DECAY ESTIMATES FOR NONLINEAR NONLOCAL DIFFUSION PROBLEMS IN THE WHOLE SPACE

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ABSTRACT. In this paper we obtain bounds for the decay rate in the $L^r(\mathbb{R}^d)$-norm for the solutions to a nonlocal and nolinear evolution equation, namely,

$$u_t(x,t) = \int_{\mathbb{R}^d} K(x,y)|u(y,t) - u(x,t)|^{p-2}(u(y,t) - u(x,t))\,dy,$$

with $x \in \mathbb{R}^d$, $t > 0$. Here we consider a kernel $K(x,y)$ of the form $K(x,y) = \psi(y - a(x)) + \psi(x - a(y))$, where ψ is a bounded, nonnegative function supported in the unit ball and a is a linear function $a(x) = Ax$. To obtain the decay rates we derive lower and upper bounds for the first eigenvalue of a nonlocal diffusion operator of the form $T(u) = -\int_{\mathbb{R}^d} K(x,y)|u(y) - u(x)|^{p-2}(u(y) - u(x))\,dy$, with $1 \le p < \infty$. The upper and lower bounds that we obtain are sharp and provide an explicit expression for the first eigenvalue in the whole $\mathbb{R}^d$:

$$\lambda_{1,p}(\mathbb{R}^d) = 2\left(\int_{\mathbb{R}^d} \psi(z)\,dz\right)\left|\frac{1}{|\det A|^{1/p}} - 1\right|^p.$$

Moreover, we deal with the $p = \infty$ eigenvalue problem studying the limit as $p \to \infty$ of $\lambda_{1,p}^{1/p}$.

1. INTRODUCTION

Recently, nonlocal problems have been widely used to model diffusion processes. In particular, for $J : \mathbb{R}^N \to \mathbb{R}$ a nonnegative, radial, continuous function with $\int_{\mathbb{R}^N} J(z)\,dz = 1$, nonlocal evolution equations of the form

$$(1.1) \qquad u_t(x,t) = (J * u - u)(x,t) = \int_{\mathbb{R}^N} J(x-y)u(y,t)\,dy - u(x,t),$$

and variations of it, have been widely used to model diffusion processes. As is stated in [19], if $u(x,t)$ is thought of as a density at the point x at time t, and $J(x - y)$ is thought of as the probability distribution of jumping from location y to location x, then $\int_{\mathbb{R}^N} J(y-x)u(y,t)\,dy = (J * u)(x,t)$ is the rate at which individuals are arriving at position x from all other places and $-u(x,t) = -\int_{\mathbb{R}^N} J(y - x)u(x,t)\,dy$ is the rate at which they are leaving location x to travel to all other sites. This consideration, in the absence of external or internal sources, leads immediately to the fact that the density u satisfies equation (1.1).

Equation (1.1) is called *nonlocal diffusion equation* since the diffusion of the density u at a point x and time t does not only depend on $u(x,t)$ and its derivatives, but on all the values of u in a neighborhood of x through the convolution term $J * u$. This equation shares many properties with the classical heat equation, $u_t = \Delta u$, such as: bounded stationary solutions are

constant, a maximum principle holds for both of them and, even if J is compactly supported, perturbations propagate with infinite speed (see [19] for more details). However, there is no regularizing effect in general.

Here we deal with a nonlinear nonlocal problem, analogous to the classical $p-$Laplacian evolution equation, $u_t = \text{div}(|\nabla u|^{p-2}\nabla u) = \Delta_p u$, namely,

$$(1.2) \quad \begin{cases} u_t(x,t) = \displaystyle\int_{\mathbb{R}^d} K(x,y)|u(y,t) - u(x,t)|^{p-2}(u(y,t) - u(x,t))\, dy, & x \in \mathbb{R}^d,\ t > 0, \\ u(x,0) = u_0(x), & x \in \mathbb{R}^d, \end{cases}$$

with an initial condition $u(x,0) = u_0(x)$.

The references [2], [3] and [4] are especially related to the nonlocal problem (1.2). In fact, this work can be viewed as a natural continuation of those papers. All these results were collected in the recent book [5]. Also, the papers [29] and [15] deal with the eigenvalue problem for a general linear nonlocal equation.

Note that here we have a kernel $K(x,y)$ which is not of convolution type. We will assume a special form for this kernel, see (1.3) below. Existence and uniqueness of solutions for $u(x,0) = u_0(x) \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ can be obtained as in [5, Chapter 6], hence our main aim here is to deal with the asymptotic behaviour as $t \to \infty$. As it is well known, to study the decay of solutions as $t \to \infty$ the first eigenvalue of the associated elliptic part plays a crucial role. Hence, one of our main purposes here is to study properties of the principal eigenvalue of nonlocal diffusion operators when the associated kernel is not of convolution type. We recall that the particular case $p = 2$ has been previously treated in [25].

Let us now sate the main assumptions we will use along this paper. We assume some structure for the kernel. Let us consider a nonnegative and bounded function ψ, supported in the unit ball of $\mathbb{R}^d$. In this work we fix the support the unit ball but any compact set can be handled in the same way. We associate with this function a kernel of the form

$$(1.3) \qquad\qquad K(x,y) = \psi(y - a(x)) + \psi(x - a(y)), \quad a(x) = Ax,$$

where A is an invertible matrix. Note that K is symmetric and any convolution type kernel also take the form (1.3) (just take $a(x) = x$). For this kernel let us look for the first eigenvalue of the associated nonlocal operator, that is,

$$\lambda_{1,p}(\mathbb{R}^d) = \inf_{u \in L^p(\mathbb{R}^d)} \frac{\displaystyle\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} K(x,y)|u(x) - u(y)|^p dxdy}{\displaystyle\int_{\mathbb{R}^d} |u|^p(x)dx}.$$

Due to the lack of compactness it is not known if the infimum is achieved. Hence we do not have an existence result for eigenfunctions, but we still call $\lambda_{1,p}(\mathbb{R}^d)$ *the first eigenvalue* for this problem because it is defined in an analogous way the local case.

The first result of this paper is the following:

Theorem 1.1. *Let $A \in \mathbb{R}^{d \times d}$ be an invertible matrix and assume that the kernel $K(x, y)$ is given by (1.3). Then, for $1 \leq p < \infty$, we have*

$$\lambda_{1,p}(\mathbb{R}^d) = 2 \left| \frac{1}{|\det A|^{1/p}} - 1 \right|^p \left(\int_{\mathbb{R}^d} \psi(x) dx \right).$$

As an immediate application of this result we observe that, when the first eigenvalue is positive, we have a decay estimate for the solutions to the associated evolution problem (1.2).

Theorem 1.2. *Let $u(x, t)$ be the solution to (1.2) with $u(x, 0) = u_0(x) \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$. Then, for any $r \in [1, \infty)$, the following hold:*

- $\|u(\cdot, t)\|_{L^r(\mathbb{R}^d)} \leq C t^{-\frac{r-1}{p-2}}$ *for $2 < p < \infty$.*

- $\|u(\cdot, t)\|_{L^r(\mathbb{R}^d)} \leq C e^{-\gamma t}$ *for $1 < p \leq 2$.*

Here, $C > 0$ and $\gamma > 0$ depend on $\|u_0\|_{L^1(\mathbb{R}^d)}$, $\|u_0\|_{L^\infty(\mathbb{R}^d)}$, r, p and K.

The extension of these results to the case of a general diffeomorphism a of $\mathbb{R}^d$ is left as an open problem.

Finally, we study the limit as $p \to \infty$ of $\lambda_{1,p}(\mathbb{R}^d)$. Indeed, note that since the limit of the L^p-norm of a function is the L^∞-norm of the function, the natural quantity to study is $\lim_{p \to \infty} [\lambda_{1,p}(\mathbb{R}^d)]^{1/p}$. So, the eigenvalue limit problem is the following:

$$\lambda_{1,\infty}(\mathbb{R}^d) = \inf \left\{ \frac{\|u(x) - u(y)\|_{L^\infty(x,y \in supp(u); K(x,y)>0)}}{\|u\|_{L^\infty(supp(u))}} : u \in L^\infty(\mathbb{R}^d), \text{ compactly supported} \right\}.$$

For this limit problem we can state the next theorem.

Theorem 1.3. *Assume that the kernel is given by (1.3), then*

$$\lim_{p \to \infty} [\lambda_{1,p}(\mathbb{R}^d)]^{1/p} = \lambda_{1,\infty}(\mathbb{R}^d) = 0.$$

Remark 1.1. *Note that we have actually proved that $\lim_{p \to \infty} \lambda_{1,p}(\mathbb{R}^d) = 0$, so it is not possible to find an uniform lower bound for $p \geq 1$.*

The paper is organized as follows: in Section 2 we show that the first eigenvalue of the whole space can be approximated with the first one of a sequence of expanding domains; in Section 3 we collect the proofs of the lower and upper bounds for the first eigenvalue and we prove Theorem 1.1; in Section 4 we apply our previous results to obtain the decay estimates for the evolution problem and prove Theorem 1.2; finally in Section 5 we estimate the first eigenvalue for $p = \infty$.

2. The limit of the first eigenvalue in expanding domains

In this section we show that the first eigenvalue of our nonlocal operator in the whole $\mathbb{R}^N$ can be approximated by the first eigenvalue in large domains. This result is not used in the rest of the article but can be of independent interest. The first eigenvalue in a bounded domain, $\lambda_{1,p}(\Omega)$, is defined as

$$\lambda_{1,p}(\Omega) = \inf_{u \in L^p(\Omega)} \frac{\displaystyle\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} K(x,y)|u(x) - u(y)|^p dxdy}{\displaystyle\int_{\Omega} |u|^p(x)dx}.$$

Here we have extended u to the whole $\mathbb{R}^d$ by zero outside Ω.

As a preliminary step, we focus our attention in the case of balls B_R that are centered at the origin with radius R.

Lemma 2.1. *Let $\lambda_{1,p}(\mathbb{R}^d)$ be the first eigenvalue in the whole space. Then*

$$(2.1) \qquad\qquad \lambda_{1,p}(\mathbb{R}^d) = \lim_{R \to \infty} \lambda_{1,p}(B_R).$$

Proof. The proof is an adaptation of the one given in [26] for the case $p = 2$, nevertheless we provide a sketch for the sake of completeness. First of all, we just remark that $\lambda_{1,p}(\Omega)$ is decreasing with Ω, that is, if $\Omega_1 \subset \Omega_2$ then $\lambda_{1,p}(\Omega_1) \geq \lambda_{1,p}(\Omega_2)$. Then we deduce that there exists the limit

$$\lim_{R \to \infty} \lambda_{1,p}(B_R) \geq 0.$$

Now, fix a function $u \in L^p(B_R)$. By the definition of $\lambda_{1,p}(\mathbb{R}^d)$, extending u by zero outside B_R, we get

$$\frac{\displaystyle\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} K(x,y)|\tilde{u}(x) - \tilde{u}(y)|^p dxdy}{\displaystyle\int_{B_R} |u|^p(x)dx} \geq \lambda_{1,p}(\mathbb{R}^d).$$

Taking the infimum in the right hand side over all functions $u \in L^p(B_R)$ we obtain that for any $R > 0$

$$\lim_{R \to \infty} \lambda_{1,p}(B_R) \geq \lambda_{1,p}(\mathbb{R}^d).$$

Now let be $\varepsilon > 0$. Then there exists $u_\varepsilon \in L^p(\mathbb{R}^d)$ such that

$$\lambda_{1,p}(\mathbb{R}^d) + \varepsilon \geq \frac{\displaystyle\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} K(x,y)|u_\varepsilon(x) - u_\varepsilon(y)|^p dxdy}{\displaystyle\int_{\mathbb{R}^d} |u_\varepsilon|^p(x)dx}.$$

We let $u_{\varepsilon,R}$ defined by

$$u_{\varepsilon,R}(x) = u_\varepsilon(x)\chi_{B_R}(x),$$

and we observe that, when $R \to +\infty$, the following limits hold

$$\int_{B_R} |u|^p_{\varepsilon,R}(x)dx \longrightarrow \int_{\mathbb{R}^d} |u_\varepsilon|^p(x)dx$$

and

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} K(x,y) |u_{\varepsilon,R}(x) - u_{\varepsilon,R}(y)|^p \, dx \, dy \longrightarrow \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} K(x,y) |u_{\varepsilon}(x) - u_{\varepsilon}(y)|^p \, dx \, dy.$$

Hence, using that $u_{\varepsilon,R}$ vanishes outside the ball B_R and the definition of $\lambda_{1,p}(B_R)$ we get

$$\frac{\displaystyle\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} K(x,y)|u_{\varepsilon,R}(x) - u_{\varepsilon,R}(y)|^p dx dy}{\displaystyle\int_{B_R} |u|^p_{\varepsilon,R}(x) dx} \geq \lambda_{1,p}(B_R).$$

Taking $R \to \infty$ we obtain

$$\frac{\displaystyle\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} K(x,y)|u_{\varepsilon}(x) - u_{\varepsilon}(y)|^p dx dy}{\displaystyle\int_{\mathbb{R}^d} |u_{\varepsilon}|^p(x) dx} \geq \lim_{R\to\infty} \lambda_{1,p}(B_R).$$

Hence, for any $\varepsilon > 0$, we have $\lambda_{1,p}(\mathbb{R}^d) + \varepsilon \geq \lim_{R\to\infty} \lambda_{1,p}(B_R)$. Thus

$$\lambda_{1,p}(\mathbb{R}^d) \geq \lim_{R\to\infty} \lambda_{1,p}(B_R),$$

and then the proof of (2.1) is finished. $\qquad\square$

It is possible to extend this result to dilatations of a domain Ω, such that $0 \in \Omega$.

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^d$ be a domain such that $0 \in \Omega$. Then,*

$$\lim_{R\to\infty} \lambda_{1,p}(R\Omega) = \lambda_{1,p}(\mathbb{R}^d).$$

Proof. Let us consider $B_{r_1} \subset \Omega \subset B_{r_2}$ then

$$\lambda_{1,p}(RB_{r_1}) \geq \lambda_{1,p}(R\Omega) \geq \lambda_{1,p}(RB_{r_2}),$$

and from the previous lemma we get that

$$\lim_{R\to\infty} \lambda_{1,p}(RB_{r_1}) = \lim_{R\to\infty} \lambda_{1,p}(RB_{r_1}) = \lambda_{1,p}(\mathbb{R}^d). \quad \square$$

3. Lower and upper bounds for the first eigenvalue

This section is mainly devoted to prove Theorem 1.1. Thus, up to the end of this section we will use the same notation as in the introduction, i.e., the kernel is given as in (1.3) and $1 \leq p < \infty$.

We first prove lower bounds for the first eigenvalue. To do that we use an elementary result. We need to compute for a given $\eta \in (0,1)$ the optimal constant θ (the biggest one) such that $|a - b|^p \geq \eta a^p + \theta b^p$ for any $a, b \in \mathbb{R}$.

Lemma 3.1. *For any $0 < \eta < 1$ and $1 \le p < \infty$ there exists $\theta(\eta, p)$ given by*

$$\theta(\eta, p) = \begin{cases} -\dfrac{\eta}{(1 - \eta^{\frac{1}{p-1}})^{p-1}}, & 1 < p < \infty, \\ -\eta & p = 1, \end{cases}$$

such that for all real numbers a and b the following inequality holds:

$$|a - b|^p \ge \eta |a|^p + \theta |b|^p.$$

Proof. We just deal with $1 < p < \infty$ since the case $p = 1$ is simpler. Also, since $|a - b| \ge ||a| - |b||$ we can treat only the case of nonnegative numbers a and b. Taking $x = a/b$ we have to find $\theta \in \mathbb{R}$ such that

$$|x - 1|^p - \eta x^p \ge \theta \qquad \text{for all } x \ge 0.$$

Therefore, the best value of θ is given by

$$\theta(\eta) = \min_{x \ge 0} |x - 1|^p - \eta x^p$$

and we are left with the computation of this minimum.

First, let us consider $0 \le x \le 1$, we have to compute the minimum of

$$f(x) = (1 - x)^p - \eta x^p.$$

Since f is the sum of two decreasing functions then its minimum on the interval $[0, 1]$ is attained at $x = 1$:

$$\min_{0 \le x \le 1} f(x) = f(1) = -\eta.$$

Now, for $x > 1$ we have to find the minimum of $g(x) = (x - 1)^p - \eta x^p$. In this case we compute the roots of its derivative

$$g'(x) = p(x - 1)^{p-1} - p\eta x^{p-1} = 0$$

and we obtain that the minimum if g in the variable x is attained at

$$x(\eta) = \frac{1}{1 - \eta^{\frac{1}{p-1}}}.$$

Since $0 < \eta < 1$, we have

$$-\frac{\eta}{(1 - \eta^{\frac{1}{p-1}})^{p-1}} < -\eta.$$

Therefore, we obtain

$$\theta(\eta, p) = -\frac{\eta}{(1 - \eta^{\frac{1}{p-1}})^{p-1}}$$

as we wanted to show. □

We now give a lower estimate for the first eigenvalue $\lambda_{1,p}(\mathbb{R}^d)$.

Lemma 3.2. *Given $1 \le p < \infty$, then the following estimate holds*

$$\lambda_{1,p}(\mathbb{R}^d) \ge 2 \left(\int_{\mathbb{R}^d} \psi(z) \, dz \right) \left| \frac{1}{|\det(A)|^{\frac{1}{p}}} - 1 \right|^p.$$

Proof. Let us first assume that $|\det(A)| \leq 1$ and $1 < p < \infty$. Using the symmetry of the kernel K, given by (1.3), we have

$$\lambda_{1,p}(\mathbb{R}^d) = 2 \inf \left\{ \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \psi(x - a(y)) \, |u(x) - u(y)|^p \, dx \, dy : u \in L^p(\mathbb{R}^d), \, \|u\|_{L^p(\mathbb{R}^d)} = 1 \right\}.$$

Given $\varepsilon > 0$, we can find a function $u \in L^p(\mathbb{R}^d)$, with $\|u\|_{L^p(\mathbb{R}^d)} = 1$, such that

$$\lambda_{1,p}(\mathbb{R}^d) + \varepsilon > 2 \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \psi(x - a(y)) \, |u(x) - u(y)|^p \, dx \, dy,$$

and, applying Lemma 3.1, we obtain

$$\lambda_{1,p}(\mathbb{R}^d) + \varepsilon > 2 \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \psi(x - a(y))(\eta |u(x)|^p + \theta(\eta, p)|u(y)|^p) \, dx \, dy$$

$$= 2 \left(\eta \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \psi(x - a(y))|u(x)|^p \, dx \, dy + \theta(\eta, p) \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \psi(x - a(y))|u(y)|^p \, dx \, dy \right)$$

$$= 2 \left(\frac{\eta}{|\det(A)|} + \theta(\eta, p) \right) \int_{\mathbb{R}^d} \psi(z) \, dz.$$

Since $\varepsilon > 0$ was arbitrary, we conclude that

$$\lambda_{1,p}(\mathbb{R}^d) \geq 2 \left(\frac{\eta}{|\det(A)|} + \theta(\eta, p) \right) \int_{\mathbb{R}^d} \psi(z) \, dz.$$

If we take

$$\eta = (1 - |\det(A)|^{\frac{1}{p}})^{p-1},$$

then

$$\theta(\eta, p) = -\frac{\eta}{(1 - \eta^{\frac{1}{p-1}})^{p-1}} = -\frac{(1 - |\det(A)|)^{\frac{1}{p}})^{p-1}}{|\det(A)|^{\frac{p-1}{p}}}.$$

Thus, we get

$$\lambda_{1,p}(\mathbb{R}^d) \geq 2 \left(\int_{\mathbb{R}^d} \psi(z) \, dz \right) \left(\frac{(1 - |\det(A)|)^{\frac{1}{p}})^{p-1}}{|\det(A)|} - \frac{(1 - |\det(A)|)^{\frac{1}{p}})^{p-1}}{|\det(A)|^{\frac{p-1}{p}}} \right)$$

$$= 2 \left(\int_{\mathbb{R}^d} \psi(z) \, dz \right) \left| \frac{1}{|\det(A)|^{\frac{1}{p}}} - 1 \right|^p.$$

The proof in the case $|\det(A)| > 1$ is analogous by interchanging the roles of x and y and of η and θ. In fact, we arrive to

$$\lambda_{1,p}(\mathbb{R}^d) \geq 2 \left(\frac{\theta(\eta, p)}{|\det(A)|} + \eta \right) \int_{\mathbb{R}^d} \psi(z) \, dz.$$

Now, we take

$$\eta = (1 - |\det(A)|^{\frac{-1}{p}})^{p-1} \quad \text{and} \quad \theta(\eta, p) = -\frac{\eta}{(1 - \eta^{\frac{1}{p-1}})^{p-1}}$$

to conclude.

For $p = 1$ the proof is the same taking $\eta = 1$ and $\theta = -1$. $\qquad\qquad\square$

We now prove upper bounds for the first eigenvalue. Note that our proof is long and technical. Hence, we include here the details.

First, we prove a lemma that gives an upper bound of the first eigenvalue in terms of the integral of ψ and an infimum involving $a(x)$.

Lemma 3.3. *Given* $1 \le p < \infty$, *then for any function* ϕ *with compact support, such that* $\|\phi\|_{L^p(\mathbb{R}^d)} = 1$, *we have*

$$\lambda_{1,p}(\mathbb{R}^d) \le 2 \left(\int_{\mathbb{R}^d} \psi(z)\, dz \right) \int_{\mathbb{R}^d} |\phi(x) - \phi(a(x))|^p\, dx.$$

Proof. First of all, recall that we have

$$\lambda_{1,p}(\mathbb{R}^d) = \inf_{u \in L^p(\mathbb{R}^d)} \frac{\displaystyle\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} K(x,y)|u(x) - u(y)|^p\, dx\, dy}{\displaystyle\int_{\mathbb{R}^d} |u|^p(x)dx}.$$

Let us choose a nonnegative smooth function ϕ, supported in the unit ball B_1, such that $\|\phi\|_{L^p(\mathbb{R}^d)} = 1$, and consider a family of test functions $u(x) = \phi(x/R)$, for $R > 0$. By a change of variable, we have

$$\lambda_{1,p}(\mathbb{R}^d) \le \frac{\displaystyle\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} K(x,y) \left| \phi\left(\frac{x}{R}\right) - \phi\left(\frac{y}{R}\right) \right|^p\, dx\, dy}{\displaystyle\int_{\mathbb{R}^d} \phi^p\left(\frac{x}{R}\right) dx}$$

$$= R^d \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} K(Rx, Ry) \left| \phi(x) - \phi(y) \right|^p\, dx\, dy.$$

Using the definition of the kernel K and the change of variable $z = Ry - a(Rx)$, we obtain

$$\lambda_{1,p}(\mathbb{R}^d) \le 2R^d \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \psi(Ry - a(Rx)) \left| \phi(x) - \phi(y) \right|^p\, dx\, dy$$

$$= 2 \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \psi(z) \left| \phi(x) - \phi\left(a(x) + \frac{z}{R} \right) \right|^p\, dz\, dx.$$

Thus, given $\varepsilon > 0$, there exists a positive constant $C(\varepsilon)$ such that

$$\lambda_{1,p}(\mathbb{R}^d) \le 2(1 + \varepsilon) \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \psi(z)|\phi(x) - \phi(a(x))|^p\, dz\, dx$$

$$+ C(\varepsilon) \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \psi(z) \left| \phi(a(x)) - \phi\left(a(x) + \frac{z}{R} \right) \right|^p\, dz\, dx.$$

Further, we have

$$
\begin{aligned}
\lambda_{1,p}(\mathbb{R}^d) \ &\leq 2(1+\varepsilon)\left(\int_{\mathbb{R}^d}\psi(z)\,dz\right)\int_{\mathbb{R}^d}|\phi(x)-\phi(a(x))|^p\,dx \\
&\quad +\frac{C(\varepsilon)}{R^p}\int_{B_1}\psi(z)\int_{\mathbb{R}^d}\int_0^1\left\|\nabla\phi\left(a(x)+\frac{sz}{R}\right)\right\|^p\|z\|^p\,ds\,dx\,dz \\
&\leq 2(1+\varepsilon)\left(\int_{\mathbb{R}^d}\psi(z)\,dz\right)\int_{\mathbb{R}^d}|\phi(x)-\phi(a(x))|^p\,dx \\
&\quad +\frac{C(\varepsilon)}{R^p}\int_{B_1}\psi(z)\,\|z\|^p\,|\det(A)|^{-1}\int_{\mathbb{R}^d}\int_0^1\left\|\nabla\phi\left(x+\frac{sz}{R}\right)\right\|^p\,ds\,dx\,dz.
\end{aligned}
$$

Therefore, letting $R\to\infty$ we obtain

$$
\lambda_{1,p}(\mathbb{R}^d)\leq 2(1+\varepsilon)\left(\int_{\mathbb{R}^d}\psi(z)\,dz\right)\int_{\mathbb{R}^d}|\phi(x)-\phi(a(x))|^p\,dx
$$

and letting $\varepsilon\to 0$

$$
\lambda_{1,p}(\mathbb{R}^d)\leq 2\left(\int_{\mathbb{R}^d}\psi(z)\,dz\right)\int_{\mathbb{R}^d}|\phi(x)-\phi(a(x))|^p\,dx,
$$

as we wanted to show. $\qquad\square$

According to Lemma 3.2, we already know the lower bound for the first eigenvalue, and as a consequence, for any ϕ with compact support we have

$$
\int_{\mathbb{R}^d}|\phi(x)-\phi(a(x))|^p\,dx\geq\left|\frac{1}{|\det(A)|^{\frac{1}{p}}}-1\right|^p.
$$

Our task is to construct a minimizing sequence. For future references, we prefer to be more specific and choose the minimizing sequence to be supported in the unit ball.

Lemma 3.4. *Let $1\leq p<\infty$ and $a:\mathbb{R}^d\to\mathbb{R}^d$ be an invertible linear map with $a(x)=Ax$. There exists a sequence of nonnegative functions $\{\phi_n\}_{n=1}^\infty\subset L^p(\mathbb{R}^d)$, supported in the unit ball with $\|\phi_n\|_{L^p(\mathbb{R}^d)}=1$, such that*

$$
(3.1)\qquad\qquad \lim_{n\to\infty}\int_{\mathbb{R}^d}|\phi_n(x)-\phi_n(a(x))|^p\,dx=\left|1-|\det(A)|^{-\frac{1}{p}}\right|^p.
$$

Our strategy for the proof of Lemma 3.4 is to construct a sequence of functions as above for each Jordan block of A and afterwards, using those functions, we define the desired sequence as a tensor product.

By the Jordan's decomposition of A, there exist C and J two $d \times d$ invertible matrices with real entries such that $A = CJC^{-1}$. Note that J is defined by Jordan blocks, i.e.,

$$(3.2) \qquad J = \begin{pmatrix} J_1(\lambda_1) & & & & & & \\ & \ddots & & & & & \\ & & J_r(\lambda_r) & & & & \\ & & & J_{r+1}(\alpha_1, \beta_1) & & & \\ & & & & \ddots & & \\ & & & & & J_{r+s}(\alpha_s, \beta_s) \end{pmatrix},$$

with $|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_r| > 0$, $\alpha_i^2 + \beta_i^2 \neq 0$, (recall that A is invertible)

$$(3.3) \qquad J_i(\lambda) = \begin{pmatrix} \lambda & 1 & & \\ & \ddots & 1 & \\ & & & \lambda \end{pmatrix}, \quad i = 1, \ldots, r,$$

and

$$(3.4) \qquad J_i(\alpha, \beta) = \begin{pmatrix} \mathbf{M} & \mathbf{I} & & \\ & \ddots & \mathbf{I} & \\ & & & \mathbf{M} \end{pmatrix}, \quad i = r+1, \ldots, r+s.$$

Here λ, α and β are real numbers, $\mathbf{M} = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$ and $\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Note that $J_i(\lambda_i)$ and $J_i(\alpha_i, \beta_i)$ are the real Jordan blocks corresponding to real and complex eigenvalues respectively.

Since the proof of Lemma 3.4 is quite large we divide it in few steps according to properties of the Jordan blocks.

First, we consider $J_i(\lambda_i)$ and $J_i(\alpha_i, \beta_i)$ when $|\lambda_i| > 1$ and $\alpha_i^2 + \beta_i^2 > 1$. Indeed we write our construction in a more general context of expansive linear maps. Recall that a linear map $a : \mathbb{R}^d \to \mathbb{R}^d$ with $a(x) = Ax$ is called expansive if all the (complex) eigenvalues of A have absolute value bigger than one. Here and in what follows we denote by $|\cdot|_d$ the Lebesgue measure of a set in $\mathbb{R}^d$.

Lemma 3.5. *Let $1 \leq p < \infty$ and $a : \mathbb{R}^d \to \mathbb{R}^d$ be an expansive linear map with $a(x) = Ax$. There exists a sequence of sets $\{E_j\}_{j \in \mathbb{N}_0} \subset B_1$ of positive measure such that*

(i) *$a^{-1}(E_j) = E_{j+1}$, $|E_j|_d = |\det(A)|^{-j}|E_0|_d$ and $E_j \cap E_l = \emptyset$, whenever $j, l \in \mathbb{N}_0$ and $j \neq l$.*
(ii) *If we choose $\sigma_n = |\det(A)|^{1/p} - \frac{1}{n}$ and*

$$\varphi_n(x) = \sum_{j=0}^{\infty} (\sigma_n)^j \chi_{E_j}(x), \qquad \text{for } n \geq 1,$$

then the functions φ_n are nonnegative and supported in the unit ball. Moreover, they belong to $L^p(\mathbb{R}^d)$ satisfying

$$\|\varphi_n\|^p_{L^p(\mathbb{R}^d)} = \frac{|E_0|_d}{1 - \sigma_n^p/|\det(A)|} \qquad \text{for } n \geq 1.$$

(iii) *The sequence $\phi_n = \frac{\varphi_n}{\|\varphi_n\|_{L^p(\mathbb{R}^d)}}$, $n \geq 1$, satisfies (3.1).*

Proof. We first construct our candidate to sequence of sets. Since a is expansive, there exists $B \subset \mathbb{R}^d$ a ball with center the origin such that $a^{-j}(B) \subset B_1$, $\forall j \in \mathbb{N}_0$. Take the following sets

$$F = \bigcup_{j=0}^{\infty} a^{-j}(B) \quad \text{and} \quad E_l = a^{-l}(F) \setminus a^{-(l+1)}(F), \quad \text{for } l \in \mathbb{N}_0.$$

Let us first prove (i). By construction, we have that F and E_l are subsets of the unit ball for all $l \in \mathbb{N}_0$. Moreover, we have that $a^{-l}(E_0) = E_l$ and hence $|E_l|_d = |\det(A)|^{-l}|E_0|_d$. Since $|\det(A)| > 1$ and $|F|_d - |\det(A)|^{-1}|F|_d > 0$ then $|E_0|_d > 0$. Hence we conclude that E_l has positive measure for any $l \geq 0$ and, by the construction of the sets E_l, it follows that $E_l \cap E_j = \emptyset$ whenever $l \neq j$. Also by construction, since $|\det A| > 1$, φ_n is nonnegative and supported in the unit ball for all $n \geq 1$. Moreover, φ_n belongs to $L^p(\mathbb{R}^d)$ and we can compute explicitly its norm

$$\|\varphi_n\|_{L^p(\mathbb{R}^d)}^p = \sum_{j=0}^{\infty} \sigma_n^{jp}|E_j|_d = |E_0|_d \sum_{j=0}^{\infty} \sigma_n^{jp}|\det(A)|^{-j} = |E_0|_d \sum_{j=0}^{\infty} \left(1 - \frac{|\det(A)|^{-1/p}}{n}\right)^{jp}$$

$$= \frac{|E_0|_d}{1 - \sigma_n^p/|\det(A)|}.$$

Finally, we prove (iii). For any positive number n we have

$$\int_{\mathbb{R}^d} |\varphi_n(x) - \varphi_n(a(x))|^p dx = \int_{\mathbb{R}^d} \left|\sum_{j=0}^{\infty} \sigma_n^j \chi_{E_j}(x) - \sum_{j=0}^{\infty} \sigma_n^j \chi_{E_j}(a(x))\right|^p dx$$

$$= \int_{\mathbb{R}^d} \left|\sum_{j=0}^{\infty} \sigma_n^j \chi_{E_j}(x) - \sum_{j=1}^{\infty} \sigma_n^{j-1} \chi_{E_{j-1}}(a(x))\right|^p dx.$$

Since for any $j \geq 0$, $a^{-1}(E_j) = E_{j+1}$, we have $\chi_{E_{j-1}}(a(x)) = \chi_{E_j}(x)$. Using also that $E_j \cap E_l = \emptyset$ if $j \neq l$, we have

$$\int_{\mathbb{R}^d} |\varphi_n(x) - \varphi_n(a(x))|^p dx = |E_0|_d + \int_{\mathbb{R}^d} \left|\sum_{j=1}^{\infty} \sigma_n^{j-1}(\sigma_n - 1)\chi_{E_j}(x)\right|^p dx$$

$$= |E_0|_d + \sum_{j=1}^{\infty} \sigma_n^{(j-1)p} |\sigma_n - 1|^p |E_j|_d.$$

Using that $|E_j|_d = |\det(A)|^{-j} |E_0|_d$, it follows that

$$
\begin{aligned}
\int_{\mathbb{R}^d} |\varphi_n(x) - \varphi_n(a(x))|^p dx &= |E_0|_d + |\sigma_n - 1|^p |E_0|_d \sum_{j=1}^{\infty} \sigma_n^{(j-1)p} |\det(A)|^{-j} \\
&= |E_0|_d + |\sigma_n - 1|^p |E_0|_d |\det(A)|^{-1} \sum_{j=0}^{\infty} \sigma_n^{jp} |\det(A)|^{-j} \\
&= |E_0|_d + |\sigma_n - 1|^p |E_0|_d |\det(A)|^{-1} \frac{1}{1 - \sigma_n^p/|\det(A)|}.
\end{aligned}
$$

Further, using the definition of the functions ϕ_n and since $\lim_{n\to\infty} \|\varphi_n\|_{L^p(\mathbb{R}^d)}^p = \infty$,

$$
\lim_{n\to\infty} \int_{\mathbb{R}^d} |\phi_n(x) - \phi_n(a(x))|^p dx = |\det(A)|^{-1} ||\det(A)|^{1/p} - 1|^p = |1 - |\det(A)|^{-1/p}|^p,
$$

which proves that (3.1) holds for any expansive map a. $\square$

Next, we consider $J_i(\lambda_i)$ and $J_i(\alpha_i, \beta_i)$ when $0 < |\lambda_i| < 1$ and $0 < \alpha_i^2 + \beta_i^2 < 1$. We write our construction in a more general context, where an invertible linear map a satisfies that a^{-1} is expansive. Using the same techniques as in the proof of Lemma 3.5 we obtain the following.

Lemma 3.6. *Let $1 \leq p < \infty$ and $a : \mathbb{R}^d \to \mathbb{R}^d$ be an invertible linear map with $a(x) = Ax$ such that a^{-1} is expansive. Then there exists a sequence of sets $\{G_j\}_{j\in\mathbb{N}_0} \subset B_1$ of positive measure such that*

(i) $a(G_j) = G_{j+1}$, $|G_j|_d = |\det(A)|^j |G_0|_d$ *and* $G_j \cap G_l = \emptyset$, *whenever* $j, l \in \mathbb{N}_0$ *and* $j \neq l$.

(ii) *If we choose* $\gamma_n = |\det(A)|^{-1/p} - \frac{1}{n}$ *and*

$$
\varphi_n(x) = \sum_{j=0}^{\infty} (\gamma_n)^j \chi_{G_j}(x), \qquad \text{for } n \geq 1,
$$

then the functions φ_n are nonnegative and supported in the unit ball. Moreover, they belong to $L^p(\mathbb{R}^d)$ satisfying

$$
\|\varphi_n\|_{L^p(\mathbb{R}^d)}^p = \frac{1}{1 - \gamma_n^p |\det(A)|}.
$$

(iii) *The sequence* $\phi_n = \frac{\varphi_n}{\|\varphi_n\|_{L^p(\mathbb{R}^d)}}$, $n \in \mathbb{N}$, *satisfies* (3.1).

In the following three lemmas, we consider linear maps whose eigenvalues have absolute value equal to one.

First, we deal with the diagonalizable (in $\mathbb{C}$) case. In the next lemma, we deal with several real Jordan blocks simultaneously that correspond to the unitary eigenvalues λ that have dimension one if λ is real and two if $\lambda \in \mathbb{C} \setminus \mathbb{R}$.

Lemma 3.7. *Let $1 \le p < \infty$ and $a : \mathbb{R}^d \to \mathbb{R}^d$, $a(x) = Ax$ with*

$$(3.5) \qquad A = \begin{pmatrix} \mathbf{M}_1(\lambda_1) & & & & & \\ & \ddots & & & & \\ & & \mathbf{M}_r(\lambda_r) & & & \\ & & & \mathbf{M}_{r+1}(\alpha_1, \beta_1) & & \\ & & & & \ddots & \\ & & & & & \mathbf{M}_{r+s}(\alpha_s, \beta_s) \end{pmatrix},$$

where λ_i, α_i and β_i are real numbers such that $|\lambda_i| = 1$ and $\alpha_i^2 + \beta_i^2 = 1$, $\mathbf{M}_i(\lambda_i) = \lambda_i$ and

$$\mathbf{M}_i(\alpha, \beta) = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}.$$

Then $\varphi = |B_1|_d^{-1/p} \chi_{B_1}$ satisfies

$$\int_{\mathbb{R}^d} |\varphi(x) - \varphi(a(x))|^p dx = 0.$$

Proof. Since a leaves invariant the unit ball, we have that $\varphi(a(x)) = \varphi(x)$ and the assertion follows. $\qquad\square$

In the next lemma, we deal with a non-diagonalizable Jordan block corresponding to a real eigenvalue with modulus one.

Lemma 3.8. *Let $1 \le p < \infty$, $\lambda \in \{\pm 1\}$ and $a : \mathbb{R}^d \to \mathbb{R}^d$, $a(x) = Ax$ be an invertible linear map such that the corresponding matrix associated to the canonical basis is given by*

$$(3.6) \qquad A = \begin{pmatrix} \lambda & 1 & & \\ & \ddots & & 1 \\ & & & \lambda \end{pmatrix}.$$

Then there exists a sequence of bounded sets $\{E_j\}_{j=0}^{\infty} \subset \mathbb{R}^d$ of positive measure such that

(i) *$|E_j|_d = |E_0|_d$, $a(E_j) = E_{j+1}$ and $E_j \cap E_l = \emptyset$, if $j, l \in \mathbb{N}_0$, $j \ne l$.*

(ii) *Given $n \in \mathbb{N}$, if we choose*

$$\varphi_n(x) = \sum_{j=1}^{n} \chi_{2^{-n} E_j}(x)$$

then, the function φ_n is nonnegative, supported in the unit ball and moreover it belongs to $L^p(\mathbb{R}^d)$ with

$$\|\varphi_n\|_{L^p(\mathbb{R}^d)}^p = 2^{-nd} n |E_0|_d.$$

(iii) *The sequence $\phi_n = \dfrac{\varphi_n}{\|\varphi_n\|_{L^p(\mathbb{R}^d)}}$, $n \in \mathbb{N}$, satisfies*

$$\lim_{n \to \infty} \int_{\mathbb{R}^d} |\phi_n(x) - \phi_n(a(x))|^p dx = 0.$$

Proof. Let us start constructing a sequence of sets which satisfies (i). Given $j \in \mathbb{N}_0$, $a^j(\mathbf{p})^t = (A^j \mathbf{p})^t = (\lambda^j + j\lambda^{j-1}, \lambda^j, 0, \ldots, 0)^t$ where $\mathbf{p} = (1, 1, 0, \ldots, 0) \in \mathbb{R}^d$. Observe that $a^j(\mathbf{p}) \neq a^l(\mathbf{p})$ if $l, j \in \mathbb{N}$ and $j \neq l$. Indeed $|a^j(\mathbf{p}) - a^l(\mathbf{p})| \geq 1$ if $j \neq l$. Here and in what follows, we use the notation $|\cdot|$ ffor the Euclidean norm in $\mathbb{R}^d$. Thus, $a^j(B_{1/4}(\mathbf{p})) \cap a^l(B_{1/4}(\mathbf{p})) = \emptyset$ if $j \neq l$, where $B_{1/4}(\mathbf{p})$ is the ball with center the point $\mathbf{p}$ and radius $1/4$.

For $j \in \mathbb{N}_0$, we take $E_j = a^j(2^{-3} B_{1/4}(\mathbf{p})) = a^j(B_{2^{-5}}(2^{-3}\mathbf{p}))$. Then the sequence $\{E_j\}_{j \in \mathbb{N}}$ satisfies (i).

We now prove see (ii). Given $n \in \mathbb{N}$, we check that the function φ_n is supported in the unit ball. If x is in the support of φ_n then there exists $j \in \{0, 1, \ldots, n\}$ such that $|x - a^j(2^{-n-3}\mathbf{p})| \leq 2^{-n-5}$ and we have

$$|x| \leq |x - a^j(2^{-n-3}\mathbf{p})| + |a^j(2^{-n-3}\mathbf{p})| \leq 2^{-n-5} + 2^{-n-3}((1+n)^2 + 1)^{1/2} \leq 2^{-n} n < 1.$$

Further, using (i)

$$\|\phi_n\|^p_{L^p(\mathbb{R}^d)} = \sum_{j=1}^{n} |2^{-n} E_j|_d = 2^{-nd} n |E_0|_d.$$

We now prove (iii). Given $n \in \mathbb{N}$, by (i) and (ii) we have

$$\int_{\mathbb{R}^d} |\varphi_n(x) - \varphi_n(a(x))|^p dx = \int_{\mathbb{R}^d} \Big| \sum_{j=1}^{n} \chi_{2^{-n} E_j}(x) - \sum_{j=1}^{n} \chi_{2^{-n} E_j}(a(x)) \Big|^p dx$$

$$= 2^{-nd} |E_n|_d + 2^{-nd} |E_0|_d = 2^{-nd+1} |E_0|_d.$$

Thus, ϕ_n satisfies

$$\lim_{n \to \infty} \int_{\mathbb{R}^d} |\phi_n(x) - \phi_n(a(x))|^p dx = \lim_{n \to \infty} \frac{2}{n} = 0,$$

as we wanted to prove. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

In the next lemma, we deal with a non-diagonalizable Jordan block corresponding to a complex eigenvalue with modulus one.

Lemma 3.9. *Let* $1 \leq p < \infty$, $n \in \mathbb{N}$ *and* $a : \mathbb{R}^d \to \mathbb{R}^d$, $a(x) = Ax$ *with*

$$(3.7) \qquad\qquad A = \begin{pmatrix} \mathbf{M} & \mathbf{I} & & \\ & \ddots & & \mathbf{I} \\ & & & \mathbf{M} \end{pmatrix},$$

where $\theta \in \mathbb{R} \setminus \{\pi k\}_{k \in \mathbb{Z}}$, $\mathbf{M} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ *and* $\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. *For any positive integer* n *there exists a finite sequence of bounded sets* $\{E_j^{(n)}\}_{j=0}^{n} \subset \mathbb{R}^d$ *of positive measure such that*

    (i) $|E_j^{(n)}|_d = |E_0^{(n)}|_d$, $a(E_j^{(n)}) = E_{j+1}^{(n)}$ *and* $E_j^{(n)} \cap E_l^{(n)} = \emptyset$, *if* $j, l \in \{0, 1, \ldots, n\}$, $j \neq l$.
    (ii) *For any positive integer* n, φ_n *defined by*

$$\varphi_n(x) = \sum_{j=1}^{n} \chi_{2^{-n} E_j^{(n)}}(x), \qquad n \in \mathbb{N}.$$

is nonnegative and supported in the unit ball. Moreover it belongs to $L^p(\mathbb{R}^d)$ and

$$\|\varphi_n\|_{L^p(\mathbb{R}^d)}^p = 2^{-nd} n |E_0^{(n)}|_d.$$

(iii) *The sequence $\phi_n = \dfrac{\varphi_n}{\|\varphi_n\|_{L^p(\mathbb{R}^d)}}$ verifies*

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} |\phi_n(x) - \phi_n(a(x))|^p dx = 0.$$

Proof. Fix $n \in \mathbb{N}$. Let us start the proof constructing $\{E_j^{(n)}\}_{j=0}^n$ a sequence of bounded sets satisfying (i). Denote $\mathbf{q} = (1,1,1,1,0\ldots 0)$. This selection can be done only if we have at least two blocks. Note that $d \geq 4$. Then, for any $j \in \mathbb{N}$,

$$a^j(\mathbf{q}) = \begin{pmatrix} \cos(j\theta) + \sin(j\theta) + j\cos((j-1)\theta) + j\sin((j-1)\theta) \\ \cos(j\theta) - \sin(j\theta) + j\cos((j-1)\theta) - j\sin((j-1)\theta) \\ \cos(j\theta) + \sin(j\theta) \\ \cos(j\theta) - \sin(j\theta) \\ 0 \\ \cdots \\ 0 \end{pmatrix}.$$

Observe that $a^j(\mathbf{q}) \neq \mathbf{q}$ if $j \in \mathbb{N}$. So $a^j(\mathbf{q}) \neq a^l(\mathbf{q})$ if $l, j \in \{0,\ldots,n\}$ and $j \neq l$. Thus, by the continuity of the linear map a, there exists $0 < r = r(n) < 1$ and $B_{r(n)}(\mathbf{q}) \subset \mathbb{R}^d$, a ball with the center at the point $\mathbf{q}$ and radius r, such that $a^j(B_{r(n)}(\mathbf{q})) \cap a^l(B_{r(n)}(\mathbf{q})) = \emptyset$ if $j, l \in \{0, 1, \ldots, n\}$, $j \neq l$.

For $j \in \{0, 1, \cdots\}$, we take $E_j = a^j(2^{-4} B_{r(n)}(\mathbf{q})) = a^j(B_{2^{-4}r(n)}(2^{-4}\mathbf{q}))$, then the sequence of set $\{E_j^{(n)}\}_{j=0}^\infty$ satisfies (i).

We now prove (ii). Given $n \in \mathbb{N}$, we check that the function φ_n, with the sets E_j defined in the previous case, is supported in the unit ball. If x is in the support of φ_n then there exists $j \in \{1, \ldots, n\}$ such that $|x - 2^{-n} a^j(2^{-4}\mathbf{q})| \leq 2^{-n-4}$ and we have

$$\begin{aligned} |x| &\leq |x - 2^{-n} a^j(2^{-4}\mathbf{q})| + |2^{-n} a^j(2^{-4}\mathbf{q})| \\ &\leq 2^{-n-4} + 2^{-n-4}(2(2+2j)^2 + 2^3)^{1/2} \leq 2^{-n} n < 1. \end{aligned}$$

Further, using (i) we obtain

$$\|\varphi_n\|_{L^p(\mathbb{R}^d)}^p = \sum_{j=0}^n |2^{-n} E_j^{(n)}|_d = 2^{-nd} n |E_0^{(n)}|_d.$$

We now prove (iii). Let $n \in \mathbb{N}$, by (i) and (ii) we have then

$$\begin{aligned} \int_{\mathbb{R}^d} |\phi_n(x) - \phi_n(a(x))|^p dx &= \frac{1}{\|\varphi_n\|_{L^p(\mathbb{R}^d)}^p} \int_{\mathbb{R}^d} |\sum_{j=0}^n \chi_{a^j(2^{-n}B)}(x) - \sum_{j=0}^n \chi_{a^j(2^{-n})}(a(x))|^p dx \\ &= \frac{1}{\|\varphi_n\|_{L^p(\mathbb{R}^d)}^p} 2^{-nd+1} |E_0^{(n)}|_d \end{aligned}$$

and the conclusion follows. $\qquad\square$

For the proof of Lemma 3.4, we also need the following three lemmas where the matrix of the linear map a contains several Jordan blocks.

In the next lemma we combine two blocks (not necessarily Jordan blocks), one of then is expansive and the other is contractive.

Lemma 3.10. *Let $1 \leq p < \infty$, $d, d_1, d_2 \in \mathbb{N}$ such that $d = d_1 + d_2$. Moreover let $a : \mathbb{R}^d \to \mathbb{R}^d$, $a(x) = Ax$ be an invertible linear map such that the corresponding matrix associated to the canonical basis is given by*

$$A = \begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix},$$

where P is a $d_1 \times d_1$ expansive matrix and Q^{-1} is a $d_2 \times d_2$ expansive matrix. Let $\{\varphi_n\}_{n \in \mathbb{N}}$ be the sequence of functions defined on $\mathbb{R}^{d_1}$ as in Lemma 3.5 when the dilation is given by P. Moreover, let $\{\theta_n\}_{n \in \mathbb{N}}$ be the sequence of functions defined on $\mathbb{R}^{d_2}$ as in Lemma 3.6 when the dilation is given by Q. Then

$$\lim_{n \to \infty} \int_{\mathbb{R}^{d_1}} \int_{\mathbb{R}^{d_2}} |\phi_n(x)\varrho_n(y) - \phi_n(P(x))\varrho_n(Q(y))|^p \, dx \, dy = \left| 1 - |\det(A)|^{-1/p} \right|^p,$$

where $\phi_n = \dfrac{\varphi_n}{\|\varphi_n\|_{L^p(\mathbb{R}^{d_1})}}$ and $\varrho_n = \dfrac{\theta_n}{\|\theta_n\|_{L^p(\mathbb{R}^{d_2})}}$

Remark 3.1. *Note that here the product $\psi_n \varphi_n$ does not have the support in the unit ball. The support is in $|(x, y)| \leq 2$ and a change of variable gives us the right support.*

Proof. For $n \in \mathbb{N}$, we choose $\varphi_n(x) = \sum_{j=0}^{\infty} (\sigma_n)^j \chi_{E_j}(x)$, where $\sigma_n = |\det(P)|^{1/p} - \frac{1}{n}$ and $\{E_j\}_{j=0}^{\infty}$ as in Lemma 3.5. Also $\theta_n(y) = \sum_{j=0}^{\infty} (\gamma_n)^j \chi_{G_j}(y)$, where $\gamma_n = |\det(Q)|^{-1/p} - \frac{1}{n}$ and $\{G_j\}_{j=-1}^{\infty}$ as in Lemma 3.6.

Using $P^{-1}(E_j) = E_{j+1}$ and $Q(G_j) = G_{j+1}$, we have

$$I_n := \int_{\mathbb{R}^{d_1}} \int_{\mathbb{R}^{d_2}} \left| \varphi_n(x)\theta_n(y) - \varphi_n(P(x))\theta_n(Q(y)) \right|^p \, dx \, dy$$

$$= \int_{\mathbb{R}^{d_1}} \int_{\mathbb{R}^{d_2}} \left| \Big(\sum_{j=0}^{\infty} \sigma_n^j \chi_{E_j}(x) \Big) \Big(\sum_{l=0}^{\infty} \gamma_n^l \chi_{G_l}(y) \Big) - \Big(\sum_{j=1}^{\infty} \sigma_n^{j-1} \chi_{E_j}(x) \Big) \Big(\sum_{l=-1}^{\infty} \gamma_n^{l+1} \chi_{G_l}(y) \Big) \right|^p \, dx \, dy$$

$$= \int_{\mathbb{R}^{d_1}} \int_{\mathbb{R}^{d_2}} \left| \varphi_n(x)\theta(y) - \Big(\frac{\varphi_n(x)}{\sigma_n} - \chi_{E_0}(x) \Big) \Big(\gamma_n \theta_n(y) - \chi_{G^{-1}}(y) \Big) \right|^p \, dx \, dy$$

$$= \int_{\mathbb{R}^{d_1}} \int_{\mathbb{R}^{d_2}} \left| \varphi_n(x)\theta_n(y) \Big(1 - \frac{\gamma_n}{\sigma_n} \Big) + \gamma_n \chi_{E_0}(x)\theta_n(y) - \Big(\frac{\varphi_n(x)}{\sigma_n} - \chi_{E_0}(x) \Big) \chi_{G_{-1}}(y) \right|^p \, dx \, dy$$

where $G_{-1} = Q^{-1} G_0$. Since $G_l \cap G_j = \emptyset$, if $l \neq j$, G_{-1} is also disjoint from G_l, $l \neq -1$, and $|G_{-1}|_{d_2} = |\det Q|^{-1} |G_0|_{d_2}$ then

$$I_n = \int_{\mathbb{R}^{d_1}} \int_{\mathbb{R}^{d_2}} \left| \varphi_n(x)\theta_n(y) \Big(1 - \frac{\gamma_n}{\sigma_n} \Big) + \gamma_n \chi_{E_0}(x)\theta_n(y) \right|^p \, dx \, dy$$

$$+ |G_{-1}|_d \int_{\mathbb{R}^{d_1}} \left| \frac{\varphi_n(x)}{\sigma_n} - \chi_{E_0}(x) \right|^p \, dx$$

$$:= I_{n1} + I_{n2}.$$

Since σ_n is bounded and $\|\theta_n\|_{L^p(\mathbb{R}^{d_2})} \to \infty$ we have that

$$\lim_{n \to \infty} \frac{I_{n2}}{\|\varphi_n\|_{L^p(\mathbb{R}^{d_1})}^p \|\theta_n\|_{L^p(\mathbb{R}^{d_2})}^p} = 0.$$

For the first term we observe that, since $\|\varphi_n\|_{L^p(\mathbb{R}^{d_1})} \to \infty$ we have that

$$\lim_{n \to \infty} \frac{\displaystyle\fint_{\mathbb{R}^{d_1}} \int_{\mathbb{R}^{d_2}} |\gamma_n \chi_{E_0}(x)\theta_n(y)|^p \, dxdy}{\|\varphi_n\|_{L^p(\mathbb{R}^{d_1})}^p \|\theta_n\|_{L^p(\mathbb{R}^{d_2})}^p} = 0.$$

Then

$$\limsup_{n \to \infty} \frac{I_{n1}^{1/p}}{\|\varphi_n\|_{L^p(\mathbb{R}^{d_1})} \|\theta_n\|_{L^p(\mathbb{R}^{d_2})}}$$

$$\leq \limsup_{n \to \infty} \frac{1}{\|\varphi_n\|_{L^p(\mathbb{R}^{d_1})} \|\theta_n\|_{L^p(\mathbb{R}^{d_2})}} \left(\int_{\mathbb{R}^{d_1}} \int_{\mathbb{R}^{d_2}} \left| \varphi_n(x)\theta_n(y)\left(1 - \frac{\gamma_n}{\sigma_n}\right) \right|^p dxdy \right)^{1/p}$$

$$+ \lim_{n \to \infty} \frac{1}{\|\varphi_n\|_{L^p(\mathbb{R}^{d_1})} \|\theta_n\|_{L^p(\mathbb{R}^{d_2})}} \left(\int_{\mathbb{R}^{d_1}} \int_{\mathbb{R}^{d_2}} \left| \gamma_n \chi_{E_0}(x)\theta_n(y) \right|^p dxdy \right)^{1/p}.$$

Thus, using that $(1 - \gamma_n/\sigma_n) \to (1 - |\det A|^{-1/p})$ we obtain that

$$\limsup_{n \to \infty} \frac{I_{n1}^{1/p}}{\|\varphi_n\|_{L^p(\mathbb{R}^{d_1})}^p \|\theta_n\|_{L^p(\mathbb{R}^{d_2})}^p} \leq 1 - |\det A|^{-1/p}.$$

Moreover,

$$\liminf_{n \to \infty} \frac{I_{n1}^{1/p}}{\|\varphi_n\|_{L^p(\mathbb{R}^{d_1})} \|\theta_n\|_{L^p(\mathbb{R}^{d_2})}}$$

$$\geq \liminf_{n \to \infty} \frac{1}{\|\varphi_n\|_{L^p(\mathbb{R}^{d_1})} \|\theta_n\|_{L^p(\mathbb{R}^{d_2})}} \left(\int_{\mathbb{R}^{d_1}} \int_{\mathbb{R}^{d_2}} \left| \varphi_n(x)\theta_n(y)\left(1 - \frac{\gamma_n}{\sigma_n}\right) \right|^p dxdy \right)^{1/p}$$

$$- \lim_{n \to \infty} \frac{1}{\|\varphi_n\|_{L^p(\mathbb{R}^{d_1})} \|\theta_n\|_{L^p(\mathbb{R}^{d_2})}} \left(\int_{\mathbb{R}^{d_1}} \int_{\mathbb{R}^{d_2}} \left| \gamma_n \chi_{E_0}(x)\theta_n(y) \right|^p dxdy \right)^{1/p}.$$

As before we get

$$\liminf_{n \to \infty} \frac{I_{n1}^{1/p}}{\|\varphi_n\|_{L^p(\mathbb{R}^{d_1})}^p \|\theta_n\|_{L^p(\mathbb{R}^{d_2})}^p} \geq 1 - |\det A|^{-1/p}.$$

and the proof is finished. $\qquad\square$

In the next lemma we combine two blocks (not necessarily Jordan blocks), one of them correspond to non-diagonalizable Jordan blocks with the eigenvalues of modulus one, while the other is an arbitrary invertible matrix. Indeed this lemma shows that we can simplify our computations neglecting the first block.

Lemma 3.11. *Let $1 \leq p < \infty$, $d, d_1, d_2 \in \mathbb{N}$ such that $d = d_1 + d_2$. Moreover let $a : \mathbb{R}^d \to \mathbb{R}^d$, $a(x) = Ax$ be an invertible linear map such that the corresponding matrix associated to the canonical basis is given by*

$$A = \begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix},$$

where P is a $d_1 \times d_1$ matrix as (3.6) or (3.7) and Q is a $d_2 \times d_2$ invertible matrix. Let $\{\varphi_n\}_{n \in \mathbb{N}}$ be the sequence of functions defined on $\mathbb{R}^{d_1}$ as in Lemma 3.8 or Lemma 3.9 when the dilation is given by P. Moreover, let $\{\theta_n\}_{n \in \mathbb{N}}$ be a sequence of functions on $\mathbb{R}^{d_2}$ such that $\|\theta_n\|_{L^p(\mathbb{R}^{d_2})}^p = 1$. If $\phi_n = \frac{\varphi_n}{\|\varphi_n\|_{L^p(\mathbb{R}^{d_1})}}$, then

$$\lim_{n \to \infty} \int_{\mathbb{R}^{d_1}} \int_{\mathbb{R}^{d_2}} |\phi_n(x)\theta_n(y) - \phi_n(P(x))\theta_n(Q(y))|^p \, dx dy = \lim_{n \to \infty} \int_{\mathbb{R}^{d_2}} |\theta_n(y) - \theta_n(Q(y))|^p \, dy$$

if both previous limits exist.

Proof. For any $n \in \mathbb{N}$, φ_n is given by $\varphi_n(x) = \sum_{j=1}^n \chi_{2^{-n} E_j^{(n)}}(x)$ where the sets $\{E_j^{(n)}\}_{j=0}^\infty$ satisfy (i) in Lemma 3.8 or in Lemma 3.9. Thus, we have

$$
\begin{aligned}
I_n &:= \int_{\mathbb{R}^{d_1}} \int_{\mathbb{R}^{d_2}} |\phi_n(x)\theta_n(y) - \phi_n(Px)\theta_n(Qy)|^p \, dxdy \\
&= \frac{1}{\|\varphi_n\|_{L^p(\mathbb{R}^{d_1})}^p} \int_{\mathbb{R}^{d_1}} \int_{\mathbb{R}^{d_2}} |(\sum_{j=1}^n \chi_{2^{-n} E_j^{(n)}}(x))\theta_n(y) - (\sum_{j=1}^n \chi_{2^{-n} E_j^{(n)}}(Px))\theta_n(Qy)|^p \, dxdy \\
&= \frac{1}{\|\varphi_n\|_{L^p(\mathbb{R}^{d_1})}^p} \int_{\mathbb{R}^{d_1}} \int_{\mathbb{R}^{d_2}} |(\sum_{j=1}^n \chi_{2^{-n} E_j^{(n)}}(x))\theta_n(y) - (\sum_{j=0}^{n-1} \chi_{2^{-n} E_j^{(n)}}(x))\theta_n(Qy)|^p \, dxdy.
\end{aligned}
$$

Since $E_j^{(n)} \cap E_l^{(n)} = \emptyset$ if $j \neq l$, then

$$(3.8) \qquad I_n \leq \int_{\mathbb{R}^{d_2}} |\theta_n(y) - \theta_n(Qy)|^p \, dy + \frac{2^{-nd+1}}{\|\varphi_n\|_{L^p(\mathbb{R}^{d_1})}^p} |E_0|_{d_1} (1 + |\det Q|^{-1}).$$

Moreover,

$$(3.9) \qquad I_n \geq \int_{\mathbb{R}^{d_2}} |\theta_n(y) - \theta_n(Qy)|^p \, dy - \frac{2^{-nd+1}}{\|\varphi_n\|_{L^p(\mathbb{R}^{d_1})}^p} |E_0|_{d_1} (1 + |\det Q|^{-1}).$$

Finally, since $\|\varphi_n\|_{L^p(\mathbb{R}^{d_1})}^p = 2^{-nd} n |E_0|_{d_1}$, the last terms in (3.8) and (3.9) go to zero and the conclusion follows. $\qquad \square$

The following lemma is analogous to the previous one but the first block is related with diagonalizable Jordan blocks with the eigenvalues of modulus one

Lemma 3.12. *Let $1 \leq p < \infty$, $d, d_1, d_2 \in \mathbb{N}$ such that $d = d_1 + d_2$. Moreover let $a : \mathbb{R}^d \to \mathbb{R}^d$, $a(x) = Ax$ with*

$$A = \begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix},$$

where P is a $d_1 \times d_1$ invertible matrix as in (3.5) and Q is a $d_2 \times d_2$ invertible matrix. Let $\phi = |B_1|^{-1/p} \chi_{B_1}$ where B_1 denotes the unitary ball on $\mathbb{R}^{d_1}$. Moreover, let θ be a function on $\mathbb{R}^{d_2}$ such that $\|\theta\|_{L^p(\mathbb{R}^{d_2})}^p = 1$. Then

$$\int_{\mathbb{R}^{d_1}} \int_{\mathbb{R}^{d_2}} |\phi(x)\theta(y) - \phi(P(x))\theta(Q(y))|^p \, dxdy = \int_{\mathbb{R}^{d_2}} |\theta(y) - \theta(Q(y))|^p \, dy.$$

Proof. Since $\phi(Px) = \phi(x)$, the statement follows. $\qquad \square$

We are ready to prove Lemma 3.4.

Proof of Lemma 3.4. We write the matrix A using a Jordan decomposition (3.2) with $d_i \times d_i$ Jordan blocks J_i as in (3.3) or (3.4).

If J_i is expansive, we construct a sequence of functions $\{\phi_n^{(i)}\}_{n=1}^\infty$ defined on variables $(x_1^{(i)}, \cdots, x_{d_i}^{(i)})$ as in Lemma 3.5. If J_i^{-1} is expansive then the sequence of functions is constructed as in Lemma 3.6. If J_i is diagonalizable with the absolute values of the eigenvalues equals to 1, we take $\phi_n(x_1^{(i)}, \cdots, x_{d_i}^{(i)}) = |B_1|^{1/p} \chi_{B_1}(x_1^{(i)}, \cdots, x_{d_i}^{(i)})$ where B_1 is the unit ball on $\mathbb{R}^{d_i}$. When J_i is not diagonalizable and the real or complex eigenvalues with absolute value 1, then the sequence of functions is constructed as in Lemma 3.8 or Lemma 3.9.

For $n \in \mathbb{N}$, we now choose

$$\Psi_n(x_1^{(1)}, \ldots, x_{d_1}^{(1)}, \ldots \ldots, x_1^{(r+s)}, \ldots, x_{d_{(r+s)}}^{(r+s)}) = \prod_{i=1}^{r+s} \phi_n^{(i)}(x_1^{(i)}, \ldots, x_{d_i}^{(i)})$$

and

$$\Phi_n(x) = d^{-d/2p} \|C^{-1}\|^{-d/p} |\det C|^{-1/p} \Psi_n(d^{-1/2} \|C^{-1}\|^{-1} C^{-1} x),$$

where $\|C^{-1}\|$ denotes the norm of C^{-1} as operator on $\mathbb{R}^d$. Observe that Φ_n is supported in B_1, $\|\Phi_n\|_{L^p(\mathbb{R}^d)} = 1$ and $\Phi_n(x)$ is nonnegative.

We now check that $\{\Phi_n\}$ defined above satisfies

(3.10) $$\lim_{n\to\infty} \int_{\mathbb{R}^d} |\Phi_n(x) - \Phi_n(a(x))|^p \, dx = \left| 1 - |\det(A)|^{-\frac{1}{p}} \right|^p.$$

After the change of variable $d^{-1/2}\|C^{-1}\|^{-1}C^{-1}x = y$, we have

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |\Phi_n(x) - \Phi_n(a(x))|^p\, dx$$
$$= d^{-d/2}\|C^{-1}\|^{-d}|\det C|^{-1}$$
$$\times \lim_{n\to\infty} \int_{\mathbb{R}^d} |\Psi_n(d^{-1/2}\|C^{-1}\|^{-1}C^{-1}x) - \Psi_n(d^{-1/2}\|C^{-1}\|^{-1}C^{-1}CJC^{-1}x)|^p\, dx$$
$$= \lim_{n\to\infty} \int_{\mathbb{R}^d} |\Psi_n(y) - \Psi_n(Jy)|^p\, dy$$
$$= \lim_{n\to\infty} \int_{\mathbb{R}^{d_1}} \cdots \int_{\mathbb{R}^{d_{r+s}}} \left| \prod_{i=1}^{r+s} \phi_n^{(i)}(z^{(i)}) - \prod_{i=1}^{r+s} \phi_n^{(i)}(J_i z^{(i)}) \right|^p dz^{(1)} \cdots dz^{(r+s)}$$

where $z^{(i)} = (x_1^{(i)}, \cdots, x_{d_i}^{(i)})$.

We study the following cases.

Case I. *The matrix A contains a single real Jordan block $J(\lambda)$.*

If $|\lambda| \neq 1$ then by Lemma 3.5 and Lemma 3.6, (3.10) holds. If λ has the absolute value equals to one, then the conclusion follows by Lemma 3.7, Lemma 3.8 or Lemma 3.9.

Case II. *The matrix A contains several real Jordan blocks.*

Assume there exists a Jordan block denoted $J_1(\lambda_1)$ that corresponds to the eigenvalue λ_1 with $|\lambda_1| = 1$. Then by Lemma 3.11 and Lemma 3.12,

$$\lim_{n\to\infty} \int_{\mathbb{R}^{d_1}} \cdots \int_{\mathbb{R}^{d_{r+s}}} \left| \prod_{i=1}^{r+s} \phi_n^{(i)}(z^{(i)}) - \prod_{i=1}^{r+s} \phi_n^{(i)}(J_i z^{(i)}) \right|^p dz^{(1)} \cdots dz^{(r+s)}$$
$$= \lim_{n\to\infty} \int_{\mathbb{R}^{d_2}} \cdots \int_{\mathbb{R}^{d_{r+s}}} \left| \prod_{i=2}^{r+s} \phi_n^{(i)}(z^{(i)}) - \prod_{i=2}^{r+s} \phi_n^{(i)}(J_i z^{(i)}) \right|^p dz^{(2)} \cdots dz^{(r+s)}.$$

Iterating the procedure for all Jordan block that corresponds to eigenvalues on the unit circle, we reduce the proof to the cases of Jordan blocks J_i that are expansive or J_i^{-1} are expansive. Indeed, without loss in the generality we can assume that we have only two Jordan blocks, one is expansive and the other has its inverse expansive. The proof finishes using Lemma 3.10. $\square$

Once we have constructed a minimizing sequence in Lemma 3.4, the proof of Theorem 1.1 follows immediately.

Proof of Theorem 1.1. By Lemma 3.2, Lemma 3.3 and Lemma 3.4 the statement follows. $\square$

4. Decay estimates for the evolution problem

Let us consider a nonnegative solution $u(x,t)$ to (1.2). Note that, since the kernel is nonnegative, there is a comparison principle for this problem. Therefore, $-u^-(x,t) \leq u(x,t) \leq$

$u^+(x,t)$, where u^- and u^+ are the solutions with initial condition the negative and the positive part of u_0 respectively. Hence, for the proof of Theorem 1.2, we may assume that the solutions are nonnegative.

Proof of Theorem 1.2. First, we assume that $r > 1$ and $p > 2$. Multiplying equation (1.2) by $u^{r-1}(x,t)$ and integrating on the x variable, we obtain the following

$$\frac{1}{r}\frac{d}{dt}\int_{\mathbb{R}^d} u^r(x,t)\,dx = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} K(x,y)|u(y,t) - u(x,t)|^{p-2}(u(y,t) - u(x,t))u^{r-1}(x,t)\,dy\,dx$$
$$= -\frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} K(x,y)|u(y,t) - u(x,t)|^{p-2}(u(y,t) - u(x,t))(u^{r-1}(y,t) - u^{r-1}(x,t))\,dy\,dx.$$

Now, we use that, for $p > 2$ and $r > 1$ there is a constant $\widetilde{C}(p,r)$ such that

$$|u(y,t) - u(x,t)|^{p-2}(u(y,t) - u(x,t))(u^{r-1}(y,t) - u^{r-1}(x,t)) \geq \widetilde{C}(p,r)|u^\alpha(y,t) - u^\alpha(x,t)|^p,$$

with

$$\alpha = \frac{r + p - 2}{p}.$$

Indeed, one can check that is a constant $\widetilde{C}(p,r)$ such that

$$|1 - z|^{p-2}(1 - z)(1 - z^{r-1}) \geq \widetilde{C}(p,r)|1 - z^\alpha|^p,$$

for all $0 \leq z \leq 1$, because

$$\lim_{z\to 1}\frac{|1 - z|^{p-2}(1 - z)(1 - z^{r-1})}{|1 - z^\alpha|^p} > 0.$$

We obtain that, for some constant $C = C(p,r)$, the following holds:

$$\frac{d}{dt}\int_{\mathbb{R}^d} u^r(x,t)\,dx \leq -C\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} K(x,y)|u^\alpha(y,t) - u^\alpha(x,t)|^p\,dy\,dx$$
$$\leq -C\lambda_1(\mathbb{R}^d)\int_{\mathbb{R}^d}|u(x,t)|^{\alpha p}\,dx.$$

Now we observe that for $p > 2$, we have $\alpha p = r + p - 2 > r$ and therefore we can use the interpolation inequality

$$\|u(\cdot,t)\|_{L^r(\mathbb{R}^d)} \leq \|u(\cdot,t)\|_{L^1(\mathbb{R}^d)}^\theta \|u(\cdot,t)\|_{L^{\alpha p}(\mathbb{R}^d)}^{1-\theta}$$

with

$$\frac{1}{r} = \theta + \frac{1 - \theta}{\alpha p}$$

that is,

$$\theta = \frac{1}{r}\frac{\alpha p - r}{\alpha p - 1}, \qquad 1 - \theta = \frac{\alpha p}{r}\frac{r - 1}{\alpha p - 1}.$$

Hence, using that the $L^1(\mathbb{R}^d)$-norm of $u(\cdot,t)$ does not increase

$$\|u(\cdot,t)\|_{L^1(\mathbb{R}^d)} \leq \|u_0(\cdot)\|_{L^1(\mathbb{R}^d)}$$

we obtain that there exists $k_1 > 0$ such that

$$\frac{d}{dt}\|u(\cdot,t)\|_{L^r(\mathbb{R}^d)}^r \leq -k_1\left(\|u(\cdot,t)\|_{L^r(\mathbb{R}^d)}^r\right)^{\frac{\alpha p}{(1-\theta)r}}.$$

Then,

$$\|u(\cdot,t)\|_{L^r(\mathbb{R}^d)}^r \le k_2 \, t^{-\frac{(1-\theta)r}{\alpha p - (1-\theta)r}},$$

that is,

$$\|u(\cdot,t)\|_{L^r(\mathbb{R}^d)} \le k_3 \, t^{-\frac{r-1}{p-2}},$$

for some $k_3 > 0$ depending on u_0, p, r and K.

Now, for $r > 1$ and $1 \le p \le 2$, using the comparison principle for (1.2) (see [5, Theorem 6.37]), we have

$$\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^d)} \le \|u_0(\cdot)\|_{L^\infty(\mathbb{R}^d)}.$$

Hence, for any $q \ge p$ there exists a constant $\widetilde{C} = 2\,\|u_0\|_{L^\infty(\mathbb{R}^d)}^{\alpha(q-p)}$ such that

$$\widetilde{C}\,|u^\alpha(y,t) - u^\alpha(x,t)|^p \ge |u^\alpha(y,t) - u^\alpha(x,t)|^q.$$

Therefore, let us choose q such that $\alpha q = r$ and perform the same computation as in the previous case. We obtain

$$
\begin{aligned}
\frac{d}{dt}\int_{\mathbb{R}^d} u^r(x,t)\,dx &\le -C \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} K(x,y)|u^\alpha(y,t) - u^\alpha(x,t)|^q \, dy\,dx \\
&\le -\gamma \int_{\mathbb{R}^d} |u(x,t)|^r \, dx.
\end{aligned}
$$

Hence an exponential decay of u in $L^r(\mathbb{R}^d)$-norm follows

$$\int_{\mathbb{R}^d} u^r(x,t)\,dx \le \left(\int_{\mathbb{R}^d} u^r(x,0)\,dx\right)\cdot e^{-\gamma t},$$

for some $\gamma > 0$ depending on K, p, r and u_0.

Finally, if $r = 1$ we just have to multiply by $\mathrm{sgn}(u)$ and the proof follows similarly as in the previous cases. $\square$

5. THE CASE $p = \infty$

Let us consider for any compact support function u the quantities:

$$Q_p(u) = \frac{\displaystyle\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} K(x,y)|u(x) - u(y)|^p dx dy}{\displaystyle\int_{\mathbb{R}^d}|u|^p(x)dx}$$

and

$$Q_\infty(u) = \frac{\|u(x) - u(y)\|_{L^\infty(x,y\in supp(u);\,K(x,y)>0)}}{\|u\|_{L^\infty(supp(u))}}.$$

We set

(5.1) $$\lambda_{1,\infty}(\mathbb{R}^d) = \inf\{Q_\infty(u) : u \in L^\infty(\mathbb{R}^d), \text{ compactly supported}\}.$$

We remark that taking $u \equiv 1$ we obtain $Q_\infty(u) = 0$ and therefore in the definition of $\lambda_{1,\infty}(\mathbb{R}^d)$ we have to consider functions u that are compactly supported in $\mathbb{R}^d$.

Lemma 5.1. *The first eigenvalue for $p = \infty$, (5.1), is a bound for the limit of the first eigenvalues as $p \to \infty$,*

$$\limsup_{p \to \infty} [\lambda_{1,p}(\mathbb{R}^d)]^{1/p} \leq \lambda_{1,\infty}(\mathbb{R}^d).$$

Proof. Fix $u \in L^\infty(\mathbb{R}^d)$ compactly supported such that

$$Q_\infty(u) - \lambda_{1,\infty}(\mathbb{R}^d) < \varepsilon.$$

Now we observe that

$$\lim_{p \to \infty} [Q_p(u)]^{1/p} = \lim_{p \to \infty} \left(\frac{\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} K(x,y) |u(x) - u(y)|^p dx dy}{\int_{\mathbb{R}^d} |u|^p(x) dx} \right)^{1/p} = Q_\infty(u) < \lambda_{1,\infty}(\mathbb{R}^d) + \varepsilon.$$

Hence

$$\limsup_{p \to \infty} [\lambda_{1,p}(\mathbb{R}^d)]^{1/p} \leq \lambda_{1,\infty}(\mathbb{R}^d).$$

$\square$

Lemma 5.2. *Let K be such that its support satisfies*

$$\mathrm{supp}(K) \subset \{|x - a(y)| \leq 1\} \cup \{|y - a(x)| \leq 1\}.$$

Then $\lambda_{1,\infty}(\mathbb{R}^d) = 0$.

Remark 5.1. *For kernels defined by (1.3), we have assumed that the function ψ is supported in the unit ball, so the hypothesis of the lemma holds. If the support of ψ is any compact set in the ball with radius R, the result remains true replacing 1 by R.*

Proof. Given $\epsilon > 0$, we just have to construct a function u_ϵ, compactly supported such that

$$\frac{\|u_\epsilon(x) - u_\epsilon(y)\|_{L^\infty(x,y \in supp(u_\epsilon); K(x,y) > 0)}}{\|u_\epsilon\|_{L^\infty(supp(u_\epsilon))}} \leq \epsilon.$$

To this end let us start with

$$u_\epsilon(x) = 1, \qquad \text{in } |x| \leq 1,$$

Next, we let

$$u_\epsilon(x) = 1 - \epsilon, \qquad \text{in } 1 < |x| \leq R_a^1$$

where R_a^1 is such that $|x - a(y)| > 1$ if $|x| \leq 1$ and $|y| \geq R_a^1$. In this way if we let $|x| \leq 1$ the points $|y| \geq R_a^1$ are such that $(x,y) \notin supp(K)$, hence for $|x| \leq 1$,

$$\|u_\epsilon(x) - u_\epsilon(y)\|_{L^\infty(y; (x,y) \in \mathrm{supp}(u_\epsilon) \text{ and } K(x,y) > 0)} \leq \epsilon.$$

Now, we continue with

$$u_\epsilon(x) = 1 - 2\epsilon \qquad \text{in } R_a^1 < |x| \leq R_a^2$$

where R_a^2 is such that $|x - a(y)| > 1$ if $|x| \leq R_a^1$ and $|y| \geq R_a^2$. Analogously as before we get

$$\|u_\epsilon(x) - u_\epsilon(y)\|_{L^\infty(y; (x,y) \in \mathrm{supp}(u_\epsilon) \text{ and } K(x,y) > 0)} \leq \epsilon,$$

for $|x| \leq R_a^1$.

Iterating this procedure a finite number of times, $N = [1/\epsilon] + 1$ we get a compactly supported function, whose support is included in $|x| \leq R_a^N$. Moreover, the $L^\infty-$norm is one and

$$\|u_\epsilon(x) - u_\epsilon(y)\|_{L^\infty(x, y \in supp(u_\epsilon); K(x,y) > 0)} \leq \epsilon.$$

The proof is now complete. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

As an immediate consequence of the previous results we get Theorem 1.3.

Proof of Theorem 1.3. We just observe that

$$0 \leq \liminf_{p \to \infty} [\lambda_{1,p}(\mathbb{R}^d)]^{1/p} \leq \limsup_{p \to \infty} [\lambda_{1,p}(\mathbb{R}^d)]^{1/p} \leq \lambda_{1,\infty}(\mathbb{R}^d) = 0.$$

and the proof is finished $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

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