Anisotropic mesh refinement in polyhedral domains: error estimates with data in $L^2(\Omega)$

Thomas Apel* Ariel L. Lombardi† Max Winkler‡

March 13, 2013

Abstract. The paper is concerned with the finite element solution of the Poisson equation with homogeneous Dirichlet boundary condition in a three-dimensional domain. Anisotropic, graded meshes from a former paper are reused for dealing with the singular behaviour of the solution in the vicinity of the non-smooth parts of the boundary. The discretization error is analyzed for the piecewise linear approximation in the $H^1(\Omega)$- and $L^2(\Omega)$-norms by using a new quasi-interpolation operator. This new interpolant is introduced in order to prove the estimates for $L^2(\Omega)$-data in the differential equation which is not possible for the standard nodal interpolant. These new estimates allow for the extension of certain error estimates for optimal control problems with elliptic partial differential equation and for a simpler proof of the discrete compactness property for edge elements of any order on this kind of finite element meshes.

Key words. Elliptic boundary value problem, edge and vertex singularities, finite element method, anisotropic mesh grading, optimal control problem, discrete compactness property.

AMS subject classifications. 65N30.

1. Introduction

We consider the homogeneous Dirichlet problem for the Laplace equation,

$$-\Delta u = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \tag{1.1}$$

where Ω is a polyhedral domain. Note that we could consider a more general elliptic equation of second order. But by a linear change of the independent variables the main part of the differential operator could be transformed to the Laplace operator in another polyhedral domain such that it is sufficient to consider the Laplace operator here.

The aim of the paper is to prove the discretization error estimate

$$\|u - u_h\|_{H^1(\Omega)} \le Ch\|f\|_{L^2(\Omega)} \tag{1.2}$$

*Institut für Mathematik und Bauinformatik, Universität der Bundeswehr München, Germany. thomas.apel@unibw.de

†Departamento de Matemática, Universidad de Buenos Aires, and Instituto de Ciencias, Universidad Nacional de General Sarmiento. Member of CONICET Argentina. aldoc7@dm.uba.ar

‡Institut für Mathematik und Bauinformatik, Universität der Bundeswehr München, Germany. max.winkler@unibw.de

for the finite element solution $u_h \in V_h$ which is constructed by using piecewise linear and continuous functions on a family of appropriate finite element meshes $\mathcal{T}_h$. Note that we assume here not more than $f \in L^2(\Omega)$ such that the L^2-error estimate

$$\|u - u_h\|_{L^2(\Omega)} \le Ch^2 \|f\|_{L^2(\Omega)} \tag{1.3}$$

follows by the Aubin–Nitsche method immediately. The generic constant C may have different values on each occurrence.

If the solution of the boundary value problem (1.1) was in $H^2(\Omega)$ then the finite element meshes could be chosen quasi-uniform, and the error estimates (1.2) and (1.3) would be standard. However, if the domain Ω is non-convex, the solution will in general contain vertex and edge singularities, that means $u \notin H^2(\Omega)$. In this case the convergence order is reduced in comparison with (1.2) and (1.3) when quasi-uniform meshes are used. As a remedy, we focus here on a priori anisotropic mesh grading techniques as they were investigated by Apel and Nicaise in [4]. In comparison with isotropic local mesh refinement, the use of anisotropic elements avoids an unnecessary refinement along the edges.

The estimate (1.2) is in general proven by using the Céa lemma (or the best approximation property of the finite element method),

$$\|u - u_h\|_{H^1(\Omega)} \le C \inf_{v_h \in V_h} \|u - v_h\|_{H^1(\Omega)}, \tag{1.4}$$

and by proving an interpolation error estimate as an upper bound for the right-hand side of (1.4). The particular difficulty is that when the Lagrange interpolant is used together with anisotropic mesh grading, then the local interpolation error estimate

$$|u - I_h u|_{W^{1,p}(T)} \le h_T |u|_{W^{2,p}(T)} \tag{1.5}$$

does not hold for $p = 2$ but only for $p > 2$, see [2]. Hence the classical proof of a finite element error estimate via

$$\|u - u_h\|_{H^1(\Omega)} \le C\|u - I_h u\|_{H^1(\Omega)} \le C \left(\sum_{T \in \mathcal{T}_h} h_T |u|^2_{H^2(T)} \right)^{1/2}$$

does not work. This problem was overcome by Apel and Nicaise, [4], by using (1.5) and related estimates in weighted spaces, as well as the Hölder inequality for the prize that $f \in L^p(\Omega)$ with $p > 2$ has to be assumed in problem (1.1). Hence estimate (1.2) cannot be proved in this way.

For prismatic domains and tensor product type meshes the problem was overcome in [1, 6] by proving local estimates for a certain quasi-interpolation operator. This work cannot be easily extended to general polyhedral domains since the orthogonality of certain edges of the elements was used there. The aim of the current paper is to construct a quasi-interpolation operator D_h such that the error estimate

$$\|u - D_h u\|_{H^1(\Omega)} \le Ch \|f\|_{L^2(\Omega)} \tag{1.6}$$

can be proved for the anisotropic meshes introduced in [4].

Quasi-interpolants were introduced by Clément [14]. The idea is to replace nodal values by certain averaged values such that non-smooth functions can be interpolated. This original idea has been modified by many authors since then. The contribution by Scott and Zhang [30] was most influential to our work.

The plan of the paper is as follows. In Section 2 we introduce notation, recall regularity results for the solution u of (1.1) and describe the finite element discretization. The main results are proved in Section 3. The paper continues with numerical results in Section 4 and ends with two sections where we describe applications which motivated us to improve the approximation result from $\|u - u_h\|_{H^1(\Omega)} \le Ch\|f\|_{L^p(\Omega)}$, $p > 2$, to $\|u - u_h\|_{H^1(\Omega)} \le Ch\|f\|_{L^2(\Omega)}$. The first one is a discretization of a distributed optimal control problem with (1.1) as the state equation. The second application consists in a simpler proof of the discrete compactness property for edge elements of any order on this kind of finite element meshes.

We finish this introduction by commenting on related work. The idea to treat singularities due to a non-smooth boundary by using graded finite element meshes is old. The two-dimensional case was investigated by Oganesyan and Rukhovets [25], Babuška [9], Raugel [27], and Schatz and Wahlbin [28]. In three dimensions we can distinguish isotropic mesh grading, see the papers by Apel and Heinrich [3] and Apel, Sändig, and Whiteman [5], and anisotropic mesh grading, see the already mentioned papers [2, 1, 6] for the special case of prismatic domains, and [4] for general polyhedral domains. This work has been extended by Băcuţă, Nistor, and Zikatanov [12] to higher order finite element approximations where naturally higher regularity of the right-hand side f has to be assumed. Boundary element methods with anisotropic, graded meshes have been considered by von Petersdorff and Stephan [26]. The main alternative to mesh grading is augmenting the finite element space with singular functions, see for example Strang and Fix [32], Blum and Dobrowolski [11], or Assous, Ciarlet Jr., and Segré [8] for various variants. It works well in two dimensions where the coefficient in front of the singular function is constant. In the case of edge singularities this coefficient is a function which can be approximated, see Beagles and Whiteman [10], or it can be treated by Fourier analysis, see Lubuma and Nicaise [22].

2. Notation, regularity, discretization

It is well known that the solution of the boundary value problem (1.1) contains edge and vertex singularities which are characterized by singular exponents. For each edge e, the corresponding leading (smallest) singular exponent λ_e is simply defined by $\lambda_e = \pi/\omega_e$ where ω_e is the interior dihedral angle at the edge e. For vertices v of Ω, the leading singular exponent $\lambda_v > 0$ has to be computed via the eigenvalue problem of the Laplace-Beltrami operator on the intersection of Ω and the unit sphere centered at v. Note that $\lambda_e > \frac{1}{2}$ and $\lambda_v > 0$. A vertex v or an edge e will be called *singular* if $\lambda_v < \frac{1}{2}$ or $\lambda_e < 1$, respectively. We exclude the case that $\frac{1}{2}$ is a singular exponent of any vertex. For a detailed discussion of edge and vertex singularities we refer to [16, Sections 2.5 and 2.6].

As in [4] we subdivide the domain Ω into a finite number of disjoint tetrahedral subdomains, subsequently called *macro-elements*,

$$\overline{\Omega} = \bigcup_{\ell=1}^{L} \overline{\Lambda_\ell}.$$

We assume that each Λ_ℓ contains at most one singular edge and at most one singular vertex. In the case that Λ_ℓ contains both a singular edge and a singular vertex, that vertex is contained in that edge. Note that the edges of Λ_ℓ are considered to have $O(1)$ length. For $\ell_1 \neq \ell_2$, the closures of the macroelements Λ_{ℓ_1} and Λ_{ℓ_2} may be disjoint or they intersect defining a *coupling face*, or a *coupling edge*, or a *coupling node*. Denote by $\mathcal{F}_c$, $\mathcal{E}_c$ and $\mathcal{N}_c$ the sets of coupling faces, edges and nodes, respectively.

For the description of the regularity of the solution u of (1.1), we set $\lambda_{\mathrm{v}}^{(\ell)} = \lambda_v$ if the macro-element Λ_ℓ contains the singular vertex v of Ω. If Λ_ℓ does not contain any singular vertex we set $\lambda_{\mathrm{v}}^{(\ell)} = +\infty$. Moreover, we set $\lambda_{\mathrm{e}}^{(\ell)} = \lambda_e$ if Λ_ℓ contains the singular edge e of Ω, otherwise we set $\lambda_{\mathrm{e}}^{(\ell)} = +\infty$. Furthermore, we define in each macro-element Λ_ℓ a Cartesian coordinate system $x^{(\ell)} = (x_1^{(\ell)}, x_2^{(\ell)}, x_3^{(\ell)})$ such that the singular vertex, if existing, is located in the origin, and the singular edge, if existing, is contained in the $x_3^{(\ell)}$-axis. We also introduce by

$$\begin{aligned}
r^{(\ell)}(x^{(\ell)}) &:= \left((x_1^{(\ell)})^2 + (x_2^{(\ell)})^2\right)^{1/2}, \\
R^{(\ell)}(x^{(\ell)}) &:= \left((x_1^{(\ell)})^2 + (x_2^{(\ell)})^2 + (x_3^{(\ell)})^2\right)^{1/2}, \\
\theta^{(\ell)}(x^{(\ell)}) &:= \frac{r^{(\ell)}(x^{(\ell)})}{R^{(\ell)}(x^{(\ell)})},
\end{aligned}$$

the distance to the $x_3^{(\ell)}$-axis, the distance to the origin, the angular distance from the $x_3^{(\ell)}$-axis, respectively.

For $k \in \mathbb{N}$ and $\beta, \delta \in \mathbb{R}$ we define the weighted Sobolev space

$$V_{\beta,\delta}^{k,2}(\Lambda_\ell) := \left\{ v \in \mathcal{D}'(\Lambda_\ell) : \|v\|_{V_{\beta,\delta}^{k,2}(\Lambda_\ell)} < \infty \right\}$$

where

$$\begin{aligned}
\|v\|^2_{V_{\beta,\delta}^{k,2}(\Lambda_\ell)} &:= \sum_{|\alpha| \le k} \int_{\Lambda_\ell} \left| R^{\beta-k+|\alpha|} \theta^{\delta-k+|\alpha|} D^\alpha v \right|^2, \\
|v|^2_{V_{\beta,\delta}^{k,2}(\Lambda_\ell)} &:= \sum_{|\alpha| = k} \int_{\Lambda_\ell} \left| R^{\beta} \theta^{\delta} D^\alpha v \right|^2
\end{aligned}$$

Here, we have used the standard multi-index notation to describe partial derivatives, and we have omitted the index (ℓ) in R and θ for simplicity.

Theorem 2.1 (Regularity). *[4, Theorem 2.10] The weak solution u of the boundary value problem (1.1) admits the decomposition*

$$u = u_{\rm r} + u_{\rm s}$$

in Λ_ℓ, $\ell = 1, \ldots, L$, where $u_{\rm r} \in H^2(\Lambda_\ell)$ and

$$\frac{\partial u_{\rm s}}{\partial x_i^{(\ell)}} \in V^{1,2}_{\beta,\delta}(\Lambda_\ell), \quad i = 1, 2, \qquad \frac{\partial u_{\rm s}}{\partial x_3^{(\ell)}} \in V^{1,2}_{\beta,0}(\Lambda_\ell),$$

for any $\beta, \delta \geq 0$ satisfying $\beta > \frac{1}{2} - \lambda_{\rm v}^{(\ell)}$ and $\delta > 1 - \lambda_{\rm e}^{(\ell)}$.

Following [4] we consider a triangulation $\mathcal{T}_h$ of Ω,

$$\overline{\Omega} = \bigcup_{T \in \mathcal{T}_h} \overline{T},$$

made up of tetrahedra which match the initial partition: if $T \cap \Lambda_\ell \neq \emptyset$ then $T \subset \Lambda_\ell$. Four cases are considered:

1. If Λ_ℓ does neither contain a singular edge nor a singular vertex then $\mathcal{T}_h|_{\Lambda_\ell}$ is assumed to be isotropic and quasi-uniform with element size h, see Figure 1, top left.

2. If Λ_ℓ contains a singular vertex but no singular edges then $\mathcal{T}_h|_{\Lambda_\ell}$ is isotropic and has a singular vertex refinement, i.e., the mesh is graded towards the singular vertex with a grading parameter $\nu_\ell \in (0, 1]$. This can be achieved by using a coordinate transformation of the vertices from Case 1, see Figure 1, top right.

3. If Λ_ℓ contains a singular edge but no singular vertices then $\mathcal{T}_h|_{\Lambda_\ell}$ is anisotropically graded towards the singular edge. The grading parameter is $\mu_\ell \in (0, 1]$. To this end, we introduce a family $\mathcal{P}_\ell$ of planes transversal to the singular edge and containing the opposite one. These planes split the macro element into strips and contain all nodes. In the planes the position of the nodes is achieved by applying a coordinate transformation to a uniform triangulation, see Figure 1, bottom left.

4. If Λ_ℓ contains both a singular vertex and a singular edge then $\mathcal{T}_h|_{\Lambda_\ell}$ is graded towards the singular edge with grading parameter $\mu_\ell \in (0, 1]$ and towards the singular vertex with grading parameter $\nu_\ell \in (0, 1]$. The mesh is topologically equivalent to the mesh of Case 3 but the planes of $\mathcal{P}_\ell$ do not divide the singular edge equidistantly but with a grading towards the singular vertex.

We point out that anisotropic elements can appear only in Cases 3 and 4, for which $\mathcal{T}_h$ contains needle elements near the singular edge and flat elements near the opposite one, see Figure 1. We further observe that if Λ_ℓ is of type 3 or 4, the elements in $\mathcal{T}_h|_{\Lambda_\ell}$ do not intersect any plane of $\mathcal{P}_\ell$.

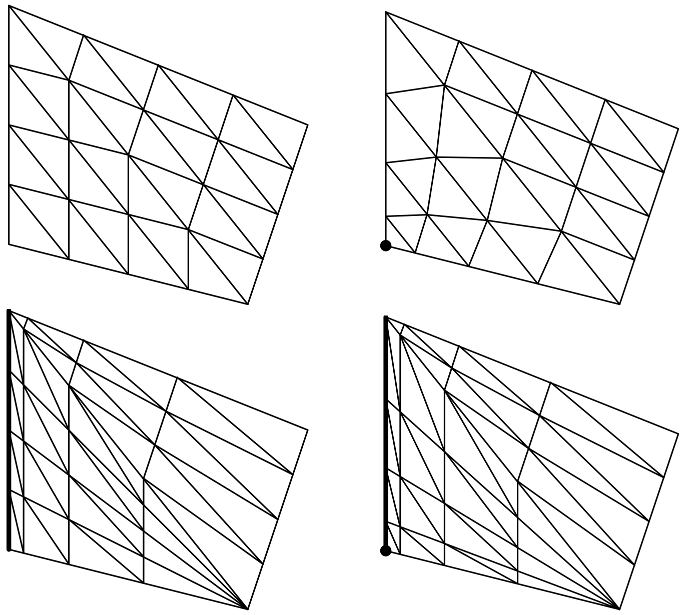

Figure 1: Macroelements of types 1, 2, 3 and 4

For each element T we introduce its lengths $h_{1,T}, h_{2,T}, h_{3,T}$ and h_T as follows. Let h_T be the diameter of T. If $T \subset \Lambda_\ell$ with Λ_ℓ of type 1 or 2, then $h_{1,T} = h_{2,T} = h_{3,T} = h_T$. If $T \subset \Lambda_\ell$ with Λ_ℓ of type 3 or 4 then $h_{3,T}$ is the length of the edge $e_{3,T}$ of T parallel to the singular edge, and $h_{1,T} = h_{2,T} = \frac{1}{2}(|e_{1,T}| + |e_{2,T}|)$ where $e_{1,T}$ and $e_{2,T}$ are the edges of T intersecting $e_{3,T}$ and each one of them is contained in some plane of $\mathcal{P}_\ell$.

By classical regularity theory, the solution u of the boundary value problem (1.1) is continuous, see e.g. [16, page page 79], such that the Lagrange interpolant u_I with respect to the subdivision $\{\Lambda_\ell\}$ is well defined. We consider the decomposition

$$u = u_I + u_R. \tag{2.1}$$

It follows that the restriction $u_R|_{\Lambda_\ell}$ has the same smoothness properties as u, see Theorem 2.1. Furthermore, u_R vanishes in coupling nodes and on singular edges. We construct now an interpolant $D_h u_R \in V_h$ which also vanishes on these nodes such that $u_I + D_h u_R \in V_h$ can be used to estimate the discretization error via (1.4).

To this end, let $\mathcal{N}$, $\mathcal{N}_{\text{in}}$, $\mathcal{N}_{\text{c}}$ and $\mathcal{N}_{\text{s}}$ be the set of all nodes of $\mathcal{T}_h$, the set of all the interior nodes, the set of coupling nodes, and the set of nodes which belong to some singular edge, respectively. The terminal points of the singular edges are included in $\mathcal{N}_{\text{s}}$. The piecewise linear nodal basis on $\mathcal{T}_h$ is denoted by $\{\phi_n\}_{n\in\mathcal{N}}$. We associate (as specified below) with each $n \in \mathcal{N} \setminus (\mathcal{N}_{\text{c}} \cup \mathcal{N}_{\text{s}})$ an edge σ_n with n as an endpoint. Note that $u|_{\sigma_n} \in L^2(\sigma_n)$ since $u \in H^s(\Omega)$ with $s > 1$. Hence the operator D_h with

$$D_h u = \sum_{n\in\mathcal{N}\setminus(\mathcal{N}_{\text{c}}\cup\mathcal{N}_{\text{s}})} (\Pi_{\sigma_n} u)(n) \cdot \phi_n(x), \tag{2.2}$$

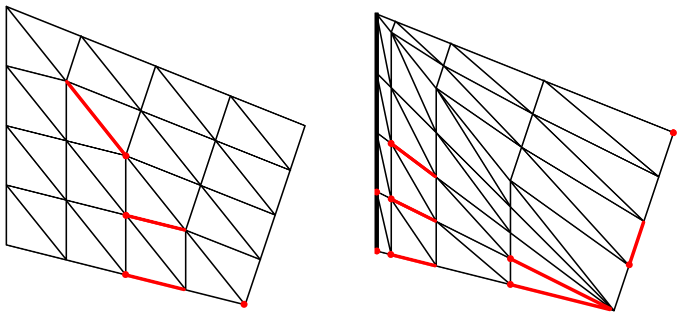

Figure 2: Illustration of the edges σ_n

is well defined when $\Pi_\sigma : L^2(\sigma) \to \mathcal{P}_1(\sigma)$ is the $L^2(\sigma)$-projection operator onto the space of polynomials of degree less than or equal to one. Note that $D_h u$ vanishes on coupling nodes and on singular edges by construction. In order to impose the boundary conditions and to be able to prove interpolation error estimates we need to select the edges σ_n in an appropriate way, compare the illustration in Figure 2. First, we demand that

- for each node $n \in \mathcal{N} \setminus (\mathcal{N}_c \cup \mathcal{N}_s)$, n and σ_n belong to the same macroelement.

  This requires in particular the following restrictions.

  - If n lays on a boundary or coupling face, then σ_n is contained in that face.
  - If n lays on a coupling edge, then σ_n is contained in that coupling edge.

Note that these requirements made the treatment of the coupling nodes via the interpolation on the initial u_I necessary. Note further that this construction leads to a preservation of the homogeneous Dirichlet boundary condition.

In order to prove the stability of D_h in the anisotropic refinement regions we also require:

- If n is a vertex of a tetrahedron contained in a macroelement Λ_ℓ of types 3 or 4, then σ_n is an edge contained on some plane of $\mathcal{P}_\ell$.
- If n_1 and n_2 belong to a macroelement Λ_ℓ of types 3 or 4 and have the same orthogonal projection onto the $x_1^{(\ell)} x_2^{(\ell)}$-plane, then the same holds for σ_{n_1} and σ_{n_2}.

In order to estimate the interpolation error we need to define for each $T \in \mathcal{T}_h$ a set S_T which should satisfy the following assumptions.

- The set S_T is a union of elements of $\mathcal{T}_h$ (plus some faces) and in particular $T \subseteq S_T$.
- The set S_T is an open connected domain, and as small as possible.
- We have $\sigma_n \subset \overline{S_T}$ for all nodes n of T.
- If $T \subset \Lambda_\ell$, then $S_T \subset \Lambda_\ell$.
- If $T \subset \Lambda_\ell$ with Λ_ℓ of type 3 of 4, then S_T is a prism where the top and bottom faces are contained in two planes of $\mathcal{P}_\ell$ (and so they are not parallel) and the other faces are parallel to the singular edge.

The following properties follow from the definitions of the edges σ_n and the sets S_T.

1. Let T be contained in a macroelement Λ_ℓ of type 3 or 4. If $\overline{T}$ intersects two planes p_1 and p_2 of $\mathcal{P}_\ell$, then $\overline{S_T}$ intersects exactly the same planes p_1 and p_2.

2. If the node n, $n \notin \mathcal{N}_\text{c} \cup \mathcal{N}_\text{s}$, belongs to a coupling face, that means that there exist tetrahedra $T_1 \subset \Lambda_{\ell_1}$ and $T_2 \subset \Lambda_{\ell_2}$ with $\ell_1 \neq \ell_2$ and $n \in \overline{T_1} \cap \overline{T_2}$, then $S_{T_1} \cap S_{T_2} = \emptyset$ but $\sigma_n \subset \overline{S_{T_1}} \cap \overline{S_{T_2}}$.

3. If T is an isotropic element then all the elements in S_T are also isotropic and of size of the same order.

The second point is essential for our proof of the approximation properties. It was the target for which we made the construction as it is.

3. Error estimates

The aim of this section is to derive error estimates for our discretization. They are based on local interpolation error estimates for our interpolant D_h. For proving these estimates we have to distinguish several cases, see also Figure 3 for an illustration:

1. T is an isotropic element without coupling node, u has full regularity,
2. T is an isotropic element with coupling node, u has full regularity,
3. T is an isotropic element with coupling node, u has reduced regularity,
4. T is an anisotropic flat element without coupling node, u has full regularity,
5. T is an anisotropic flat element with coupling node, u has full regularity,
6. T is an anisotropic needle element without node on the singular edge, u has full regularity,
7. T is an anisotropic needle element with node on the singular edge, u has reduced regularity.

In Lemma 3.1 we present the general approach for the proof of the local interpolation error estimate by considering isotropic elements with and without coupling nodes (cases 1 and 2). We proceed with Lemmas 3.2 where we introduce for isotropic elements how to cope with the weighted norms in the case of reduced regularity (case 3). The interpolated function is only from a weighted Sobolev space but we will see that this even simplifies some parts of the proof.

For anisotropic elements the use of an inverse inequality (as was done in the previous lemmas) has to be avoided; instead we use the structure of the meshes in the macroelements of types 3 and 4. We start with a stability estimate of $\partial_3 D_h u$ which allows immediately the treatment of anisotropic flat elements (cases 4 and 5) in Lemma 3.4. Then we prove stability estimates for the remaining derivatives and continue with the interpolation error estimates for needle elements. Lemma 3.7 is devoted to case 6, and Lemma 3.9 to case 7.

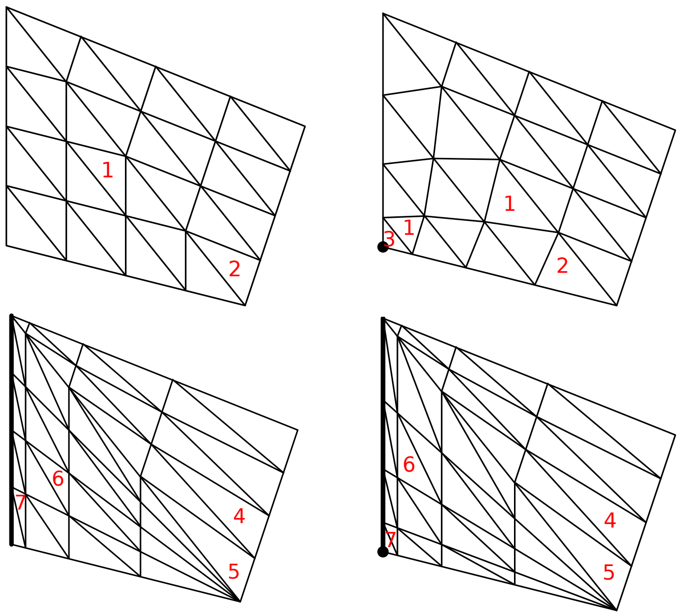

Figure 3: Illustration of the cases that have to considered for the interpolation error estimates

All these local estimates can then be combined to prove the global interpolation error estimate, see Theorem 3.10, and the finite element error estimates, see Corollary 3.12.

Lemma 3.1 (isotropic element, full regularity)**.** *If T is an isotropic element then the local interpolation error estimate*

$$|u - D_h u|_{H^1(T)} \leq C h_T |u|_{H^2(S_T)} \tag{3.1}$$

holds provided that $u \in H^2(S_T)$ and $u(n) = 0$ for all $n \in \mathcal{N}_c$.

Proof. Following the explanations in [30, page 486] and [1, page 1156], an explicit representation of $D_h u$ from (2.2) can be given by introducing the unique function $\psi_n \in V_h|_{\sigma_n}$ with $\int_{\sigma_n} \psi_n \phi_j = \delta_{nj}$ for all $j \in \mathcal{N}$ such that

$$(\Pi_{\sigma_n} u)(n) = \int_{\sigma_n} u \psi_n \tag{3.2}$$

and

$$D_h u|_T = \sum_{n \in \mathcal{N}_T} \left(\int_{\sigma_n} u \psi_n \right) \cdot \phi_n \tag{3.3}$$

where we denote by $\mathcal{N}_T$ the set of nodes of T without the coupling nodes. Note that

$$\|\psi_n\|_{L^\infty(\sigma_n)} = C \, |\sigma_n|^{-1}, \tag{3.4}$$

compare [1, page 1157]. (By some calculation one can even specify that $C = 4$.) With (3.3), the direct computation

$$|\phi_n|_{H^1(T)} \le C h_T^{-1} |T|^{1/2}, \tag{3.5}$$

the trace theorem

$$\|u\|_{L^1(\sigma_n)} \le C|\sigma_n||S_T|^{-1/2}(\|u\|_{L^2(S_T)} + h_T|u|_{H^1(S_T)} + h_T^2|u|_{H^2(S_T)}), \tag{3.6}$$

and $|S_T| \le C|T|$ we obtain

$$\begin{aligned} |D_h u|_{H^1(T)} &\le C \sum_{n\in\mathcal{N}_T} \|u\|_{L^1(\sigma_n)} \|\psi_n\|_{L^\infty(\sigma_n)} |\phi_n|_{H^1(T)} \\ &\le C h_T^{-1}(\|u\|_{L^2(S_T)} + h_T|u|_{H^1(S_T)} + h_T^2|u|_{H^2(S_T)}). \end{aligned} \tag{3.7}$$

If $\mathcal{N}_T$ does not contain a node $n \in \mathcal{N}_c$ we find that $D_h w = w$ for all $w \in \mathcal{P}_1$ such that we get by using the triangle inequality and the stability estimate (3.7)

$$\begin{aligned} |u - D_h u|_{H^1(T)} &= |(u-w) - D_h(u-w)|_{H^1(T)} \quad \forall w \in \mathcal{P}_1 \\ &\le |u-w|_{H^1(T)} + |D_h(u-w)|_{H^1(T)} \\ &\le C\left(h_T^{-1}\|u-w\|_{L^2(S_T)} + |u-w|_{H^1(S_T)} + h_T|u|_{H^2(S_T)}\right). \end{aligned}$$

We use now a Deny–Lions type argument (see e.g. [15]) and conclude estimate (3.1).

In the case when $\mathcal{N}_T$ contains a node $n \in \mathcal{N}_c$ we do not have the property that $D_h w = w$ for all $w \in \mathcal{P}_1$ but we can use that $u(n) = 0$. Let σ_n be an edge contained in T having n as an endpoint, and let ϕ_n be the Lagrange basis function associated with n. (Note that we deal here with nodes n which are not used in the definition of D_h. Therefore we can assume that σ_n is local in Λ_ℓ.) Consequently, we have with the previous argument that

$$|u - (D_h u + (\Pi_{\sigma_n} u)(n)\phi_n)|_{H^1(T)} \le C h_T |u|_{H^2(S_T)}. \tag{3.8}$$

Let $I_T u$ be the linear Lagrange interpolation of u on T. Since $I_T u|_{\sigma_n}$ is linear, we have have $(\Pi_{\sigma_n} I_T u)(n) = 0$. From this fact and using (3.2)–(3.6) as in the derivation of (3.7) (here with the specific T instead of S_T since $\sigma_n \subset \overline{T}$), we have

$$\begin{aligned} |(\Pi_{\sigma_n} u)(n)\phi_n|_{H^1(T)} &= |(\Pi_{\sigma_n}(u - I_T u))(n)\phi_n|_{H^1(T)} \\ &\le C h_T^{-1}\left(|u - I_T u|_{L^2(T)} + h_T|u - I_T u|_{H^1(T)} + h_T^2|u|_{H^2(T)}\right) \\ &\le C h_T |u|_{H^2(T)} \end{aligned}$$

where we used standard estimates for the Lagrange interpolant in the last step. With (3.8) and the triangle inequality we conclude estimate (3.1) also in this case. □

Lemma 3.2 (isotropic element, reduced regularity)**.** *If T is an isotropic element then the local interpolation error estimate*

$$|u - D_h u|_{H^1(T)} \le C h_T^{1-\beta} \|u\|_{V^{2,2}_{\beta,0}(S_T)} \tag{3.9}$$

holds provided that $u \in V^{2,2}_{\beta,0}(S_T)$, $\beta \in [0,1)$.

Proof. We start as in the proof of Lemma 3.1 but use the sharper trace theorem

$$\|u\|_{L^1(\sigma_n)} \leq C|\sigma_n||S_T|^{-1}(\|u\|_{L^1(S_T)} + h_T|u|_{W^{1,1}(S_T)} + h_T^2|u|_{W^{2,1}(S_T)}).$$

With (3.3), (3.4), (3.5), and $|S_T| \leq C|T|$ we obtain

$$\begin{aligned}
|D_h u|_{H^1(T)} &\leq C \sum_{n\in\mathcal{N}_T} \|u\|_{L^1(\sigma_n)}\|\psi_n\|_{L^\infty(\sigma_n)}|\phi_n|_{H^1(T)} \\
&\leq C|S_T|^{-1/2}(h_T^{-1}\|u\|_{L^1(S_T)} + |u|_{W^{1,1}(S_T)} + h_T|u|_{W^{2,1}(S_T)}) \\
&\leq C(h_T^{-1}\|u\|_{L^2(S_T)} + |u|_{H^1(S_T)} + |S_T|^{-1/2}h_T|u|_{W^{2,1}(S_T)})
\end{aligned}$$

and hence via the triangle inequality

$$|u - D_h u|_{H^1(T)} \leq C(h_T^{-1}\|u\|_{L^2(S_T)} + |u|_{H^1(S_T)} + |S_T|^{-1/2}h_T|u|_{W^{2,1}(S_T)}). \tag{3.10}$$

For the first two terms we just use that $R \leq h_T$, hence $1 \leq h_T R^{-1}$, to get

$$\begin{aligned}
\|u\|_{L^2(S_T)} &\leq h_T^{2-\beta}\|u\|_{V^{0,2}_{\beta-2,0}(S_T)}, \\
|u|_{H^1(S_T)} &\leq h_T^{1-\beta}|u|_{V^{1,2}_{\beta-1,0}(S_T)}.
\end{aligned}$$

To estimate the third term we use the Cauchy–Schwarz inequality and again $R \leq h_T$, to obtain for $|\alpha| = 2$

$$\begin{aligned}
|D^\alpha u|_{L^1(S_T)} &\leq \|R^{-\beta}\|_{L^2(S_T)}\|R^\beta D^\alpha u\|_{L^2(S_T)} \\
&\leq C|S_T|^{1/2}h_T^{-\beta}|u|_{V^{2,2}_{\beta,0}(S_T)}
\end{aligned}$$

where $\|R^{-\beta}\|_{L^2(S_T)} \leq C|S_T|^{1/2}h_T^{-\beta}$ is obtained by executing the integration and using that $\beta < \frac{3}{2}$. All these estimates imply estimate (3.9). □

In order to prove interpolation error estimates for the anisotropic elements we derive stability estimates for D_h where we avoid the use of the inverse inequality. Let x_1, x_2 and x_3 be a Cartesian coordinate system with the x_3-direction parallel to the singular edge of Λ. We will estimate separately the L^2-norm of the derivatives of $D_h u$.

Let T be an anisotropic element with the characteristic lengths $h_{1,T} = h_{2,T}$ and $h_{3,T}$. We will not use that $h_{3,T} \geq h_{j,T}$, $j = 1, 2$, in the next lemma in order to use this estimate both for the needle and the flat elements.

Lemma 3.3 (Stability in direction of the singular edge)**.** *For any anisotropic element T the estimate*

$$\|\partial_3 D_h u\|_{L^2(T)} \leq C|S_T|^{-1/2} \sum_{|\alpha|\leq 1} h_T^\alpha \|D^\alpha \partial_3 u\|_{L^1(S_T)}$$

holds provided that $\partial_3 u \in W^{1,1}(S_T)$.

Proof. We observe that T has an edge e_T parallel to the singular edge, and so, parallel to the x_3-axis. Since D_hu is linear on T, we have $\partial_3 D_h u|_T = \partial_3 D_h u|_{e_T}$. If e_T is contained on the singular edge, then $\partial_3 D_h u|_T = 0$ since $D_h u|_{e_T} = u|_{e_T} = 0$ and we are done. Now, consider the case that e_T is not contained in a singular edge and denote its endpoints by n_1 and n_2 such that $\partial_3\phi_{n_1}\big|_T = -h_{3,T}^{-1}$ and $\partial_3\phi_{n_2}\big|_T = h_{3,T}^{-1}$. Then we have

$$\partial_3 D_h u = h_{3,T}^{-1}\left[\int_{\sigma_{n_2}} u\psi_{n_2} - \int_{\sigma_{n_1}} u\psi_{n_1}\right]$$

We observe now that by our assumptions σ_{n_1} and σ_{n_2} have the same projection σ_T into the x_1x_2-plane and hence form two opposite edges of a plane quadrilateral which is parallel to the x_3-axis and which we will denote by F_T. We note further that ψ_{n_1} and ψ_{n_2} can be considered as the same function ψ_T defined on σ_T and $\|\psi_T\|_{L^\infty(\sigma_T)} = C|\sigma_T|^{-1}$. With this insight we obtain

$$\begin{aligned}|\partial_3 D_h u| &= h_3^{-1}\left|\int_{\sigma_{n_2}} u\psi_{n_2} - \int_{\sigma_{n_1}} u\psi_{n_1}\right| = h_3^{-1}\left|\int_{F_T}\partial_3 u\psi_T\right| \\ &\le C h_3^{-1}|\sigma_T|^{-1}\|\partial_3 u\|_{L^1(F_T)} \le C|F_T|^{-1}\|\partial_3 u\|_{L^1(F_T)}.\end{aligned}$$

We integrate this estimate over T, apply the standard trace theorem

$$\|v\|_{L^1(F_T)} \le C|F_T||S_T|^{-1}\sum_{|\alpha|\le 1} h_T^\alpha \|D^\alpha v\|_{L^1(S_T)}$$

and obtain the desired estimate. □

We are now prepared to estimate the interpolation error for the flat elements occurring far away from the singular edge in cases 3 and 4.

Lemma 3.4 (anisotropic flat element, full regularity)**.** *If T is an anisotropic flat element ($h_{3,T} \le h_{1,T} = h_{2,T}$) then the local interpolation error estimate*

$$|u - D_h u|_{H^1(T)} \le C h_T |u|_{H^2(S_T)} \tag{3.11}$$

holds provided that $u \in H^2(S_T)$. (Remember that $h_T = \mathrm{diam}(T)$.)

Proof. The proof for $\partial_3(u - D_h u)$ can be done on the basis of Lemma 3.3. Assume for the moment that the element T does not contain a coupling node. Similar to the proof of Lemma 3.1 we obtain for any $w \in \mathcal{P}_1$

$$\begin{aligned}\|\partial_3(u - D_h u)\|_{L^2(T)} &= \|\partial_3(u - w) - \partial_3 D_h(u - w)\|_{L^2(T)} \\ &\le C\|\partial_3(u - w)\|_{L^2(S_T)} + C\sum_{|\alpha|=1} h_T^\alpha \|D^\alpha \partial_3 u\|_{L^2(S_T)}.\end{aligned}$$

We choose now $w \in \mathcal{P}_1$ such that the constant $\partial_3 w$ satisfies $\int_{S_T} \partial_3(u-w) = 0$ and such that we can conclude by using the Poincaré–Friedrichs inequality (or again a Deny–Lions type argument)

$$\|\partial_3(u-w)\|_{L^2(S_T)} \leq C \sum_{|\alpha|=1} h_T^\alpha \|D^\alpha \partial_3 u\|_{L^2(S_T)}$$

and hence

$$\|\partial_3(u - D_h u)\|_{L^2(T)} \leq C \sum_{|\alpha|=1} h_T^\alpha \|D^\alpha \partial_3 u\|_{L^2(S_T)} \leq C h_T |u|_{H^2(S_T)}.$$

Note that the polynomial w can be chosen such that it vanishes in three nodes of T. It is completely described by choosing the appropriate value at one endpoint of the edge of T which is parallel to the x_3-axis. Since a possible coupling node is not an endpoint of this edge, the argument above can also be used in the case of coupling nodes.

For the other directions we can proceed as in the proof of Lemma 3.1. In the case of coupling nodes the interpolation error estimate $|u - I_T u|_{H^1(T)} \leq C h_T |u|_{H^2(T)}$ is used there which does not hold for anisotropic elements. However, the estimate $\|\partial_i(u - I_T u)\|_{L^2(T)} \leq C h_T |u|_{H^2(T)}$, $i = 1, 2$, does hold, see for example [2]. □

It remains to prove interpolation error estimates for needle elements such that we will assume $h_{1,T} = h_{2,T} \leq C h_{3,T}$ for the next lemmas.

Lemma 3.5 (Stability in direction perpendicular to singular edge, anisotropic needle element away from singular edge)**.** *Assume that the element T does not contain a node $n \in \mathcal{N}_\mathrm{s}$ and that $h_{1,T} = h_{2,T} \leq C h_{3,T}$. Then for $i = 1, 2$ we have*

$$\|\partial_i D_h u\|_{L^2(T)} \leq C \left(|u|_{H^1(S_T)} + h_{3,T} |\partial_3 u|_{H^1(S_T)}\right) \tag{3.12}$$

provided that $u \in H^1(S_T)$ and $\partial_3 u \in H^1(S_T)$.

Proof. For each node $n \in \mathcal{N}_T$ we denote by $F_{n,T}$ the top or bottom face of the prismatic domain S_T such that $n \in \overline{F}_{n,T}$. Observe that we have $\sigma_n \subset \overline{F}_{n,T} \subset \overline{S}_T$ for all $n \in \mathcal{N}_T$. Observe further that $F_{n,T}$ is isotropic with diameter of order $h_{1,T}$ and recall the standard trace inequality

$$\|v\|_{L^1(\sigma_n)} \leq C |\sigma_n| |F_{n,T}|^{-1} \left(\|v\|_{L^1(F_{n,T})} + h_{1,T} |v|_{W^{1,1}(F_{n,T})}\right) \tag{3.13}$$

for all $v \in W^{1,1}(F_{n,T})$. We need also the trace inequality

$$\|v\|_{L^1(F_{n,T})} \leq C |F_{n,T}| |S_T|^{-1} \left(\|v\|_{L^1(S_T)} + h_{3,T} \|\partial_3 v\|_{L^1(S_T)}\right) \tag{3.14}$$

which can be proved by using Lemma A.1 from page 23 and the facts that S_T is a union of prisms, and $F_{n,T}$ is a face of S_T.

Let s_T be one of the short edges of T and denote its endpoints by n^1 and n^2. We use the same notation s_T for the direction of this edge in order to denote by $\partial_{s_T} v = \nabla v \cdot s_T / |s_T|$

the directional derivative. In the following we first estimate $\|\partial_{s_T} D_h u\|_{L^2(T)}$. After that, the desired estimates (3.12) easily follow as we will show.

Notice that if $n \in \mathcal{N}_T \setminus \{n^1, n^2\}$ we have $\partial_{s_T}\phi_n = 0$, and if $n \in \{n^1, n^2\}$ then $\|\partial_{s_T}\phi_n\|_{L^\infty(T)} = |s_T|^{-1} \le C h_{1,T}^{-1}$. For all $w \in P_0(S_T)$ we have (and here we use that the element does not contain a node $n \in \mathcal{N}_{\rm c} \cup \mathcal{N}_{\rm s}$)

$$
\begin{aligned}
\|\partial_{s_T} D_h u\|_{L^2(T)} &= \|\partial_{s_T} D_h (u-w)\|_{L^2(T)} \\
&\le \sum_{n\in\mathcal{N}_T\cap s_T} \left| \int_{\sigma_n} (u-w)\psi_n \right| \|\partial_{s_T}\phi_n\|_{L^2(T)} \\
&\le C h_{1,T}^{-1}|T|^{1/2} \sum_{n\in\mathcal{N}_T\cap s_T} \|u-w\|_{L^1(\sigma_n)} \|\psi_n\|_{L^\infty(\sigma_n)} \\
&\le C h_{1,T}^{-1}|T|^{1/2} \sum_{n\in\mathcal{N}_T\cap s_T} |\sigma_n|^{-1} \|u-w\|_{L^1(\sigma_n)}.
\end{aligned} \tag{3.15}
$$

From the trace inequality (3.13) we have for each $n \in \mathcal{N}_T \cap s_T$

$$
\|u-w\|_{L^1(\sigma_n)} \le C|\sigma_n||F_{n,T}|^{-1} \left(\|u-w\|_{L^1(F_{n,T})} + h_{1,T}|u|_{W^{1,1}(F_{n,T})} \right).
$$

Since the definition of $F_{n,T}$ implies $F_{n^1} = F_{n^2} =: F_T$, we have

$$
\|u-w\|_{L^1(\sigma_n)} \le C|\sigma_n||F_T|^{-1} \left(\|u-w\|_{L^1(F_T)} + h_{1,T}|u|_{W^{1,1}(F_T)} \right).
$$

Now we choose w as the average of u on F_T and use a Poincaré type inequality on F_T to get

$$
\|u-w\|_{L^1(\sigma_n)} \le C|\sigma_n||F_T|^{-1} h_{1,T}|u|_{W^{1,1}(F_T)}.
$$

Therefore we arrive at

$$
\begin{aligned}
\|\partial_{s_T} D_h u\|_{L^2(T)} &\le C|T|^{1/2}|F_T|^{-1}|u|_{W^{1,1}(F_T)} \\
&\le C|T|^{1/2}|S_T|^{-1} \left(|u|_{W^{1,1}(S_T)} + h_{3,T}|\partial_3 u|_{W^{1,1}(S_T)} \right) \\
&\le C|S_T|^{-1/2} \left(|u|_{W^{1,1}(S_T)} + h_{3,T}|\partial_3 u|_{W^{1,1}(S_T)} \right) \\
&\le C \left(|u|_{H^1(S_T)} + h_{3,T}|\partial_3 u|_{H^1(S_T)} \right)
\end{aligned} \tag{3.16}
$$

where we used again the trace inequality (3.14).

Now, let $s_{1,T}$ and $s_{2,T}$ be two different short edges (edge vectors) of T such that the determinant of the matrix made up of $\frac{s_{1,T}}{|s_{1,T}|}$, $\frac{s_{2,T}}{|s_{2,T}|}$ and $\mathbf{e_3}$ as columns is greater than a constant depending only the maximum angle of T. Note that this is possible due to the maximal angle condition, see [18]. Then, if the canonical vector e_i, $i = 1, 2$, is expressed as

$$
\mathrm{e}_i = c_{1,i}\frac{s_{1,T}}{|s_{1,T}|} + c_{2,i}\frac{s_{2,T}}{|s_{2,T}|} + c_{3,i}\mathrm{e}_3,
$$

it follows that $c_{1,i}$, $c_{2,i}$ and $c_{3,i}$ are bounded by above by a constant depending only on the maximum angle condition. Since

$$
\partial_i = c_{1,i}\partial_{s_{1,T}} + c_{2,i}\partial_{s_{2,T}} + c_{3,i}\partial_3
$$

we obtain (3.12) from (3.16) with $s_T = s_{1,T}$ and $s_T = s_{2,T}$, Lemma 3.3, and recalling that $h_{1,T} = h_{2,T} \le C h_{3,T}$. □

Lemma 3.6 (Stability in direction perpendicular to singular edge, anisotropic needle element at the singular edge)**.** *Assume that the element T contains at least one node $n \in \mathcal{N}_s$ and that $h_{1,T} = h_{2,T} \le C h_{3,T}$. Then we have for $i = 1, 2$*

$$\|\partial_i D_h u\|_{L^2(T)} \le C|S_T|^{-1/2} \left(|u|_{W^{1,1}(S_T)} + \frac{h_{3,T}}{h_{i,T}} \|\partial_3 u\|_{L^1(S_T)} + \sum_{|\alpha|=1} h_T^\alpha |D^\alpha u|_{W^{1,1}(S_T)} \right) \tag{3.17}$$

provided that $u \in W^{2,1}(S_T)$.

Proof. For each node $n \in \mathcal{N}_s$ of T we select one short edge σ_n with an endpoint at n and contained in the same macroelement as T such that we can apply Lemma 3.5. We have for $i = 1, 2$

$$\left\| \partial_i \left(D_h u + \sum_{n \in \mathcal{N}_s \cap \overline{T}} (\Pi_{\sigma_n} u)(n)\, \phi_n \right) \right\|_{L^2(T)} \le C|S_T|^{-1/2} \left(|u|_{W^{1,1}(S_T)} + h_{3,T} |\partial_3 u|_{W^{1,1}(S_T)} \right). \tag{3.18}$$

Now we deal with $\|\partial_i[(\Pi_{\sigma_n} u)(n)\phi_n]\|_{L^2(T)}$ which is first estimated by

$$\|\partial_i[(\Pi_{\sigma_n} u)(n)\phi_n]\|_{L^2(T)} \le C \|\partial_i \phi_n\|_{L^2(T)} |\sigma_n|^{-1} \|u\|_{L^1(\sigma_n)} \tag{3.19}$$

for each $n \in \mathcal{N}_s \cap \overline{T}$.

Let $n \in \mathcal{N}_s \cap \overline{T}$ and be $F_{n,T}$ be the face of S_T having σ_n as an edge and another edge on the singular edge. Let $P_{n,T}$ be the greatest parallelogram contained in $F_{n,T}$ and having σ_n as an edge. So, $P_{n,T}$ is parallel to the x_3-axis, and its area is comparable with the area of $F_{n,T}$ since opposite edges of the trapezoid $F_{n,T}$ have equivalent length. Using a trace inequality we have

$$\|u\|_{L^1(\sigma_n)} \le C|\sigma_n||F_{n,T}|^{-1} (\|u\|_{L^1(P_{n,T})} + h_{3,T} \|\partial_3 u\|_{L^1(P_{n,T})}).$$

But, since $u = 0$ on the edge of $P_{n,T}$ contained on the singular edge we can use the Poincaré inequality to obtain

$$\|u\|_{L^1(\sigma_n)} \le C|\sigma_n||F_{n,T}|^{-1} (|\sigma_n| \|\partial_{\sigma_n} u\|_{L^1(P_{n,T})} + h_{3,T} \|\partial_3 u\|_{L^1(P_{n,T})}). \tag{3.20}$$

From Lemma A.2 we have for all $v \in W^{1,1}(S_T)$

$$\|v\|_{L^1(P_{n,T})} \le C|F_{n,T}||S_T|^{-1} \left(\|v\|_{L^1(S_T)} + |s_{1,T}| \|\partial_{s_{1,T}} v\|_{L^1(S_T)} + |s_{2,T}| \|\partial_{s_{2,T}} v\|_{L^1(S_T)} \right). \tag{3.21}$$

Using twice (3.21) we obtain from (3.20)

$$\begin{aligned} \|u\|_{L^1(\sigma_n)} &\le C|\sigma_n|^2 |S_T|^{-1} \left(\|\partial_{\sigma_n} u\|_{L^1(S_T)} + |s_{1,T}| \|\partial_{s_{1,T}\sigma_n} u\|_{L^1(S_T)} + |s_{2,T}| \|\partial_{s_{2,T}\sigma_n} u\|_{L^1(S_T)} \right) + \\ &\quad + C|\sigma_n||S_T|^{-1} h_{3,T} \left(\|\partial_3 u\|_{L^1(S_T)} + |s_{1,T}| \|\partial_{s_{1,T}3} u\|_{L^1(S_T)} + |s_{2,T}| \|\partial_{s_{2,T}3} u\|_{L^1(S_T)} \right). \end{aligned} \tag{3.22}$$

With the estimates

$$\begin{aligned} \|\partial_{\sigma_n} u\|_{L^1(S_T)} &\le |u|_{W^{1,1}(S_T)}, \\ \|\partial_{s_{i,T}\sigma_n} u\|_{L^1(S_T)} &\le |u|_{W^{2,1}(S_T)}, \qquad i=1,2, \\ \|\partial_{s_{i,T}3} u\|_{L^1(S_T)} &\le |\partial_3 u|_{W^{1,1}(S_T)}, \qquad i=1,2, \end{aligned}$$

the inequality

$$\|\partial_i \phi_n\|_{L^2(T)} \le C h_{i,T}^{-1} |T|^{1/2},$$

and $|\sigma_n| \sim h_{i,T}$ $(i = 1, 2)$ we obtain from (3.19)

$$\begin{aligned} \|\partial_i[(\Pi_{\sigma_n} u)(n)\phi_n]\|_{L^2(T)} &\le C|S_T|^{-1/2} \left(|u|_{W^{1,1}(S_T)} + (h_{1,T} + h_{2,T})|u|_{W^{2,1}(S_T)}\right) \\ &\quad + C|S_T|^{-1/2} \frac{h_{3,T}}{h_{i,T}} \|\partial_3 u\|_{L^1(S_T)}. \end{aligned} \tag{3.23}$$

Finally, taking into account that, since $h_{1,T} = h_{2,T} \le C h_{3,T}$, we have

$$(h_{1,T} + h_{2,T})|u|_{W^{2,1}(S_T)} + h_{3,T}\|\partial_3 u\|_{W^{1,1}(S_T)} \le C \sum_{|\alpha|=1} h_T^\alpha |D^\alpha u|_{W^{1,1}(S_T)},$$

inequality (3.17) follows from (3.18) and (3.23). □

We are now prepared to estimate the interpolation error for needle elements.

Lemma 3.7 (anisotropic needle element, full regularity)**.** *If T is an anisotropic element with $h_{1,T} = h_{2,T} \le C h_{3,T}$ then the local interpolation error estimates*

$$|u - D_h u|_{H^1(T)} \le C \sum_{|\alpha|=1} h_T^\alpha |D^\alpha u|_{H^1(S_T)} \tag{3.24}$$

hold provided that $u \in H^2(S_T)$.

Remark 3.8. *The estimate* (3.24) *does not hold for the Lagrange interpolant, see [2].*

Proof. **(Lemma 3.7)** Since the needle elements with full regularity do not contain a coupling node we can apply both Lemmas 3.3 and 3.5. That means we have shown that

$$\begin{aligned} |D_h u|_{H^1(T)} &\le C \sum_{|\alpha|\le 1} h_T^\alpha \|D^\alpha \partial_3 u\|_{L^2(S_T)} + C\left(|u|_{H^1(S_T)} + h_{3,T}|\partial_3 u|_{H^1(S_T)}\right) \\ &\le C \sum_{|\alpha|\le 1} h_T^\alpha \, |D^\alpha u|_{H^1(S_T)}. \end{aligned}$$

We exploit now that $D_h w = w$ for all $w \in \mathcal{P}_1$. Consequently, we get

$$\begin{aligned} |u - D_h u|_{H^1(T)} &= |(u - w) - D_h(u - w)|_{H^1(T)} \quad \forall w \in \mathcal{P}_1 \\ &\le |u - w|_{H^1(T)} + |D_h(u - w)|_{H^1(T)} \\ &\le C \sum_{|\alpha|\le 1} h_T^\alpha \, |D^\alpha(u - w)|_{H^1(S_T)}. \end{aligned}$$

We use now again a Deny–Lions type argument where the form of Lemma 1 in [1] best suits our needs, and conclude the desired estimate. □

Lemma 3.9 (anisotropic needle element, reduced regularity)**.** *Let T be an anisotropic element with $h_{1,T} = h_{2,T} \le Ch_{3,T}$ and let S_T have zero distance to the singular edge. Then the local interpolation error estimate*

$$|u - D_h u|_{H^1(T)} \le Ch_{1,T}^{1-\delta} \sum_{i=1}^{2} \|\partial_i u\|_{V^{1,2}_{\delta,\delta}(S_T)} + Ch_{3,T}\|\partial_3 u\|_{V^{1,2}_{0,0}(S_T)} \tag{3.25}$$

holds provided that u has the regularity demanded by the right-hand sides of the estimates and $\delta \in [0, 1)$. If T is an element with $h_{1,T} = h_{2,T} \le Ch_{3,T}$ and S_T has zero distance to both a singular vertex and a singular edge then the local interpolation error estimate

$$|u - D_h u|_{H^1(T)} \le Ch_{1,T}^{1-\beta-\delta} h_{3,T}^{\delta} \sum_{i=1}^{2} \|\partial_i u\|_{V^{1,2}_{\beta,\delta}(S_T)} + Ch_{1,T}^{-\beta} h_{3,T}\|\partial_3 u\|_{V^{1,2}_{\beta,0}(S_T)} \tag{3.26}$$

hold provided that u has the regularity demanded by the right-hand sides of the estimates and $\beta, \delta \in [0, 1)$, $\beta + \delta < 1$.

Proof. As in the proof of Lemma 3.7 we distinguish between the derivatives $\partial_3 D_h u$ and the derivatives along directions perpendicular to the x_3-axis. From Lemma 3.3 we obtain by using the triangle inequality and $|S_T|^{-1/2}\|\partial_3 u\|_{L^1(S_T)} \le \|\partial_3 u\|_{L^2(T)}$

$$\begin{aligned}
\|\partial_3(u - D_h u)\|_{L^2(T)} &\le \|\partial_3 u\|_{L^2(T)} + C|S_T|^{-1/2} \sum_{|\alpha|\le 1} h^\alpha \|D^\alpha \partial_3 u\|_{L^1(S_T)} \\
&\le C\|\partial_3 u\|_{L^2(S_T)} + C|S_T|^{-1/2} \sum_{|\alpha|=1} h^\alpha \|D^\alpha \partial_3 u\|_{L^1(S_T)}.
\end{aligned}$$

For the estimate of $\partial_i D_h u$, $i = 1, 2$, we use Lemma 3.6, from which we conclude that

$$\begin{aligned}
&\|\partial_i(u - D_h u)\|_{L^2(T)} \\
&\le C|u|_{H^1(S_T)} + C|S_T|^{-1/2} \left(\frac{h_{3,T}}{h_{i,T}} \|\partial_3 u\|_{L^1(S_T)} + \sum_{|\alpha|=1} h_T^\alpha |D^\alpha u|_{W^{1,1}(S_T)} \right).
\end{aligned}$$

These two estimates can be summarized by using $h_{1,T} \le Ch_{3,T}$ to

$$\begin{aligned}
&|u - D_h u|_{H^1(T)} \\
&\le C|u|_{H^1(S_T)} + C|S_T|^{-1/2} \left(\frac{h_{3,T}}{h_{1,T}} \|\partial_3 u\|_{L^1(S_T)} + \sum_{|\alpha|=1} h_T^\alpha |D^\alpha u|_{W^{1,1}(S_T)} \right).
\end{aligned} \tag{3.27}$$

It remains to estimate the terms against the weighted norms. Firstly, we have

$$\begin{aligned}
|u|_{H^1(S_T)} &\le \sum_{i=1}^{2} \|R^{1-\beta}\theta^{1-\delta} \cdot R^{\beta-1}\theta^{\delta-1}\partial_i u\|_{L^2(T)} + \|R^{1-\beta}\theta \cdot R^{\beta-1}\theta^{-1}\partial_3 u\|_{L^2(T)} \\
&\le \sum_{i=1}^{2} \max_{S_T} R^{1-\beta}\theta^{1-\delta} \, \|\partial_i u\|_{V^{1,2}_{\beta,\delta}(S_T)} + \max_{S_T} R^{1-\beta} \, \|\partial_3 u\|_{V^{1,2}_{\beta,0}(S_T)}.
\end{aligned}$$

With $R^{1-\beta}\theta^{1-\delta} = r^{1-\delta}R^{-\beta}R^{\delta} \le r^{1-\beta-\delta}R^{\delta} \le Ch_{1,T}^{1-\beta-\delta}h_{3,T}^{\delta}$ (where we used the assumption $\beta+\delta \le 1$) and $R^{1-\beta} \le h_{3,T}^{1-\beta} \le h_{1,T}^{-\beta}h_{3,T}$ we derive

$$|u|_{H^1(S_T)} \le Ch_{1,T}^{1-\beta-\delta}h_{3,T}^{\delta}\sum_{i=1}^{2}\|\partial_i u\|_{V_{\beta,\delta}^{1,2}(S_T)} + Ch_{1,T}^{-\beta}h_{3,T}\|\partial_3 u\|_{V_{\beta,0}^{1,2}(S_T)}.$$

With $R^{1-\delta}\theta^{1-\delta} = r^{1-\delta} \le r^{1-\delta} \le Ch_{1,T}^{1-\delta}$ (using that the exponent is positive) we derive also

$$|u|_{H^1(S_T)} \le Ch_{1,T}^{1-\delta}\sum_{i=1}^{2}\|\partial_i u\|_{V_{\delta,\delta}^{1,2}(S_T)} + Ch_{3,T}\|\partial_3 u\|_{V_{0,0}^{1,2}(S_T)}.$$

Secondly, for T intersecting the singular edge, but no singular vertices, we have

$$\frac{h_{3,T}}{h_{1,T}}\|\partial_3 u\|_{L^1(S_T)} \le \frac{h_{3,T}}{h_{1,T}}\|\partial_3 u\|_{V_{0,0}^{1,2}(S_T)}\|r\|_{L^2(S_T)} \le h_{3,T}|S_T|^{1/2}\|\partial_3 u\|_{V_{0,0}^{1,2}(S_T)}.$$

If T has also a singular vertex, then we have with $R^{\beta-1}\theta^{-1} = R^{\beta}r^{-1}$

$$\frac{h_{3,T}}{h_{1,T}}\|\partial_3 u\|_{L^1(S_T)} \le \frac{h_{3,T}}{h_{1,T}}\|\partial_3 u\|_{V_{\beta,0}^{1,2}(S_T)}\|R^{-\beta}r\|_{L^2(S_T)} \le h_{3,T}h_{1,T}^{-\beta}|S_T|^{1/2}\|\partial_3 u\|_{V_{\beta,0}^{1,2}(S_T)}$$

where we used that

$$\|R^{-\beta}r\|_{L^2(S_T)} \le \|r^{1-\beta}\|_{L^2(S_T)} \le Ch_{1,T}^{1-\beta}|S_T|^{1/2} \tag{3.28}$$

which can be obtained by integration. The second derivatives in estimate (3.27) are treated in a similar way. For $i = 1, 2, 3$ we get

$$\|\partial_{i3}u\|_{L^1(S_T)} \le \|R^{-\beta}\|_{L^2(S_T)}\|R^{\beta}\partial_{i3}u\|_{L^2(S_T)} \le h_{1,T}^{-\beta}|S_T|^{1/2}\|\partial_3 u\|_{V_{\beta,0}^{1,2}(S_T)}.$$

For $i, j = 1, 2$ and supposing that T does not have singular vertices we have

$$\begin{aligned} h_{i,T}\|\partial_{ij}u\|_{L^1(S_T)} &\le h_{1,T}\|R^{-\delta}\theta^{-\delta}\|_{L^2(S_T)}\|R^{\delta}\theta^{\delta}\partial_{ij}u\|_{L^2(S_T)} \\ &\le h_{1,T}^{1-\delta}|S_T|^{1/2}|\partial_i u|_{V_{\delta,\delta}^{1,2}(S_T)}, \end{aligned}$$

where we used again an argument as in (3.28). If T has a singular vertex, then

$$h_{i,T}\|\partial_{ij}u\|_{L^1(S_T)} \le h_{1,T}\|R^{-\beta}\theta^{-\delta}\|_{L^2(S_T)}\|R^{\beta}\theta^{\delta}\partial_{ij}u\|_{L^2(S_T)}.$$

But, $R^{-\beta}\theta^{-\delta} = R^{-\beta+\delta}r^{-\delta} \le R^{\delta}r^{-\beta-\delta} \le h_{3,T}^{\delta}r^{-\beta-\delta}$, and so, since $\beta+\delta < 1$, a similar argument as in (3.28) give us

$$\|R^{-\beta}\theta^{-\delta}\|_{L^2(S_T)} \le h_{3,T}^{\delta}\|r^{-\beta-\delta}\|_{L^2(S_T)} \le Ch_{1,T}^{-\beta-\delta}h_{3,T}^{\delta}|S_T|^{1/2}.$$

Hence we have

$$h_{i,T}\|\partial_{ij}u\|_{L^1(S_T)} \le Ch_{1,T}^{1-\beta-\delta}h_{3,T}^{\delta}|S_T|^{1/2}\|\partial_i u\|_{V_{\beta,\delta}^{1,2}(S_T)}.$$

Therefore, the desired estimates are proved. □

Theorem 3.10 (global interpolation error estimate). *Let u be the solution of the boundary value problem (1.1) with $f \in L^2(\Omega)$, and let u_I, u_R be the functions obtained from the splitting (2.1). Assume that the refinement parameters μ_ℓ and ν_ℓ satisfy the conditions*

$$\mu_\ell < \lambda_{\rm e}^{(\ell)}, \tag{3.29}$$

$$\nu_\ell < \lambda_{\rm v}^{(\ell)} + \frac{1}{2}, \tag{3.30}$$

$$\frac{1}{\nu_\ell} + \frac{1}{\mu_\ell}\left(\lambda_{\rm v}^{(\ell)} - \frac{1}{2}\right) > 1, \tag{3.31}$$

$\ell = 1, \dots, L$. Then the global interpolation error estimate

$$|u_R - D_h u_R|_{H^1(\Lambda_\ell)} \le Ch\|f\|_{L^2(\Lambda_\ell)} \tag{3.32}$$

is satisfied.

Proof. The proof can be carried out following the lines of the proof of Theorem 5.1 in [4] with the setting $p = 2$. Note that only a finite number (independent of h) of the S_T overlap at any point. □

Remark 3.11. *The refinement conditions (3.29)–(3.31) were discussed in [4] already: The conditions (3.29) and (3.30) balance the edge and vertex singularities. The third condition, (3.31), follows from (3.30) in the case $\mu_\ell = \nu_\ell$; only in the case $\mu_\ell < \nu_\ell$ it imposes a condition between μ_ℓ and ν_ℓ limiting the anisotropy of the mesh. For the Fichera example treated in Section 4 we have $\lambda_{\rm v}^{(\ell)} \approx 0.454$ and $\lambda_{\rm e}^{(\ell)} = \frac{2}{3}$. With the choice $\nu_\ell = 0.9$ the conditions (3.29) and (3.31) imply the choice $0.414 < \mu_\ell < \frac{2}{3}$. For $\nu_\ell = 0.8$ we would get the weaker condition $0.184 < \mu_\ell < \frac{2}{3}$.*

Note also that in the absence of singularities we have set $\lambda_{\rm e}^{(\ell)} = \infty$ and/or $\lambda_{\rm v}^{(\ell)} = \infty$. In these cases we can set $\mu_\ell = 1$ and/or $\nu_\ell = 1$.

Corollary 3.12 (H^1 and L^2 finite element error estimate). *Let u be the solution of the boundary value problem (1.1), and let u_h be the corresponding finite element solution on a finite element mesh as constructed in Section 2 with grading parameters satisfying the conditions (3.29)–(3.31). Then the discretization error can be estimated by*

$$\|u - u_h\|_{H^1(\Omega)} \le Ch\|f\|_{L^2(\Omega)}, \tag{3.33}$$

$$\|u - u_h\|_{L^2(\Omega)} \le Ch^2\|f\|_{L^2(\Omega)}. \tag{3.34}$$

Proof. We choose $v_h = u_I + D_h u_R$ in estimate (1.4) and observe that $u - v_h = u_R - D_h u_R$. With Lemma 3.10 we obtain the estimate (3.33). The L^2-error estimate can be derived by the standard Aubin–Nitsche method. □

Remark 3.13. *A trivial conclusion from (3.33) is the stability estimate*

$$\|u_h\|_{H^1(\Omega)} \le C\|f\|_{L^2(\Omega)} \tag{3.35}$$

which we will need in Section 5.

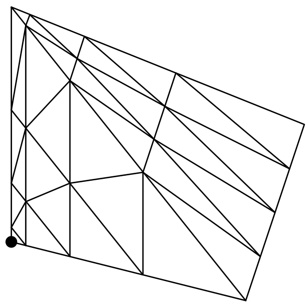

Figure 4: Modification of macroelement of type 4

Remark 3.14. *In macroelements of type 4 with $\mu_\ell = \nu_\ell < 1$, Apel and Nicaise suggested in [4] the use of a more elegant refinement strategy as depicted in Figure 4. Our proof cannot be transfered to this kind of mesh immediately since there may be elements T where S_T is not prismatic as it was exploited in the proof of Lemmas 3.5 and 3.6. We conjecture that the assertion still holds but do not pursue this further in this paper.*

4. Numerical test

As in [4] we consider the Poisson problem (1.1) in the "Fichera domain" $\Omega := (-1,1)^3 \setminus [0,1]^3$ and choose the right-hand side $f = 1 + R^{-3/2}\ln^{-1}(R/4)$ which is in $L^2(\Omega)$ but not in $L^p(\Omega)$ for $p > 2$. For this problem we have $\lambda_v \approx 0.45$ for the concave vertex [29] and $\lambda_e = \frac{\pi}{\omega_0} = \frac{2}{3}$ for the three concave edges. All other edges and vertices are non-singular.

This boundary value problem was solved on quasi-uniform and on graded meshes with our refinement strategy using $\mu = \nu = 0.5 < \min\{\lambda_e, \lambda_v + \frac{1}{2}\}$, where types 1, 2 and 4 occur. Additionally we include the strategy where the macros of type 4 are replaced by type 5, compare Remark 3.14. Pictures of such meshes can be found in [4]. The refinement strategies and an a posteriori error estimator of residual type [31] were implemented into the finite element package MooNMD [19]. The estimated error norms are plotted against the number of unknowns in Figure 5. We see that the theoretical approximation order $h^1 \sim N^{-1/3}$ from Corollary 3.12 can be verified in the practical calculation for both refinement strategies. The error with the second strategy is slightly smaller. We denoted by N the number of nodes.

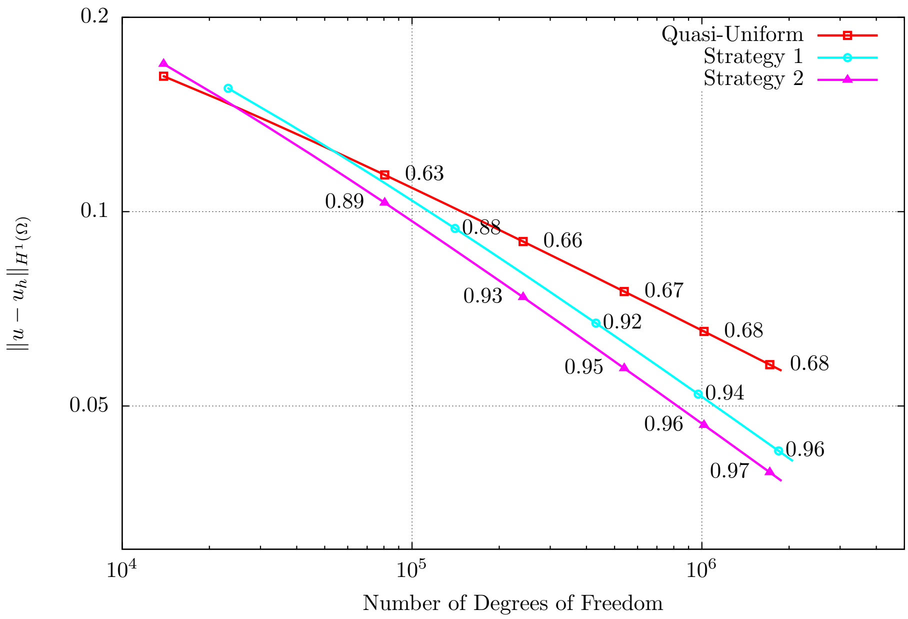

Figure 5: Plot of the estimated error against the number of unknowns. The labels at the curve denote the estimated convergence order in terms of $h \sim N^{-1/3}$.

5. Discretization error estimates for a distributed optimal control problem

Hinze introduced the variational discretization concept for linear-quadratic control constrained optimal control problems in [17]. We follow here this concept in a special case. Consider the the optimal control problem

$$\min_{(y,u)\in H_0^1(\Omega)\times U^{\mathrm{ad}}} J(y,u) := \frac{1}{2}\|y - y_d\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|u\|^2_{L^2(\Omega)},$$

where the state $y \in H_0^1(\Omega)$ is the weak solution of the Poisson problem

$$-\Delta y = u \quad \text{in } \Omega, \qquad y = 0 \quad \text{on } \partial\Omega, \tag{5.1}$$

and the control u is constrained by constant bounds $a_a, u_b \in \mathbb{R}$, this means that the set of admissible controls is defined by

$$U^{\mathrm{ad}} := \{u \in L^2(\Omega) : u_a \le u \le u_b \text{ a.e. } \Omega\}.$$

The regularization parameter α is a fixed positive number and $y_d \in L^2(\Omega)$ is the desired state. It is well known that this problem has a unique optimal solution $(\bar{y}, \bar{u})$. There is an optimal adjoint state $\bar{p} \in H_0^1(\Omega)$, and the triplet $(\bar{y}, \bar{u}, \bar{p})$ satisfies the first order optimality conditions

$$\begin{aligned}
(\nabla \bar{y}, \nabla v)_{L^2(\Omega)} &= (\bar{u}, v)_{L^2(\Omega)} && \forall v \in H_0^1(\Omega), \\
(\nabla \bar{p}, \nabla v)_{L^2(\Omega)} &= (\bar{y} - y_d, v)_{L^2(\Omega)} && \forall v \in H_0^1(\Omega), \\
(\alpha \bar{u} + \bar{p}, u - \bar{u})_{L^2(\Omega)} &\geq 0 && \forall u \in U^{\mathrm{ad}}.
\end{aligned}$$

With the variational discretization concept the approximate solution is obtained by replacing $H_0^1(\Omega)$ by a finite element space $V_h \subset H_0^1(\Omega)$ and searching $(\bar{y}_h, \bar{u}_h, \bar{p}_h) \in V_h \times U^{\mathrm{ad}} \times V_h$ such that

$$\begin{aligned}
(\nabla \bar{y}_h, \nabla v_h)_{L^2(\Omega)} &= (\bar{u}_h, v_h)_{L^2(\Omega)} && \forall v_h \in V_h, \\
(\nabla \bar{p}_h, \nabla v_h)_{L^2(\Omega)} &= (\bar{y}_h - y_d, v_h)_{L^2(\Omega)} && \forall v_h \in V_h, \\
(\alpha \bar{u}_h + \bar{p}_h, u - \bar{u}_h)_{L^2(\Omega)} &\geq 0 && \forall u \in U^{\mathrm{ad}}.
\end{aligned}$$

Note that the control space is not discretized; nevertheless $\bar{u}_h$ can be obtained by the projection of $-\bar{p}_h/\alpha$ onto U^{ad}, see [17]. The discretization error estimate

$$\|\bar{u} - \bar{u}_h\|_{L^2(\Omega)} + \|\bar{y} - \bar{y}_h\|_{L^2(\Omega)} + \|\bar{p} - \bar{p}_h\|_{L^2(\Omega)} \leq Ch^2 \left(\|\bar{u}\|_{L^2(\Omega)} + \|y_d\|_{L^2(\Omega)} \right)$$

can be concluded from (3.34) and (3.35), see [17, 7]. With the proof of Corollary 3.12 we have established this result for anisotropic discretizations of the state equation (5.1) in the case of three-dimensional polyhedral domains.

6. Discrete compactness property for edge elements

The Discrete Compactness Property is a useful tool to study the convergence of finite element discretizations of the Maxwell equations, both for eigenvalue and source problems. It was first introduced by Kikuchi [20] and proved for Nédélec edge elements of lowest order on tetrahedral shape regular meshes. We refer to the monograph by Monk [23] and the references therein for further analysis on isotropic meshes. The property was also analyzed on anisotropically refined tetrahedral meshes on polyhedra for edge elements of lowest order by Nicaise [24] (excluding corner singularities) and by Buffa, Costabel, and Dauge [13].

Lombardi [21] extended this result to edge elements of arbitrary order, also including corners and edge singularities. The proof is based on two tools: 1) interpolation error estimates for edge elements on meshes satisfying the maximum angle condition, and 2) interpolation error estimates for a piecewise linear interpolation operator defined on $W^{2,p}(\Omega) \cap H_0^1(\Omega)$, $p \geq 2$, preserving boundary conditions. For the latter, the Lagrange interpolation was used (implying $p > 2$) together the results of Apel and Nicaise [4], giving some artificial restrictions on the grading parameters defining the allowed anisotropically graded meshes. Using now estimate (3.33) of Corollary 3.12 we can extend the result of [21] allowing little more general meshes.

In what follows we define a family of edge element spaces and introduce the DCP for this family. We refer to [21] for further definitions and notation. First we introduce the divergence-free space

$$X = \{\mathbf{v} \in H_0(\mathbf{curl}, \Omega) : \operatorname{div} \mathbf{v} = 0 \text{ on } \Omega\}.$$

Then we introduce discretizations of this space where the divergence-free condition is weakly imposed. Let I be a denumerable set of positive real numbers having 0 as the only limit point. From now till the end of this section, we assume that $h \in \mathrm{I}$. For each h, let $\mathcal{T}_h$ be the mesh on the polyhedron Ω constructed in Section 2. Given an integer $k \geq 1$, let X_h be the space defined as

$$X_h = \{\mathbf{v}_h \in H_0(\mathbf{curl}, \Omega) : \mathbf{v}_h|_T \in \mathcal{N}_k(T)\, \forall T \in \mathcal{T}_h, (\nabla p_h, \mathbf{v}_h) = 0\, \forall p_h \in S_h\}$$

where $\mathcal{N}_k(T)$ is the space of edge elements of order k on T, and

$$S_h = \left\{p_h \in H_0^1(\Omega) : p_h|_T \in \mathcal{P}_k(T)\, \forall T \in \mathcal{T}_h\right\}.$$

We say that the family of spaces $\{X_h\}_{h\in\mathrm{I}}$ satisfies the discrete compactness property if for each sequence $\{\mathbf{v}_h\}_{h\in\mathrm{J}}$, $\mathrm{J} \subseteq \mathrm{I}$, verifying for a constant C

$$\begin{aligned} &\mathbf{v}_h \in X_h, && \forall h \in \mathrm{J}, \\ &\|\mathbf{v}_h\|_{H_0(\mathbf{curl},\Omega)} \leq C, && \forall h \in \mathrm{J}, \end{aligned}$$

there exists a function $\mathbf{v} \in X$ and a subsequence $\{\mathbf{v}_{h_n}\}_{n\in\mathbb{N}}$ such that (for $n \to \infty$)

$$\begin{aligned} &\mathbf{v}_{h_n} \to \mathbf{v} && \text{in } L^2(\Omega) \\ &\mathbf{v}_{h_n} \rightharpoonup \mathbf{v} && \text{weakly in } H_0(\mathbf{curl}, \Omega). \end{aligned}$$

Theorem 6.1. *If the grading parameters defining the meshes $\mathcal{T}_h$ satisfy the conditions (3.29)–(3.31), then the family of spaces $\{X_h\}_{h>0}$ verifies the discrete compactness property.*

Proof. Follow exactly the arguments used to prove Theorem 5.2 of [21] taking into account that the inequality (4.21) of that paper is now a consequence of estimate (3.33). □

A. Proof of trace inequalities

Lemma A.1. *Let P be a triangular prism with vertices v_i, $i = 1, \ldots, 6$, where the face $v_1v_2v_3$ is opposite to the face $v_4v_5v_6$, and where the edges v_1v_4, v_2v_5, and v_3v_6 are parallel to the x_3-axis, see Figure 6. Denote by F the face $v_1v_2v_3$. Then for all $v \in W^{1,p}(P)$, $p \in [1, \infty)$, we have*

$$\|v\|_{L^p(F)}^p \leq \frac{C_{reg}}{\cos\gamma} \cdot h_3^{-1} \left(\|v\|_{L^p(P)}^p + h_3^p \|\partial_3 v\|_{L^p(P)}^p\right),$$

where h_3 is length of the shortest vertical edge, and γ is the angle between the x_1x_2-plane and the plane containing the face F. The constant C_{reg} depends only on the minimum angle of the face F.

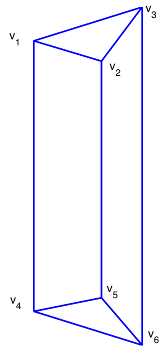

Figure 6: Illustration of the prism

Proof. We can assume $v_1 = (0,0,0)$ and $v_4 = (0,0,h_3)$. Suppose $v_2 = (a_2, b_2, c_2)$, $v_3 = (a_3, b_3, c_3)$. Let s, t such that

$$\begin{aligned} a_2 s + b_2 t &= c_2 \\ a_3 s + b_3 t &= c_3. \end{aligned}$$

It is clear that there exist such s and t since v_1, v_2, and v_3 do not lay on one line. Then the map $f(\tilde{x}) = B\tilde{x}$ with

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ s & t & 1 \end{pmatrix}$$

sends $\widetilde{P}$ to P where $\widetilde{P}$ is a prism with three vertical edges and some of its vertices are $\tilde{v}_1 = (0,0,0), \tilde{v}_2 = (a_2, b_2, 0), \tilde{v}_3 = (a_3, b_3, 0)$ and $\tilde{v}_4 = (0,0,h_3)$. Let $\widetilde{F}$ be the face $\tilde{v}_1\tilde{v}_2\tilde{v}_3$ of $\widetilde{P}$.

Let $\tilde{v}$ be defined by $\tilde{v}(\tilde{x}) = v(x)$ if $x = B\tilde{x}$. Then we have

$$\|v\|^p_{L^p(F)} = \frac{1}{\cos\gamma} \|\tilde{v}\|^p_{L^p(\widetilde{F})}.$$

Now, if $\widetilde{Q}$ is the right prism with vertices $\tilde{v}_1, \ldots, \tilde{v}_4$, (a_2, b_2, h_3) and (a_3, v_3, h_3), then we have using a trace inequality on $\widetilde{Q}$ and noting that $\widetilde{Q} \subseteq \widetilde{P}$ that

$$\begin{aligned} \|\tilde{v}\|^p_{L^p(\widetilde{F})} &\leq C_p h_3^{-1} \left(\|\tilde{v}\|^p_{L^p(\widetilde{Q})} + h_3^p \|\tilde{\partial}_3 \tilde{v}\|^p_{L^p(\widetilde{Q})} \right) \\ &\leq C_p h_3^{-1} \left(\|\tilde{v}\|^p_{L^p(\widetilde{P})} + h_3^p \|\tilde{\partial}_3 \tilde{v}\|^p_{L^p(\widetilde{P})} \right) \end{aligned}$$

with C_p depending only on p. Therefore, we have

$$\begin{aligned} \|v\|^p_{L^p(F)} &= \frac{C_p}{\cos\gamma} h_3^{-1} \left(\|\tilde{v}\|^p_{L^p(\widetilde{P})} + h_3^p \|\tilde{\partial}_3 \tilde{v}\|^p_{L^p(\widetilde{P})} \right) \\ &= \frac{C_{reg}}{\cos\gamma} h_3^{-1} |B| \left(\|v\|^p_{L^p(P)} + h_3^p \|\partial_3 v\|^p_{L^p(P)} \right) \end{aligned}$$

where we used that $\tilde{\partial}_3 \tilde{v}(\tilde{x}) = \partial_3 v(x)$. Since $|B| = 1$ we obtain the desired result. $\square$

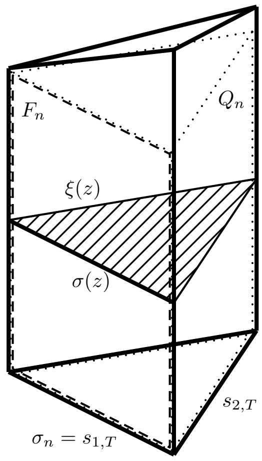

Figure 7: Illustration of the notation used in Lemma A.2. The dotted lines indicate the prism Q_n, dashed lines the parallelogram P_n while the triangle $\xi(z)$ is hatched. Note that $\sigma(z) = \overline{\xi(z)} \cap F_n$.

Lemma A.2. *Let T be an anisotropic element with the node n on the singular edge and let σ_n be a short edge. Let $P_n \subset \overline{S_T}$ be a parallelogram of maximal area having σ_n as an edge and another edge on the singular edge, see Figure 7. And let F_n the face of S_T containing P_n. Then $|P_n| \geq C|F_n|$, and for all $v \in W^{1,1}(S_T)$ we have*

$$\|v\|_{L^1(P_n)} \leq C|F_n||S_T|^{-1} \left(\|v\|_{L^1(S_T)} + |s_{1,T}|\|\partial_{s_{1,T}} v\|_{L^1(S_T)} + |s_{2,T}|\|\partial_{s_{2,T}} v\|_{L^1(S_T)}\right).$$

where $s_{1,T}$ and $s_{2,T}$ are two short edges of T.

Proof. The inequality $|P_n| \geq C|F_n|$ follows from our assumptions on the mesh, in particular from the comparable length of opposite edges of F_n. For proving the estimate choose the coordinate system such that $n = (0,0,0)$.

Assume first v is regular. We have

$$\begin{aligned} \|v\|_{L^1(P_n)} &\leq C \int_0^{h_{3,P_n}} \int_0^{|\sigma_n|} |v((0,0,z) + t\sigma_n)| \, dt \, dz \\ &= \int_0^{h_{3,P_n}} \int_{\sigma(z)} |v| \, ds \, dz \end{aligned}$$

where $\sigma(z)$ is the segment parallel to σ_n and with the same length and passing through $(0,0,z)$. If $\xi(z)$ is the triangle contained in S_T having $\sigma(z)$ as an edge and being parallel to the bottom face of S_T, then since we can assume $v|_{\xi(z)}$ is regular (because v is it), by a trace inequality we have

$$\int_{\sigma(z)} |v| \leq C \frac{|\sigma_n|}{|\xi|} \int_{\xi(z)} (|v| + |s_{1,T}||\partial_{s_{1,T}} v| + |s_{2,T}||\partial_{s_{2,T}} v|)$$

where $|s_{1,T}|$ and $|s_{2,T}|$ are the lengths of two small edges of T and $|\xi| = |\xi(0)|$. So we have

$$\begin{aligned}
\|v\|_{L^1(P_n)} &\leq C\frac{|\sigma_n|}{|\xi|}\int_0^{h_{3,P_n}}\int_{\xi(z)}(|v| + |s_{1,T}||\partial_{s_{1,T}}v| + |s_{2,T}||\partial_{s_{2,T}}v|) \\
&\leq C\frac{|F_n|}{|S_T|}\int_0^{h_{3,P_n}}\int_{\xi(z)}(|v| + |s_{1,T}||\partial_{s_{1,T}}v| + |s_{2,T}||\partial_{s_{2,T}}v|) \\
&\leq C\frac{|F_n|}{|S_T|}\int_{Q_n}(|v| + |s_{1,T}||\partial_{s_{1,T}}v| + |s_{2,T}||\partial_{s_{2,T}}v|) \\
&\leq C\frac{|F_n|}{|S_T|}\int_{S_T}(|v| + |s_{1,T}||\partial_{s_{1,T}}v| + |s_{2,T}||\partial_{s_{2,T}}v|)
\end{aligned}$$

where Q_n is the prism formed by the union of $\xi(z)$ with $z \in [0, h_{3,P_n}]$ that is contained in S_T.

If $v \in W^{1,1}(S_T)$, let $\{v_k\}_k$ be a sequence of C^∞ functions converging to v in $W^{1,1}(S_T)$. For each k we have

$$\|v_k\|_{L^1(P_n)} \leq C|F_n||S_T|^{-1}\left(\|v_k\|_{L^1(S_T)} + |s_{1,T}|\|\partial_{s_{1,T}}v_k\|_{L^1(S_T)} + |s_{2,T}|\|\partial_{s_{2,T}}v_k\|_{L^1(S_T)}\right).$$

Now, the proof concludes by taking limit $k \to \infty$. □

Acknowledgement. The work of all authors was supported by DFG (German Research Foundation), IGDK 1754. The work of the second author is also supported by ANPCyT (grant PICT 2010-1675 and PICTO 2008-00089) and by CONICET (grant PIP 11220090100625). This support is gratefuly acknowledged.

References

[1] Th. Apel. Interpolation of non-smooth functions on anisotropic finite element meshes. *Math. Modeling Numer. Anal.*, 33:1149–1185, 1999.

[2] Th. Apel and M. Dobrowolski. Anisotropic interpolation with applications to the finite element method. *Computing*, 47:277–293, 1992.

[3] Th. Apel and B. Heinrich. Mesh refinement and windowing near edges for some elliptic problem. *SIAM J. Numer. Anal.*, 31:695–708, 1994.

[4] Th. Apel and S. Nicaise. The finite element method with anisotropic mesh grading for elliptic problems in domains with corners and edges. *Math. Methods Appl. Sci.*, 21:519–549, 1998.

[5] Th. Apel, A.-M. Sändig, and J. R. Whiteman. Graded mesh refinement and error estimates for finite element solutions of elliptic boundary value problems in non-smooth domains. *Math. Methods Appl. Sci.*, 19:63–85, 1996.

[6] Th. Apel and D. Sirch. L^2-error estimates for Dirichlet and Neumann problems on anisotropic finite element meshes. *Appl. Math.*, 56:177–206, 2011.

[7] Th. Apel and D. Sirch. A priori mesh grading for distributed optimal control problems. In G. Leugering, S. Engell, A. Griewank, M. Hinze, R. Rannacher, V. Schulz, M. Ulbrich, and S. Ulbrich, editors, *Constrained Optimization and Optimal Control for Partial Differential Equations*, volume 160 of *International Series of Numerical Mathematics*, pages 377–389. Springer, Basel, 2011.

[8] F. Assous, P. Ciarlet, Jr., and J. Segré. Numerical solution to the time-dependent Maxwell equations in two-dimensional singular domains: the Singular Complement Method. *J. Comput. Phys.*, 161:218–249, 2000.

[9] I. Babuška. Finite element method for domains with corners. *Computing*, 6:264–273, 1970.

[10] A. E. Beagles and J. R. Whiteman. Finite element treatment of boundary singularities by augmentation with non-exact singular functions. *Numer. Methods Partial Differential Equations*, 2:113–121, 1986.

[11] H. Blum and M. Dobrowolski. On finite element methods for elliptic equations on domains with corners. *Computing*, 28:53–63, 1982.

[12] C. Băcuţă, V. Nistor, and L. T. Zikatanov. Improving the rate of convergence of high-order finite elements in polyhedra II: mesh refinements and interpolation. *Numer. Funct. Anal. Optimization*, 28:775–824, 2007.

[13] A. Buffa, M. Costabel, and M. Dauge. Algebraic convergence for anisotropic edge elements in polyhedral domains. *Numer. Math.*, 101:29–65, 2005.

[14] P. Clément. Approximation by finite element functions using local regularization. *RAIRO Anal. Numer.*, 2:77–84, 1975.

[15] T. Dupont and R. Scott. Polynomial approximation of functions in Sobolev spaces. *Math. Comp.*, 34:441–463, 1980.

[16] P. Grisvard. *Singularities in boundary value problems*, volume 22 of *Research Notes in Applied Mathematics*. Springer, New York, 1992.

[17] M. Hinze. A variational discretization concept in control constrained optimization: The linear-quadratic case. *Comput. Optim. Appl.*, 30:45–61, 2005.

[18] P. Jamet. Estimations d'erreur pour des éléments finis droits presque dégénérés. *R.A.I.R.O. Anal. Numér.*, 10:43–61, 1976.

[19] V. John and G. Matthies. MooNMD–a program package based on mapped finite element methods. *Computing and Visualization in Science*, 6:163–169, 2004.

[20] F. Kikuchi. On a discrete compactness property for the nédélec finite elements. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, 36:479–490, 1989.

[21] A. L. Lombardi. The discrete compactness property for anisotropic edge elements on polyhedral domains. *ESAIM: M2AN*, 47:169–181, 2013.

[22] J. M.-S. Lubuma and S. Nicaise. Dirichlet problems in polyhedral domains II: approximation by FEM and BEM. *J. Comp. Appl. Math.*, 61:13–27, 1995.

[23] P. Monk. *Finite Element Methods for Maxwell's Equations.* Oxford University Press, New York, 2003.

[24] S. Nicaise. Edge elements on anisotropic meshes and approximation of the Maxwell equations. *SIAM J. Numer. Anal.*, 39:784–816, 2001.

[25] L. A. Oganesyan and L. A. Rukhovets. Variational-difference schemes for linear second-order elliptic equations in a two-dimensional region with piecewise smooth boundary. *Zh. Vychisl. Mat. Mat. Fiz.*, 8:97–114, 1968. In Russian. English translation in USSR Comput. Math. and Math. Phys., 8 (1968) 129–152.

[26] T. von Petersdorff and E. P. Stephan. Regularity of mixed boundary value problems in $\mathbb{R}^3$ and boundary element methods on graded meshes. *Math. Methods Appl. Sci.*, 12:229–249, 1990.

[27] G. Raugel. *Résolution numérique de problèmes elliptiques dans des domaines avec coins.* PhD thesis, Université de Rennes, 1978.

[28] A. H. Schatz and L. B. Wahlbin. Maximum norm estimates in the finite element method on plane polygonal domains. Part 2: Refinements. *Math. Comp.*, 33(146):465–492, 1979.

[29] H. Schmitz, K. Volk, and W. L. Wendland. On three-dimensional singularities of elastic fields near vertices. *Numer. Methods Partial Differential Equations*, 9:323–337, 1993.

[30] L. R. Scott and S. Zhang. Finite element interpolation of non-smooth functions satisfying boundary conditions. *Math. Comp.*, 54:483–493, 1990.

[31] K. Siebert. An a posteriori error estimator for anisotropic refinement. *Numer. Math.*, 73:373–398, 1996.

[32] G. Strang and G. Fix. *An analysis of the finite element method.* Prentice–Hall, Englewood Cliffs, NJ, 1973.