ON THE TOPOLOGICAL ENTROPY OF THE IRREGULAR PART OF V-STATISTICS MULTIFRACTAL SPECTRA

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Abstract. Let (X,d) be a compact metric space and $f : X \to X$, if X^r is the product of $r-$copies of X, $r \geq 1$, and $\Phi : X^r \to \mathbf{R}$, then the multifractal decomposition for $V-$statistics is given by
$E_{\Phi}(\alpha) = \left\{ x : \lim_{n\to\infty} \frac{1}{n^r} \sum_{0\leq i_1 \leq \ldots \leq i_r \leq n-1} \Phi\left(f^{i_1}(x), ..., f^{i_r}(x)\right) = \alpha \right\}$. The irregular part, or historic set, of the spectrum is the set points $x \in X$, for which the limit does not exist.

In this article we prove that for dynamical systems with specification, the irregular part of the $V-$statistics spectrum has topological entropy equal to that of the whole space X.

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1. INTRODUCTION

Motivated by the problems on convergence of multiple ergodic averages Fan, Schmeling and Wu[5], treated the problem of multifractal analysis of V-statistics.

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In the present paper, we would like to study the irregular part of the multifractal decomposition.

Let us consider a topological dynamical system (X, f), with X a compact metric space and f a continuous map. Let $X^r = X \times ... \times X$ be the product of $r-$copies of X with $r \geq 1$, if $\Phi : X^r \to \mathbf{R}$ is a continuous map, then let

$$V_\Phi (n, x) = \frac{1}{n^r} \sum_{1 \leq i_1, \ldots, i_r \leq n} \Phi \left(f^{i_1} (x), ..., f^{i_r} (x) \right). \tag{1}$$

These averages are called the *$V-$statistics of order r with kernel Φ*. For the idea of V-statistics from a Statistical point of view and its relationship with the U-statistics see section 2 of[5]

Ergodic limits of the form

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \Phi \left(f^{i_1} (x), ..., f^{i_r} (x) \right),$$

were studied among others by Furstenberg[8], Bergelson[2] and Bourgain[3].

The multifractal spectra of $V-$statistics are specified by the decomposition sets

$$E_\Phi (\alpha) = \left\{ x : \lim_{n \to \infty} V_\Phi (n, x) = \alpha \right\}.$$

Fan, Schemeling and Wu[5] treated the problem of measuring the sizes of the multifractal sets $E_\Phi (\alpha)$. They established the following variational principle:

$$h_{top}(E_\Phi (\alpha)) = \sup \left\{ h_\mu (f) : \int \Phi d\mu^{\otimes r} = \alpha \right\},$$

where h_μ is the measure-theoretic entropy of μ. This formula is valid for dynamical systems with the specification property. This generalizes the variational formula obtained by Takens and Verbitski for $r = 1$[9].

The *irregular part* of the spectrum, or *historic set*, is the set of points x for which $\lim_{n \to \infty} V_\Phi (n, x)$ does not exist. We denote this set by E_Φ^∞, so that the space X can be decomposed as

$$X = \bigcup_{\alpha \in \mathbf{R}} E_\Phi (\alpha) \cup E_\Phi^\infty$$

An important problem in Multifractal Analysis is to determine the dimension of the irregular part. For $r = 1$ the irregular part of the spectrum has been extensively studied. Fan, Feng and Wu, in reference [6], did it for topological mixing subshifts. Barreira and Schmeling[1] obtained a similar result than [6] but for Hölder continuous maps. More recently the irregular part was studied by Thompson [11] and by Zhou and Chen [13]. Here we propose the study of the irregular part of the spectrum for multiple ergodic averages. The result to be proved is

Theorem: Let (X, f) be a dynamical system with the property of specification, let $\Phi \in C(X^r)$, $r \geq 1$, if the irregular part E_Φ^∞ of the spectrum of multiple ergodic averages $V_\Phi(n, x)$ is non-empty then it has the same topological entropy as the whole space X.

The case $E_\Phi^\infty = \varnothing$ can occur in situations like for instance Φ cohomologous to 0, or when the ergodic limits $V_\Phi(n, x)$ have the same value for any x.

2. Preliminary definitions

Firstly let us recall the Bowen definition of topological entropy of sets: Let $f : X \to X$, with X a compact metric space, for $n \geq 1$ the dynamical metric, or Bowen metric, is $d_n(x, y) = \max\left\{d\left(f^i(x), f^i(y)\right) : i = 0, 1, ..., n-1\right\}$. We denote by $B_{n,\varepsilon}(x)$ the ball of centre x and radius ε in the metric d_n. Let $Z \subset X$ and let $\mathcal{C}(n, \varepsilon, Z)$ be the collection of finite or countable coverings of the set Z by balls $B_{m,\varepsilon}(x)$ with $m \geq n$. Let

$$M(Z, s, n, \varepsilon) = \inf_{\mathcal{B} \in \mathcal{C}(n,\varepsilon,Z)} \sum_{B_{m,\varepsilon}(x) \in \mathcal{B}} \exp(-sm),$$

and set

$$M(Z, s, \varepsilon) = \lim_{n\to\infty} M(Z, s, n, \varepsilon).$$

There is an unique number $\overline{s}$ such that $M(Z, s, \varepsilon)$ jumps from $+\infty$ to 0. Let

$$H(Z, \varepsilon) = \overline{s} = \sup\{s : M(Z, s, \varepsilon) = +\infty\} = \inf\{s : M(Z, s, \varepsilon) = 0\},$$

and

$$h_{top}(Z) = \lim_{\varepsilon\to 0} H(Z, \varepsilon). \tag{2}$$

The number $h_{top}(Z)$ is the *topological entropy* of Z.

A dynamical system (X, f) has the *specification property* if the following condition holds: for $\varepsilon > 0$, there is an integer $M(\varepsilon)$ such that for any finite disjoint collection of integer intervals $I_1 = [a_1, b_1], ..., I_k = [a_k, b_k]$, of length $\geq M(\varepsilon)$ and for any points $x_1, x_2, ..., x_k \in X$, there is a point $z \in X$ which $\varepsilon-$shadows the sequence $\{x_1, x_2, ..., x_k\}$, i.e. $d(f^{a_j+n}(z), f^n(x_j)) \leq \varepsilon$, for any $n = 0, .., b_j - a_j$ and $j = 0, 1, ..., k$.

By $\mathcal{M}(X)$ we denote the space of measures in X, and by $\mathcal{M}_{inv}(X, f)$ the space of $f-$invariant measures on X. The space $\mathcal{M}(X)$ can be endowed with a metric D compatible with the metric in X, in the sense that $D(\delta_x, \delta_y) = d(x, y)$, where δ is the point mass measure. More precisely the metric considered in $\mathcal{M}(X)$ will be

$$D(\mu, \nu) =_{n=1}^{\infty} \frac{\left|\int \varphi_n d\mu - \int \varphi_n d\nu\right|}{2^n \left\|\varphi_n\right\|_{\infty}},$$

where $\{\varphi_n\}$ is a dense set in $C(X)$. We denote by $B_R(\mu)$ the ball of center μ and radius R in the above metric. The topology induced by this metric is the weak star topology, and if X is compact then $\mathcal{M}(X)$ is compact in the weak topology. The weak star convergence is the convergence in the metric which induces the weak star topology.

The so called empirical measures on X associated to the dynamical system (X, f) are

$$\mathcal{E}_n(x) = \frac{1}{n} \sum_{i=0}^{n-1} \delta_{f^i(x)}.$$

We denote the weak limits of the sequence$\{\mathcal{E}_n(x)\}$ by $V(x)$. Since X is compact, $V(x) \neq \varnothing$. If μ is a measure on X then a point $x \in X$ is $\mu-$generic if $V(x) = \{\mu\}$, by $G(\mu)$ is denoted the set of $\mu-$generic points. A result by Bowen[4] is that if μ is ergodic then

$$h_{top}(G(\mu)) = h_{\mu}(f).$$

For general measures, not necessarily ergodic, the equality holds for dynamical systems with the specification property[7]. This result is the key point in the proof of variational theorem of Fan, Schemeling and Wu[5].

3. Proof of the theorem

Let

$$\mathcal{M}_{\Phi}(\alpha)=\left\{\mu\in\mathcal{M}_{inv}(X):\int\Phi d\mu^{\otimes r}=\alpha\right\},$$

and let

$$G_{\Phi}(\alpha)=\left\{x:\text{ there is }\{n_k\}\text{ such that }w^{*}-\lim_{k\to\infty}\mathcal{E}_{n_k}(x)=\mu\in\mathcal{M}_{\Phi}\ (\alpha)\right\},$$

here $w^{*}-$ means weak star convergence.

For $\alpha_1\neq\alpha_2\in\mathbf{R}$, we shall find a set $G\subset G_{\Phi}(\alpha_1)\cap G_{\Phi}(\alpha_2)$.

Before proving the theorem we give some lemmas.

Lemma 1: If $\alpha_1\neq\alpha_2$ then $G_{\Phi}(\alpha_1)\cap G_{\Phi}(\alpha_2)\subset E_{\Phi}^{\infty}$.

Proof: In [5] was established, as a consequence of the Stone-Weierstrass theorem, that for any $\Phi\in C(X^r)$ and for any $\varepsilon>0$ there is a map $\widetilde{\Phi}:X^r\to\mathbf{R}$ of the form

$$\widetilde{\Phi}=\sum_j\varphi_j^{(1)}\otimes...\otimes\varphi_j^{(r)},$$

with $\varphi_j^{(i)}\in C(X)$ such that $\left\|\Phi-\widetilde{\Phi}\right\|_{\infty}<\varepsilon$. Let $x\in G_{\Phi}(\alpha_1)\cap G_{\Phi}(\alpha_2)$, so there are sequences $\{n_k\}$, $\{m_k\}$ such that

$$\text{(3)}\qquad \begin{aligned}\mu&=w^{*}-\lim_{k\to\infty}\mathcal{E}_{n_k}(x)\,;\mu\in\mathcal{M}_{\Phi}(\ \alpha_1)\\ \nu&=w^{*}-\lim_{k\to\infty}\mathcal{E}_{m_k}(x)\,;\nu\in\mathcal{M}_{\Phi}(\ \alpha_2),\end{aligned}$$

We have

$$\text{(4)}\qquad V_{\widetilde{\Phi}}(n,x)=\sum_j\prod_{i=1}^{r}\frac{1}{n}S_n\left(\varphi_j^{(i)}(x)\right),$$

where $S_n\left(\varphi_j^{(i)}(x)\right)=\sum\limits_{k=0}^{n-1}\varphi_j^{(i)}\left(f^k(x)\right)$. Therefore, by Eqs.(3-4)

$$\lim_{k\to\infty} V_{\widetilde{\Phi}}\left(n_k,x\right)=\int \widetilde{\Phi}d\mu^{\otimes r}$$
$$\lim_{k\to\infty} V_{\widetilde{\Phi}}\left(m_k,x\right)=\int \widetilde{\Phi}d\nu^{\otimes r}.$$

By the above argument of approximation we get in the same way of [5] that

$$\lim_{k\to\infty} V_{\Phi}\left(n_k,x\right)=\int \Phi d\mu^{\otimes r}=\alpha_1 \text{ and } \lim_{k\to\infty} V_{\Phi}\left(m_k,x\right)\int \Phi d\nu^{\otimes r}=\alpha_2, \text{ with } \alpha_1\neq\alpha_2.$$

Then $x\in E_{\Phi}^{\infty}$.

■

We have that

$$G_{\Phi}\left(\alpha\right)\subset\left\{x:\exists\;\mu\in V(x),\text{ such that } h_{\mu}\left(f\right)\leq\sup\left\{h_{\mu}\left(f\right):\int\Phi d\mu^{\otimes r}=\alpha\right\}\right\},$$

and so, by the Bowen lemma

$$h_{top}(G_{\Phi}\left(\alpha\right))\leq\sup\left\{h_{\mu}\left(f\right):\int\Phi d\mu^{\otimes r}=\alpha\right\}$$

For $\rho_1,\rho_2,...,\rho_k\in\mathcal{M}(X)$ and positive numbers $R_1,R_2,...,R_k$, let $x_1,x_2,...,x_k\in X$, $n_1,n_2,...,n_k\in\mathbf{N}$ such that $\mathcal{E}_{n_j}\left(x_j\right)\in B_{R_j}\left(\rho_j\right)$, $j=1,2,...,k.$, for a given $\rho_1,\rho_2,...,\rho_k\in\mathcal{M}(X)$ and $R_1,R_2,...,R_k$. Let $\varepsilon_1>0$, $\varepsilon_2>0,...,$ $\varepsilon_k>0$, if $n_i>M\left(\varepsilon_i\right)$ (the number of specification), $i=1,2,...,k$, then by specification property

$$\bigcap_{j=1}^{k} f^{-M_{j-1}}\left(B_{n_j,\varepsilon_j}\left(x_j\right)\right)\neq\varnothing,\text{with } M_j=n_1+n_2+...+n_j.$$

Lemma 2: Let $z\in\bigcap_{j=1}^{k} f^{-M_{j-1}}\left(B_{n_j,\varepsilon_j}\left(x_j\right)\right)$, then for any $\rho\in\mathcal{M}(X)$ holds

$$D\left(\mathcal{E}_{M_k}\left(z\right),\rho\right)\leq\frac{1}{M_k}\sum_{j=1}^{k}n_j\left(\overline{R_j}+D\left(\rho_j,\rho\right)\right),$$

where $\overline{R_j}=R_j+\varepsilon_j$, $j=1,2,...,k..,$

Remark: It can replace an uniform ε for all balls $B_{n_j,\varepsilon}\left(x_j\right)$, by the ε_1, ε_2, ..., ε_k, following a trick used in the proof of the proposition 2.1 in [10].

Proof: We have

$$\mathcal{E}_{M_k}(z) = \frac{1}{M_k}\sum_{j=1}^{k} n_j \mathcal{E}_{n_j}\left(f^{M_{j-1}}(z)\right),$$

and

$$D(\mathcal{E}_{n_j}(x_j), \mathcal{E}_{n_j}\left(f^{M_{j-1}}(z)\right)) \leq \frac{1}{n_j}\sum_{l=0}^{n_j-1} d\left(f^l(x_j), f^{-M_{j-1}-l}(z)\right).$$

Therefore

$$\begin{aligned} & D\left(\mathcal{E}_{M_k}(z), \rho\right) \\ \leq\ & \frac{1}{M_k}\sum_{j=1}^{k}\left[D(\mathcal{E}_{n_j}(x_j),_j, \mathcal{E}_{n_j}\left(f^{M_{j-1}}(z)\right)) + D(\mathcal{E}_{n_j}(x_j), \rho_j) + D(\rho_j, \rho)\right] \\ \leq\ & \frac{1}{M_k}\sum_{j=1}^{k}\left[R_j + \varepsilon_j + D(\rho_j, \rho)\right] \end{aligned}$$

■.

Lemma 3: Let $\alpha_1 \neq \alpha_2$ with $\mathcal{M}_\Phi(\alpha_1) \neq \varnothing$, $\mathcal{M}_\Phi(\alpha_2) \neq \varnothing$ then

$$h_{top}(G_\Phi(\alpha_1) \cap G_\Phi(\alpha_2)) = \min\left\{h_{top}(G_\Phi(\alpha_1)),\ h_{top}(G_\Phi(\alpha_2))\right\}.$$

Proof: Since $G_\Phi(\alpha_1) \cap G_\Phi(\alpha_2) \subset G_\Phi(\alpha_1)$ and $G_\Phi(\alpha_1) \cap G_\Phi(\alpha_2) \subset G_\Phi(\alpha_2)$, by the monotonicity of the entropy we have

$$h_{top}(G_\Phi(\alpha_1) \cap G_\Phi(\alpha_2)) \leq \min\left\{h_{top}(G_\Phi(\alpha_1)),\ h_{top}(G_\Phi(\alpha_2))\right\}$$

. To prove the other inequality we shall find a set $G \subset\ G_\Phi(\alpha_1) \cap G_\Phi(\alpha_2)$ with $h_{top}(G) \geq \min\left\{h_{top}(G_\Phi(\alpha_1)),\ h_{top}(G_\Phi(\alpha_2))\right\}$.

To construct G, let us choose sequences $\{n_k\}$, $\{R_k\}$, $\{\varepsilon_k\}$ with $R_k \searrow 0$ and $\varepsilon_k \searrow 0$ and, for a given sequence $\{\rho_1, \rho_2, ..., \rho_k\} \subset \mathcal{M}(X)$, for $,\overline{\varepsilon} > \varepsilon_1$.let us consider $(n_k, \overline{\varepsilon})$ –sets $\Gamma_k \subset \{x : \mathcal{E}_{n_k}(x) \in B_{R_k}(\rho_k)\}$, so that (by the Lemma 2)

$$x \in \Gamma_k, z \in B_{n_k,\varepsilon_k}(x) \Longrightarrow \mathcal{E}_{n_k}(z) \in B_{R_k+\varepsilon_k}(\rho_k).$$

Let us choose now a strictly increasing sequence $\{N_k\}$ such that

$$n_{k+1} \leq R_k \sum_{j=1}^{k} n_j N_j$$

and

$$\sum_{j=1}^{k-1} n_j N_j \leq R_k \sum_{j=1}^{k} n_j N_j.$$

We consider stretched sequences $\left\{ n_j^{'} \right\}, \left\{ \varepsilon_j^{'} \right\}, \left\{ \Gamma_j^{'} \right\}$ such that if $j = N_1 + ... + N_{k-1} + q$ with $1 \leq q \leq N_k$ then $n_j^{'} = n_k$, $\varepsilon_j^{'} = \varepsilon_k$ and $\Gamma_j^{'} = \Gamma_k$.

Finally, we can define

$$G_k := \bigcap_{j=1}^{k} \left(\bigcup_{x_j \in \Gamma_j^{'}} f^{-M_{j-1}} \left(B_{n_j^{'}, \varepsilon_j^{'}} (x_j) \right) \right), \tag{5}$$

with $M_j = n_1^{'} + n_2^{'} + ... + n_j^{'}$ and

$$G := \bigcap_{k \geq 1} G_k. \tag{6}$$

Any element of G can be labelled by a sequence $x_1\, x_2 ...,$ with $x_j \in \Gamma'_j$. According to Pfister and Sullivan [10] the following holds: Let $x_j, y_j \in \Gamma'_j$, $x_j \neq y_j$, if $x \in B_{n_j, \epsilon_j}(x_j)$, $y \in B_{n_j, \epsilon_j}(y_j)$ then $\max \left\{ d\left(f^k(x), f^k(y)\right) : k = 0, ..., n_j - 1 \right\} > 2\varepsilon$, with $\varepsilon > \varepsilon_1/4$.

We see that $G \subset G_\Phi(\alpha_1) \cap G_\Phi(\alpha_2)$. Let $z \in G$, and let $\mu_0 \in \mathcal{M}_\Phi(\alpha_1)$, $\nu_0 \in \mathcal{M}_\Phi(\alpha_2)$, it can be considered sequences[13] $\{\mu_k\}, \{\nu_k\}$ such that$D(\mu_0, \mu_k) < R_k$ and $D(\nu_0, \nu_k) < R_k$, then form the sequence

$$\{\rho_k\} = \{\mu_1, \mu_1, \nu_1, \nu_1, \mu_2, \mu_2, \nu_2, \nu_2, ...\}.$$

Let $\rho \in \{\mu_0, \nu_0\}$, and $\sum_{l=1}^{j} n_l N_l \leq M_k \leq \sum_{l=1}^{j+1} n_l N_l$, thus

$$D\left(\mathcal{E}_{M_k}(z), \rho\right) \leq \frac{1}{M_k} \sum_{l=1}^{j-1} n_l N_l D\left(\mathcal{E}_{\sum_{l=1}^{j-1} n_l N_l}(z), \rho \right) + \frac{n_j N_j}{M_k} D\left(\mathcal{E}_{n_j N_j}(z), \rho\right) +$$
$$\frac{M_k - \sum_{l=1}^{j} n_l N_{lj}}{M_k} D\left(\mathcal{E}_{n_{j+1} N_{j+1}}(z), \rho\right).$$

Therefore

$$\begin{aligned} & D\left(\mathcal{E}_{M_k}(z), \rho\right) \\ \leq\ & R_j + D\left(\mathcal{E}_{n_j N_j}(z), \rho_j\right) + D\left(\rho_j, \rho\right) + D\left(\mathcal{E}_{n_{j+1} N_{j+1}}(z), \rho\right) + D\left(\rho_{j+1}, \rho\right) \\ \leq\ & 2R_j + \varepsilon_j + D\left(\rho_j, \rho\right) + D\left(\rho_{j+1}, \rho\right). \end{aligned}$$

Thus, choosing subsequences $t_k = 4k+1$ and $s_k = 4k+3$, we get

$$\mu_0 = w^* - \lim_{k\to\infty}\ \mathcal{E}_{M_{t_k}}(z)$$
$$\nu_0 = w^* - \lim_{k\to\infty}\ \mathcal{E}_{M_{s_k}}(z),$$

so that $z \in G_\Phi(\alpha_1) \cap G_\Phi(\alpha_2)$.

To complete the proof it must be proved that

$$h_{top}(G) \geq \min\left\{h_{top}(G_\Phi(\alpha_1)),\ h_{top}(G_\Phi(\alpha_2))\right\}.$$

For this, we follow [10]. Let $s < \overline{h} := \min\left\{h_{top}(G_\Phi(\alpha_1)),\ h_{top}(G_\Phi(\alpha_2))\right\}$, the set G is closed, and so it is compact, let us consider a finite covering $\mathcal{U}$ by balls $B_{m,\varepsilon}(x)$ having non-empty intersection with G. Now

$$M(G,s,N,\varepsilon) = \inf_{\mathcal{U}\in\mathcal{C}(n,\varepsilon,G)} \sum_{B_{m,\varepsilon}(x)\in\mathcal{U}} \exp(-sm).$$

For any finite covering $\mathcal{U}$ of G, we can construct a covering $\mathcal{U}_0$ in the following way: each ball $B_{m,\varepsilon}(x)$ is replaced by a ball $B_{M_{rr},\varepsilon}(x)$ with $M_r \leq m \leq M_{r+1}$. Thus

$$M(G,s,N,\varepsilon) = \inf_{\mathcal{U}\in\mathcal{C}(n,\varepsilon,G)} \sum_{B_{m,\varepsilon}(x)\in\mathcal{U}} \exp(-sm) \geq \inf_{\mathcal{U}\in\mathcal{C}(N,\varepsilon,G)} \sum_{B_{M_r,\varepsilon}\in\mathcal{U}_0} \exp(-sM_{r+1}).$$

Now we can consider a covering $\mathcal{U}_0$ in which

$$m = \max\left\{r : \text{there is a ball } B_{M_r,\varepsilon}(x) \in \mathcal{U}_0\right\}.$$

We set

$$W_k := \prod_{i=1}^{k} \Gamma_i, \quad \overline{W_m} = \bigcup_{k=1}^{m} W_k.$$

Let $x_j,\, y_j \in \Gamma\acute{}_j,\, x_j \neq y_j$, as we pointed out earlier, if $x \in B_{N\acute{}_j,\varepsilon\acute{}_j}(x_j)$, $y \in B_{N\acute{}_j,\varepsilon\acute{}_j}(y_j)$ then $d\left(f^l(x), f^l(y)\right) > 2\varepsilon$

for any $l = 0, ..., N_j - 1$, and with $\varepsilon > \varepsilon_1/4$. Now for any $x \in B_{M_r,\varepsilon}(z) \cap G$ there is a, uniquely determined $z = z(x) \in W_r$. A word $\overline{w} \in W_j$, with $j = 1, 2, ..., k$, is a called a prefix of a word $w \in W_k$ if the first $j-$letters of $\overline{w}$ agree with the first $j-$letters of w.The number of times that each $w \in W_k$ is a prefix of a word in W_m is

$cardW_m/cardW_k$, thus if W is a subset of $\overline{W_m}$ then

$$\sum_{k=1}^{m} \frac{card\,(W \cap W_k)}{card\,(W_k)} \geq card\,(W_m)\,.$$

If each word in W_m has a prefix contained in a $W \subset \overline{W_m}$ then

$$\sum_{k=1}^{m} \frac{card\,(W \cap W_k)}{card\,(W_k)} \geq 1,$$

and since $\mathcal{U}_0$ is a covering each point of W_m has a prefix associated to a ball in $\mathcal{U}_0$. By this and because $cardW_k \geq \exp\left(\overline{h}M_r\right)$, we obtain

$$\sum_{B_{M_r,\varepsilon} \in \mathcal{U}_0} \exp\left(-sM_r\right) \geq 1.$$

Thus if r is taken such that $k \geq r$ then $sM_{k+1} \leq \overline{h}M_k$, for $N \geq M_r, \mathcal{U} \in \mathcal{G}\,(N, \varepsilon, G)\,.$

Therefore

$$\sum_{B_{m,\varepsilon}(x) \in \mathcal{U}} \exp\left(-sm\right) \geq 1,$$

and so

$$M\,(G, s, N, \varepsilon) \geq 1.$$

By this $h_{top}(G) \geq \overline{h}$.

■

We are now in condition of giving the proof of the theorem. Let

$$\Psi = \Psi_{r,\Phi} : \mathcal{M}(X) \to \mathbf{R}$$
$$\Psi(\mu) = \int \Phi d\mu^{\otimes r}$$

and let

$h = h_{top}(X)$ be the topological entropy of the whole space X. By the classical variational principle and by the variational principle of [5]

$$h = \sup\{h_\mu(f) : \mu \in \mathcal{M}_{inv}(X,f)\} = \sup_{\alpha\in Im(\Psi)} \{h_\mu(f) : \mu \in \mathcal{M}_\Phi(\alpha)\ \} =$$
$$\sup_{\alpha\in Im(\Psi)} \{h_{top}(E_\Phi(\alpha))\ \}.$$

We must show that $h_{top}(E_\Phi^\infty)\ \geq h$. For any $\gamma > 0$, there is an $\alpha_1 \in Im\Psi$ such that $h_{top}(E_\Phi(\alpha_1)) > h - \gamma$, let $\alpha_2 \in Im\Psi$ and let $\mu_1,\ \mu_2 \in \mathcal{M}(X,f)$ with $\Psi(\mu_1) = \alpha_1$, $\Psi(\mu_2) = \alpha_2$. The map $\lambda \longmapsto \Psi((1-\lambda)\mu_1 + \lambda\mu_2)$ is continuous.Recall that

$$\begin{aligned} & h_{top}(G_\Phi(\alpha_1) \cap\ G_\Phi((1-\lambda)\alpha_1 + \lambda\alpha_2)) \\ = \quad & \min\{h_{top}(G_\Phi(\alpha_1), h_{top}(\ G_\Phi((1-\lambda)\alpha_1 + \lambda\alpha_2))\}, \end{aligned}$$

then, by the continuity of Ψ as a function of λ, we have

$$h_{top}(E_\Phi^\infty)\ \geq \lim_{\lambda\to 0}\ h_{top}(G_\Phi(\alpha_1) \cap G_\Phi((1-\lambda)\alpha_1 + \lambda\alpha_2)) \geq$$
$$h_{top}(G_\Phi(\alpha_1) \geq h_{top}(E_\Phi(\alpha_1)) > h - \gamma.$$

Since γ is arbitrary the result follows.

■

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