivated and it is shown that it is suitable to deal with questions posed in a nite-dimensional real or complex Hilbert space, reinforcing the idea that this concept of duality solves the question about an appropriate de nition of dual fusion frames. It is shown that for overcomplete fusion frames there always exist duals di erent from the canonical one. Conditions that assure the uniqueness of duals are given. The relation of dual fusion frame systems with dual frames and dual projective reconstruction systems is established. Optimal dual fusion frames for the reconstruction in case of erasures of subspaces, and optimal dual fusion frame systems for the reconstruction in case of erasures of local frame vectors are determined. Examples that illustrate the obtained results are exhibited.

Key words: Frames, Fusion frames, Dual fusion frames, G-frames, Projective reconstruction systems, Erasures, Optimal dual fusion frames.

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1. Introduction

A *frame* [2, 3, 7, 13] for a separable Hilbert space $\mathcal{H}$ is a family of vectors in $\mathcal{H}$ which allows stable and not necessarily unique representations of the elements of $\mathcal{H}$ via the so-called *dual frames*. Frames are useful in areas such as signal processing, coding theory, communication theory and sampling theory, among others.

In many applications such as distributing sensing, parallel processing and packet encoding, a distributed processing by combining locally data vectors has to be implemented. *Fusion frames* (or *frames of subspaces*) [4, 6] (see also [3, Chapter 13]) are a generalization of frames and provide a mathematical framework suitable for these applications. They are collections of closed subspaces and weights, and permit the reconstruction of each element of $\mathcal{H}$ from packets of coefficients.

1.1. Duality in fusion frames.

Given a frame, the set of dual frames plays a crucial role in designing suitable reconstruction strategies. In the attempt to define dual fusion frames appears a technical difficulty related to the domain of the synthesis operator. A new concept of dual fusion frame has been proposed by the first author of this paper, which extends the "canonical" notion used so far and overcomes this technical difficulty.

In [9] properties and examples in infinite-dimensional separable Hilbert spaces are provided. There the focus is set on questions related to the boundedness of the operators involved in the definition of duality, and examples of dual fusion frames are given in $L^2(\mathbb{R})$.

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In the present paper we consider instead the finite-dimensional case studying aspects not addressed in [9]. In applications, finite-dimensional Hilbert spaces and finite fusion frames play a main role [3]. They avoid the approximation problems related to the truncation needed in the infinite-dimensional case. It is worth to mention that there are questions which only make sense in the finite-dimensional situation. This is the case for example for the study of optimal reconstructions under erasures (see, e. g., [5]), that is considered in the present paper.

1.2. **Previous approaches.** Other approaches can be considered to study duality of fusion frames. One of them are the alternate dual fusion frames introduced in [8]. We show that the reconstruction formula provided by these duals can be obtained using the new concept. One advantage of the new dual fusion frames with respect to alternate dual frames is that they can be easily obtained from the left inverses of the analysis operator of the fusion frames, or from dual frames.

Fusion frames can be viewed as a particular case of g-frames [17], so one attempt could be to study duality of fusion frames in the context of dual g-frames. For example, in [1] dual g-frames with respect to the same family of subspaces are considered, but this would have no sense applied to the study of duality of fusion frames. Reconstruction systems are g-frames in finite-dimensional Hilbert spaces. In this setting, duality of fusion frames was studied viewing them as projective reconstruction systems [15, 16]. But projective reconstruction systems are not closed under duality, more precisely, there exist projective reconstruction systems with non-projective canonical dual or without any projective dual [16]. This drawback also appears in the setting considered in [10]. We note that these problems are not present if we use the new definition of dual fusion frames.

1.3. **Optimal reconstruction under erasures.** In real implementations often some of the data vectors, or part of them, are lost or erased, and it is necessary to perform the reconstruction with the partial information at hand.

One approach to address this situation is to derive sufficient conditions for a fusion frame to be robust to such erasures, and construct fusion frames that are optimally robust. Here robustness is understood as a certain minimizing reconstruction error property. This approach is considered, e. g., in [5] for tight fusion frames using the canonical dual for the reconstruction.

In applications there might be several restrictions when selecting fusion frames for encoding, that make it impossible to find one that is optimally robust. The new concept of dual fusion frames allows another approach, studying how to select optimal dual fusion frames for a fixed fusion frame. In particular, in this article we analyze this question when a blind reconstruction process is used, in a similar way as it was done in [12, 11] for frames and in [16] for projective reconstruction systems. As in these works, we obtain, under certain conditions, a unique optimal dual fusion frame of a given fusion frame. We note that in [16], it is shown that the optimal dual reconstruction system is not necessarily projective, so it can not always be viewed as a fusion frame.

1.4. **Contents.** In Section 2 we briefly review frames, fusion frames and fusion frame systems.

In Section 3 we present the new concept of dual fusion frame. Then we consider two special cases: block-diagonal and component preserving duals, for which the reconstruction formula has a simpler expression. We present a characterization of component preserving dual fusion frames in terms of the left inverses of the analysis operator of the original fusion frame. We then refer to the duals defined in [4]. These duals are component preserving and we call them canonical. We prove that for overcomplete fusion frames with non-trivial subspaces, there always exist component preserving dual fusion frames different from the canonical ones. The new definition of dual fusion frames is a generalization of conventional dual frames and it provides more flexibility. For instance, a Riesz fusion basis can have only one component preserving dual but we show that it can have more than one non-component dual, unless additional conditions are imposed.

In Section 4, we introduce a linear transformation that links the analysis operator of a fusion frame system with the analysis operator of its associated frame. Using this transformation, we define dual fusion frame systems, which are block-diagonal. We establish the close relation of dual fusion frame systems with dual frames and dual projective reconstruction systems, showing that the new definition of dual fusion frames arises naturally.

In Section 5, we determine the duals that minimize the mean square error and the worst case error in the presence of erasures when a blind reconstruction process is used. In both cases, we determine optimal dual fusion frames for the reconstruction in case of erasures of subspaces and optimal dual fusion frame systems for the reconstruction in case of erasures of local frame vectors.

Finally, in Section 6, we show that the reconstruction formula provided by the alternate dual fusion frames introduced in [8] can be obtained using the new concept of dual fusion frame. We also present examples that illustrate the results described before.

2. Preliminaries

In this section we review the concepts of frame [2, 3, 7, 13], fusion frame and fusion frame system [4, 6] (see also [3, Chapter 13]). We refer to the mentioned works for more details. We begin introducing some notation.

2.1. **Notation.** Let $\mathcal{H}, \mathcal{K}$ be finite-dimensional Hilbert spaces over $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$. Let $L(\mathcal{H}, \mathcal{K})$ be the space of linear transformations from $\mathcal{H}$ to $\mathcal{K}$ (we write $L(\mathcal{H})$ for $L(\mathcal{H}, \mathcal{H})$). Given $T \in L(\mathcal{H}, \mathcal{K})$ we write $R(T)$, $N(T)$ and T^* to denote the image, the null space and the adjoint of T, respectively. If $T \in L(\mathcal{H}, \mathcal{K})$ is injective, $\mathfrak{L}_T$ denotes the set of left inverses of T.

The inner product and the norm in $\mathcal{H}$ will be denoted by $\langle .,. \rangle_{\mathcal{H}}$ and $\|.\|_{\mathcal{H}}$, respectively. If $T \in L(\mathcal{H}, \mathcal{K})$, then $\|T\|_F$ and $\|T\|_{sp}$ denote the Frobenius and the spectral norms of T, respectively.

If $V \subset \mathcal{H}$ is a subspace, $\pi_V \in L(\mathcal{H})$ denotes the orthogonal projection onto V.

Let $m, n, d \in \mathbb{N}$ and $\mathbf{n} = (n_1, \dots, n_m) \in \mathbb{N}^m$. In the sequel, $\mathcal{H}$ will be a finite-dimensional Hilbert space over $\mathbb{F}$ of dimension d. For $J \subseteq \{1, \dots, m\}$ let $\chi_J : \{1, \dots, m\} \to \{0, 1\}$ be the characteristic function of J. We abbreviate $\chi_{\{j\}} = \chi_j$. For $p \in \mathbb{N} \cup \{\infty\}$ let $\|.\|_p$ denote the p-norm in $\mathbb{F}^n$.

2.2. **Frames.**

Definition 2.1. Let $\mathcal{F} = \{f_i\}_{i=1}^m \subset \mathcal{H}$.

(1) The *synthesis operator* of $\mathcal{F}$ is
$$T_{\mathcal{F}} : \mathbb{F}^m \to \mathcal{H},\ T_{\mathcal{F}}(x_i)_{i=1}^m = \textstyle\sum_{i=1}^m x_i f_i$$
and the *analysis operator* is
$$T_{\mathcal{F}}^* : \mathcal{H} \to \mathbb{F}^m,\ T_{\mathcal{F}}^* f = (\langle f, f_i \rangle)_{i=1}^m.$$
(2) $\mathcal{F}$ is a *frame* for $\mathcal{H}$ if span $\mathcal{F} = \mathcal{H}$.
(3) If $\mathcal{F}$ is a frame for $\mathcal{H}$,
$$S_{\mathcal{F}} = T_{\mathcal{F}} T_{\mathcal{F}}^*,\ S_{\mathcal{F}} f = \textstyle\sum_{i=1}^m \langle f, f_i \rangle f_i,$$
is the *frame operator* of $\mathcal{F}$.

The set $\mathcal{F} = \{f_i\}_{i=1}^m \subset \mathcal{H}$ is a frame for $\mathcal{H}$ if and only if there exist $\alpha, \beta > 0$ such that

$$\alpha \|f\|^2 \leq \sum_{i=1}^m |\langle f, f_i \rangle|^2 \leq \beta \|f\|^2 \text{ for all } f \in \mathcal{H}. \tag{2.1}$$

We call α and β the *frame bounds*. The *optimal lower frame bound* is $\|S_{\mathcal{F}}^{-1}\|^{-1}$ and the *optimal upper frame bound* is $\|S_{\mathcal{F}}\| = \|T_{\mathcal{F}}\|^2$. The set $\mathcal{F}$ is an α-*tight frame*, if in (2.1) the constants α and β can be chosen so that $\alpha = \beta$, or equivalently, $S_{\mathcal{F}} = \alpha I_{\mathcal{H}}$. If $\alpha = \beta = 1$, $\mathcal{F}$ is a *Parseval frame*.

In frame theory each $f \in \mathcal{H}$ is represented by the collection of scalar coefficients $\langle f, f_i \rangle$, $i = 1, \ldots, m$, that can be thought as a measure of the projection of f onto each frame vector. From these coefficients f can be recovered using a reconstruction formula via the so-called dual frames.

Definition 2.2. Let $\mathcal{F} = \{f_i\}_{i=1}^m$ and $\widetilde{\mathcal{F}} = \{\tilde{f}_i\}_{i=1}^m$ be frames for $\mathcal{H}$. Then $\widetilde{\mathcal{F}}$ is a *dual frame* of $\mathcal{F}$ if the following reconstruction formula holds

$$\sum_{i=1}^m \langle f, f_i \rangle \widetilde{f}_i, \text{ for all } f \in \mathcal{H}, \tag{2.2}$$

or equivalently,

$$T_{\widetilde{\mathcal{F}}} T_{\mathcal{F}}^* = I_{\mathcal{H}}. \tag{2.3}$$

Let $\mathcal{F} = \{f_i\}_{i=1}^m$ be a frame for $\mathcal{H}$. Then $\{S_{\mathcal{F}}^{-1} f_i\}_{i=1}^m$ is the *canonical dual frame* of $\mathcal{F}$.

2.3. **Fusion frames and fusion frame systems.** Fusion frames generalize the concept of frames. The representation of each $f \in \mathcal{H}$ via fusion frames is given by projections onto multidimensional subspaces, which also satisfy some stability conditions.

Definition 2.3. Let $\{W_i\}_{i=1}^m$ be a family of subspaces of $\mathcal{H}$, and let $\{w_i\}_{i=1}^m$ be a family of weights, i.e., $w_i > 0$ for $i = 1, \ldots, m$. Then $\{(W_i, w_i)\}_{i=1}^m$ is called a *Bessel fusion sequence* for $\mathcal{H}$.

We will denote $\{W_i\}_{i=1}^m$ with $\mathbf{W}$, $\{w_i\}_{i=1}^m$ with $\mathbf{w}$ and $\{(W_i, w_i)\}_{i=1}^m$ with $(\mathbf{W}, \mathbf{w})$. If $T \in L(\mathcal{H}, \mathcal{K})$ we will write $(T\mathbf{W}, \mathbf{w})$ for $\{(TW_i, w_i)\}_{i=1}^m$.

Let $\mathcal{W} := \bigoplus_{i=1}^m W_i = \{(f_i)_{i=1}^m : f_i \in W_i\}$ be the Hilbert space with $\langle (f_i)_{i=1}^m, (g_i)_{i=1}^m \rangle_{\mathcal{W}} = \sum_{i=1}^m \langle f_i, g_i \rangle$.

Definition 2.4. Let $(\mathbf{W}, \mathbf{w})$ be a Bessel fusion sequence.

(1) $(\mathbf{W}, \mathbf{w})$ is called *w-uniform*, if $w_i = w$ for all $i \in \{1, \ldots, m\}$. In this case we write $(\mathbf{W}, w)$.
(2) $(\mathbf{W}, \mathbf{w})$ is called *n-equi-dimensional*, if $\dim(W_i) = n$ for all $i \in \{1, \ldots, m\}$.
(3) The *synthesis operator* of $(\mathbf{W}, \mathbf{w})$ is
$$T_{\mathbf{W},\mathbf{w}} : \mathcal{W} \to \mathcal{H},\ T_{\mathbf{W},\mathbf{w}} (f_i)_{i=1}^m = \textstyle\sum_{i=1}^m w_i f_i.$$
The *analysis operator* is
$$T_{\mathbf{W},\mathbf{w}}^* : \mathcal{H} \to \mathcal{W},\ T_{\mathbf{W},\mathbf{w}}^* f = (w_i \pi_{W_i}(f))_{i=1}^m.$$
(4) $(\mathbf{W}, \mathbf{w})$ is called a *fusion frame* for $\mathcal{H}$ if $\operatorname{span} \bigcup_{i=1}^m W_i = \mathcal{H}$.
(5) $(\mathbf{W}, \mathbf{w})$ is a *Riesz fusion basis* if $\mathcal{H}$ is the direct sum of the W_i, and $(\mathbf{W}, 1)$ an *orthonormal fusion basis* if $\mathcal{H}$ is the orthogonal sum of the W_i.
(6) If $(\mathbf{W}, \mathbf{w})$ is a fusion frame for $\mathcal{H}$, the operator
$$S_{\mathbf{W},\mathbf{w}} = T_{\mathbf{W},\mathbf{w}} T_{\mathbf{W},\mathbf{w}}^* : \mathcal{H} \to \mathcal{H}\ ,\ S_{\mathbf{W},\mathbf{w}}(f) = T_{\mathbf{W},\mathbf{w}} T_{\mathbf{W},\mathbf{w}}^*(f) = \textstyle\sum_{i=1}^m w_i^2 \pi_{W_i}(f)$$
is called the *fusion frame operator* of $(\mathbf{W}, \mathbf{w})$.

A Bessel fusion sequence $(\mathbf{W}, \mathbf{w})$ is a fusion frame for $\mathcal{H}$ if and only if $T_{\mathbf{W},\mathbf{w}}$ is onto, or equivalently, if and only if there exist constants $0 < \alpha \leq \beta < \infty$ such that

$$\alpha \|f\|^2 \leq \sum_{i=1}^m w_i^2 \|\pi_{W_i}(f)\|^2 \leq \beta \|f\|^2 \text{ for all } f \in \mathcal{H}. \tag{2.4}$$

We call α and β the *fusion frame bounds*. A fusion frame $(\mathbf{W}, \mathbf{w})$ is called an *α-tight fusion frame* if in (2.4) the constants α and β can be chosen so that $\alpha = \beta$, or equivalently, $S_{\mathbf{W},\mathbf{w}} = \alpha I_{\mathcal{H}}$. If $\alpha = \beta = 1$ we say that it is a *Parseval fusion frame*.

The use of fusion frames permits furthermore local processing in each of the subspaces. For this, it is useful to have a set of local frames for its subspaces:

Definition 2.5. Let $(\mathbf{W}, \mathbf{w})$ be a fusion frame for $\mathcal{H}$, and let $\{f_i^l\}_{l\in L_i}$ be a frame for W_i for $i = 1, \ldots, m$. Then we call $\{(W_i, w_i, \{f_i^l\}_{l\in L_i})\}_{i=1}^m$ a *fusion frame system* for $\mathcal{H}$.

From now on we denote $\mathcal{F}_i = \{f_i^l\}_{l\in L_i}$, $\mathcal{F} = \{\mathcal{F}_i\}_{i=1}^m$, $\mathbf{w}\mathcal{F} = \{w_i\mathcal{F}_i\}_{i=1}^m$, and we will abbreviate $\{(W_i, w_i, \{f_i^l\}_{l\in L_i})\}_{i=1}^m$ with $(\mathbf{W}, \mathbf{w}, \mathcal{F})$. If $T \in L(\mathcal{H}, \mathcal{K})$ we use the notation $T\mathcal{F}$ for $\{\{Tf_i^l\}_{l\in L_i}\}_{i=1}^m$.

Remark 2.6. Clearly, $\mathbf{w}\mathcal{F}$ is a frame for $\mathcal{H}$ if and only if $(\mathcal{W}, \mathbf{w}, \mathcal{F})$ is a fusion frame system for $\mathcal{H}$.

3. Dual fusion frames

One of the most important properties of frames is that they permit different representations for each element of $\mathcal{H}$, which are provided by the duals via the reconstruction formula (2.2). Taking this into account, our purpose is to have a notion of dual fusion frame as we have it in the classical frame theory, and that furthermore leads to analogous results. We note that for frames the duality condition can be expressed in two forms: (2.2) and (2.3). So, it is natural to try to generalize these expressions to the context of fusion frames in order to obtain a definition of dual fusion frame.

Let $(\mathbf{W}, \mathbf{w})$ be a fusion frame. Since $S_{\mathbf{W},\mathbf{w}}^{-1} S_{\mathbf{W},\mathbf{w}} = I_{\mathcal{H}}$, we have the following reconstruction formula

$$f = \sum_{i=1}^m w_i^2 S_{\mathbf{W},\mathbf{w}}^{-1} \pi_{W_i}(f), \text{ for all } f \in \mathcal{H}, \tag{3.1}$$

that is analogous to (2.2). The family $(S_{\mathbf{W},\mathbf{w}}^{-1}\mathbf{W}, \mathbf{w})$ is a fusion frame which in [4, Definition 3.19] is called the dual fusion frame of $(\mathbf{W}, \mathbf{w})$, and is similar to the canonical dual frame in the classical frame theory. As it is pointed out in [3, Chapter 13], (3.1) - in contrast to the analogous one for frames - does not lead automatically to a dual fusion frame concept.

So, instead of trying to generalize (2.2), we can try with (2.3). But in this case we find the following obstacle. Given $(\mathbf{W}, \mathbf{w})$ and $(\mathbf{V}, \mathbf{v})$ two fusion frames for $\mathcal{H}$, with $\mathbf{W} \neq \mathbf{V}$, the corresponding synthesis operators $T_{\mathbf{W},\mathbf{w}}$ and $T_{\mathbf{V},\mathbf{v}}$ have different domains. Therefore the composition of $T_{\mathbf{V},\mathbf{v}}$ with $T_{\mathbf{W},\mathbf{w}}^*$ is not possible. The next definition overcomes this problem, extends the notion introduced in [4] (see Section 3.1) and, as we are going to see, leads to the properties that we would desire a dual fusion frame to have.

Definition 3.1. Let $(\mathbf{W}, \mathbf{w})$ and $(\mathbf{V}, \mathbf{v})$ be two fusion frames for $\mathcal{H}$. Then $(\mathbf{V}, \mathbf{v})$ is a dual fusion frame of $(\mathbf{W}, \mathbf{w})$ if there exists a $Q \in L(\mathcal{W}, \mathcal{V})$ such that

$$T_{\mathbf{V},\mathbf{v}} Q T_{\mathbf{W},\mathbf{w}}^* = I_{\mathcal{H}}. \tag{3.2}$$

If we need to do an explicit reference to the linear transformation Q we say that $(\mathbf{V}, \mathbf{v})$ is a Q-dual fusion frame of $(\mathbf{W}, \mathbf{w})$. Note that if $(\mathbf{V}, \mathbf{v})$ is a Q-dual fusion frame of $(\mathbf{W}, \mathbf{w})$, then $(\mathbf{W}, \mathbf{w})$ is a dual Q^*-dual fusion frame of $(\mathbf{V}, \mathbf{v})$.

Note that in (2.3) the operator "between" $T_{\widetilde{\mathcal{F}}}$ and $T_{\mathcal{F}}^*$ is $I_{\mathbb{F}^m}$, which is hidden. In view of this, (3.2) can be seen as a generalization of (2.3).

Now we introduce two particular types of linear transformations Q for which the reconstruction formula obtained from (3.2) is simpler. For this, we consider the selfadjoint operator $M_{J,\mathbf{W}} : \mathcal{W} \to \mathcal{W}$, $M_{J,\mathbf{W}}(f_j)_{j=1}^m = (\chi_J(j) f_j)_{j=1}^m$. We simply write M_J if it clear to which $\mathbf{W}$ we refer to. We abbreviate $M_{\{j\},\mathbf{W}} = M_{j,\mathbf{W}}$ and $M_{\{j\}} = M_j$.

Definition 3.2. Let $Q \in L(\mathcal{W}, \mathcal{V})$.

(1) If $QM_{j,\mathbf{W}}\mathcal{W} \subseteq M_{j,\mathbf{V}}\mathcal{V}$ for each $j \in \{1, \ldots, m\}$, Q is called *block-diagonal*.
(2) If $QM_{j,\mathbf{W}}\mathcal{W} = M_{j,\mathbf{V}}\mathcal{V}$ for each $j \in \{1, \ldots, m\}$, Q is called *component preserving*.

Observe that Q is block-diagonal if and only if $QM_{J,\mathbf{W}} = M_{J,\mathbf{V}}Q$ for each $J \subseteq \{1,\dots,m\}$, or equivalently, $QM_{j,\mathbf{W}} = M_{j,\mathbf{V}}Q$ for each $j \in \{1,\dots,m\}$. If Q is block-diagonal, then Q^* is block-diagonal. If in Definition 3.1 Q is block-diagonal (component preserving) we say that $(\mathbf{V},\mathbf{v})$ is a *block-diagonal dual fusion frame* (*component preserving dual fusion frame*) of $(\mathbf{W},\mathbf{w})$.

It is important to note, as we will see in Theorem 3.5, that Q is component preserving for dual fusion frames obtained from the left inverses of $T^*_{\mathbf{W},\mathbf{w}}$. Also, Q is block-diagonal for dual fusion frame systems (see Definition 4.1 and Remark 4.2).

The reconstruction formula following from (3.2) has the form

$$f = \sum_{j=1}^{m} v_j M_j Q(w_i \pi_{W_i} f)_{i=1}^m \ , \ \forall f \in \mathcal{H}. \tag{3.3}$$

If Q is block-diagonal, (3.3) becomes

$$f = \sum_{j=1}^{m} v_j w_j Q_j \pi_{W_j} f \ , \ \forall f \in \mathcal{H}, \tag{3.4}$$

where $Q_j \pi_{W_j} f := QM_j(\pi_{W_i} f)_{i=1}^m$.

The main advantage of (3.4) over (3.3) is that in (3.4) the j-th term is obtained using only the projection onto W_j, whereas in (3.3) all the projections onto W_i for $i = 1,\dots,m$ are involved. This fact is particularly useful for truncation purposes. In this case we can consider an index subset J in (3.4), to obtain an approximate reconstruction formula where only the subspaces W_j (and V_j) for $j \in J$ are used.

The linear transformation Q has in many cases a very simple expression and consequently the reconstruction formula is very simple too (see examples in Section 3.1 and Section 6).

Now we are going to relate the duals of a fusion frame with the left inverses of its analysis operator, in a similar fashion as for frames (see, e. g., [7, Lemma 5.6.3.]). For this, given $A \in L(\mathcal{W},\mathcal{H})$ and $\mathbf{v}$ a collection of weights, we consider the subspaces $V_i = AM_i\mathcal{W}$, for each $i = 1,\dots,m$, and the linear transformation

$$Q_{A,\mathbf{v}} : \mathcal{W} \to \mathcal{V} \ , \ Q_{A,\mathbf{v}}(f_j)_{j=1}^m = \left(\frac{1}{v_i} AM_i(f_j)_{j=1}^m\right)_{i=1}^m .$$

Its adjoint is $Q^*_{A,\mathbf{v}} : \mathcal{V} \to \mathcal{W}$, $Q^*_{A,\mathbf{v}}(g_j)_{j=1}^m = \sum_{i=1}^m \frac{1}{v_i} M_i A^* g_i$.

To simplify the exposition, we just formulate the next lemmas which proofs are straightforward.

Lemma 3.3. *Let $(\boldsymbol{W},\mathbf{w})$ be a Bessel fusion sequence for $\mathcal{H}$, $A \in L(\mathcal{W},\mathcal{H})$, $\mathbf{v}$ a collection of weights and $V_i = AM_i\mathcal{W}$, for each $i \in \{1,\dots,m\}$. Then $Q_{A,\mathbf{v}}$ is component preserving and $A = T_{\mathbf{V},\mathbf{v}}Q_{A,\mathbf{v}}$.*

Lemma 3.4. *Let $(\boldsymbol{W},\mathbf{w})$, $(\mathbf{V},\mathbf{v})$ and $(\widetilde{\mathbf{V}},\widetilde{\mathbf{v}})$ be Bessel fusion sequences for $\mathcal{H}$. Let $Q \in L(\mathcal{W},\mathcal{V})$ and $\widetilde{Q} \in L(\mathcal{W},\widetilde{\mathcal{V}})$. If $T_{\mathbf{V},\mathbf{v}}Q = T_{\widetilde{V},\widetilde{v}}\widetilde{Q}$, then the following assertions hold:*

1. *If Q and $\widetilde{Q}$ are component preserving then $V_i = \widetilde{V}_i$ for $i = 1,\dots,m$.*
2. *Let $D : \mathcal{V} \to \mathcal{V}$, $D(g_i)_{i=1}^m = (\frac{\widetilde{v}_i}{v_i} g_i)_{i=1}^m$. If Q and $\widetilde{Q}$ are block-diagonal and $\mathbf{V} = \widetilde{\mathbf{V}}$, then $Q = D\widetilde{Q}$.*

The next theorem characterizes the component preserving dual fusion frames of $(\mathbf{W},\mathbf{w})$ in terms of the left inverses of $T^*_{\mathbf{W},\mathbf{w}}$:

Theorem 3.5. *Let $(\boldsymbol{W},\mathbf{w})$ be a fusion frame for $\mathcal{H}$. Then $(\mathbf{V},\mathbf{v})$ is a Q-component preserving dual fusion frame of $(\boldsymbol{W},\mathbf{w})$ if and only if $V_i = AM_i\mathcal{W}$ for each $i \in \{1,\dots,m\}$ and $Q = Q_{A,\mathbf{v}}$, for some $A \in \mathfrak{L}_{T^*_{W,\mathbf{w}}}$. Moreover, any element of $\mathfrak{L}_{T^*_{W,\mathbf{w}}}$ is of the form $T_{\mathbf{V},\mathbf{v}}Q$ where $(\mathbf{V},\mathbf{v})$ is some Q-component preserving dual fusion frame of $(\boldsymbol{W},\mathbf{w})$.*

Proof. Let $(\mathbf{V},\mathbf{v})$ be a Q-component preserving dual fusion frame of $(\mathbf{W},\mathbf{w})$. Since $T_{\mathbf{V},\mathbf{v}}QT^*_{\mathbf{W},\mathbf{w}} = I_{\mathcal{H}}$, $A := T_{\mathbf{V},\mathbf{v}}Q \in \mathfrak{L}_{T^*_{\mathbf{W},\mathbf{w}}}$. Since Q is component preserving, $AM_i\mathcal{W} = T_{\mathbf{V},\mathbf{v}}QM_i\mathcal{W} = V_i$. By Lemma 3.3, $Q_{A,\mathbf{v}}$ is component preserving and $T_{\mathbf{V},\mathbf{v}}Q = A = T_{\mathbf{V},\mathbf{v}}Q_{A,\mathbf{v}}$. By Lemma 3.4 (2), $Q = Q_{A,\mathbf{v}}$.

Let now $A \in \mathfrak{L}_{T^*_{\mathbf{W},\mathbf{w}}}$ and $V_i = AM_i\mathcal{W}$ for each $i \in \{1,...,m\}$. By Lemma 3.3, $Q_{A,\mathbf{v}}$ is component preserving and $A = T_{\mathbf{V},\mathbf{v}}Q_{A,\mathbf{v}}$. Since $A \in \mathfrak{L}_{T^*_{\mathbf{W},\mathbf{w}}}$, $T_{\mathbf{V},\mathbf{v}}Q_{A,\mathbf{v}}T^*_{\mathbf{W},\mathbf{w}} = I_{\mathcal{H}}$. So $(\mathbf{V},\mathbf{v})$ is a $Q_{A,\mathbf{v}}$-component preserving dual fusion frame of $(\mathbf{W},\mathbf{w})$.

The last assertion of the theorem follows from the previous steps of the proof. □

Remark 3.6. By Theorem 3.5, we can always associate to any Q-dual fusion frame $(\mathbf{V},\mathbf{v})$ of $(\mathbf{W},\mathbf{w})$ the $Q_{A,\tilde{\mathbf{v}}}$-component preserving dual fusion frame $\{(AM_i\mathcal{W},\tilde{v}_i)\}_{i=1}^m$ where $A = T_{\mathbf{V},\mathbf{v}}Q$ and $\{\tilde{v}_i\}_{i=1}^m$ are arbitrary weights. Moreover if Q is block-diagonal, then $Q_{T_{\mathbf{V},\mathbf{v}}Q,\mathbf{v}}(f_j)_{j=1}^m = Q(f_j)_{j=1}^m$ for each $(f_j)_{j=1}^m \in \mathcal{W}$.

3.1. **The canonical dual fusion frame.** If $(\mathbf{W},\mathbf{w})$ is a fusion frame, then $(S^{-1}_{\mathbf{W},\mathbf{w}}\mathbf{W},\mathbf{w})$ is the dual fusion frame of $(\mathbf{W},\mathbf{w})$ in the sense of [4].

By Theorem 3.5, $(S^{-1}_{\mathbf{W},\mathbf{w}}\mathbf{W},\mathbf{v})$ is a $Q_{A,\mathbf{v}}$-component preserving dual with $A = S^{-1}_{\mathbf{W},\mathbf{w}}T_{\mathbf{W},\mathbf{w}}$, $\mathbf{v}$ a family of arbitrary weights and $Q_{A,\mathbf{v}} : \mathcal{W} \to \bigoplus_{i=1}^m S^{-1}_{\mathbf{W},\mathbf{w}}W_i$, $Q_{A,\mathbf{v}}(f_j)_{j=1}^m = (\frac{w_i}{v_i}S^{-1}_{\mathbf{W},\mathbf{w}}f_i)_{i=1}^m$. In the sequel we refer to this $Q_{S^{-1}_{\mathbf{W},\mathbf{w}}T_{\mathbf{W},\mathbf{w}},\mathbf{v}}$-dual fusion frame as the *canonical dual with weights* $\mathbf{v}$.

Note that with a canonical dual fusion frame we have the reconstruction formula (3.1) that can be written as

$$f = T_{S^{-1}_{\mathbf{W},\mathbf{w}}W,\mathbf{v}}Q_{S^{-1}_{\mathbf{W},\mathbf{w}}T_{\mathbf{W},\mathbf{w}},\mathbf{v}}T^*_{W,\mathbf{w}}f, \ \forall f \in \mathcal{H}, \tag{3.5}$$

whereas with another dual fusion frame, (3.2) provides other alternatives for the reconstruction.

3.2. **Existence of non-canonical dual fusion frames.** A Bessel fusion sequence $(\mathbf{W},\mathbf{w})$ is a Riesz fusion basis if and only if $T^*_{\mathbf{W},\mathbf{w}}$ is bijective, or equivalently, $T^*_{\mathbf{W},\mathbf{w}}$ has a unique left inverse. So, if $(\mathbf{W},\mathbf{w})$ is a Riesz fusion basis, the only component preserving duals of $(\mathbf{W},\mathbf{w})$ are the canonical ones, and if we fix the weights $\mathbf{v}$, by Lemma 3.4(2), the unique Q for this dual is $Q_{S^{-1}_{\mathbf{W},\mathbf{w}}T_{\mathbf{W},\mathbf{w}},\mathbf{v}}$. A Riesz fusion basis can have other dual fusion frames (see Example 6.2). We will give now conditions that guarantee the uniqueness of the subspaces of the duals of a Riesz fusion basis.

Proposition 3.7. *Let* $(\boldsymbol{W},\mathbf{w})$ *be a Riesz fusion basis and* $\mathbf{v}$ *a family of weights. The following assertions hold:*

1. *Let* $(\mathbf{V},\mathbf{v})$ *be a block-diagonal dual fusion frame of* $(\boldsymbol{W},\mathbf{w})$*. Then, for each* $i = 1,\dots,m$, $S^{-1}_{\boldsymbol{W},\mathbf{w}}W_i \subseteq V_i$.
2. *If* $(\mathbf{V},\mathbf{v})$ *is a Riesz fusion basis which is a block-diagonal dual fusion frame of* $(\boldsymbol{W},\mathbf{w})$*, then* $V_i = S^{-1}_{\boldsymbol{W},\mathbf{w}}W_i$ *for* $i = 1,\dots,m$.

Proof. (1) Let $f_i \in W_i$. Using that $T_{\mathbf{V},\mathbf{v}}QT^*_{\mathbf{W},\mathbf{w}} = I_{\mathcal{H}}$ and that $T^*_{\mathbf{W},\mathbf{w}}S^{-1}_{\mathbf{W},\mathbf{w}}T_{\mathbf{W},\mathbf{w}} = I_{\mathcal{W}}$ (the last equation holds since $(\mathbf{W},\mathbf{w})$ is a Riesz fusion basis), we obtain
$S^{-1}_{\mathbf{W},\mathbf{w}}f_i = T_{\mathbf{V},\mathbf{v}}QT^*_{\mathbf{W},\mathbf{w}}S^{-1}_{\mathbf{W},\mathbf{w}}T_{\mathbf{W},\mathbf{w}}M_i(\frac{1}{w_i}f_j)_{j=1}^m = T_{\mathbf{V},\mathbf{v}}QM_i(\frac{1}{w_i}f_j)_{j=1}^m = T_{\mathbf{V},\mathbf{v}}M_iQ(\frac{1}{w_i}f_j)_{j=1}^m \in V_i$.
So $S^{-1}_{\mathbf{W},\mathbf{w}}W_i \subseteq V_i$.

(2) By (1), $S^{-1}_{\mathbf{W},\mathbf{w}}W_i \subseteq V_i$ for $i = 1,\dots,m$. If there exists $i_0 \in \{1,\dots,m\}$ such that $S^{-1}_{\mathbf{W},\mathbf{w}}W_{i_0} \subset V_{i_0}$, then $\sum_{i=1}^m \dim(S^{-1}_{\mathbf{W},\mathbf{w}}W_i) < \sum_{i=1}^m \dim(V_i)$, which is a contradiction, since $(S^{-1}_{\mathbf{W},\mathbf{w}}W,\mathbf{v})$ and $(\mathbf{V},\mathbf{v})$ are Riesz fusion bases. Thus the conclusion follows. □

Remark 3.8. Note that Proposition 3.7(2) implies that if $(\mathbf{W}, \mathbf{w})$ is an orthonormal fusion basis and $(\mathbf{V}, \mathbf{v})$ is a block-diagonal dual fusion frame which is an orthonormal fusion basis, then $W_i = V_i$ for $i = 1, \dots, m$.

Although it could seem surprising that a Riesz fusion basis can have more than one dual, it is absolutely reasonable taking into account the generality of the new definition. Moreover observe that uniqueness of the dual of a Riesz basis in classical frame theory can be understood in the sense of Proposition 3.7 (1) considering that each element of the Riesz basis and of its dual generate subspaces of dimension 1.

The following result asserts that if $(\mathbf{W}, \mathbf{w})$ is an overcomplete fusion frame (i.e. a fusion frame which is not a Riesz fusion basis) with non-trivial subspaces, there always exist component preserving dual fusion frames which differ from the canonical ones.

Proposition 3.9. *Let $(\boldsymbol{W}, \mathbf{w})$ be an overcomplete fusion frame such that $W_i \neq \{0\}$ for every $i \in \{1, \dots, m\}$. Then there exist component preserving dual fusion frames of $(\boldsymbol{W}, \mathbf{w})$ different from $(S^{-1}_{\boldsymbol{W},\boldsymbol{w}} \boldsymbol{W}, \mathbf{v})$ for any family of weights $\mathbf{v}$.*

Proof. Since $(\mathbf{W}, \mathbf{w})$ is not a Riesz fusion basis, there exists $i_0 \in \{1, \dots, m\}$ such that $W_{i_0} \cap \operatorname{span}\{W_i : i \neq i_0\} \neq \{0\}$. Observe that $(\widetilde{\mathbf{W}}, \mathbf{w})$ given by $\widetilde{W}_i = W_i$ for $i \neq i_0$ and $\widetilde{W}_{i_0} = W_{i_0} \cap (\operatorname{span}\{W_i : i \neq i_0\})^{\perp}$ is a fusion frame for $\mathcal{H}$. Define $V_i = S^{-1}_{\widetilde{\mathbf{W}},\mathbf{w}} \widetilde{W}_i$ for $i \in \{1, \dots, m\}$. Consider the component preserving $\widetilde{Q} \in L(\mathcal{W}, \mathcal{V})$, given by $\widetilde{Q}(f_i)_{i=1}^m = (S^{-1}_{\widetilde{\mathbf{W}},\mathbf{w}} \pi_{\widetilde{W}_i} f_i)_{i=1}^m$.

Let $f \in \mathcal{H}$. Since $\pi_{\widetilde{W}_{i_0}} \pi_{W_{i_0}} f = \pi_{\widetilde{W}_{i_0}} f$, we obtain $T_{\mathbf{V},\mathbf{w}} \widetilde{Q} T^*_{\mathbf{W},\mathbf{w}} f = \sum_{i=1}^m w_i^2 S^{-1}_{\widetilde{\mathcal{W}},\mathbf{w}} \pi_{\widetilde{W}_i}(f) = f$.

Now, $\dim(S^{-1}_{\widetilde{\mathcal{W}},\mathbf{w}} \widetilde{W}_{i_0}) = \dim(\widetilde{W}_{i_0})$ and $\widetilde{W}_{i_0} \subseteq W_{i_0}$. Assume that $\widetilde{W}_{i_0} = W_{i_0}$. Then $W_{i_0} \subseteq (\operatorname{span}\{W_i : i \neq i_0\})^{\perp}$ which is a contradiction since $W_{i_0} \cap \operatorname{span}\{W_i : i \neq i_0\} \neq \{0\}$. Hence $\dim(\widetilde{W}_{i_0}) < \dim(W_{i_0}) = \dim(S^{-1}_{\mathbf{W},\mathbf{w}} W_{i_0})$ and so $V_{i_0} \neq S^{-1}_{\mathbf{W},\mathbf{w}} W_{i_0}$ obtaining the desired result. $\square$

4. Dual fusion frame systems and their relation with dual frames and dual projective reconstruction systems

We begin this section by defining dual fusion frame systems. We first introduce a linear transformation that provides the fundamental link between the synthesis operator of a fusion frame system with the synthesis operator of its associated frame.

Let $(\mathbf{W}, \mathbf{w})$ be a Bessel fusion sequence and $\mathcal{F}_i$ be a frame for W_i. Let

$$C_{\mathcal{F}} : L(\bigoplus_{i=1}^m \mathbb{F}^{|L_i|} \to \mathcal{W}), \;\; C_{\mathcal{F}}((x_i^l)_{l \in L_i})_{i=1}^m = (T_{\mathcal{F}_i}(x_i^l)_{l \in L_i})_{i=1}^m.$$

Then $C_{\mathcal{F}}$ is surjective and $C^*_{\mathcal{F}} : \mathcal{W} \to \bigoplus_{i=1}^m \mathbb{F}^{|L_i|}$, $C^*_{\mathcal{F}}(g_i)_{i=1}^m = (T^*_{\mathcal{F}_i} g_i)_{i=1}^m$. The left inverses of $C^*_{\mathcal{F}}$ are all $C_{\widetilde{\mathcal{F}}} \in L(\bigoplus_{i=1}^m \mathbb{F}^{|L_i|}, \mathcal{W})$ with $\widetilde{\mathcal{F}}_i$ a dual frame of $\mathcal{F}_i$ for $i = 1, \dots, m$. Note that

$$T_{\mathbf{w}\mathcal{F}} = T_{\mathbf{W},\mathbf{w}} C_{\mathcal{F}} \;\; \text{and} \;\; T_{\mathbf{W},\mathbf{w}} = T_{\mathbf{w}\mathcal{F}} C^*_{\widetilde{\mathcal{F}}}.$$

Definition 4.1. Let $(\mathbf{W}, \mathbf{w}, \mathcal{F})$ and $(\mathbf{V}, \mathbf{v}, \mathcal{G})$ be two fusion frame systems for $\mathcal{H}$. Then $(\mathbf{V}, \mathbf{v}, \mathcal{G})$ is a dual fusion frame system of $(\mathbf{W}, \mathbf{w}, \mathcal{F})$ if $(\mathbf{V}, \mathbf{v})$ is a $C_{\mathcal{G}} C^*_{\mathcal{F}}$-dual fusion frame of $(\mathbf{W}, \mathbf{w})$.

Remark 4.2. Observe that $C_{\mathcal{G}} C^*_{\mathcal{F}} : \mathcal{W} \to \mathcal{V}$, $C_{\mathcal{G}} C^*_{\mathcal{F}}(f_i)_{i=1}^m = (T_{\mathcal{G}_i} T^*_{\mathcal{F}_i} f_i)_{i=1}^m$ is block-diagonal. Whereas, $M_i \mathcal{V} \subseteq C_{\mathcal{G}} C^*_{\mathcal{F}} M_i \mathcal{W}$ if and only if $T_{\mathcal{G}_i}(R(T^*_{\mathcal{G}_i})) \subseteq T_{\mathcal{G}_i}(R(T^*_{\mathcal{F}_i}))$, or equivalently, $R(T^*_{\mathcal{G}_i}) = \pi_{R(T^*_{\mathcal{G}_i})} R(T^*_{\mathcal{F}_i})$.

If in Definition 4.1 $C_{\mathcal{G}} C^*_{\mathcal{F}}$ is component preserving we say that $(\mathbf{V}, \mathbf{v}, \mathcal{G})$ is a *component preserving dual fusion frame system* of $(\mathbf{W}, \mathbf{w}, \mathcal{F})$. Moreover, if in Definition 4.1 $(\mathbf{V}, \mathbf{v})$ is a canonical dual fusion frame of $(\mathbf{W}, \mathbf{w})$ we say that $(\mathbf{V}, \mathbf{v}, \mathcal{G})$ is a *canonical dual fusion frame system* of $(\mathbf{W}, \mathbf{w}, \mathcal{F})$.

In the following subsections we show that there is a close relation of dual fusion frame systems with dual frames and dual projective reconstruction systems, a fact that supports the idea that Definition 4.1 is the proper definition of dual fusion frame systems. Consequently, this close relation also reveals that the inclusion of a "Q" in a definition of duality for fusion frames as in Definition 3.1 is natural.

4.1. **Dual fusion frame systems and dual frames.** The following theorem gives the link between the concepts of dual fusion frame system and dual frame.

Theorem 4.3. *Let* $(\boldsymbol{W}, \mathbf{w})$, $(\mathbf{V}, \mathbf{v})$ *be Bessel fusion sequences,* $|L_i| \geq \max\{\dim(W_i), \dim(V_i)\}$, $\mathcal{F}_i$ *be a frame for* W_i *and* $\mathcal{G}_i$ *be a frame for* V_i*. The following conditions are equivalent:*

(1) $\mathbf{w}\mathcal{F}$ *and* $\mathbf{v}\mathcal{G}$ *are dual frames for* $\mathcal{H}$*.*
(2) $(\mathbf{V}, \mathbf{v}, \mathcal{G})$ *is a dual fusion frame system of* $(\boldsymbol{W}, \mathbf{w}, \mathcal{F})$*.*

Proof. By Remark 2.6 it only remains to see the duality condition, and this follows from $T_{\mathbf{v}\mathcal{G}}T^*_{\mathbf{w}\mathcal{F}} = T_{\mathbf{V},\mathbf{v}}C_{\mathcal{G}}C^*_{\mathcal{F}}T^*_{\mathbf{W},\mathbf{w}}$. □

The next corollary shows how to construct component preserving dual fusion frame systems from a given fusion frame using local dual frames for each subspace and a left inverse of its analysis operator.

Corollary 4.4. *Let* $(\boldsymbol{W}, \mathbf{w})$ *be a fusion frame for* $\mathcal{H}$, $A \in \mathfrak{L}_{T^*_{\mathbf{W},\mathbf{w}}}$ *and* $\mathbf{v}$ *be a collection of weights. For each* $i = 1, \ldots, m$, *let* $V_i = AM_i\mathcal{W}$, $\{f_i^l\}_{l\in L_i}$ *and* $\{\tilde{f}_i^l\}_{l\in L_i}$ *be dual frames for* W_i, *and* $\tilde{\alpha}_i$ *and* $\tilde{\beta}_i$ *frame bounds of* $\{\tilde{f}_i^l\}_{l\in L_i}$, *and* $\mathcal{G}_i = \{\frac{1}{v_i}A(\chi_i(j)\tilde{f}_i^l)_{j=1}^m\}_{l\in L_i}$*. Then*

(1) $\mathcal{G}_i$ *is a frame for* V_i *with frame bounds* $\|T^*_{\mathbf{W},\mathbf{w}}\|^{-2}\frac{\tilde{\alpha}_i}{v_i^2}$ *and* $\|A\|^2\frac{\tilde{\beta}_i}{v_i^2}$*.*
(2) $(\mathbf{V}, \mathbf{v}, \mathcal{G})$ *is a component preserving dual fusion frame system of* $(\boldsymbol{W}, \mathbf{w}, \mathcal{F})$*. In particular, if* $A = S^{-1}_{\mathbf{W},\mathbf{w}}T_{\mathbf{W},\mathbf{w}}$ *then* $(\mathbf{V}, \mathbf{v}, \mathcal{G})$ *is a canonical dual fusion frame system of* $(\boldsymbol{W}, \mathbf{w}, \mathcal{F})$*.*

Proof. The set $\{(\chi_i(j)\tilde{f}_i^l)_{j=1}^m\}_{l\in L_i}$ is a frame for $M_i\mathcal{W}$. Thus, by [7, Corollary 5.3.2], (1) holds. If $(h_i)_{i=1}^m \in \mathcal{W}$, then $Q_{A,\mathbf{v}}(h_i)_{i=1}^m = (\frac{1}{v_i}AM_i(h_j)_{j=1}^m)_{i=1}^m = (\frac{1}{v_i}AM_i(\sum_{l\in L_j} < h_j, f_j^l > \tilde{f}_j^l)_{j=1}^m)_{i=1}^m = (\sum_{l\in L_i} < h_i, f_i^l > \frac{1}{v_i}A(\chi_i(j)\tilde{f}_i^l)_{j=1}^m)_{i=1}^m = C_{\mathcal{G}}C^*_{\mathcal{F}}(h_i)_{i=1}^m$. So (2) follows from (1) and Theorem 3.5. □

The next result exhibits a way to construct component preserving dual fusion frame systems from a given frame using a left inverse of its analysis operator.

Corollary 4.5. *Let* $\mathbf{w}$ *and* $\mathbf{v}$ *be two collections of weights. Let* $\mathbf{w}\mathcal{F}$ *be a frame for* $\mathcal{H}$ *with local frame bounds* α_i, β_i, $A \in \mathfrak{L}_{T^*_{\mathbf{w}\mathcal{F}}}$ *and* $\{\{e_i^l\}_{l\in L_i}\}_{i=1}^m$ *be the standard basis for* $\mathbb{F}^{\sum_{i=1}^m |L_i|}$*. For each* $i \in \{1, \ldots, m\}$, *let* $W_i = span\{f_i^l\}_{l\in L_i}$ *and* $V_i = span\{\frac{1}{v_i}Ae_i^l\}_{l\in L_i}$*. Set* $\mathcal{G} = \{\{\frac{1}{v_i}Ae_i^l\}_{l\in L_i}\}_{i=1}^m$*. Then*

(1) $\{\frac{1}{v_i}Ae_i^l\}_{l\in L_i}$ *is a frame for* V_i *with frame bounds* $\|T^*_{\mathbf{w}\mathcal{F}}\|^{-2}\frac{\alpha_i}{v_i^2}$ *and* $\|A\|^2\frac{\beta_i}{v_i^2}$*.*
(2) $(\mathbf{V}, \mathbf{v}, \mathcal{G})$ *is a dual fusion frame system of* $(\boldsymbol{W}, \mathbf{w}, \mathcal{F})$*.*

Proof. Part (1) follows from [7, Corollary 5.3.2]. By [7, Lemma 5.6.3], $\mathbf{w}\mathcal{F}$ and $\mathbf{v}\mathcal{G}$ are dual frames for $\mathcal{H}$. So, part (2) follows from Theorem 4.3. □

Let $f \in \mathcal{H}$. For a fusion frame system $(\mathbf{W}, \mathbf{w}, \mathcal{F})$ for $\mathcal{H}$ with local frame bounds α, β and associated local dual frames $\{\tilde{f}_i^l\}_{l\in L_i}$, $i = 1, \ldots, m$, in [6] it is considered the *centralized reconstruction*

$$f = \sum_{i=1}^m \sum_{l\in L_i} \langle f, w_i f_i^l \rangle (S^{-1}_{\mathbf{w}\mathcal{F}} w_i f_i^l) \tag{4.1}$$

and the *distributed reconstruction*

$$f = \sum_{i=1}^{m} \sum_{l \in L_i} \langle f, w_i f_i^l \rangle (S_{\mathbf{W},\mathbf{w}}^{-1} w_i \tilde{f}_i^l). \tag{4.2}$$

Let now $(\mathbf{V}, \mathbf{v}, \mathcal{G})$ be any dual fusion frame system of $(\mathbf{W}, \mathbf{w}, \mathcal{F})$. We have the reconstruction formula

$$f = \sum_{i=1}^{m} \sum_{l \in L_i} \langle f, w_i f_i^l \rangle v_i g_i^l. \tag{4.3}$$

By Corollary 4.5 with $A = S_{\mathbf{w}\mathcal{F}}^{-1} T_{\mathbf{w}\mathcal{F}}$, $(\mathbf{W}, \mathbf{w}, \mathcal{F})$ and $(S_{\mathbf{w}\mathcal{F}}^{-1}\mathbf{W}, 1, S_{\mathbf{w}\mathcal{F}}^{-1}\mathbf{w}\mathcal{F})$ are dual fusion frame systems for $\mathcal{H}$, and hence (4.1) turns out to be a particular case of (4.3). On the other hand, by Corollary 4.4 with $A = S_{\mathbf{W},\mathbf{w}}^{-1} T_{\mathbf{W},\mathbf{w}}$, $(\mathbf{W}, \mathbf{w}, \mathcal{F})$ and $(S_{\mathbf{W},\mathbf{w}}^{-1}\mathbf{W}, 1, S_{\mathbf{W},\mathbf{w}}^{-1}\mathbf{w}\widetilde{\mathcal{F}})$ are dual fusion frame systems for $\mathcal{H}$, and so (4.2) can also be seen as a particular case of (4.3). The advantage of reconstruction (4.3) is that now we can give many different representations of f according to our needs, since we have more freedom for the choice of $\{g_i^l\}_{l \in L_i}$, $i = 1, \dots, m$.

4.2. **Dual fusion frame systems and dual projective reconstruction systems.** The concept of reconstruction systems for finite-dimensional Hilbert spaces was introduced in [14]. Previously in [17] reconstruction systems for any separable Hilbert spaces were called g-frames.

Definition 4.6. Let $T_i \in L(\mathbb{F}^{n_i}, \mathcal{H})$ for $i = 1, \dots m$.

(1) The *synthesis operator* of $(T_i)_{i=1}^m$ is $T : \bigoplus_{i=1}^m \mathbb{F}^{n_i} \rightarrow \mathcal{H}$, $T(x_i)_{i=1}^m = \sum_{i=1}^m T_i x_i$ and the *analysis operator* is $T^* : \mathcal{H} \rightarrow \bigoplus_{i=1}^m \mathbb{F}^{n_i}$, $T^* f = (T_i^* f)_{i=1}^m$.
(2) The sequence $(T_i)_{i=1}^m$ is an $(m, \mathbf{n}, \mathcal{H})$*-reconstruction system* if $\operatorname{span} \cup_{i=1}^m R(T_i) = \mathcal{H}$.
(3) In case $(T_i)_{i=1}^m$ is an $(m, \mathbf{n}, \mathcal{H})$-reconstruction system, $S = \sum_{i=1}^m T_i T_i^*$ is called the *reconstruction system operator* of $(T_i)_{i=1}^m$.

An $(m, 1, \mathcal{H})$-reconstruction system is a frame. The set of $(m, \mathbf{n}, \mathcal{H})$-reconstruction systems is denoted with $\mathcal{RS}(m, \mathbf{n}, \mathcal{H})$. If $n_i = n$ for $i = 1, \dots m$, we write $(m, n, \mathcal{H})$-reconstruction system.

Definition 4.7. (cf. [15, Definition 2.5]) Let $(T_i)_{i=1}^m, (\widetilde{T}_i)_{i=1}^m \in \mathcal{RS}(m, \mathbf{n}, \mathcal{H})$. Then $(T_i)_{i=1}^m$ and $(\widetilde{T}_i)_{i=1}^m$ are dual reconstruction systems if $\widetilde{T} T^* = I_{\mathcal{H}}$.

In [15] the relation between reconstruction systems and fusion frames is established via *projective* reconstruction systems.

Definition 4.8. $(T_i)_{i=1}^m \in \mathcal{RS}(m, \mathbf{n}, \mathcal{H})$, is said to be *projective* if there exists a sequence of weights $\mathbf{w} = (w_i)_{i=1}^m \in \mathbb{R}_+^m$ such that $T_i^* T_i = w_i^2 I_{\mathbb{F}^{n_i}}$, $i = 1, \dots, m$.

If $(T_i)_{i=1}^m \in \mathcal{RS}(m, \mathbf{n}, \mathcal{H})$ is projective, then $w_i = \|T_i\|_{sp}$, $T_i T_i^* = w_i^2 P_{R(T_i)}$, $S = \sum_{i=1}^m w_i^2 P_{R(T_i)}$ and $\{(R(T_i), \|T_i\|_{sp})\}_{i=1}^m$ is a fusion frame for $\mathcal{H}$. Conversely, if $(\mathbf{W}, \mathbf{w})$ is a fusion frame for $\mathcal{H}$, then there exists a (non-unique) projective $(T_i)_{i=1}^m \in \mathcal{RS}(m, \mathbf{n}, \mathcal{H})$ such that $R(T_i) = W_i$ and $\|T_i\|_{sp} = w_i$.

The next corollary gives the relation between dual fusion frame systems and dual reconstruction systems in the projective case.

Corollary 4.9. *Let* $(\boldsymbol{W}, \mathbf{w})$ *and* $(\mathbf{V}, \mathbf{v})$ *be fusion frames for* $\mathcal{H}$*, and* $(T_i)_{i=1}^m$ *and* $(\widetilde{T}_i)_{i=1}^m$ *be projective* $(m, \mathbf{n}, \mathcal{H})$*-reconstruction systems for* $\mathcal{H}$ *such that* $R(T_i) = W_i$*,* $\|T_i\|_{sp} = w_i$*, and* $R(\widetilde{T}_i) = V_i$ *and* $\|\widetilde{T}_i\|_{sp} = v_i$*, respectively. For* $\{e_{n_i}^l\}_{l=1}^{n_i}$ *an orthonormal basis for* $\mathbb{F}^{n_i}$*, set* $\mathcal{F}_i = \{\frac{1}{w_i} T_i e_{n_i}^l\}_{l=1}^{n_i}$ *and* $\mathcal{G}_i = \{\frac{1}{v_i} \widetilde{T}_i e_{n_i}^l\}_{l=1}^{n_i}$ *for* $l = 1, \dots, n_i$*. Then the following conditions are equivalent.*

(1) $(\mathbf{V}, \mathbf{v}, \mathcal{G})$ *is a dual fusion frame system of* $(\boldsymbol{W}, \mathbf{w}, \mathcal{F})$.

(2) $(\widetilde{T}_i)_{i=1}^m$ *is a dual* $(m, \mathbf{n}, \mathcal{H})$*-reconstruction system of* $(T_i)_{i=1}^m$.

Proof. We have that $\mathcal{F}_i$ is a frame for W_i, $\mathcal{G}_i$ is a frame for V_i and $\sum_{i=1}^m \widetilde{T}_i T_i^* = T_{w\mathcal{G}} T_{w\mathcal{F}}^*$. Then the conclusion follows from Definition 4.7, Definition 2.2 and Theorem 4.3. □

In view of the relation between fusion frames and projective reconstruction systems, the study of duality of fusion frames can be done in the context of projective reconstruction systems using Definition 4.7. This approach is considered in [15] and [16]. But a dual reconstruction system of a projective reconstruction system is not always projective. In [16] the authors provide examples of projective reconstruction systems with non-projective canonical dual or without projective duals at all. These projective reconstruction systems, along with their associated fusion frames, are considered in examples 6.2 and 6.3 below. It is worth to note that with Definition 3.1 the dual of a fusion frame is always a fusion frame. Moreover, as it was shown in Section 3.1, a fusion frame has always a canonical dual fusion frame. Similar considerations for fusion frame systems follow from Definition 4.1 and Corollary 4.4.

5. Optimal dual fusion frames for erasures

Having different duals is convenient in many applications e.g. in the theory of *optimal dual fusion frames for erasures* that will be discussed in this section. In this case, (3.2) (or (4.3)) can give a reconstruction that behaves better than the ones given in (3.5) ((4.1) or (4.2)).

Let $(\mathbf{W}, \mathbf{w})$ be a fusion frame for $\mathcal{H}$. In applications an element $f \in \mathcal{H}$ (e. g. a signal) is converted into the data vectors $T_{\mathbf{W},\mathbf{w}}^* f$. In an ideal setting these vectors are transmitted and f can be reconstructed by the receiver using $f = T_{\mathbf{V},\mathbf{v}} Q T_{\mathbf{W},\mathbf{w}}^* f$, where $(\mathbf{V}, \mathbf{v})$ is some Q-dual of $(\mathbf{W}, \mathbf{w})$. But in real implementations, sometimes some of the data vectors, or part of them, are lost or erased, and it is necessary to reconstruct f with the partial information at hand. There are several approaches to study this problem, here we consider optimal dual fusion frames for a fixed fusion frame when a blind reconstruction process is used, in a similar way as in [12, 11] for frames and in [16] for projective reconstruction systems.

5.1. Optimal dual fusion frames for erasures of subspaces.

Let $J \subseteq \{1, \dots, m\}$ and suppose that the data vectors corresponding to the subspaces $\{W_j\}_{j\in J}$ are lost. The reconstruction then gives $T_{\mathbf{V},\mathbf{v}} Q M_{\{1,\dots,m\}\setminus J} T_{\mathbf{W},\mathbf{w}}^* f$. So we need to find those dual fusion frames of $(\mathbf{W}, \mathbf{w})$ that are in some sense optimal for this situation.

Fix $r \in \{1, \dots, m\}$. Let $\mathcal{P}_r^m := \{J \subseteq \{1, \dots, m\} : |J| = r\}$. Noting that $M_J = I_{\mathcal{W}} - M_{\{1,\dots,m\}\setminus J}$, given a Q-dual fusion frames $(\mathbf{V}, \mathbf{v})$ of $(\mathbf{W}, \mathbf{w})$ we consider the vector error

$$e(r, (\mathbf{W}, \mathbf{w}), (\mathbf{V}, \mathbf{v}), Q) = (\|T_{\mathbf{V},\mathbf{v}} Q M_J T_{\mathbf{W},\mathbf{w}}^*\|_F)_{J\in\mathcal{P}_r^m}.$$

For $p \in \mathbb{N} \cup \{\infty\}$ we define inductively:

$e_1^{(p)}(\mathbf{W}, \mathbf{w}) = \inf\{\|e(1, (\mathbf{W}, \mathbf{w}), (\mathbf{V}, \mathbf{v}), Q)\|_p : (\mathbf{V}, \mathbf{v})$ is a Q-dual fusion frame of $(\mathbf{W}, \mathbf{w})\}$,

$\mathcal{D}_1^{(p)}(\mathbf{W}, \mathbf{w})$ as the set of $((\mathbf{V}, \mathbf{v}), Q)$ where $(\mathbf{V}, \mathbf{v})$ is a Q-dual fusion frame of $(\mathbf{W}, \mathbf{w})$ and

$$\|e(1, (\mathbf{W}, \mathbf{w}), (\mathbf{V}, \mathbf{v}), Q)\|_p = e_1^{(p)}(\mathbf{W}, \mathbf{w}),$$

$$e_r^{(p)}(\mathbf{W}, \mathbf{w}) = \inf\{\|e(r, (\mathbf{W}, \mathbf{w}), (\mathbf{V}, \mathbf{v}), Q)\|_p : ((\mathbf{V}, \mathbf{v}), Q) \in \mathcal{D}_{r-1}^{(p)}(\mathbf{W}, \mathbf{w})\},$$

$$\mathcal{D}_r^{(p)}(\mathbf{W}, \mathbf{w}) = \{((\mathbf{V}, \mathbf{v}), Q) \in \mathcal{D}_{r-1}^{(p)}(\mathbf{W}, \mathbf{w}) : \|e(r, (\mathbf{W}, \mathbf{w}), (\mathbf{V}, \mathbf{v}), Q)\|_p = e_r^{(p)}(\mathbf{W}, \mathbf{w})\},$$

in case each $\mathcal{D}_r^{(p)}(\mathbf{W}, \mathbf{w})$, called the set of (r, p)-loss optimal dual fusion frames for $(\mathbf{W}, \mathbf{w})$, is non-empty.

Note that $M_i T_{\mathbf{W},\mathbf{w}}^* T_{\mathbf{W},\mathbf{w}} M_i^* = w_i^2 M_i$. Let $A \in L(\mathcal{W}, \mathcal{H})$, then

$$\|A M_i T_{\mathbf{W},\mathbf{w}}^*\|_F^2 = w_i^2 \|A M_i\|_F^2. \tag{5.1}$$

5.1.1. *The mean square error.* Consider the mean square error,
$$\|e(r,(\mathbf{W},\mathbf{w}),(\mathbf{V},\mathbf{v}),Q)\|_2 = (\textstyle\sum_{J\in\mathcal{P}_r^m}\|T_{\mathbf{V},\mathbf{v}}QM_JT^*_{\mathbf{W},\mathbf{w}}\|_F^2)^{1/2}.$$

The next theorem describes an $(r,2)$-loss optimal component preserving dual fusion frame of a given fusion frame. It also asserts that the reconstruction formula provided by this dual coincides with the reconstruction formula provided by any other $(r,2)$-loss optimal dual fusion frame. Furthermore, it shows that it is the only $(r,2)$-loss optimal component preserving dual fusion frame and when it coincides with a canonical dual.

Theorem 5.1. *Let* $(\boldsymbol{W},\mathbf{w})$ *be a fusion frame for* $\mathcal{H}$. *Let* $D:\mathcal{W}\to\mathcal{W}$, $D(f_i)_{i=1}^m = (\frac{1}{w_iv_i}f_i)_{i=1}^m$ *and* $S_D = T_{\boldsymbol{W},\mathbf{v}}DT^*_{\boldsymbol{W},\mathbf{w}}$. *Then* S_D *is a selfadjoint positive invertible operator and if* $Q_D:\mathcal{W}\to\bigoplus_{j=1}^m S_D^{-1}W_j$, $Q_D(f_j)_{j=1}^m = (\frac{1}{w_jv_j}S_D^{-1}f_j)_{j=1}^m$ *then*

1. $(S_D^{-1}\boldsymbol{W},\mathbf{v})$ *is an* $(r,2)$*-loss optimal component preserving* Q_D*-dual for* $(\boldsymbol{W},\mathbf{w})$.
2. *If* $(\mathbf{V},\mathbf{v})$ *is an* $(r,2)$*-loss optimal* Q*-dual fusion frame of* $(\boldsymbol{W},\mathbf{w})$, *then* $T_{\mathbf{V},\mathbf{v}}Q = T_{S_D^{-1}\boldsymbol{W},\mathbf{v}}Q_D$.
3. *If* $(\mathbf{V},\mathbf{v})$ *is an* $(r,2)$*-loss optimal* Q*-component preserving dual fusion frame of* $(\boldsymbol{W},\mathbf{w})$ *then* $\mathbf{V} = S_D^{-1}\boldsymbol{W}$ *and* $Q = Q_D$.
4. *If* $w_i = w$ *for* $i = 1,\ldots,m$, *then* $S_D^{-1}\boldsymbol{W} = S^{-1}_{\boldsymbol{W},w}\boldsymbol{W}$ *and* $Q_D = Q_{S^{-1}_{\boldsymbol{W},w}T_{\boldsymbol{W},w},\mathbf{v}}$.

Proof. Clearly, $S_D = T_{\mathbf{W},\mathbf{v}}DT^*_{\mathbf{W},\mathbf{w}} = \sum_{i=1}^m \pi_{W_i}$ is selfadjoint.

Let $f\in\mathcal{H}$. If $\alpha>0$ is the lower fusion frame bound of $(\mathbf{W},\mathbf{w})$, then
$$\langle S_Df,f\rangle = \sum_{i=1}^m w_i^{-2}w_i^2\langle\pi_{W_i}f,f\rangle \geq (\min_{i\in\{1,\ldots,m\}} w_i^{-2})\langle S_{\mathbf{W},\mathbf{w}}f,f\rangle \geq (\min_{i\in\{1,\ldots,m\}} w_i^{-2})\alpha\|f\|^2.$$

So S_D is positive and invertible. Since S_D^{-1} is linear, it is easy to see that Q_D is component preserving.

We have
$$T_{S_D^{-1}\mathbf{W},\mathbf{v}}Q_DT^*_{\mathbf{W},\mathbf{w}} = \textstyle\sum_{i=1}^m S_D^{-1}\pi_{W_i} = S_D^{-1}S_D = I_{\mathcal{H}},$$
thus $(S_D^{-1}\mathbf{W},\mathbf{v})$ is a Q_D-component preserving dual fusion frame of $(\mathbf{W},\mathbf{w})$.

Let $(\mathbf{V},\mathbf{v})$ be a fusion frame for $\mathcal{H}$. Using (5.1),
$$\begin{aligned}\|T_{\mathbf{V},\mathbf{v}}QM_iT^*_{\mathbf{W},\mathbf{w}}\|_F^2 =& w_i^2\|T_{\mathbf{V},\mathbf{v}}QM_i\|_F^2\\ =& w_i^2\|T_{S_D^{-1}\mathbf{W},\mathbf{v}}Q_DM_i\|_F^2 + w_i^2\|T_{\mathbf{V},\mathbf{v}}QM_i - T_{S_D^{-1}\mathbf{W},\mathbf{v}}Q_DM_i\|_F^2\\ &+ 2\mathrm{Re}(w_i^2\mathrm{tr}[(T_{\mathbf{V},\mathbf{v}}QM_i - T_{S_D^{-1}\mathbf{W},\mathbf{v}}Q_DM_i)M_i^*Q_D^*T^*_{S_D^{-1}\mathbf{W},\mathbf{v}}]).\end{aligned}\tag{5.2}$$

Suppose that $(\mathbf{V},\mathbf{v})$ is a Q-dual fusion frame of $(\mathbf{W},\mathbf{w})$. Using $T_{S_D^{-1}\mathbf{W},\mathbf{v}}Q_DM_i = w_i^{-2}S_D^{-1}T_{\mathbf{W},\mathbf{w}}M_i$ and $T_{\mathbf{V},\mathbf{v}}QT^*_{\mathbf{W},\mathbf{w}} = T_{S_D^{-1}\mathbf{W},\mathbf{v}}Q_DT^*_{\mathbf{W},\mathbf{w}} = I_{\mathcal{H}}$ we have
$$\begin{aligned}\sum_{i=1}^m w_i^2\mathrm{tr}[(T_{\mathbf{V},\mathbf{v}}QM_i - T_{S_D^{-1}\mathbf{W},\mathbf{v}}Q_DM_i)M_i^*Q_D^*T^*_{S_D^{-1}\mathbf{W},\mathbf{v}}] =&\\ =& \sum_{i=1}^m w_i^2w_i^{-2}\mathrm{tr}[(T_{\mathbf{V},\mathbf{v}}QM_i - T_{S_D^{-1}\mathbf{W},\mathbf{v}}Q_DM_i)M_i^*T^*_{\mathbf{W},\mathbf{w}}S_D^{-1}]\\ =& \mathrm{tr}[(T_{\mathbf{V},\mathbf{v}}Q - T_{S_D^{-1}\mathbf{W},\mathbf{v}}Q_D)T^*_{\mathbf{W},\mathbf{w}}S_D^{-1}] = 0.\end{aligned}\tag{5.3}$$

By (5.2), (5.1) and (5.3) we obtain

$$\sum_{i=1}^{m} \|T_{\mathbf{V},\mathbf{v}} Q M_i T^*_{\mathbf{W},\mathbf{w}}\|_F^2 = \sum_{i=1}^{m} \|T_{S_D^{-1}\mathbf{W},\mathbf{v}} Q_D M_i T^*_{\mathbf{W},\mathbf{w}}\|_F^2 \tag{5.4}$$
$$+ \sum_{i=1}^{m} w_i^2 \|T_{\mathbf{V},\mathbf{v}} Q M_i - T_{S_D^{-1}\mathbf{W},\mathbf{v}} Q_D M_i\|_F^2.$$

Thus, $\|e(1,(\mathbf{W},\mathbf{w}),(\mathbf{V},\mathbf{v}),Q)\|_2 \geq \|e(1,(\mathbf{W},\mathbf{w}),(S_D^{-1}\mathbf{W},\mathbf{v}),Q_D)\|_2$. So $(S_D^{-1}\mathbf{W},\mathbf{v}) \in \mathcal{D}_1^{(2)}(\mathbf{W},\mathbf{w})$.

If $(\mathbf{V},\mathbf{v}) \in \mathcal{D}_1^{(2)}(\mathbf{W},\mathbf{w})$, then $\|e(1,(\mathbf{W},\mathbf{w}),(\mathbf{V},\mathbf{v}),Q)\|_2 = \|e(1,(\mathbf{W},\mathbf{w}),(S_D^{-1}\mathbf{W},\mathbf{v}),Q_D)\|_2$, and by (5.4)

$$T_{\mathbf{V},\mathbf{v}} Q = T_{S_D^{-1}\mathbf{W},\mathbf{v}} Q_D. \tag{5.5}$$

Suppose now that Q is component preserving. By (5.5) and Lemma 3.4, $\mathbf{V} = S_D^{-1}\mathbf{W}$ and $Q = Q_D$.

By the hierarchical definitions of $\mathcal{D}_r^{(p)}(\mathbf{W},\mathbf{w})$ for $r \geq 1$, the conclusions (1)-(3) follow.

If $w_i = v_i = w$ for $i = 1,\dots,m$, then $S_D = w^{-2} S_{\mathbf{W},w}$. Thus, (4) follows. □

5.1.2. *The worst-case error.* Consider $p = \infty$. In this case, we obtain the worst-case error
$$\|e(r,(\mathbf{W},\mathbf{w}),(\mathbf{V},\mathbf{v}),Q)\|_\infty = \max_{J \in \mathcal{P}_r^m} \|T_{\mathbf{V},\mathbf{v}} Q M_{J,\mathbf{W}} T^*_{\mathbf{W},\mathbf{w}}\|_F.$$

To prove Theorem 5.3 we need the following proposition that gives some properties of the set of elements in $\mathfrak{L}_{T^*_{\mathbf{W},\mathbf{w}}}$ satisfying certain optimality condition.

Proposition 5.2. *Let $(\boldsymbol{W},\mathbf{w})$ be a fusion frame for $\mathcal{H}$. Then*

$$\{A \in \mathfrak{L}_{T^*_{\boldsymbol{W},\mathbf{w}}} : \max_{1\leq i\leq m} \|AM_i T^*_{\boldsymbol{W},\mathbf{w}}\|_F = \min_{B \in \mathfrak{L}_{T^*_{\boldsymbol{W},\mathbf{w}}}} \max_{1\leq i\leq m} \|BM_i T^*_{\boldsymbol{W},\mathbf{w}}\|_F\} \tag{5.6}$$

is non-empty, compact and convex.

Proof. The map $\|.\|_{\mathbf{W},\mathbf{w}} : L(\mathcal{W},\mathcal{H}) \to \mathbb{R}^+$, $\|A\|_{\mathbf{W},\mathbf{w}} = \max_{1\leq i\leq m} \|AM_i T^*_{\mathbf{W},\mathbf{w}}\|_F$ is a norm in $L(\mathcal{W},\mathcal{H})$. To see this, let $A \in L(\mathcal{W},\mathcal{H})$ such that $\|A\|_{\mathbf{W},\mathbf{w}} = 0$. Then $AM_i T^*_{\mathbf{W},\mathbf{w}} = 0$ for $i = 1,\dots,m$, and since $\mathrm{R}(M_i T^*_{\mathbf{W},\mathbf{w}}) = M_i\mathcal{W}$, it follows that $AM_i\mathcal{W} = \{0\}$ for $i = 1,\dots,m$. Thus $A = \sum_{i=1}^m AM_i = 0$. The other norm properties are immediate.

Since the set $\mathfrak{L}_{T^*_{\mathbf{W},\mathbf{w}}}$ is closed in $L(\mathcal{W},\mathcal{H})$ under the usual norm and all norms in a finite-dimensional Hilbert space are equivalent, $\mathfrak{L}_{T^*_{\mathbf{W},\mathbf{w}}}$ is a closed subset of $L(\mathcal{W},\mathcal{H})$ under the norm $\|.\|_{\mathbf{W},\mathbf{w}}$. Given $B_0 \in \mathfrak{L}_{T^*_{\mathbf{W},\mathbf{w}}}$, $B_0 \neq 0$, there exists an A_0 in the non-empty compact set $\{A \in \mathfrak{L}_{T^*_{\mathbf{W},\mathbf{w}}} : \|A\|_{\mathbf{W},\mathbf{w}} \leq \|B_0\|_{\mathbf{W},\mathbf{w}}\}$ where the continuous map $\|.\|_{\mathbf{W},\mathbf{w}}$ attains its minimum. So, $\|A_0\|_{\mathbf{W},\mathbf{w}} = \min_{A \in \mathfrak{L}_{T^*_{\mathbf{W},\mathbf{w}}}} \|A\|_{\mathbf{W},\mathbf{w}}$, and the set (5.6) is non-empty and compact.

Since $\mathfrak{L}_{T^*_{\mathbf{W},\mathbf{w}}}$ is convex and $\|.\|_{\mathbf{W},\mathbf{w}}$ is a convex map, the set (5.6) is convex. □

Given a fusion frame, Theorem 5.3 gives sufficient conditions that guarantee that there exists a unique (up to weights) (r,∞)-loss optimal component preserving dual fusion frame.

Theorem 5.3. *Let $(\boldsymbol{W},\mathbf{w})$ be a fusion frame for $\mathcal{H}$. If $A_0 \in \mathfrak{L}_{T^*_{\boldsymbol{W},\mathbf{w}}}$ is such that $w_i\|A_0 M_i\|_F = c$ for each $i = 1,\dots,m$, then*

1. *A_0 is the unique element of the set (5.6).*
2. *The only (r,∞)-loss optimal component preserving dual fusion frames of $(\boldsymbol{W},\mathbf{w})$ are the $Q_{A_0,\mathbf{v}}$-dual fusion frames with arbitrary vector of weights $\mathbf{v}$.*

Proof. (1) By Proposition 5.2, there exists $A \in \mathfrak{L}_{T^*_{\mathbf{W},\mathbf{w}}}$ such that
$$\max_{1\leq i\leq m} \|AM_i T^*_{\mathbf{W},\mathbf{w}}\|_F = \min_{B \in \mathfrak{L}_{T^*_{\mathbf{W},\mathbf{w}}}} \max_{1\leq i\leq m} \|BM_i T^*_{\mathbf{W},\mathbf{w}}\|_F.$$

So,

$$\max_{1\leq i\leq m}\|AM_iT^*_{\mathbf{W},\mathbf{w}}\|_F \leq \max_{1\leq i\leq m}\|A_0M_iT^*_{\mathbf{W},\mathbf{w}}\|_F,$$

and then, by hypothesis and (5.1),

$$\|AM_i\|_F \leq \|A_0M_i\|_F, \text{ for each } i \in \{1,\dots,m\}. \tag{5.7}$$

Using $\|AM_i\|_F^2 = \|A_0M_i\|_F^2 + \|(A-A_0)M_i\|_F^2 + 2\mathrm{Re}(\mathrm{tr}[(A-A_0)M_iT^*_{\mathbf{W},\mathbf{w}}S^{-1}_{\mathbf{W},\mathbf{w}}])$, by (5.7),

$$\|(A-A_0)M_i\|_F^2 + 2\mathrm{Re}(\mathrm{tr}[(A-A_0)M_iT^*_{\mathbf{W},\mathbf{w}}S^{-1}_{\mathbf{W},\mathbf{w}}]) \leq 0. \tag{5.8}$$

Since $AT^*_{\mathbf{W},\mathbf{w}} = A_0T^*_{\mathbf{W},\mathbf{w}} = I_{\mathcal{H}}$,

$$\sum_{i=1}^m \mathrm{tr}[(A-A_0)M_iT^*_{\mathbf{W},\mathbf{w}}S^{-1}_{\mathbf{W},\mathbf{w}}] = \mathrm{tr}[(A-A_0)T^*_{\mathbf{W},\mathbf{w}}S^{-1}_{\mathbf{W},\mathbf{w}}] = 0.$$

Thus, by (5.8), $\sum_{i=1}^m \|(A-A_0)M_i\|_F^2 \leq 0$, and consequently, $AM_i = A_0M_i$ for every $i \in \{1,\dots,m\}$, or equivalently, $A = A_0$.

(2) It follows from part (1), Theorem 3.5, Lemma 3.4 and the inductive definition of (r,∞)-loss optimal dual fusion frames. □

Corollary 5.4 gives sufficient conditions that assure that the only (r,∞)-loss optimal component preserving dual fusion frames are the canonical ones.

Corollary 5.4. *Let $(\boldsymbol{W},\mathbf{w})$ be a fusion frame for $\mathcal{H}$. If $w_i\|S^{-1}_{\mathbf{W},\mathbf{w}}\pi_{W_i}\|_F = c$ for each $i = 1,\dots,m$, then*

1. *$S^{-1}_{\boldsymbol{W},\mathbf{w}}T_{\boldsymbol{W},\mathbf{w}}$ is the unique element of the set (5.6).*
2. *The only (r,∞)-loss optimal component preserving dual fusion frames of $(\boldsymbol{W},\mathbf{w})$ are the canonical ones $(S^{-1}_{\boldsymbol{W},\mathbf{w}}\boldsymbol{W},\mathbf{v})$ with arbitrary vector of weights $\mathbf{v}$.*

Proof. Since $\|S^{-1}_{\mathbf{W},\mathbf{w}}T_{\mathbf{W},\mathbf{w}}M_iT^*_{\mathbf{W},\mathbf{w}}\|_F = w_i\|S^{-1}_{\mathbf{W},\mathbf{w}}\pi_{W_i}\|_F$, the proof follows from Theorem 5.1. □

Corollary 5.5. *Let $(\boldsymbol{W},\mathbf{w})$ be a Parseval fusion frame for $\mathcal{H}$. If $w_i\dim(W_i)^{1/2} = c$ for each $i = 1,\dots,m$, then*

1. *$T_{\boldsymbol{W},\mathbf{w}}$ is the unique element of the set (5.6).*
2. *The only (r,∞)-loss optimal component preserving dual fusion frames of $(\boldsymbol{W},\mathbf{w})$ are the canonical ones $(\boldsymbol{W},\mathbf{v})$ with arbitrary vector of weights $\boldsymbol{v}$.*

Proof. By hypothesis, $\|T_{\mathbf{W},\mathbf{w}}M_iT^*_{\mathbf{W},\mathbf{w}}\|_F = w_i\|\pi_{W_i}\|_F = w_i\dim(W_i)^{1/2} = c$ for each $i = 1,\dots,m$, so the proof follows from the previous corollary. □

Remark 5.6. By Corollary 5.5, the only (r,∞)-loss optimal component preserving dual fusion frames of a uniform equi-dimensional Parseval fusion frame $(\mathbf{W},\mathbf{w})$ are the canonical ones $(\mathbf{W},\mathbf{v})$ with arbitrary vector of weights $\mathbf{v}$.

In Example 6.3, we are going to see a fusion frame that has a unique (up to weights) loss optimal Q-component preserving dual fusion frame with the same subspaces as the canonical dual, but with $Q \neq Q_{S^{-1}_{\mathbf{W},\mathbf{w}}T_{\mathbf{W},\mathbf{w}},\mathbf{v}}$ and therefore it gives another reconstruction formula.

5.2. **Optimal dual fusion frame systems for erasures of local frame vectors.** We will analyze now the situation where some local frame vectors are lost. In this case we consider $J_i \subseteq L_i$, for each $i = 1,\dots,m$, $\mathcal{J} = (J_1,\dots,J_m)$ and $|\mathcal{J}| = \sum_{i=1}^m |J_i|$. Let $M_{\mathcal{J}} \in L(\mathbb{F}^{\sum_{i=1}^m |L_i|}, \mathbb{F}^{\sum_{i=1}^m |L_i|})$ be the self-adjoint operator given by $M_{\mathcal{J}}((x_i^l)_{l\in L_i})_{i=1}^m = ((\chi_{J_i}(l)x_i^l)_{l\in L_i})_{i=1}^m$.

Fix $r \in \{1,\dots,\sum_{i=1}^m |L_i|\}$. Let $\mathcal{P}_r^{\times_{i=1}^m L_i} = \{\mathcal{J} : |\mathcal{J}| = r\}$. By similar considerations to those in section 5.1, we consider the vector error

$$e(r,(\mathbf{W},\mathbf{w},\mathcal{F}),(\mathbf{V},\mathbf{v},\mathcal{G})) = (\|T_{\mathbf{V},\mathbf{v}}C_{\mathcal{G}}M_{\mathcal{J}}C^*_{\mathcal{F}}T^*_{\mathbf{W},\mathbf{w}}\|_F)_{\mathcal{J}\in\mathcal{P}_r^{\times_{i=1}^m L_i}},$$

and define inductively

$$e_1^{(p)}(\mathbf{W},\mathbf{w},\mathcal{F}) = \inf\{\|e(1,(\mathbf{W},\mathbf{w},\mathcal{F}),(\mathbf{V},\mathbf{v},\mathcal{G}))\|_p : (\mathbf{V},\mathbf{v},\mathcal{G}) \text{ is a dual fusion frame system of } (\mathbf{W},\mathbf{w},\mathcal{F})\},$$

$\mathcal{D}_1^{(p)}(\mathbf{W},\mathbf{w},\mathcal{F})$ as the set of dual fusion frame systems $(\mathbf{V},\mathbf{v},\mathcal{G})$ of $(\mathbf{W},\mathbf{w},\mathcal{F})$ with

$$\|e(1,(\mathbf{W},\mathbf{w},\mathcal{F}),(\mathbf{V},\mathbf{v},\mathcal{G}))\|_p = e_1^{(p)}(\mathbf{W},\mathbf{w},\mathcal{F}),$$

$$e_r^{(p)}(\mathbf{W},\mathbf{w},\mathcal{F}) = \inf\{\|e(r,(\mathbf{W},\mathbf{w},\mathcal{F}),(\mathbf{V},\mathbf{v},\mathcal{G}))\|_p : (\mathbf{V},\mathbf{v},\mathcal{G}) \in \mathcal{D}_{r-1}^{(p)}(\mathbf{W},\mathbf{w},\mathcal{F})\},$$

$$\mathcal{D}_r^{(p)}(\mathbf{W},\mathbf{w},\mathcal{F}) = \{(\mathbf{V},\mathbf{v},\mathcal{G}) \in \mathcal{D}_{r-1}^{(p)}(\mathbf{W},\mathbf{w},\mathcal{F}) : \|e(r,(\mathbf{W},\mathbf{w},\mathcal{F}),(\mathbf{V},\mathbf{v},\mathcal{G}))\|_p = e_r^{(p)}(\mathbf{W},\mathbf{w},\mathcal{F})\},$$

in case each $\mathcal{D}_r^{(p)}(\mathbf{W},\mathbf{w},\mathcal{F})$, called the set of (r,p)-loss optimal dual fusion frames for $(\mathbf{W},\mathbf{w},\mathcal{F})$, is non-empty.

In the following we consider the cases $p=2$ and $p=\infty$ obtaining results that are analogous to the ones viewed in Section 5.1. For this, let $\mathcal{J} \in \mathcal{P}_1^{\times_{i=1}^m L_i}$ where $J_l = \emptyset$ if $l \neq i$ and $J_i = \{j\}$. Then $M_{\mathcal{J}} C_{\mathcal{F}}^* T_{\mathbf{W},\mathbf{w}}^* T_{\mathbf{W},\mathbf{w}} C_{\mathcal{F}} M_{\mathcal{J}}^* = w_i^2 \|f_i^j\|^2 M_{\mathcal{J}}$, and if $A \in L(\mathcal{W},\mathcal{H})$,

$$\|AM_{\mathcal{J}} C_{\mathcal{F}}^* T_{\mathbf{W},\mathbf{w}}^*\|_F^2 = w_i^2 \|f_i^j\|^2 \|AM_{\mathcal{J}}\|_F^2. \tag{5.9}$$

5.2.1. *The mean square error.* Considering $p=2$ we obtain the mean square error,

$$\|e(r,(\mathbf{W},\mathbf{w},\mathcal{F}),(\mathbf{V},\mathbf{v},\mathcal{G}))\|_2 = \left(\sum_{\mathcal{J}\in\mathcal{P}_r^{\times_{i=1}^m L_i}} \|T_{\mathbf{V},\mathbf{v}} C_{\mathcal{G}} M_{\mathcal{J}} C_{\mathcal{F}}^* T_{\mathbf{W},\mathbf{w}}^*\|_F^2\right)^{1/2}.$$

The following theorem about optimal $(r,2)$-loss optimal dual fusion frame systems is similar to Theorem 5.1.

Theorem 5.7. *Let $(\boldsymbol{W},\mathbf{w},\mathcal{F})$ be a fusion frame system for $\mathcal{H}$ where each element in $\mathcal{F}$ has norm equal to 1. Let $\mathcal{G}_c = \{\frac{1}{w_i v_i} S_{\mathcal{F}}^{-1}\mathcal{F}_i\}_{i=1}^m$ and $\mathcal{G}_{c,i} = \frac{1}{w_i v_i} S_{\mathcal{F}}^{-1}\mathcal{F}_i$. Then*

1. *$(S_{\mathcal{F}}^{-1}\boldsymbol{W},\mathbf{v},\mathcal{G}_c)$ is an $(r,2)$-loss optimal component preserving dual fusion frame system for $(\boldsymbol{W},\mathbf{w},\mathcal{F})$.*
2. *If $(\mathbf{V},\mathbf{v},\mathcal{G})$ is an $(r,2)$-loss optimal dual fusion frame system of $(\boldsymbol{W},\mathbf{w},\mathcal{F})$, then $T_{\mathbf{V},\mathbf{v}}C_{\mathcal{G}} = T_{S_{\mathcal{F}}^{-1}\mathbf{W},\mathbf{v}} C_{\mathcal{G}_c}$.*
3. *If $(\mathbf{V},\mathbf{v},\mathcal{G})$ is an $(r,2)$-loss optimal component preserving dual fusion frame system of $(\boldsymbol{W},\mathbf{w},\mathcal{F})$ then $\mathbf{V} = S_{\mathcal{F}}^{-1}\boldsymbol{W}$ and $\mathcal{G} = \mathcal{G}_c$.*

Proof. By Remark 2.6, $\mathbf{w}\mathcal{F}$ is a frame for $\mathcal{H}$, so $\mathcal{F}$ is also a frame for $\mathcal{H}$ and $S_{\mathcal{F}}$ is a selfadjoint positive invertible operator.

Note that $T_{\mathcal{G}_{c,i}} = \frac{1}{w_i v_i} S_{\mathcal{F}}^{-1} T_{\mathcal{F}_i}$, so $R(T_{\mathcal{G}_{c,i}}^*) \subseteq R(T_{\mathcal{F}_i}^*)$ and, by Remark 4.2, $C_{\mathcal{G}_c} C_{\mathcal{F}}^*$ is component preserving.

We also have $T_{S_{\mathcal{F}}^{-1}\mathbf{W},\mathbf{v}} C_{\mathcal{G}_c} C_{\mathcal{F}}^* T_{\mathbf{W},\mathbf{w}}^* = \sum_{i=1}^m S_{\mathcal{F}}^{-1} T_{\mathcal{F}_i} T_{\mathcal{F}_i}^* = S_{\mathcal{F}}^{-1} S_{\mathcal{F}} = I_{\mathcal{H}}$, thus $(S_{\mathcal{F}}^{-1}\mathbf{W},\mathbf{v})$ is a component preserving dual fusion frame system of $(\mathbf{W},\mathbf{w})$, so (1) follows.

Using (5.9), the rest of the proof is similar to that of Theorem 5.1. □

5.2.2. *The worst case error.* For $p=\infty$ we have the worst-case error,

$$\|e(r,(\mathbf{W},\mathbf{w},\mathcal{F}),(\mathbf{V},\mathbf{v},\mathcal{G}))\|_\infty = \max_{\mathcal{J}\in\mathcal{P}_r^{\times_{i=1}^m L_i}} \|T_{\mathbf{V},\mathbf{v}} C_{\mathcal{G}} M_{\mathcal{J}} C_{\mathcal{F}}^* T_{\mathbf{W},\mathbf{w}}^*\|_F.$$

The following results are analogous to Proposition 5.2, Theorem 5.3, Corollary 5.4 and Corollary 5.5, respectively. Their proofs follow similar lines, so we omit them.

Proposition 5.8. *Let $(\boldsymbol{W},\mathbf{w},\mathcal{F})$ be a fusion frame system for $\mathcal{H}$. Then*

$$\{A \in \mathfrak{L}_{C_{\mathcal{F}}^* T_{\boldsymbol{W},\mathbf{w}}^*} : \max_{\mathcal{J}\in\mathcal{P}_1^{\times_{i=1}^m L_i}} \|AM_{\mathcal{J}} C_{\mathcal{F}}^* T_{\boldsymbol{W},\mathbf{w}}^*\|_F = \min_{B\in\mathfrak{L}_{T_{\boldsymbol{W},\mathbf{w}}^*}} \max_{\mathcal{J}\in\mathcal{P}_1^{\times_{i=1}^m L_i}} \|BM_{\mathcal{J}} C_{\mathcal{F}}^* T_{\boldsymbol{W},\mathbf{w}}^*\|_F\} \tag{5.10}$$

is non-empty, compact and convex.

Theorem 5.9. *Let $(\boldsymbol{W},\mathbf{w},\mathcal{F})$ be a fusion frame system for $\mathcal{H}$. If $(\mathbf{V},\mathbf{v},\mathcal{G})$ is a dual fusion frame system of $(\boldsymbol{W},\mathbf{w},\mathcal{F})$ such that $w_i\|f_{ij}\|\ \|T_{V,\mathbf{v}} C_{\mathcal{G}} M_{\mathcal{J}}\|_F = c$ for each $\mathcal{J} \in \mathcal{P}_1^{\times_{i=1}^m L_i}$ with $J_l = \emptyset$ if $l \neq i$ and $J_i = \{j\}$, then*

1. *$T_{\mathbf{V},\mathbf{v}} C_{\mathcal{G}}$ is the unique element of the set (5.10).*
2. *$(\mathbf{V},\mathbf{v},\mathcal{G})$ is the unique (r,∞)-loss optimal dual fusion frame system of $(\boldsymbol{W},\mathbf{w},\mathcal{F})$.*

Corollary 5.10. *Let $(\boldsymbol{W}, \mathbf{w}, \mathcal{F})$ be a fusion frame system for $\mathcal{H}$. If $w_i\|f_i^j\| \, \|S_{\mathbf{w}\mathcal{F}}^{-1}\pi_{span\{f_i^j\}}\|_F = c$ for each $i = 1, \ldots, m$, then*

(1) *$T_{S_{\mathbf{w}\mathcal{F}}^{-1}W,\mathbf{w}}C_{S_{\mathbf{w}\mathcal{F}}^{-1}\mathcal{F}}$ is the unique element of the set (5.10).*
(2) *$(S_{\mathbf{w}\mathcal{F}}^{-1}\boldsymbol{W}, \mathbf{w}, S_{\mathbf{w}\mathcal{F}}^{-1}\mathcal{F})$ is the unique (r, ∞)-loss optimal component preserving dual fusion frame system of $(\boldsymbol{W}, \mathbf{w}, \mathcal{F})$.*

Corollary 5.11. *Let $(\boldsymbol{W}, \mathbf{w}, \mathcal{F})$ be a fusion frame for $\mathcal{H}$. If $\mathbf{w}\mathcal{F}$ is Parseval and $w_i\|f_i^j\| = c$ for each $i = 1, \ldots, m$, then*

(1) *$T_{\boldsymbol{W},\mathbf{w}}C_{\mathcal{F}}$ is the unique element of the set (5.10).*
(2) *$(\boldsymbol{W}, \mathbf{v}, \mathcal{F})$ is the unique (r, ∞)-loss optimal component preserving dual fusion frame system of $(\boldsymbol{W}, \mathbf{w}, \mathcal{F})$.*

A fusion frame system that has a unique $(r, 2)$-loss optimal Q-component preserving dual fusion frame system with the same subspaces as the canonical dual, but with $Q \neq Q_{S_{\mathbf{W},\mathbf{w}}^{-1}T_{\mathbf{W},\mathbf{w}},\mathbf{v}}$, will be given in Example 6.3. In this case the optimal dual provides another reconstruction formula than the canonical dual.

In [12] and [11], the spectral norm is used instead of the Frobenius norm in the definition of the worst-case error. We prefer the Frobenius norm in accordance with the study made in Section 5.1. Both worst-case errors coincide for $r = 1$. So, by the used hierarchical definition and the relation between dual fusion frame systems and dual frames, provided by Theorem 4.3, we conclude that we can obtain examples for Theorem 5.9, Corollary 5.10 and Corollary 5.11 from the examples for the corresponding results in [12] and [11] (see Example 6.4).

6. Examples

Example 6.1. In [8] a Bessel fusion sequence $(\mathbf{V}, \mathbf{v})$ is called an alternate dual of the fusion frame $(\mathbf{W}, \mathbf{w})$ if for all $f \in \mathcal{H}$

$$f = \sum_{i=1}^{m} w_i v_i \pi_{V_i} S_{\mathbf{W},\mathbf{w}}^{-1} \pi_{W_i}(f). \tag{6.1}$$

If $A : \mathcal{W} \to \mathcal{H}$, $A(f_i)_{i=1}^m = \sum_{i=1}^m v_i \pi_{V_i} S_{\mathbf{W},\mathbf{w}}^{-1} f_i$, then by (6.1) $A \in \mathfrak{L}_{T_{\mathbf{W},\mathbf{w}}^*}$. We have $\widetilde{V}_i := AM_i\mathcal{W} = \pi_{V_i}S_{\mathbf{W},\mathbf{w}}^{-1}W_i$ and $Q_{A,\mathbf{v}} : \mathcal{W} \to \widetilde{\mathcal{V}}$, $Q_{A,\mathbf{v}}(f_i)_{i=1}^m = (\pi_{V_i}S_{\mathbf{W},\mathbf{w}}^{-1}f_i)_{i=1}^m$. By Theorem 3.5, $(\widetilde{\mathbf{V}}, \mathbf{v})$ is a $Q_{A,\mathbf{v}}$-component preserving dual fusion frame of $(\mathbf{W}, \mathbf{w})$. By Lemma 3.3, (6.1) can be written using this dual fusion frame as $f = T_{\widetilde{\mathbf{V}},\mathbf{v}}Q_{A,\mathbf{v}}T_{\mathbf{W},\mathbf{w}}^*f$.

Example 6.2. Let $\mathcal{H} = \mathbb{C}^4$, $w_1 > 0$, $w_2 > 0$, $W_1 = \{(x_1, x_2, 0, 0) : x_1, x_2 \in \mathbb{C}\}$ and $W_2 = \{(0, x_2, x_3, -x_2) : x_2, x_3 \in \mathbb{C}\}$. Then $(\mathbf{W}, \mathbf{w})$ is a 2-equi-dimensional Riesz fusion basis for $\mathbb{C}^4$ and so its unique component preserving duals are the canonical ones. Set $w_1 = w_2 = 1$.

(a) Although $(\mathbf{W}, 1)$ is a Riesz fusion basis, it is possible to construct a dual fusion frame which is different from the canonical ones. For this, let

$$\mathcal{F}_1 = \{(1,0,0,0), (0,1,0,0), (1,0,0,0)\},\ \mathcal{F}_2 = \{(0,1,0,-1), (0,0,1,0), (0,0,1,0)\},$$
$$\mathcal{G}_1 = \{(\tfrac{1}{2}, \tfrac{1}{2}, -\tfrac{1}{2}, 0), (0,1,0,1), (\tfrac{1}{2}, -\tfrac{1}{2}, \tfrac{1}{2}, 0)\},\ \mathcal{G}_2 = \{(0,0,0,-1), (\tfrac{1}{2}, -\tfrac{1}{2}, \tfrac{1}{2}, 0), (-\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, 0)\}.$$

Then $\{\mathbf{W}, \mathbf{w}, \mathcal{F}\}$ is a fusion frame system for $\mathbb{C}^4$ and $\mathcal{G}$ is a dual frame of $\mathcal{F}$ that is not the canonical one.

Let $V_i = \text{span}\mathcal{G}_i$, $i = 1, 2$. By Theorem 4.3, $(\mathbf{V}, 1, \mathcal{G})$ is a dual fusion frame system of $(\mathbf{W}, 1, \mathcal{F})$.

Note that $C_{\mathcal{G}}C_{\mathcal{F}}^* : \mathcal{W} \to \mathcal{V}, C_{\mathcal{G}}C_{\mathcal{F}}^*((x_1, x_2, 0, 0), (0, y_2, y_3, -y_2)) = ((x_1, x_2, 0, x_2), (0, 0, y_3, -2y_2))$ is block-diagonal but not component preserving.

Since $\dim(V_i) = 3 > \dim(W_i) = 2$, $i = 1, 2$, $(\mathbf{V}, 1)$ gives a dual fusion frame which is different from the canonical one, moreover, it is not a Riesz fusion basis.

(b) Consider $T_1 : \mathbb{C}^2 \to \mathbb{C}^4$, $T_1(x_1, x_2) = (x_1, x_2, 0, 0)$ and $T_2 : \mathbb{C}^2 \to \mathbb{C}^4$, $T_2(x_1, x_2) = (0, \frac{1}{\sqrt{2}}x_2, x_1, -\frac{1}{\sqrt{2}}x_2)$. Then (T_1, T_2) is a projective Riesz $(2, 2, \mathbb{C}^4)$-reconstruction system associated with $(\mathbf{W}, \mathbf{w})$ with a unique dual, the canonical one, that is not projective (see [16], Example 5.4).

Example 6.3. Let $\mathcal{H} = \mathbb{F}^3$, $W_1 = \{(1, 0, 0)\}^\perp$, $W_2 = \{(0, 1, 0)\}^\perp$, $w_1 > 0$ and $w_2 > 0$. Then $(\mathbf{W}, \mathbf{w})$ is a 2-equi-dimensional fusion frame for $\mathbb{F}^3$ with $S^{-1}_{\mathbf{W},\mathbf{w}}(x_1, x_2, x_3) = (\frac{x_1}{w_2^2}, \frac{x_2}{w_1^2}, \frac{x_3}{w_1^2+w_2^2})$.

(a) Any $A \in \mathfrak{L}_{T^*_{\mathbf{W},\mathbf{w}}}$ is given by

$$A\left((0, x_2, x_3), (y_1, 0, y_3)\right) =$$
$$= \left(r_{11}x_3 + \frac{y_1}{w_2} + r_{12}y_3, \frac{x_2}{w_1} + r_{21}x_3 + r_{22}y_3, (\frac{w_1}{w_1^2 + w_2^2} + r_{31})x_3 + (\frac{w_2}{w_1^2 + w_2^2} + r_{32})y_3\right),$$

where $r_{i1}w_1 + r_{i2}w_2 = 0$ for $i = 1, 2, 3$.

By Theorem 3.5, any $Q_{A,\mathbf{v}}$-component preserving dual fusion frame has arbitrary weights v_1, v_2 and subspaces

$$V_1 = AM_1\mathcal{W} = \text{span}\{(0, 1, 0), (r_{11}, r_{21}, \tfrac{w_1}{w_1^2+w_2^2} + r_{31})\},$$
$$V_2 = AM_2\mathcal{W} = \text{span}\{(1, 0, 0), (r_{12}, r_{22}, \tfrac{w_2}{w_1^2+w_2^2} + r_{32})\}.$$

We have $Q_{S^{-1}_{\mathbf{W},\mathbf{w}}T_{\mathbf{W},\mathbf{w}},\mathbf{v}}((0, x_2, x_3), (y_1, 0, y_3)) = (\frac{1}{v_1}(0, \frac{x_2}{w_1}, \frac{w_1x_3}{w_1^2+w_2^2}), \frac{1}{v_2}(\frac{y_1}{w_2}, 0, \frac{w_2y_3}{w_1^2+w_2^2}))$. We note that $\{(W_1, w_1), (W_2, w_2)\}$ is not Parseval and has the same subspaces as its canonical dual.

(b) Let $Q : \mathcal{W} \to \mathcal{V}$, $Q((0, x_2, x_3), (y_1, 0, y_3)) = (\frac{1}{v_1}(0, \frac{x_2}{w_1}, \frac{x_3}{2w_1}), \frac{1}{v_2}(\frac{y_1}{w_2}, 0, \frac{y_3}{2w_2}))$.

If S_D and Q_D are as in Theorem 5.1, then $S_D^{-1}\mathbf{W} = \mathbf{W}$ and $Q_D = Q$. By Theorem 5.1(1), $(\mathbf{W}, \mathbf{v})$ is an $(r, 2)$-loss optimal Q-component preserving dual fusion frame of $(\mathbf{W}, \mathbf{w})$.

The unique element in the set (5.6) is given by $A((0, x_2, x_3), (y_1, 0, y_3)) = (\frac{y_1}{w_2}, \frac{x_2}{w_1}, \frac{1}{2}(\frac{x_3}{w_1} + \frac{y_3}{w_2}))$. In this case, $V_1 = W_1$, $V_2 = W_2$ and $Q_{A,\mathbf{v}} = Q$. Therefore, $(\mathbf{W}, \mathbf{v})$ is the unique (up to weights) (r, ∞)-loss optimal Q-component preserving dual fusion frame of $(\mathbf{W}, \mathbf{w})$.

(c) Let $\mathcal{F}_1 = \{(0, 0, 1), (0, \frac{3}{2}, -\frac{1}{2}), -(0, \frac{\sqrt{3}}{2}, \frac{1}{2})\}$ and $\mathcal{F}_2 = \{(0, 0, 1), (\frac{\sqrt{3}}{2}, 0, -\frac{1}{2}), -(\frac{\sqrt{3}}{2}, 0, \frac{1}{2})\}$. Then $\mathcal{F}_i$ is a unit norm $\frac{3}{2}$-tight frame for W_i, $i = 1, 2$, $(\mathbf{W}, \mathbf{w}, \mathcal{F})$ is a fusion frame system for $\mathbb{F}^4$ and $\mathcal{F}$ is a frame for $\mathbb{F}^4$ with $S_\mathcal{F}(x_1, x_2, x_3) = (\frac{3}{2}x_1, \frac{3}{2}x_2, 3x_3)$.

Let $\mathcal{G}_c = \{\{g_i^l\}_{l=1}^3\}_{i=1}^2$ be as in Theorem 5.7, i.e.

$$g_1^1 = \tfrac{1}{w_1v_1}(0, 0, \tfrac{1}{3}),\ g_1^2 = \tfrac{1}{w_1v_1}(0, \tfrac{\sqrt{3}}{3}, \tfrac{-1}{6}),\ g_1^3 = \tfrac{1}{w_1v_1}(0, \tfrac{-\sqrt{3}}{3}, \tfrac{-1}{6}),$$
$$g_2^1 = \tfrac{1}{w_2v_2}(0, 0, \tfrac{1}{3}),\ g_2^2 = \tfrac{1}{w_2v_2}(\tfrac{\sqrt{3}}{3}, 0, \tfrac{-1}{6}),\ g_2^3 = \tfrac{1}{w_2v_2}(\tfrac{-\sqrt{3}}{3}, 0, \tfrac{-1}{6}).$$

By Theorem 5.7, $(\mathbf{W}, \mathbf{v}, \mathcal{G}_c)$ is the unique (up to weights) $(r, 2)$-loss optimal component preserving dual fusion frame system of $(\mathbf{W}, \mathbf{w}, \mathcal{F})$ and $C_{\mathcal{G}_c}C^*_\mathcal{F} = Q$.

If $w_1 \neq w_2$, then $Q \neq Q_{S^{-1}_{\mathbf{W},\mathbf{w}}T_{\mathbf{W},\mathbf{w}},\mathbf{v}}$ and $T_{\mathbf{W},\mathbf{v}}Q \neq T_{\mathbf{W},\mathbf{v}}Q_{S^{-1}_{\mathbf{W},\mathbf{w}}T_{\mathbf{W},\mathbf{w}},\mathbf{v}}$. So, by the analysis done in (b) and (c), $(\mathbf{W}, \mathbf{w})$ has loss optimal duals with the same subspaces as the canonical ones but that provide different reconstruction formulas than the canonical duals.

(d) Let $T_1 : \mathbb{F}^2 \to \mathbb{F}^3$, $T_1(x_1, x_2) = (0, x_1, x_2)$, and $T_2 : \mathbb{F}^2 \to \mathbb{F}^3$, $T_2(x_1, x_2) = (x_1, 0, x_2)$. Then (T_1, T_2) is a projective $(2, 2, \mathbb{F}^3)$-reconstruction system associated with $(\mathbf{W}, \mathbf{w})$. This reconstruction system is considered in [16, Example 5.1], where it is shown that if $\mathbb{F} = \mathbb{C}$, (T_1, T_2) has projective duals but the canonical dual is not projective, and if $\mathbb{F} = \mathbb{R}$, (T_1, T_2) has not projective duals.

Example 6.4. Let $\mathcal{H} = \mathbb{F}^3$, $\mathcal{F}_1 = \{(1, 0, 0), (0, 1, 0), (-2, 1, 1)\}$, $W_1 = \text{span}\,\mathcal{F}_1$, $\mathcal{F}_2 = \{(1, -2, -1)\}$, $W_2 = \text{span}\,\mathcal{F}_2$, $\mathcal{G}_1 = \{(\frac{22-\sqrt{74}}{20}, \frac{2-\sqrt{74}}{20}, \frac{3}{2}), (\frac{2-\sqrt{74}}{20}, \frac{22-\sqrt{74}}{20}, -\frac{3}{2}), (\frac{2-\sqrt{74}}{20}, \frac{2-\sqrt{74}}{20}, \frac{1}{2})\}$, $V_1 = \text{span}\,\mathcal{G}_1$, $\mathcal{G}_2 = \{(\frac{2-\sqrt{74}}{20}, \frac{2-\sqrt{74}}{20}, -\frac{1}{2})\}$ and $V_2 = \text{span}\,\mathcal{G}_2$.

Then $(\mathbf{W}, 1, \mathcal{F})$ is a fusion frame system for $\mathcal{H}$, $V_1 = S_{\mathbf{W},1}^{-1} W_1$ and $V_2 \neq S_{\mathbf{W},1}^{-1} W_2$. By [11, Example 3.5], $(\mathbf{V}, 1, \mathcal{G})$ is the (r, ∞)-loss optimal dual fusion frame system of $(\mathbf{W}, 1, \mathcal{F})$, and it is different from the canonical ones.

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