A NEW PROOF OF THE FLAT WALL THEOREM

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12 June 2012, revised 8 August 2016.

[1]Supported by JST, ERATO, Kawarabayashi Large Graph Project.
[2]Partially supported by NSF under Grants No. DMS-0701077 and DMS-1202640.
[3]Partially supported by the European Research Council under the European Union's Seventh Framework Programme (FP7/2007-2013)/ERC Grant Agreement no. 279558.

ABSTRACT

We give an elementary and self-contained proof, and a numerical improvement, of a weaker form of the excluded clique minor theorem of Robertson and Seymour, the following. Let $t, r \geq 1$ be integers, and let $R = 49152t^{24}(40t^2 + r)$. An r-wall is obtained from a $2r \times r$-grid by deleting every odd vertical edge in every odd row and every even vertical edge in every even row, then deleting the two resulting vertices of degree one, and finally subdividing edges arbitrarily. The vertices of degree two that existed before the subdivision are called the pegs of the r-wall. Let G be a graph with no K_t minor, and let W be an R-wall in G. We prove that there exist a set $A \subseteq V(G)$ of size at most $12288t^{24}$ and an r-subwall W' of W such that $V(W') \cap A = \emptyset$ and W' is a flat wall in $G - A$ in the following sense. There exists a separation (X, Y) of $G - A$ such that $X \cap Y$ is a subset of the vertex set of the cycle C' that bounds the outer face of W', $V(W') \subseteq Y$, every peg of W' belongs to X and the graph $G[Y]$ can almost be drawn in the unit disk with the vertices $X \cap Y$ drawn on the boundary of the disk in the order determined by C'. Here almost means that the assertion holds after repeatedly removing parts of the graph separated from $X \cap Y$ by a cutset Z of size at most three, and adding all edges with both ends in Z. Our proof gives rise to an algorithm that runs in polynomial time even when r and t are part of the input instance. The proof is self-contained in the sense that it uses only results whose proofs can be found in textbooks.

1 Introduction

All graphs in this paper are finite, and may have loops and parallel edges. A graph is a *minor* of another if the first can be obtained from a subgraph of the second by contracting edges. An H *minor* is a minor isomorphic to H. There is an ever-growing collection of so-called excluded minor theorems in graph theory. These are theorems which assert that every graph with no minor isomorphic to a given graph or a set of graphs has a certain structure. The best known such theorem is perhaps Wagner's reformulation of Kuratowski's theorem [20], which says that a graph has no K_5 or $K_{3,3}$ minor if and only if it is planar. One can also characterize graphs that exclude only one of those minors. To state such a characterization for excluded K_5 we need the following definition. Let H_1 and H_2 be graphs, and let J_1 and J_2 be complete subgraphs of H_1 and H_2, respectively, with the same number of vertices. Let G be obtained from the disjoint union of $H_1 - E(J_1)$ and $H_2 - E(J_2)$ by choosing a bijection between $V(J_1)$ and $V(J_2)$ and identifying the corresponding pairs of vertices. We say that G is a *clique-sum* of H_1 and H_2. Since we allow parallel edges, the set that results from the identification of $V(J_1)$ and $V(J_2)$ may include edges of the clique-sum. For instance, the graph obtained from K_4 by deleting an edge can be expressed as a clique-sum of two smaller graphs, where one is a triangle and the other is a triangle with a parallel edge added. By V_8 we mean the graph obtained from a cycle of length eight by adding an edge joining every pair of vertices at distance four in the cycle. The characterization of graphs with no K_5 minor, due to Wagner [19], reads as follows.

Theorem 1.1 *A graph has no K_5 minor if and only if it can be obtained by repeated clique-sums, starting from planar graphs and V_8.*

There are many other similar theorems; a survey can be found in [3]. Theorem 1.1 is very elegant, but attempts at extending it run into difficulties. For instance, no characterization is known for graphs with no K_6 minor, and there is evidence suggesting that such a characterization would be fairly complicated. Even if a characterization of graphs with no K_6 is found, there is no hope in finding one for excluding K_t for larger values of t.

Thus when excluding an H minor for a general graph H we need to settle for a less ambitious goal—a theorem that gives a necessary condition for excluding an H minor, but not necessarily a sufficient one. However, for such a theorem to be meaningful, the structure it describes must be sufficient to exclude some other, possibly larger graph H'. For planar graphs H this has been done by Robertson and Seymour [10]. To state their theorem we need to recall that the *tree-width* of a graph G is the least integer k such that G can be obtained by repeated clique-sums, starting from graphs on at most $k + 1$ vertices.

Theorem 1.2 *For every planar graph H there exists an integer k such that every graph with no H minor has tree-width at most k. If H is not planar, then no such integer exists.*

This is a very satisfying theorem, because it is best possible in at least two respects. Not only is there no such integer when H is not planar, but no graph of tree-width k has a minor isomorphic to the $(k + 1) \times (k + 1)$-grid.

But how about excluding a non-planar graph? Robertson and Seymour have an answer to that question as well, but in order to motivate it we need to digress a bit.

1.1 The Two Disjoint Paths Problem

Let C be a cycle in a graph G. We say that a *C-cross* in G is a pair of disjoint paths P_1, P_2 with ends s_1, t_1 and s_2, t_2, respectively, such that s_1, s_2, t_1, t_2 occur on C in the order listed, and the paths are otherwise disjoint from C.

Let G be a graph, and let $s_1, s_2, t_1, t_2 \in V(G)$. The TWO DISJOINT PATHS PROBLEM asks whether there exist two disjoint paths P_1, P_2 in G such that P_i has ends s_i and t_i. There is a beautiful characterization of the feasible instances, which we now describe. First of all, let us assume that G has a cycle C with vertex-set $\{s_1, s_2, t_1, t_2\}$ in order. This we can assume, because the edges of C can be added without changing the feasibility status of the problem. It follows that the TWO DISJOINT PATHS PROBLEM is feasible if and only if the graph G has a C-cross. Thus we will study the more general problem of when a graph has a C-cross.

Now if G can be drawn in the plane with C bounding a face, then it has no C-cross. (Proof. Add a new vertex in the face bounded by C and join it by an edge to every vertex of C. The new graph is planar, and yet if the C-cross existed, they would give rise to a K_5 minor in G.) So this gives one class of obstructions, but there is another one. A *separation* in a graph G is a pair (A, B) of subsets of vertices such that $A \cup B = V(G)$, and there is no edge of G with one end in $A \setminus B$ and the other in $B \setminus A$. The order of the separation (A, B) is $|A \cap B|$. Now if there exists a separation (A, B) of G of order at most three with $V(C) \subseteq A$, then the vertices in $B \setminus A$ are not very useful. Let H be the graph obtained from G by deleting $B \setminus A$ and instead adding an edge joining every pair of vertices in $A \cap B$. It follows that if G has a C-cross, then so does H. Furthermore, if we choose (A, B) so that some component of $G[B \setminus A]$ includes a neighbor of every vertex in $A \cap B$, then the converse holds as well. Let us turn this observation into a definition.

Definition Let G be a graph, and let $X \subseteq V(G)$. Let (A, B) be a separation of G of order at most three with $X \subseteq A$ and such that there exist $|A \cap B|$ paths from some vertex $v \in B \setminus A$ to X that are disjoint except for v. Let H be the graph obtained from $G[A]$ by adding an edge joining every pair of distinct vertices in $A \cap B$. We say that H is an *elementary X-reduction* of G, and we say that it is an *elementary X-reduction determined by* (A, B). We say that a graph J is an *X-reduction* of G if it can be obtained from G be a series of elementary X-reductions. If C is a subgraph of G, then by an (elementary) C-reduction we mean an (elementary) $V(C)$-reduction.

Thus taking C-reductions does not change whether there exists a C-cross, and as we are about to see, when no C-reduction is possible, the only obstruction to the existence of a C-cross is topological, namely that G can be drawn in the plane with C bounding a face. The first version of the promised theorem, obtained in various forms by Jung [6], Robertson and Seymour [11], Seymour [15], Shiloach [16], and Thomassen [18] reads as follows.

Theorem 1.3 *Let G be a graph, and let C be a cycle in G. Then G has no C-cross if and only if some C-reduction of G can be drawn in the plane with C bounding a face.*

Since Theorem 1.3 is not as well-known as it should be, and its proof is not entirely trivial, we give a proof in the Appendix. An additional reason for including a proof of Theorem 1.3 is to validate our claim that we only use results that can be found in textbooks. For applications it is desirable to have a representation of the entire graph G as opposed to some unspecified C-reduction. Formalizing this idea is the subject to the next two definitions.

Definition If X is a set in a topological space, we define $\widetilde{X} := \overline{X} \setminus X$. A *painting* in a surface Σ is a pair $\Gamma = (U, N)$, where $N \subseteq U \subseteq \Sigma$, N is finite, $U \setminus N$ has finitely many arcwise-connected components, called *cells*, and for every cell c, the closure $\overline{c}$ is a closed disk and $\widetilde{c} = \overline{c} \cap N \subseteq \mathrm{bd}(\overline{c})$ satisfies $|\widetilde{c}| \leq 3$. We define $N(\Gamma) := N$, $U(\Gamma) := U$ and denote the set of cells of Γ by $C(\Gamma)$. Thus the cells of a painting define a hypergraph with hyperedges of cardinality at most of three by saying that c is incident with the elements $\widetilde{c}$.

Definition Let G be a graph, and let Ω be a cyclic permutation of a set $V(\Omega) \subseteq V(G)$. By an Ω-*rendition* of G we mean a triple (Γ, σ, π), where

- Γ is painting in the unit disk Δ,
- σ assigns to each cell $c \in C(\Gamma)$ a subgraph $\sigma(c)$ of G, and
- $\pi : N(\Gamma) \to V(G)$ is an injection

such that

(P1) $G = \bigcup(\sigma(c) \,:\, c \in C(\Gamma))$,

(P2) $\sigma(c)$ and $\sigma(c')$ are edge-disjoint for distinct $c, c' \in C(\Gamma)$,

(P3) $\pi(\widetilde{c}) \subseteq V(\sigma(c))$ for every cell $c \in C(\Gamma)$,

(P4) $V(\sigma(c) \cap \bigcup(\sigma(c') \,:\, c' \in C(\Gamma) \setminus \{c\})) \subseteq \pi(\widetilde{c})$ for every cell $c \in C(\Gamma)$, and

(P5) the image under π of $N(\Gamma) \cap \mathrm{bd}(\Delta)$ is $V(\Omega)$, mapping the cyclic order of $\mathrm{bd}(\Delta)$ to the cyclic order of Ω.

A cycle C defines a cyclic permutation of $V(C)$, and so we may speak of a *C-rendition*.

Using the above definitions we can extend Theorem 1.3 as follows.

Theorem 1.4 *Let G be a graph, and let C be a cycle in G. Then the following conditions are equivalent:*

(1) *G has no C-cross,*

(2) *some C-reduction of G can be drawn in the plane with C bounding a face, and*

(3) *G has a C-rendition.*

Proof. The implication (1)⇒(2) holds by Theorem 1.3. Let us now prove that (2)⇒(3) by induction on $|V(G)|$. To that end let us assume that some C-reduction of G can be drawn in the plane with C bounding a face, and that the implication (2)⇒(3) holds for all graphs on strictly fewer than $|V(G)|$ vertices. We may assume that G has no isolated vertices, because otherwise the implication follows by induction by deleting them. Let us assume first that G can be drawn in the plane with C bounding a face. We may assume that $V(C)$ is drawn on the boundary of the unit disk Δ, and that the rest of G is drawn in the interior of Δ. We now construct a C-rendition as follows. Let $F \subseteq E(G)$ be the set of all edges $e \in E(G)$ such that e is not contained in the closed disk bounded by a loop edge other than e, and let V be the set of vertices $v \in V(G)$ that do not belong to the open disk bounded by a loop edge of G. For every edge $e \in F$ we "fatten" e into a disk D_e in such a way that $e \subseteq D_e \subseteq \Delta$, D_e includes the two ends of e in its boundary and is otherwise disjoint from V and $F \setminus \{e\}$, and for distinct edges $e, e' \in F$ the intersection $D_e \cap D_{e'}$ consists of common end(s) of e and e'. Let Γ be the painting defined by $U(\Gamma) = \bigcup_{e \in F} D_e$ and $N(\Gamma) = V$. Thus each cell c of Γ includes an edge $e \in E(G)$ and we define $\sigma(c)$ to be the graph consisting of all vertices contained in c and all edges contained in c and their ends. Thus if e is not a loop, then it is the only edge contained in c. Finally, we define $\pi : N(\Gamma) \to V(G)$ to be the identity. Then (Γ, σ, π) is a C-rendition of G, and hence (3) holds. This completes the case when G can be drawn in the plane with C bounding a face.

We may assume now that some elementary C-reduction G' of G has a C-reduction that can be drawn in the plane with C bounding a face. Let (A, B) be the separation of G giving rise to the C-reduction G'. By the induction hypothesis G' has a C-rendition (Γ', σ', π'). If $A \cap B \subseteq \sigma'(c)$ for some $c \in C(\Gamma')$, then by adding $G[B]$ to $\sigma'(c)$ we obtain a C-rendition of G, and so we may assume that $A \cap B \not\subseteq \sigma'(c)$ for all $c \in C(\Gamma')$. It follows that $|A \cap B| = 3$, that $\pi(X) = A \cap B$ for some set $X \subseteq N(\Gamma)$ of size three, and that every pair of elements of X are incident with a cell of Γ'. Thus there exists a closed disk D whose boundary intersects $U(\Gamma')$ in X only. Let the painting Γ be defined by saying that $U(\Gamma) = U(\Gamma') \cup D$ and that $N(\Gamma)$ consists of all $n \in N(\Gamma')$ that do not belong to the interior of D. Thus the cells of Γ are $D \setminus X$ and all the cells of Γ' that are disjoint from D. We define $\sigma(D \setminus X)$ to be the union of $G[B]$ and $\sigma'(c)$ over all cells $c \in C(\Gamma')$ contained in D, and for cells $c \in C(\Gamma')$ that are disjoint from D we define $\sigma(c) = \sigma'(c)$. Finally, let π be the restriction of π' to $N(\Gamma)$. Then (Γ, σ, π) is a C-rendition of G. This completes the proof of the implication (2)⇒(3).

It remains to prove (3)⇒(1). We again proceed by induction on $|V(G)|$. Let (Γ, σ, π) be a C-rendition of G, and assume that the implication (3)⇒(1) holds for all graphs on strictly fewer than $|V(G)|$ vertices. Let us say that a cell $c \in C(\Gamma)$ is *slim* if $V(\sigma(c)) \subseteq \pi(\widetilde{c})$. If every cell of Γ is slim, then it is easy to convert the C-rendition into a drawing of G in the plane with C bounding a face, and hence G has no C-cross, as desired. We may therefore assume that there exists a cell $c \in C(\Gamma)$ that is not slim. Let G' be obtained from G by deleting $V(\sigma(c)) - \pi(\widetilde{c})$ and adding an edge joining every pair of vertices in $\pi(\widetilde{c})$. It is easy to convert (Γ, σ, π) to a C-rendition of G'. By induction the graph G' has no C-cross, and it follows from the definition of G' that neither does G, as desired. □

1.2 The Flat Wall Theorem

We are now ready to formulate the weaker version of the excluded K_t theorem of Robertson and Seymour [12, Theorem 9.8]. Let us begin by describing it informally. We use $[r]$ to denote $\{1, 2, \dots, r\}$. Let $r, s \geq 2$ be integers. An $r \times s$-grid is the graph with vertex-set $[r] \times [s]$ in which (i, j) is adjacent to (i', j') if and only if $|i - i'| + |j - j'| = 1$. An *elementary r-wall* is obtained from the $2r \times r$-grid by deleting all edges with ends $(2i - 1, 2j - 1)$ and $(2i - 1, 2j)$ for all $i = 1, 2, \dots, r$ and $j = 1, 2, \dots, \lfloor r/2 \rfloor$ and all edges with ends $(2i, 2j)$ and $(2i, 2j + 1)$ for all $i = 1, 2, \dots, r$ and $j = 1, 2, \dots, \lfloor (r - 1)/2 \rfloor$ and then deleting the two resulting vertices of degree one. An *r-wall* is any graph obtained from an elementary r-wall by subdividing edges. In other words each edge of the elementary r-wall is replaced by a path. Figure 1 shows an elementary 4-wall. Walls are harder to describe than grids, but they are easier to work with; moreover, if a graph has a $2r \times 2r$-grid minor, then it has a subgraph isomorphic to an r-wall. Let W be an r-wall, where W is a subdivision of an elementary wall Z. Let X be the set of vertices of W that correspond to vertices (i, j) of Z with $j = 1$, and let Y be the set of vertices of W that correspond to vertices (i, j) of Z with $j = r$. There is a unique set of r disjoint paths $Q_1, Q_2, \dots, Q_r$ in W, such that each has one end in X and one end in Y, and no other vertex in $X \cup Y$. We may assume that the paths are numbered so that the first coordinates of their vertices are increasing. We say that $Q_1, Q_2, \dots, Q_r$ are the *vertical paths* of W. There is a unique set of r disjoint paths with one end in Q_1, the other end in Q_r, and otherwise disjoint from $Q_1 \cup Q_r$. Those will be called the *horizontal paths* of W. Let $P_1, P_2, \dots, P_r$ be the horizontal paths numbered in the order of increasing second coordinates. Then $P_1 \cup Q_1 \cup P_r \cup Q_r$ is a cycle, and we will call it the *outer cycle* of W. If W is drawn as a plane graph in the obvious way, then this is indeed the cycle bounding the outer face. The sets $V(P_1 \cap Q_1)$, $V(P_1 \cap Q_r)$, $V(P_r \cap Q_1)$, and $V(P_r \cap Q_r)$ each include exactly one vertex of W; those vertices will be called the *corners* of W. In Figure 1 the four corners are circled. The vertices of W that correspond to vertices of Z of degree two will be called the *pegs* of W. Thus given W as a graph the corners and pegs are not necessarily uniquely determined. Finally let W, W' be walls such that W' is a subgraph of W. We say that W' is a *subwall* of W if every horizontal path of W' is a subpath of a horizontal path of W, and every vertical path of W' is a subpath of a vertical path of W.

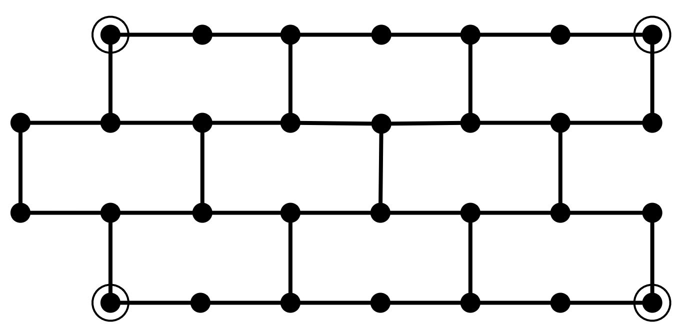

Figure 1: An elementary 4-wall.

Now let W be a large wall in a graph G with no K_t minor. The Flat Wall Theorem asserts that there exist a set of vertices $A \subseteq V(G)$ of bounded size and a reasonably big subwall W' of W that is disjoint from A and has the following property. Let C' be the outer cycle of W'. The property we want is that C' separates the graph $G - A$ into two graphs, and the one containing W', say H, can be drawn in the plane with C' bounding a face. However, as the discussion of the previous subsection attempted to explain, the latter condition is too strong. The most we can hope for is for the graph H to be C'-flat. That is, in spirit, what the theorem will guarantee, except that we cannot guarantee that all of C' be part of a planar C'-reduction of H. The correct compromise is that some subset of $V(C')$ separates off the wall W', and it is that subset that is required to be incident with one face of the planar drawing. Here is the formal definition.

Definition Let G be a graph, and let W be a wall in G with outer cycle D. Let us assume that there exists a separation (A, B) such that $A \cap B \subseteq V(D)$, $V(W) \subseteq B$, and there is a choice of pegs of W such that every peg belongs to A. If some $A \cap B$-reduction of $G[B]$ can be drawn in a disk with the vertices of $A \cap B$ drawn on the boundary of the disk in the order determined by D, then we say that the wall W is *flat in G*. It follows that it is possible to choose the corners of W is such a way that every corner belongs to A.

We need one more definition. Given a wall W in a graph G we will (sometimes) produce a K_t minor in G. However, this K_t will not be arbitrary; it will be very closely related to the wall W. To make this notion precise we first notice that a K_t minor in G is determined by t pairwise disjoint sets $X_1, X_2, \ldots, X_t$ such that each induces a connected subgraph and every two of the sets are connected by an edge of G. We say that $X_1, X_2, \ldots, X_t$ form a *model* of a K_t minor and we will refer to the sets X_i as the *branch-sets* of the model. Often we will shorten this to a model of K_t. Let $P_1, P_2, \ldots, P_r$ be the horizontal paths and $Q_1, Q_2, \ldots, Q_r$ the vertical paths of W. We say that a model of a K_t minor in G is *grasped* by the wall W if for every branch-set X_k of the model there exist distinct indices $i_1, i_2, \ldots, i_t \in \{1, 2, \ldots, r\}$ and distinct indices $j_1, j_2, \ldots, j_t \in \{1, 2, \ldots, r\}$ such that $V(P_{i_l} \cap Q_{j_l}) \subseteq X_k$ for all $l = 1, 2, \ldots, t$. Let us remark, for those familiar with the literature, that if a wall grasps a model of K_t, then the tangle determined by W controls it in the sense of [13]. The notion of control is important in applications, but since the stronger property is a consequence of the proof, we state the theorem that way.

We can now formulate the Flat Wall Theorem. It first appeared in a slightly weaker form in [12, Theorem 9.8] with an unspecified bound on R in terms of t and r.

Theorem 1.5 *Let $r, t \geq 1$ be integers, let $R = 49152t^{24}(40t^2 + r)$, let G be a graph, and let W be an R-wall in G. Then either G has a model of a K_t minor grasped by W, or there exist a set $A \subseteq V(G)$ of size at most $12288t^{24}$ and an r-subwall W' of W such that $V(W') \cap A = \emptyset$ and W' is a flat wall in $G - A$.*

We can use Theorem 1.5 to obtain an approximate characterization of graphs with no large clique minor, as follows.

Theorem 1.6 *Let $r, t \geq 1$ be integers, let $R = 49152t^{24}(40t^2 + r)$, and let G be a graph. If G has no K_t minor, then for every R-wall W in G there exist a set $A \subseteq V(G)$ of size at most*

$12288t^{24}$ and an r-subwall W' of W such that $V(W')\cap A=\emptyset$ and W' is a flat wall in $G-A$. Conversely, if $t\ge 2$ and $r\ge 80t^{12}$ and for every R-wall W in G there exist a set $A\subseteq V(G)$ of size at most $12288t^{24}$ and an r-subwall W' of W such that $V(W')\cap A=\emptyset$ and W' is a flat wall in $G-A$, then G has no $K_{t'}$ minor, where $t'=2R^2$.

Proof. The first part of the theorem follows immediately from Theorem 1.5. To prove the converse suppose for a contradiction that $r\ge 123t^{12}$ and that G has a $K_{t'}$ minor, and yet for every R-wall W in G there exist a set $A\subseteq V(G)$ of size at most $12288t^{24}$ and an r-subwall W' of W such that $V(W')\cap A=\emptyset$ and W' is a flat wall in $G-A$. Let W_0 be the elementary R-wall. We may assume that G has a $K_{t'}$ minor with model $(X_v : v\in V(W_0))$. Since W_0 has maximum degree at most three, it follows that there exists a a subgraph W of G isomorphic to a subdivision of W_0 such that

- for every vertex $v\in V(W_0)$ the corresponding vertex of W belongs to X_v, and
- for every edge $uv\in E(W_0)$ the vertex-set of the corresponding path of W is contained in $X_u\cup X_v$.

Thus W is an R-wall in G, and hence there exist a set $A\subseteq V(G)$ of size at most $12288t^{24}$ and an r-subwall W' of W such that $V(W')\cap A=\emptyset$ and W' is a flat wall in $G-A$. Let W_0' be the elementary subwall of W_0 that corresponds to W'. Since $t\ge 2$ and $r\ge 80t^{12}$ there exist five distinct vertices $v_1,v_2,\ldots,v_5\in V(W_0')$ such that none of them belongs to the outer cycle of W_0' and $X_{v_i}\cap A=\emptyset$ for all $i=1,2,\ldots,5$. Let (X,Y) be a separation of $G-A$ as in the definition of a flat wall. Since $X\cap Y$ is a subset of the vertex-set of the outer cycle of W' we deduce that $X_i\cap X\cap Y=\emptyset$, and hence $X_i\subseteq Y$ for all $i=1,2,\ldots,5$. Thus $X_{v_1},X_{v_2},\ldots,X_{v_5}$ is a model of a K_5 minor in $G[Y]$. Furthermore, by considering the vertices v_1,v_2,v_3,v_4,v_5 of the wall W' we conclude that there exist four disjoint paths in W', and hence in $G[Y]$, such that the i-th path has one end in $X_{v_{j_i}}$ and the other end a peg of W', and none of the paths has an internal vertex in any of the sets X_{v_j}, where j_1,j_2,j_3,j_4 are pairwise distinct. However, the existence of the K_5 minor and the four disjoint paths contradict the fact that some $X\cap Y$-reduction of $G[Y]$ can be drawn in a disk. □

An earlier version of this paper as well as other articles refer to Theorem 1.5 as the Weak Structure Theorem. However, we prefer the current name, because it gives a more accurate description of the result.

By Theorem 1.2 every graph of sufficiently large tree-width has an R-wall. It follows from [7] that in Theorem 1.5 the hypothesis that G have an R-wall can be replaced by the assumption that G have tree-width at least $t^{\Omega(t^2\log t)}R$, and by [2] it can be replaced by the assumption that G have tree-width at least $\Omega(R^{19}\text{ poly }\log R)$.

We prove Theorem 1.5 in Section 5. Our proof is self-contained, but it is inspired by the Graph Minors series of Robertson and Seymour. Giannopoulou and Thilikos [5] improved the bound on the size of A to the best possible bound of $|A|\le t-5$. Their proof uses Theorem 1.10, and therefore does not give an explicit bound on R as a function of t. In Section 6 we deduce the bound of $|A|\le t-5$ from Theorem 1.5 by an elementary argument with an explicit bound on R, as follows.

Theorem 1.7 *Let $t \geq 5$ and $r \geq 3\lceil\sqrt{t}\rceil$ be integers. Let $n = 12288t^{24}$, $R = r^{2^n}$ and $R_0 = 49152t^{25}(40t + R)$. Let G be a graph, and let W_0 be an R_0-wall in G. Then either G has a model of a K_t minor grasped by W_0, or there exist a set $A \subseteq V(G)$ of size at most $t - 5$ and an r-subwall W of W_0 such that $V(W) \cap A = \emptyset$ and W is a flat wall in $G - A$.*

In fact, in Theorem 6.2 we prove a stronger result asserting that the set A and subwall W may be chosen in such a way that every vertex of A attaches throughout the wall W. In Section 6 we prove another variation, where in the second outcome we are able to conclude that if (X, Y) is a separation that witnesses that W is a flat wall, then $G[Y]$ has bounded tree-width (or, equivalently, has no big wall). That conclusion is useful in algorithmic applications, but in order to obtain it we need to drop the conditions that the K_t minor is grasped by the wall W_0 and that the desired wall W is a subwall of W_0.

Theorem 1.8 *Let $r \geq 2$ and $t \geq 5$ and be integers, let $n = 12288t^{24}$ and $R_0 = 49152t^{24}(40t^2 + (rt)^{2^n})$ and let G be a graph with no K_t minor. If G has an R_0-wall, then there exist a set $A \subseteq V(G)$ of size at most $t - 5$ and an r-wall W in G such that $V(W) \cap A = \emptyset$ and W is a flat wall in $G - A$. Furthermore, if (X, Y) is a separation as in the definition of flat wall, then the graph $G[Y]$ has no $(R_0 + 1)$-wall.*

In Section 7 we convert the proof of Theorem 1.5 into a polynomial-time algorithm, as follows.

Theorem 1.9 *There is an algorithm with the following specifications.*
Input: *A graph G on n vertices and m edges, integers $r, t \geq 1$, and an R-wall W in G, where $R = 49152t^{24}(60t^2 + r)$.*
Output: *Either a model of a K_t minor in G grasped by W, or a set $A \subseteq V(G)$ of size at most $12288t^{24}$ and an r-subwall W' of W such that $V(W') \cap A = \emptyset$ and W' is a flat wall in $G - A$.*
Running time: $O(t^{24}m + n)$.

In the second alternative the algorithm also returns a separation (A, B) as in the definition of flat wall, and a certificate that the separation is as desired. The details are in the version stated as Theorem 7.7.

1.3 The Excluded Clique Minor Theorem

Theorem 1.5 is a step toward a more comprehensive excluded minor theorem of Robertson and Seymour [13].

Theorem 1.10 *For every finite graph H there exists an integer k such that every graph with no H minor can be obtained by repeated clique-sums, starting from graphs that k-near embed in a surface in which H cannot be embedded.*

Since we do not need Theorem 1.10, let us omit the precise definition of k-near embedding. Instead, let us describe it informally. A graph G can be k-near embedded in a surface Σ if there exists a set $A \subseteq V(G)$ of size at most k such that $G - A$ can be almost drawn in Σ, except for at most k areas of non-planarity, where crossings are permitted, but the graph is restricted in a different way. Here almost (similarly as in the abstract) means that we are not drawing the graph G itself, but some C-reduction instead, where now C is a large wall in G. We refer to [13] for a precise statement.

We believe that we have found a much simpler proof of Theorem 1.10 with a significantly improved bound on k. We will report on it soon.

The paper is organized as follows. In the next three sections we prove auxiliary lemmas, and in Section 5 we prove Theorem 1.5. In Section 6 we prove Theorems 1.7 and 1.8. In Section 7 we convert the proof of Theorem 1.5 to a polynomial-time algorithm to construct either a K_t minor or a flat wall. In order to keep the paper self-contained we give a proof of Theorem 1.3 in the Appendix.

2 Disjoint M-paths with distance constraints

Let G be a graph, and let M be a subgraph of G. By an *M-path* we mean a path in G with at least one edge, both ends in $V(M)$ and otherwise disjoint from M. The objective of this section is to study M-paths that are "long" in the sense that their ends are at least some specified distance apart according to a metric on $V(M)$. We prove an Erdős-Pósa-type result that says that either there are many long M-paths, or all long M-paths can be destroyed by deleting a restricted set of vertices. In fact, we prove two closely related results along the same lines. It turns out that for these lemmas the distance need not be given by a metric—all that is needed is the knowledge of which pairs of vertices are far apart. We capture that using the relation R below.

Definition Let G be a graph, let M be a subgraph of G, and let R be a reflexive and symmetric relation on $V(M)$. We say that pairwise disjoint M-paths $P_1, \dots, P_k$ are *R-semi-dispersed* if it is possible to label the ends of P_i as x_i and y_i such that $(x_i, y_i) \notin R$ and $(x_i, x_j) \notin R$ for all distinct indices $i, j \in \{1, 2, \dots, k\}$. Thus no restriction is placed on the relative position of the vertices $y_1, y_2, \dots, y_k$. For $x \in V(M)$ we define $R(x)$, the *ball around* x, as the set of all $y \in V(M)$ such that $(x, y) \in R$.

Let us recall that a collection of paths $\mathcal{P}$ are *internally disjoint* if every vertex that belongs to two distinct members of $\mathcal{P}$ is an end of both.

Lemma 2.1 *Let G be a graph, let M be a subgraph of G, let R be a reflexive and symmetric relation on $V(M)$, and let $k \geq 0$ be an integer. Then either there exist pairwise disjoint M-paths $P_1, \dots, P_k$ which are R-semi-dispersed, or, alternatively, the following holds. There exist sets $A \subseteq V(G)$ and $Z \subseteq V(M)$ with $|A| \leq k-1$ and $|Z| \leq 3k-3$ such that every M-path P in $G - A$ with ends x and y either satisfies $(x, y) \in R$ or both $x, y \in \bigcup_{z \in Z} R(z)$.*

Proof. For the duration of the proof, we will say that an M-path P is *long* if the ends x and y of P satisfy $(x, y) \notin R$. Let $P_1, \dots, P_s$ be disjoint M-paths with the ends of P_i labeled x_i

and y_i satisfying the requirements in the definition of R-semi-dispersed. Let $0 \leq p \leq s$ be an integer, and let $Q_1, \ldots, Q_p$ be disjoint paths with the ends of Q_i equal to a_i and w_i satisfying the following for all distinct integers $i, j \in \{1, 2, \ldots, p\}$:

(a) $w_i \in V(M) \setminus \bigcup_{m=1}^{s}(R(x_m) \cup R(y_m))$,

(b) $a_i \in V(P_i)$,

(c) Q_i is internally disjoint from $V(M) \cup \bigcup_{m=1}^{s} V(P_m)$ and $E(Q_i) \cap E(M) = \emptyset$, and

(d) $(w_i, w_j) \notin R$.

We may assume that these paths are chosen so that s is maximum, and, subject to that, p is maximum. We may assume that $s < k$, for otherwise the first outcome of the lemma holds. We will show that the sets $A := \{a_1, a_2, \ldots, a_p\}$ and $Z := \{w_1, w_2, \ldots, w_p, x_1, y_1, \ldots, x_s, y_s\}$ satisfy the second outcome of the lemma.

To that end let $W := \bigcup_{i=1}^{p} R(w_i) \cup \bigcup_{i=1}^{s} (R(x_i) \cup R(y_i))$. We may assume for a contradiction that there exists a long M-path S in $G - A$ which has an end in $V(M) \setminus W$. If S is disjoint from $P_1, \ldots, P_s$, we see that $S, P_1, \ldots, P_s$ satisfy the definition of R-semi-dispersed, contrary to the maximality of s. Thus S intersects one of the paths P_i, and hence we may let y be the first vertex of $\bigcup_{i=1}^{p} V(Q_i) \cup \bigcup_{i=1}^{s} V(P_i)$ which we encounter when traversing the path S beginning at an end in $x \in V(M) \setminus W$.

There are now several different cases, depending on where the vertex y lies. As the first case, assume $y \in V(Q_i)$ for some $1 \leq i \leq p$. It follows that $S \cup Q_i$ contains a long M-path, call it P', which has x as an end and is disjoint from $P_1, \ldots, P_s$. Then the paths $P', P_1, \ldots, P_s$ are R-semi-dispersed, contrary to the maximality of s. As the next case, assume $y \in V(P_i)$ for some $1 \leq i \leq p$. Then $S \cup Q_i \cup P_i$ contains two disjoint long M-paths, call them P' and P'', such that P' has x as an end and P'' has w_i as an end. Note that here we are using the property that $y \neq a_i$ to ensure that P' and P'' can be chosen disjoint. Then the paths $P_1, \ldots, P_{i-1}, P_{i+1}, \ldots, P_s, P', P''$ are R-semi-dispersed, again contrary to the maximality of s. As the final case, consider when $y \in V(P_i)$ for some index i with $p < i \leq s$. We may assume, by swapping the paths P_{p+1} and P_i, that $i = p + 1$. Then the paths $Q_1, \ldots, Q_p, S$ contradict the maximality of p.

This completes the analysis of the possible cases, proving the lemma. □

We also need the following closely related lemma. Let G be a graph, let M be a subgraph of G, and let R be a reflexive and symmetric relation on $V(M)$. We say that pairwise disjoint M-paths $P_1, P_2, \ldots, P_k$ are *R-dispersed* if $(x, y) \notin R$ for every two distinct vertices x, y such that each is an end of one of the paths P_i.

Lemma 2.2 *Let G be a graph, let M be a subgraph of G, let R be a reflexive and symmetric relation on $V(M)$, and let $k \geq 0$ be an integer. Then either there exist pairwise disjoint R-dispersed M-paths $P_1, \ldots, P_k$, or, alternatively, the following holds. There exist sets $A \subseteq V(G)$ and $Z \subseteq V(M)$ with $|A| \leq k - 1$ and $|Z| \leq 3k - 3$ such that for every M-path P in $G - A$ its ends can be denoted by x and y such that either $(x, y) \in R$ or $x \in \bigcup_{z \in Z} R(z)$.*

Proof. This follows by the same argument as Lemma 2.1, with the following differences. Instead of choosing the paths P_i to be R-semi-dispersed we choose them to be R-dispersed. We choose the path S to be an $(M - A - W)$-path in $G - A$; if such a choice is not possible, then the lemma holds. We then derive a contradiction as in the proof of Lemma 2.1. □

3 Meshes and clique minors

In this section we introduce the notion of a mesh—a common generalization of walls and grids. It will allow us to reduce problems about walls to problems about grids, which is useful, because grids are easier to work with. We also introduce a distance function on a mesh.

Definition Let $r, s \geq 2$ be positive integers, let M be a graph, and let $P_1, P_2, \ldots, P_r$, $Q_1, Q_2, \ldots, Q_s$ be paths in M such that the following conditions hold for all $i = 1, 2, \ldots, r$ and $j = 1, 2, \ldots, s$:

(1) $P_1, P_2, \ldots, P_r$ are pairwise vertex disjoint, $Q_1, Q_2, \ldots, Q_s$ are pairwise vertex disjoint, and $M = P_1 \cup P_2 \cup \cdots \cup P_r \cup Q_1 \cup Q_2 \cup \cdots \cup Q_s$,

(2) $P_i \cap Q_j$ is a path, and if $i \in \{1, s\}$ or $j \in \{1, r\}$ or both, then $P_i \cap Q_j$ has exactly one vertex,

(3) P_i has one end in Q_1 and the other end in Q_s, and when traversing P_i the paths $Q_1, Q_2, \ldots, Q_s$ are encountered in the order listed,

(4) Q_j has one end in P_1 and the other end in P_r, and when traversing Q_j the paths $P_1, P_2, \ldots, P_r$ are encountered in the order listed.

In those circumstances we say that M is an $r \times s$ *mesh*. We will refer to $P_1, P_2, \ldots, P_r$ as *horizontal paths* and to $Q_1, Q_2, \ldots, Q_s$ as *vertical paths*. Thus every $r \times s$ grid is an $r \times s$ mesh, and every planar graph obtained from an $r \times s$ grid by subdividing edges and splitting vertices is an $r \times s$ mesh. In particular, every r-wall is an $r \times r$-mesh.

We wish to define a distance function on a mesh, but we first do it for a grid. Let H be the $r \times s$ grid, so that $V(H) = [r] \times [s]$. We regard H as a plane graph, using the obvious straight-line drawing. For $v_1 = (x_1, y_1)$ and $v_2 = (x_2, y_2)$ we define $d(v_1, v_2) := k - 1$, where k is the least integer such that every curve in the plane joining v_1 and v_2 intersects H at least k times. (We may clearly restrict ourselves to curves intersecting H only in vertices.) This distance can be calculated from the knowledge of the coordinates. Indeed, it is easy to check that $d(v_1, v_2)$ is equal to the minimum of $\max\{|x_1 - x_2|, |y_1 - y_2|\}$ and $\min\{x_1, y_1, r + 1 - x_1, s + 1 - y_1\} + \min\{x_2, y_2, r + 1 - x_2, s + 1 - y_2\} - 1$.

We now extend this definition to meshes as follows. Let M be a mesh with horizontal paths $P_1, P_2, \ldots, P_r$ and vertical paths $Q_1, Q_2, \ldots, Q_s$ as above. Then M has an H minor, where H is the $r \times s$ grid, as in the previous paragraph. Thus there exists a surjective mapping

$f : V(M) \to V(H)$ such that $f^{-1}(u)$ is a branch-set of the H minor for every $u \in V(H)$. Furthermore, if $u = (i, j)$, then the set $f^{-1}(u)$ includes $V(P_i) \cap V(Q_j)$. If d_H denotes the distance function on H from the previous paragraph, then we define $d(u, v) := d_H(f(u), f(v))$. We say that d is a *distance function on* M. The function d is a pseudometric; that is, it is symmetric and satisfies the triangle inequality, but there may be distinct vertices u, v with $d(u, v) = 0$. The function d is not unique; it depends on the choice of the function f.

Definition The definition of grasping extends to meshes almost verbatim, as follows. Let M be an $r \times s$-mesh in a graph G with horizontal paths $P_1, P_2, \ldots, P_r$ and vertical paths $Q_1, Q_2, \ldots, Q_s$. We say that a model of a K_t minor in G is *grasped* by M if for every branch-set X of the model there exist distinct indices $i_1, i_2, \ldots, i_t \in \{1, 2, \ldots, r\}$ and distinct indices $j_1, j_2, \ldots, j_t \in \{1, 2, \ldots, s\}$ such that $V(P_{i_l} \cap Q_{j_l}) \subseteq X$ for all $l = 1, 2, \ldots, t$.

Let G be a graph and M a mesh in G. We first extend the definition of subwall to meshes in the natural way.

Definition Let the horizontal and vertical paths of M be $\mathcal{H}$ and $\mathcal{V}$, respectively. A mesh M' with horizontal and vertical paths $\mathcal{H}'$ and $\mathcal{V}'$ is a *submesh* of M if every element of $\mathcal{H}'$ is a subpath of a distinct element of $\mathcal{H}$ and similarly, every element of $\mathcal{V}'$ is a subpath of a distinct element of $\mathcal{V}$.

Definition Let G' be a minor of G and M' a mesh in G'. We say that M' is *compatible* with a mesh M in G if there exist a subset $Z \subseteq E(G)$ and a submesh $\bar{M}$ of M such that G' is obtained from a subgraph of G by contracting Z and M' is obtained from $\bar{M}$ by contracting $Z \cap E(\bar{M})$.

Lemma 3.1 *Let G be a graph, let M be a mesh in G, let G' be a minor of G, and let M' be a mesh in G' compatible with M. If for some integer $t \geq 0$ the graph M' grasps a K_t minor of G', then M grasps a K_t minor of G.*

The proof is clear and we omit it.

Let $r \geq 1$ be an integer, and let H_{2r} be the $2r \times 2r$-grid with vertex-set $[2r] \times [2r]$, as usual. The graph H^1_{2r} is defined as the graph obtained from H_{2r} by adding all edges with ends (i, r) and $(i + 1, r + 1)$, and all edges with ends $(i, r + 1)$ and $(i + 1, r)$ for all $i = 1, 2, \ldots, 2r - 1$. In other words, H^1_{2r} is constructed from the $2r \times 2r$-grid by adding a pair of crossing edges in each face of the middle row of faces. We will refer to the grid H_{2r} as the *underlying grid* of H^1_{2r}

Lemma 3.2 *Let $t \geq 2$ be an integer. The graph $H^1_{t(t-1)}$ has a K_t minor grasped by the underlying grid.*

Proof. The proof is by induction on t. Let the vertices of $H^1_{t(t-1)}$ be labeled as in the definition, and let L be the set of vertices of $H^1_{t(t-1)}$ with the second coordinate one. We actually prove a slightly stronger statement, to facilitate the induction. We show that $H^1_{t(t-1)}$ has a K_t minor grasped by the underlying grid such that every branch set contains a vertex in

L. The statement clearly holds for $t = 2$, and so we assume that $t > 2$ and that the statement holds for $t - 1$.

Let H' be the subgraph of $H^1_{t(t-1)}$ induced by vertices (x, y), where $1 \le x \le (t-1)(t-2)$ and $t \le y \le (t-1)^2$, and let L' be the set of vertices of H' with second coordinate $(t-1)^2$. Then $H^1_{(t-1)(t-2)}$ is isomorphic to H' by an isomorphism that maps the first row of $H^1_{(t-1)(t-2)}$ onto L'. By the induction hypothesis the graph H' has a K_{t-1} minor with branch sets $X'_1, X'_2, \dots, X'_{t-1}$ such that $X'_i \cap L' \neq \emptyset$ for all $i = 1, 2, \dots, t-1$. Let $i \in \{1, 2, \dots, t-1\}$. Let x_i be such that $(x_i, (t-1)^2) \in X'_i \cap L'$. We may assume that $x_1 > x_2 > \cdots > x_{t-1}$. We define X_i to consist of X'_i, the vertices $(x_i, (t-1)^2 + i)$, $((t-1)(t-2) + 2i - 1, (t-1)^2 + i)$, $((t-1)(t-2) + 2i - 1, t(t-1)/2 + 1)$, $((t-1)(t-2) + 2i, t(t-1)/2)$, $((t-1)(t-2) + 2i, 1)$, and the vertices of vertical and horizontal paths of the underlying grid connecting those vertices, making each X_i induce a connected subgraph of $H^1_{t(t-1)}$. Finally we define X_t as the set containing all the vertices $((t-1)(t-2)+2i-1, t(t-1)/2)$ and $((t-1)(t-2)+2i, t(t-1)/2+1)$ for all $i = 1, 2, \dots, t-1$, and the vertices of the vertical path connecting $((t-1)(t-2)+1, t(t-1)/2)$ to $((t-1)(t-2)+1, 1)$. This is illustrated in Figure 2. In order to satisfy the definition of grasping, we also add to X_t the vertices $((t-1)(t-2)+1-i, t-i)$ and $((t-1)(t-2)+2-i, t-i)$ for all $i = 1, 2, \dots, t-1$ and the path joining them. It follows that $X_1, X_2, \dots, X_t$ are the branch sets of a K_t minor, and each branch set intersects L. Hence each branch set X_i satisfies

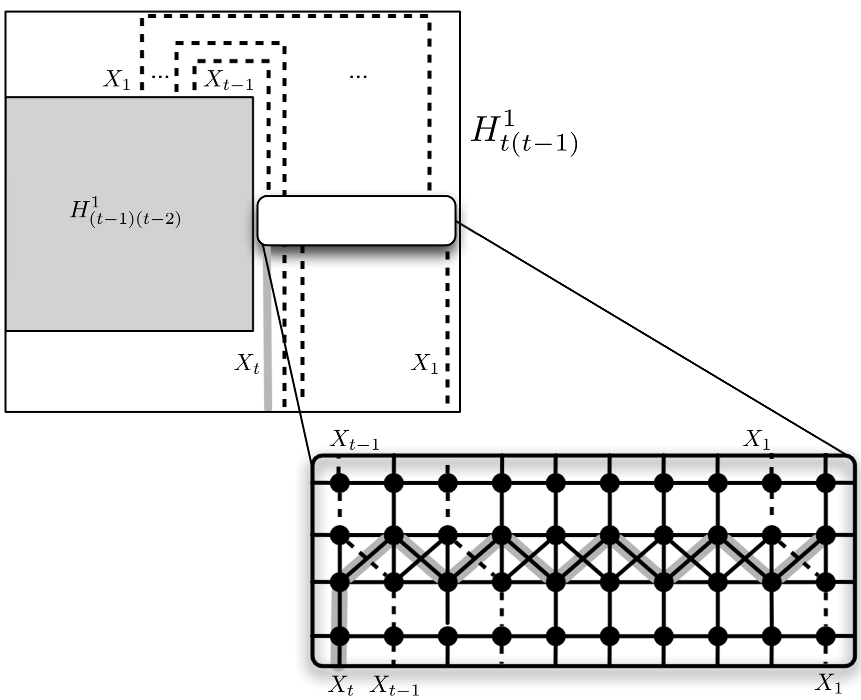

Figure 2: Finding a K_t minor in $H^1_{t(t-1)}$.

the definition of grasping. We deduce that the minor is grasped by the underlying grid, as required. □

4 Disjoint paths attaching to a mesh

The goal of this section is to show that given a mesh M in a graph G, either G has a K_t minor grasped by M, or there exist bounded number of vertices and bounded number of balls in M of bounded radius such that after deleting those vertices and balls, every M-path has its ends close to each other.

We will need two classic lemmas going forward.

Lemma 4.1 *Let $k, r, s \geq 1$ be integers, and let $\mathcal{I}$ be a set of k intervals on the real line. If $k \geq (r-1)(s-1)+1$, then either $\mathcal{I}$ has a subset of r pairwise disjoint intervals, or $\mathcal{I}$ has a subset of s intervals that have non-empty intersection.*

Lemma 4.2 (Erdős and Szekerés) *Let $r, s \geq 1$ be integers. Every sequence of $k \geq (r-1)(s-1)+1$ real numbers has either a non-decreasing subsequence of length r, or a non-increasing subsequence of length s.*

Lemma 4.3 *Let $t \geq 2$ be a positive integer, let $k = 32(t(t-1))^6$, let G be a graph, let M be a mesh in G with distance function d, let $X \subseteq V(M)$ with $|X| = 2k$ such that $d(x, y) \geq 2t(t-1)$ for all $x, y \in X$, and let $F \subseteq E(G) - E(M)$ be a matching of size k with vertex-set X. Then the graph G has a K_t minor grasped by M.*

Proof. The definition of distance function involves a grid minor of M. Let H be a grid minor of M that gives rise to the distance function d. Then H is obtained from M by contracting a set of edges. Let G' be the minor obtained from G by contracting the same set of edges. Then F gives rise to a matching F' in G' of size k. Given the way we defined the distance function on a mesh, the ends of the edges in F' are pairwise at distance at least $2t(t-1)$ with respect to the distance function on H. If G' has a K_t minor grasped by H, then G has a K_t minor grasped by M by Lemma 3.1. Thus it suffices to prove the lemma when M is grid.

We therefore assume for the rest of the proof that M is a grid. Let the vertices of M be labeled (x, y) for $1 \leq x \leq s$, $1 \leq y \leq r$. We number the edges in F as $e_1, e_2, \ldots$ and denote the ends of e_i by (x_i, y_i) and (u_i, v_i). There is at most one edge of F which has an end with distance at most $t(t-1)-1$ from a vertex of the outer cycle of M. We discard such an edge from F if it exists. The remaining edges e_i therefore satisfy

(1) if (x, y) is an end of e_i, then $t(t-1) < x < s+1-t(t-1)$ and $t(t-1) < y < r+1-t(t-1)$.

We may temporarily assume that for every i either $x_i < u_i$, or $x_i = u_i$ and $y_i > v_i$. By reducing F to no less than half its original size we may assume that either $y_i \leq v_i$ for all i, or $y_i > v_i$ for all i. In the former case it follows that $x_i < u_i$ for all i. In the latter case we reverse the second coordinate and then swap the coordinates (formally we map each vertex (x, y) to $(r+1-y, x)$) and conclude that we may assume that for at least half the indices i

$(*)$ $x_i < u_i$ and $y_i \leq v_i$.

By restricting ourselves to a subset of F of size $4(t(t-1))^3$ we may assume that either $x_i \neq x_j$ for all remaining pairs of distinct edges e_i, e_j, or that $x_i = x_j$ for all such pairs. In the latter case notice that $|y_i - y_j| \geq 2t(t-1) \geq 4$, because (x_i, y_i) and (x_j, y_j) are at distance at least $2t(t-1)$. In the latter case we swap the coordinates one more time to arrive at a set $\{e_1, e_2, \ldots, e_l\} \subseteq F$ such that for all distinct indices $i, j = 1, 2, \ldots, l$ condition (1) holds and

(2) either $x_i < u_i$, or $x_i = u_i$ and $|x_i - x_j| \geq 4$,

(3) $x_i \neq x_j$, and

(4) $l \geq 4(t(t-1))^3$.

We apply Lemma 4.1 to the set of intervals $\{[x_i, u_i] : 1 \leq i \leq l\}$. We conclude that either there exists a set $I \subseteq \{1, 2, \ldots, l\}$ of size at least $t(t-1)$ such that the intervals $\{[x_i, u_i] : i \in I\}$ are pairwise disjoint, or there exist a set $J \subseteq \{1, 2, \ldots, l\}$ of size at least $4(t(t-1))^2$ and an integer z such that $x_i \leq z \leq u_i$ for all $i \in J$.

Assume first that I exists. We claim that the graph obtained from M by adding the edges $\{e_i : i \in I\}$ has an $H^1_{t(t-1)}$ minor, where the underlying grid of $H^1_{t(t-1)}$ is compatible with M. To see this we use the first and last $t(t-1)$ vertical and horizontal paths of M (notice that by (1) for $i \in I$ no end of e_i belongs to any of those paths), and use the edges e_i to obtain the crossings in the middle row of faces. The i^{th} crossing will use vertices (x, y) with $t(t-1) \leq y \leq r + 1 - t(t-1)$ and $x_i \leq x \leq u_i$ if $x_i < u_i$ and $x_i - 1 \leq x \leq x_i + 1$ if $x_i = u_i$. Condition (2) guarantees that the crossings will be pairwise disjoint. By Lemma 3.2 the graph $H^1_{t(t-1)}$ has a K_t minor grasped by the underlying grid of $H^1_{t(t-1)}$. By Lemma 3.1 the graph G has a K_t minor grasped by M, as desired. This completes the case when I exists.

We may therefore assume that J and z exist. By renumbering the indices we may assume that $x_1 < x_2 < \cdots < x_{4(t(t-1))^2} < z$ and $u_i \geq z$ for all $1 \leq i \leq 4(t(t-1))^2$. Let M_1 be the subgraph of M induced by vertices (x, y) with $1 \leq x < z$ and $1 \leq y \leq r$, and let M_2 be the subgraph of M induced by vertices (x, y) with $z \leq x \leq s$ and $1 \leq y \leq r$. We see that $(u_i, v_i) \in V(M_2)$ for all $1 \leq i \leq 4(t(t-1))^2$. Let P be a path in M_2 covering the vertices of M_2. The edges e_i for $1 \leq i \leq 4(t(t-1))^2$ each have one end in P and one end in $V(M_1)$. By Lemma 4.2 there exists a sequence $1 \leq i_1 < i_2 < \cdots < i_{2t(t-1)}$ such that the ends of $e_{i_1}, e_{i_2}, \ldots, e_{i_{2t(t-1)}}$ occur on P in the order listed. For $j = 1, 2, \ldots, t(t-1)$ we make use of the edges $e_{i_{2j-1}}, e_{i_{2j}}$ and the subpath of P connecting the ends of $e_{i_{2j-1}}$ and $e_{i_{2j}}$ to construct an M_1-path with ends $x_{i_{2j-1}}$ and $x_{i_{2j}}$. The paths just constructed are pairwise vertex-disjoint, and, similarly as in the previous paragraph, can be used to deduce that G has an $H^1_{t(t-1)}$ minor, where the underlying grid is compatible with M_1, and hence with M. By Lemma 3.2 the graph $H^1_{t(t-1)}$ has a K_t minor grasped by the underlying grid of $H^1_{t(t-1)}$. By Lemma 3.1 the graph G has a K_t minor grasped by M, as desired. □

Lemma 4.4 *Let B be a connected graph of maximum degree at most four, and let $Y \subseteq V(B)$. Then there exist at least $(|Y| - 1)/4$ disjoint paths in B, each with at least one edge and with both ends in Y.*

Proof. By a *leaf* of B we mean a vertex of degree one. We may assume for a contradiction that the conclusion does not hold and, subject to that, $|E(B)|$ is minimum. Then B is a tree, every leaf belongs to Y, $|Y| \geq 6$ and (by contracting the incident edge we see that) the unique neighbor of every leaf belongs to Y. Let L be the set of leaves of B. Since $|Y| \geq 6$ the graph $B - L$ is a tree on at least two vertices, and therefore we may select a leaf t of $B - L$. Since t has degree at most four and $|Y| \geq 6$, the vertex t is adjacent to at most three leaves of B. Let B' be the graph obtained from B by deleting t and all leaves of B adjacent to it, and let $Y' := Y \cap V(B')$. By the minimality of B there exist at least $(|Y'| - 1)/4 \geq (|Y| - 1)/4 - 1$ disjoint paths in B', each with at least one edge and both ends in Y'. By adding the path with vertex-set $\{t, t'\}$, where t' is a leaf of B adjacent to t, we obtain a collection as required in the lemma, a contradiction. □

Before the next lemma, let us remark that $3 \times 2^{12} = 12288$.

Lemma 4.5 *Let $t \geq 1$ be an integer, let $k_0 := 12288(t(t-1))^{12}$, let G be a graph, let M be a mesh in G with distance function d, and assume that G has a set $\mathcal{P}$ of cardinality k_0 of pairwise disjoint M-paths with the property that the ends of every path $P \in \mathcal{P}$ can be denoted by $x(P)$ and $y(P)$ in such a way that $x(P)$ and $y(P)$ are at distance at least $10t(t-1)$ for every $P \in \mathcal{P}$, and $x(P)$ and $x(P')$ are at distance at least $10t(t-1)$ for every two distinct paths $P, P' \in \mathcal{P}$. Then G has a K_t minor grasped by M.*

Proof. Let us define a relation R on $V(M)$ by saying that $(x, y) \in R$ if $d(x, y) < 2t(t-1)$. By Lemma 2.2 applied to the relation R, graph M and integer $k = 32(t(t-1))^6$ we deduce that one of the two outcomes holds. If the first outcome holds, then G has K_t minor grasped by M by Lemma 4.3, and hence our lemma holds. Thus we may assume that the second outcome of Lemma 2.2 holds, and hence there exist sets $A \subseteq V(G)$ and $Z \subseteq V(M)$ with $|A| \leq k - 1$ and $|Z| \leq 3k - 3$ such that

(1) *for every M-path P in $G - A$ its ends can be denoted by x and y such that either $d(x, y) < 2t(t-1)$ or $d(x, z) < 2t(t-1)$ for some $z \in Z$.*

The set $\mathcal{P}$ has a subset of size at least $k_0 - k$ such that each member is disjoint from A. By (1) there exists $z \in Z$ and a subset of the latter set of paths of size at least $(k_0 - k)/(3k - 3)$ such that every member P of the latest set has the property that one of $x(P), y(P)$ is at distance at most $2t(t-1)$ from z. Let B denote the subgraph of M induced by vertices of M at distance at most $2t(t-1)$ from z. Since the vertices $x(P)$ are pairwise at distance at least $10t(t-1)$, we deduce that $x(P) \in V(B)$ for at most one of those paths P. By omitting that path we obtain a set $\mathcal{P}' \subseteq \mathcal{P}$ of disjoint M-paths in $G - A$ with $y(P) \in V(B)$ for every $P \in \mathcal{P}'$ and such that $\mathcal{P}'$ has cardinality at least $(k_0 - k)/(3k - 3) - 1 \geq 128(t(t-1))^6$.

Let $P_1, P_2, \ldots, P_r$ be the vertical paths of M, and let $Q_1, Q_2, \ldots, Q_s$ be the horizontal paths of M. Let H be a grid minor of M that gave rise to the distance function d on M, and let $f : V(M) \to V(H)$ be the corresponding surjection as in the definition of distance function. We define $\mathcal{Q}$ to be the set of vertical and horizontal paths P of M such that P is not a subgraph of B and there is no vertex x of P such that $f(x)$ and $f(z)$ are connected by a curve that intersects H at most $2t(t-1)$ times and does not use the outer face of H. (If z is at distance at least $2t(t-1)$ from $P_1 \cup P_r \cup Q_1 \cup Q_s$, then this is equivalent to saying that $\mathcal{Q}$ is the set of vertical and horizontal paths of M that are disjoint from B; otherwise we need this more complicated definition.) We define a submesh M' consisting of subpaths of members of $\mathcal{Q}$ as follows. Let I, J be such that $\mathcal{Q}$ consists of P_i and Q_j for all $i \in I$ and all $j \in J$. Let $i_0 := \min I$, $i_1 := \max I$, $j_0 := \min J$, and $j_1 := \max J$. For $i \in I$ let P'_i be the shortest subpath of P_i from Q_{j_0} to Q_{j_1}, and for $j \in J$ let Q'_j be the shortest subpath of Q_j from P'_{i_0} to P'_{i_1}. Let M' be the union of P'_i and Q'_j for all $i \in I$ and $j \in J$. It is not hard to see that M' is a mesh. We now select a distance function on M' as follows. Starting with M' we first contract all edges that were contracted during the production of H from M, and then contract edges arbitrarily until we arrive at a grid H'. We use H' in order to define a distance function d' on M'. It follows that

(2) $\quad d'(x,y) \geq d(x,y) - 8t(t-1)$ *for all* $x, y \in V(M')$.

Let $P \in \mathcal{P}'$, and let $x = x(P)$. We wish to define a path $\phi(P)$ with one end x. If $x \in V(M')$, then $\phi(P)$ is defined to be the path with vertex-set $\{x\}$; otherwise we proceed as follows. By symmetry between the paths P_i and Q_j we may assume that $x \in V(P_i)$. We claim that $P_i \notin \mathcal{Q}$. To prove this claim suppose to the contrary that $P_i \in \mathcal{Q}$. Since $x \notin V(M')$ it follows that when traversing P_i starting from Q_0 we either encounter x strictly before Q_{i_0}, or we encounter x strictly after Q_{j_0}. In either case it follows that $x \in V(B)$, a contradiction. This proves our claim that $P_i \notin \mathcal{Q}$. Let j be such that either $x \in V(Q_j)$, or when traversing P_i as above we encounter Q_j, then x, and then Q_{j+1}. Then at least one of Q_j, Q_{j+1} belongs to $\mathcal{Q}$, for otherwise $x \in V(B)$, a contradiction (if $x \in V(Q_j)$, then $Q_j \in \mathcal{Q}$). If $Q_j \in \mathcal{Q}$, then let $\phi(P)$ be the shortest subpath of P_i from x to $x' \in V(Q_j)$; otherwise let $\phi(P)$ be the shortest subpath of P_i from x to $x' \in V(Q_{j+1})$. The argument used above to show that $P_i \notin \mathcal{Q}$ now implies that $x' \in V(M')$.

Let Y be the set of all vertices $y(P)$ over all paths $P \in \mathcal{P}'$. Since the graph B is connected, by Lemma 4.4 there exists a set $\mathcal{R}$ of at least $\lceil (|\mathcal{P}'|-1)/4 \rceil \geq 32(t(t-1)^6$ disjoint subpaths of B, each with distinct ends in Y. For each $R \in \mathcal{R}$ with ends y_1 and y_2 we define an M'-path by taking the union $R \cup P_1 \cup \phi(P_1) \cup P_2 \cup \phi(P_2)$, where $P_i \in \mathcal{P}'$ satisfies $y(P_i) = y_i$. These paths are pairwise vertex-disjoint. Since for distinct paths $P, P' \in \mathcal{P}'$ the vertices $x(P), x(P')$ are at distance at least $10t(t-1)$ in M, they are at distance at least $2t(t-1)$ in M' by (2). By Lemma 4.3 the graph G has a K_t minor grasped by M', and hence it has a K_t minor grasped by M by Lemma 3.1, as desired. $\square$

Lemma 4.6 *Let $t \geq 1$ be an integer, let $k := 12288(t(t-1))^{12}$, let G be a graph, and let M be a mesh in G with distance function d. Then either G has a K_t minor grasped by M, or there exist sets $A \subseteq V(G)$ and $Z \subseteq V(M)$ such that $|A| \leq k-1$, $|Z| \leq 3k-3$, and if x, y are the ends of an M-path in $G - A$, then either $d(x,y) < 10t(t-1)$, or each of x, y lies at distance at most $10t(t-1)-1$ from some vertex of Z.*

Proof. We define a relation R on $V(M)$ by saying that $(x,y) \in R$ if $d(x,y) < 10t(t-1)$ and apply Lemma 2.1 to the relation R, mesh M and integer k. If the second outcome holds, then the second outcome of the current lemma holds, and so we may assume that the first outcome of Lemma 2.1 holds. Thus there exists a set $\mathcal{P}$ of k pairwise disjoint M-paths that are R-semi-dispersed. The set $\mathcal{P}$ satisfies the hypothesis of Lemma 4.5, and hence that lemma implies that the graph G has a K_t minor grasped by M, as desired. □

5 Proof of the Flat Wall Theorem

We need a somewhat technical lemma before we can begin the proof of Theorem 1.5. In the proof of Theorem 1.5, we will find a large flat wall W' in a subgraph, say G', of the original graph G. To find a flat wall in G itself, we then consider a smaller subwall W'' of W'. It is intuitive that W'' should be flat as well; the near-planarity of W' should ensure this. However, to rigorously consider the sequence of reductions certifying that a subwall of W' is flat requires some care. The next lemma allows us to do so. It is the main lemma we need for the proof of the Flat Wall Theorem.

Lemma 5.1 *Let G be a graph and let C be a cycle in G such that some C-reduction of G can be drawn in the plane with C bounding a face. Let W be a subgraph of G and let D be a cycle in W such that $W - V(D)$ is connected and there exist four internally disjoint paths from $V(W) \setminus V(D)$ to $V(C)$ with distinct ends in $V(C)$ such that each intersects D in a (non-null) path. Then there exists a separation (A, B) in G such that*

(1) *$A \cap B \subseteq V(D)$,*

(2) *$V(W) \subseteq B$,*

(3) *$V(C) \subseteq A$, and*

(4) *some $A \cap B$-reduction of $G[B]$ can be drawn in a disk with $A \cap B$ drawn on the boundary of the disk in the order determined by D.*

Proof. By Theorem 1.4 there exists a C-rendition (Γ, σ, π) of G, as defined prior to Theorem 1.4. In particular, Γ is a painting in the unit disk Δ. We now define a set $X \subseteq \Delta$ homeomorphic to the unit circle. The existence of the four internally disjoint paths from $V(W) \setminus V(D)$ to $V(C)$ implies that D is not a subgraph of $\sigma(c)$, for all $c \in C(\Gamma)$. Therefore D can be written as $P_1 \cup P_2 \cup \cdots \cup P_n$, where $n \geq 2$ and each P_i is a path with both ends and no internal vertex in $\pi(N(\Gamma))$. For each $i = 1, 2, \ldots, n$ the path P_i is a subgraph of $\sigma(c_i)$ for a unique $c_i \in C(\Gamma)$. Let $\pi(x)$ and $\pi(y)$ be the ends of P_i. Let X_i be the closure of a component of $\mathrm{bd}(c_i) \setminus \{x, y\}$ that is disjoint from $N(\Gamma)$. Thus if $|\widetilde{c_i}| = 3$, then this component is unique, whereas if $|\widetilde{c_i}| = 2$, then there are two such components. If $|\widetilde{c_i}| = 3$, then let $\mathrm{bd}(c_i) \setminus \{x, y\} = \{z_i\}$; otherwise z_i is undefined. Finally let $X = X_1 \cup X_2 \cup \cdots \cup X_n$. We will refer to X as the *track* of D.

Let Δ' be the closed disk bounded by X, let A be the union of $V(\sigma(c))$ over all $c \in C(\Gamma)$ such that $c \not\subseteq \Delta'$, and let B' be the union of $V(\sigma(c))$ over all $c \in C(\Gamma)$ such that $c \subseteq \Delta'$. Then

(A, B') is a separation of G that satisfies (1) and (3). Let B be the union of B' and $V(P_i)$ for all $i = 1, 2, \ldots, n$ such that $\sigma(c_i) \not\subseteq \Delta'$. Then (A, B) is also a separation of G, and it also satisfies (1) and (3).

We claim that (A, B) satisfies the conclusion of the theorem. To prove that we must show that (A, B) satisfies (2) and (4), and we begin with (4). For $i = 1, 2, \ldots, n$ such that $\sigma(c_i) \not\subseteq \Delta'$ let d_i be a closed disk with $d_i = \overline{c_i}$ if $|\widetilde{c_i}| = 2$ and $c_i \cap X \subseteq d_i \subseteq \overline{c_i} \setminus \{z_i\}$ otherwise. Let Δ'' be the union of Δ' and all the disks d_i. Then Δ'' is a closed disk. Let Γ' be the painting defined by $N(\Gamma') = N(\Gamma) \cap \Delta'$ and $U(\Gamma') = U(\Gamma) \cap \Delta''$. Thus every cell $c \in C(\Gamma')$ is either a subset of Δ', in which case $c \in C(\Gamma)$, or there exists $i = 1, 2, \ldots, n$ such that $\sigma(c_i) \not\subseteq \Delta'$ and $c \subseteq d_i$. In the former case we define $\sigma'(c) = \sigma(c)$, and in the latter case we define $\sigma'(c) = P_i$. We define π' to be the restriction of π to $N(\Gamma')$. Then (Γ', σ', π') is an Ω-rendition of $G[B]$, where Ω is a cyclic ordering of $A \cap B$ and the cyclic order is determined by the order on D. It follows from Theorem 1.4 that (A, B) satisfies (4).

To prove that (A, B) satisfies (2) we first note that $V(D) \subseteq B$, and so it remains to show that $V(W) \setminus V(D) \subseteq B$. To that end suppose for a contradiction that $V(W) \setminus V(D) \not\subseteq B$. Since $W - V(D)$ is connected, it follows that $W - V(D)$ is a subgraph of

$$\bigcup(\sigma(c)\ :\ c \in C(\Gamma) \text{ and } c \not\subseteq \Delta').$$

Let P_1, P_2, P_3 be three of the four paths guaranteed by the hypothesis of the lemma. We may assume that they have a common end in $W - V(D)$. By considering the "tracks" of P_1, P_2, P_3 (defined similarly as above), we obtain a planar drawing of the graph $H' := C \cup D \cup P_1 \cup P_2 \cup P_3$ in which both C and D bound faces. Let H be obtained from H' by adding a new vertex in the face bounded by D and joining it by an edge to every vertex of D, and adding a new vertex in the face bounded by C and joining it by an edge to every vertex of C. Then H is planar, and yet it has a $K_{3,3}$ subdivision (because each P_i intersects D), a contradiction. This proves that (A, B) satisfies (2), and hence it satisfies the conclusion of the lemma. □

Let H be a subgraph of a graph G. An H-*bridge* in G is a connected subgraph B of G such that $E(B) \cap E(H) = \emptyset$ and either $E(B)$ consists of a unique edge with both ends in H, or for some component C of $G \backslash V(H)$ the set $E(B)$ consists of all edges of G with at least one end in $V(C)$. The vertices in $V(B) \cap V(H)$ are called the *attachments* of B.

We are now ready to prove the Flat Wall Theorem, which we restate.

Theorem 5.2 *Let $r, t \geq 1$ be integers, let r be even, let $R = 49152t^{24}(40t^2 + r)$, let G be a graph, and let W be an R-wall in G. Then either G has a model of a K_t minor grasped by W, or there exist a set $A \subseteq V(G)$ of size at most $12288t^{24}$ and an r-subwall W' of W such that $V(W') \cap A = \emptyset$ and W' is a flat wall in $G - A$.*

Proof. Let $t, r \geq 1$, and W be given, where W is an R-wall in G, and $R \geq 4 \cdot 12288t^{24}(40t(t-1)) + r)$. Let d be a distance function on W. By Lemma 4.6 applied to the mesh W and distance function d we may assume that there exist sets $A \subseteq V(G)$ and $Z \subseteq V(M)$ such that

(1) $|A| \leq 12288(t(t-1))^{12}$, $|Z| \leq 3 \cdot 12288(t(t-1))^{12}$, *and if x, y are the ends of a W-path in $G - A$, then either $d(x, y) < 10t(t-1)$, or each of x, y lies at distance at most*

$10t(t-1)-1$ *from some vertex of* Z.

Let the horizontal paths of W be $P_0,\dots,P_R$ and the vertical paths $Q_0,\dots,Q_R$. A *strip* of W is a subgraph of W consisting of $40t(t-1)+r$ consecutive horizontal paths of W, say $P_{i+1},\dots,P_{i+40t(t-1)+r}$, along with every subpath Q of a vertical path of W such that Q has both ends in $V(P_{i+1})\cup\dots\cup V(P_{i+40t(t-1)+r})$. By our choice of R, there exists a strip S consisting of paths numbered as above such that S contains no vertex of $Z\cup A$. We conclude that there exist subwalls $W_1,\dots,W_{t(t-1)}$ contained in S satisfying the following for all distinct integers $i,j=1,2,\dots,t(t-1)$:

(2) W_i is a $20t(t-1)+r$-wall such that the horizontal paths of W_i are subpaths of the horizontal paths of the strip S and the vertical paths of W_i are subpaths of the vertical paths of S,

(3) if $x\in V(W_i)$ and $y\in V(W_j)$, then $d(x,y)\geq 10t(t-1)$, and

(4) W_i is disjoint from the first and last $10t(t-1)$ horizontal paths of S.

See Figure 3.

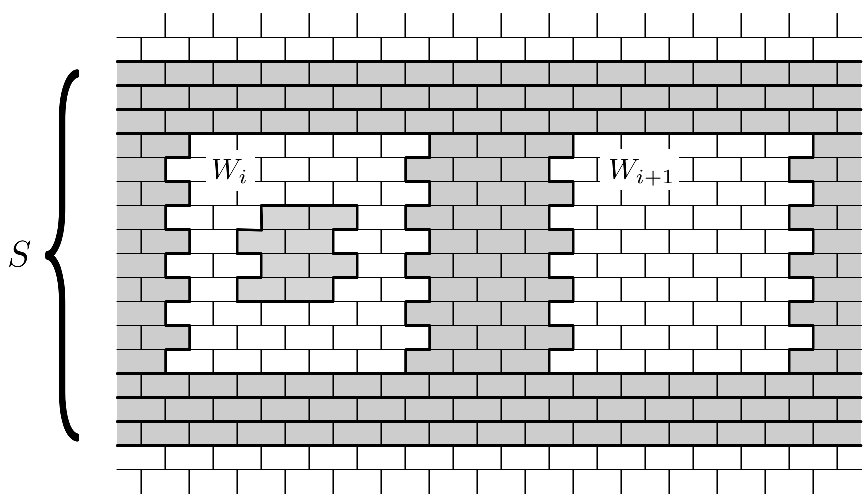

Figure 3: Subwalls of a strip.

For $i=1,2,\dots,t(t-1)$ we define a graph H_i. Let us recall that the corners of a wall were defined at the end of the first paragraph of Subsection 1.2. Let C_i be a cycle with vertex-set the four corners of the wall W_i in the order of their appearance on the outer cycle of W_i. In other words, the cycle C_i may be obtained from the outer cycle of W_i by suppressing all vertices, except the four corners of W_i. Let B be a $(W-A)$-bridge in the graph $G-A$ with at least one attachment in $V(W_i)$, and let B' be obtained from B by deleting all its attachments that do not belong to $V(W_i)$. The graph H_i is defined as the union of the wall W_i, the cycle C_i and all graphs B' as above. We claim that the subgraphs H_i are pairwise disjoint. To see this, if there exist indices i and j with $i \neq j$ such that H_i and H_j share a vertex, then there exists a $(W - A)$-bridge with an attachment $x \in V(W_i)$ and $y \in V(W_j)$. However then there exists a W-path in $G - A$ with ends x and y, contrary to (1) and (3), because by (4) both x and y are at distance at least $10t(t - 1)$ from every vertex of Z.

If for all $i = 1, 2, \ldots, t(t - 1)$ the graph H_i has a C_i-cross, then the graph G has a $H^1_{t(t-1)}$ minor such that the underlying grid is compatible with the original wall W. By Lemma 3.2 the graph $H^1_{t(t-1)}$ has a K_t minor grasped by the underlying grid of $H^1_{t(t-1)}$, and hence G has a K_t minor grasped by W by Lemma 3.1, as desired.

We conclude that we may assume that there exists an index i such that the graph H_i does not have a C_i-cross. By Theorem 1.3 some C_i-reduction of H_i can be drawn in the plane with C_i bounding a face. Let W' be the r-wall obtained from W_i by deleting the first and final $10t(t-1)$ of both the horizontal and vertical paths of W_i, and let D be the outer cycle of W'. By Lemma 5.1 applied to the graph H_i, wall W', cycle D in W' and the cycle C_i there exists a separation (X', Y) of H_i satisfying (1)–(4) of Lemma 5.1. Let $X := X' \cup (V(G) \setminus A \setminus V(H_i))$. Then $X \cap Y = X' \cap Y \subseteq V(D)$, $V(W') \subseteq V(Y)$, $V(C_i) \subseteq X$, and some $X \cap Y$-reduction of $G[Y]$ can be drawn in a disk with $X \cap Y$ drawn on the boundary of the disk.

We claim that (X, Y) is a separation of $G - A$. To prove this claim suppose for a contradiction that $x \in X \setminus Y$ is adjacent to $y \in Y \setminus X$. Then $y \in V(H_i)$ and $x \notin V(H_i)$, because (X', Y) is a separation of H_i. We have $y \notin V(W_i) \setminus V(W')$, for otherwise $W_i - V(W')$ includes a path from y to $V(C_i)$ disjoint from $V(D)$, contrary to the facts that $V(C_i) \subseteq X'$, $y \in Y$ and $X' \cap Y \subseteq V(D)$. It follows that the edge joining x and y belongs to a $(W - A)$-bridge of $G - A$, and hence x is an attachment of that $(W - A)$-bridge outside W_i. It follows that this $(W - A)$-bridge includes a W-path with one end x and the other end say $x' \in V(W')$. It follows that x' is at distance at least $10t(t - 1)$ from x and every vertex in Z, contrary to (1). This proves that (X, Y) is a separation of G.

We may choose the pegs of W' in such a way that for every peg x of W' there exists a path P in W with one end in C_i, the other end x, and otherwise disjoint from W'. It follows that $V(P) \setminus \{x\} \subseteq X \setminus Y$, and hence $x \in X$, as desired.

Thus the separation (X, Y) is a witness that W' is a flat wall in $G - A$. We have $V(W') \cap A = \emptyset$, because W' is a subgraph of the strip S, and S was chosen disjoint from A. □

6 A flat wall theorem with few apex vertices

In this section we prove Theorems 1.7 and 1.8. The first gives an improved bound on the size of the subset A of vertices. It also ensures that the subset A of vertices is highly connected to the resulting wall W', which is useful in applications. First we need a lemma and a definition.

Lemma 6.1 *Let G be a graph, let W be a flat wall in G, and let W' be a subwall of W disjoint from the outer cycle of W. Then W' is a flat wall in G.*

Proof. Let (A, B) be a separation witnessing that W is a flat wall in G. Let C be the cycle with vertex-set $A \cap B$ such that the cyclic order of its vertex-set is the one inherited from the cyclic order of the outer cycle of W. Thus some C-reduction of $G[B] \cup C$ can be drawn in the plane with C bounding a face. By Lemma 5.1 applied to the graph $G[B] \cup C$, wall W', the outer cycle of W' and the cycle C there exists a separation (X, Y) satisfying (1)–(4) of Lemma 5.1. We may select the pegs of W' in such a way that for every peg x of W' there exists a path with one end x and the other end in C that is disjoint from $W' - x$. Given this choice it follows that every peg of W' belongs to X, and hence the separation (X, Y) shows that the wall W' is flat in G. □

The next definition makes explicit what we mean by the set A being highly connected to the wall.

Definition Let W be an r-wall in a graph G for some positive integer $r \geq 2$. A *brick* of W is a cycle C which forms the boundary of a finite face (that is, a face other than the outer face) in the natural embedding of W in the plane. Let $A \subseteq V(G)$ and assume $V(W) \cap A = \emptyset$. A subset $A' \subseteq A$ is *apex-universal for the pair* (W, A) if for all $a \in A'$ and for all bricks C of W, there exists a path with one end in $V(C)$, one end equal to a which is internally disjoint from $V(W) \cup A$. If A is apex-universal for (W, A), then we just say that A is *apex-universal* for W.

We now give the strengthening of Theorem 1.5.

Theorem 6.2 *Let $t \geq 5$ and $r \geq 3\lceil\sqrt{t}\rceil$ be integers. Let $n = 12288t^{24}$, $R = r^{2^n}$ and $R_0 = 49152t^{25}(40t + R)$. Let G be a graph, and let W_0 be an R_0-wall in G. Then either G has a model of a K_t minor grasped by W_0, or there exist a set $A \subseteq V(G)$ of size at most $t-5$ and an r-subwall W of W_0 such that $V(W) \cap A = \emptyset$, W is a flat wall in $G - A$ and A is apex-universal for W.*

Proof. By Theorem 1.5 we may assume that there exists a set $A_0 \subseteq V(G)$ of size at most $n = 12288t^{24}$ and an Rt-subwall W_1 of W_0 such that $V(W_1) \cap A_0 = \emptyset$ and W_1 is a flat wall in $G - A_0$. Let W be a subwall of W_1 obtained by selecting every t^{th} horizontal and every t^{th} vertical path of W_1.

We fix a subwall W' of W and subsets $A' \subseteq \bar{A} \subseteq A_0$ such that

(1) W' is a $r^{2^{|\bar{A}|-|A'|}}$-subwall of W,

(2) W' is flat in $G - \bar{A}$, and

(3) the subset A' is apex-universal for $(W', \bar{A})$.

Moreover, we pick W', A', and $\bar{A}$ satisfying (1)-(3) to minimize $|\bar{A}| - |A'|$. Note that such a choice exists by setting $W' = W$, $A' = \emptyset$, and $\bar{A} = A$.

We claim that $\bar{A} = A'$. To prove that assume for a contradiction that $\bar{A} \neq A'$. We define a subwall W^* of W' as follows. Let $k = r^{2^{|\bar{A}|-|A|-1}}$; thus W' is a k^2-wall. Let the vertical and horizontal paths of W' be $V_1, \dots, V_{k^2}$ and $H_1, \dots, H_{k^2}$, respectively. Let W^* be the k-subwall of W' whose horizontal and vertical paths are subpaths of $\{H_{2+i(k-1)} : 1 \le i \le k\}$ and $\{V_{2+i(k-1)} : 1 \le i \le k\}$. Note that W^* does not intersect the outer cycle of W, which will allow us to apply Lemma 6.1 later. Exactly one component of $W' - V(W^*)$ contains the outer cycle of W', and every brick of W^* is the outer cycle of a k-subwall of W'. Let $W_1, \dots, W_{(k-1)^2}$ be these k-subwalls of W'.

Fix a vertex $a \in \bar{A} \setminus A'$. Assume, as a case, that for all $i \in \{1, 2, \dots, (k-1)^2\}$, there exists a path P_i with one end equal to a, one end in $V(W_i)$ and internally disjoint from $V(W') \cup \bar{A}$. Then we claim that $A' \cup \{a\}$ is apex-universal for $(W^*, \bar{A})$. Fix a brick C of W^*; let $i \in \{1, 2, \dots, (k-1)^2\}$ be such that C is the outer cycle of W_i. Thus, by extending P_i through W_i, we can find a path from $V(C)$ to a with no internal vertex in $V(W^*) \cup A$. Similarly, if $a' \in A'$, then there exists a path P' from a' to some (in fact, every) brick of W_i with no internal vertex in $V(W) \cup A$. Thus, again we can extend P' through W_i to find a path from $V(C)$ to a' which has no internal vertex in $V(W^*) \cup A$. We conclude that $A' \cup \{a\}$ is apex-universal for $(W^*, \bar{A})$. It follows now by Lemma 6.1 that W^*, $A' \cup \{a\}$, and $\bar{A}$ satisfy (1)–(3), contrary to our choice to minimize $|\bar{A}| - |A'|$.

Thus there exists an index $i \in \{1, 2, \dots, (k-1)^2\}$ such that there does not exist a path with one end equal to a and one end in $V(W_i)$ which is internally disjoint from $V(W') \cup \bar{A}$. As every brick of W_i is a brick of W', we see that A' is apex-universal for $(W_i, \bar{A} \setminus \{a\})$. We claim as well that W_i is flat in $G - (\bar{A} \setminus \{a\})$. By Lemma 6.1, W_i is flat in $G - \bar{A}$. Let (X, Y) be a separation of $G - \bar{A}$ as in the definition of flat wall, chosen with $|Y|$ minimum. The minimality of Y implies that for every $y \in Y \setminus X$ there exists a path in $G[Y] - X$ with one end y and the other end in $V(W_i)$. Note that $W' - V(W_i)$ is connected; it follows that $V(W') \setminus V(W_i)$ is contained in X. We conclude that a has no neighbor in $Y \setminus X$, lest there exist a path from a to W_i avoiding the vertices of $V(W') \setminus V(W_i)$. Consequently, $(X \cup \{a\}, Y)$ is a separation of $G - (\bar{A} \setminus \{a\})$ that proves that the wall W_i is flat in $G - (\bar{A} \setminus \{a\})$. It follows that W_i, A', and $\bar{A} \setminus \{a\}$ satisfy (1)–(3), again contrary to our choice. This proves our claim that $\bar{A} = A'$.

We conclude that W' is an r-subwall of W which is flat in $G - \bar{A}$. To complete the proof, it suffices to show that $|\bar{A}| \le t - 5$. Assume not, and that $|\bar{A}| \ge t - 4$; let $a_1, \dots, a_{t-4}$ be $t - 4$ distinct vertices in $\bar{A}$. By the assumption that $r \ge 3\lceil\sqrt{t}\rceil$ we can choose bricks $C_1, C_2, \dots, C_t$ in W' such that each of them is disjoint from the outer cycle of W' and every two distinct bricks in the family are separated by a vertical or horizontal path of W'. For $i \in \{1, 2, \dots, t\}$ and $x \in \bar{A}$ there exists a path P^i_x from x to $V(C_i)$, internally disjoint from $V(W') \cup \bar{A}$. For $i \in \{1, 2, \dots, t\}$ let X'_i be the union of $V(C_i)$ and all the sets $V(P^i_x) \setminus \bar{A}$ for $x \in \bar{A}$. For $i \in \{1, 2, \dots, t-4\}$ let $X_i = X'_i \cup \{a_i\}$, and for $i \in \{t-3, t-2, t-1, t\}$ let $X_i = X'_i$. The sets X_i induce connected graphs, and we claim that they are pairwise disjoint. Indeed, to see that it suffices to argue that for distinct $i, j \in \{1, 2, \dots, t\}$ and not necessarily distinct $x, y \in \bar{A}$ the paths $P^i_x - x$ and $P^j_y - y$ are disjoint. But if those two paths intersect, then there exists a path P in $G - \bar{A}$ from C_i to C_j that is internally disjoint from W'. However, the existence of P contradicts the flatness of W'. To see this, let (X, Y) be a separation of $G - \bar{A}$ as in the definition of flat wall, and let $s_1, s_2, t_1, t_2 \in X \cap Y$ be distinct vertices appearing on the outer cycle of W' in the order listed. It follows that $W' \cup P$ has two disjoint paths, one with ends s_1 and t_1, and the other with ends s_2 and t_2. However, that contradicts the fact that some $X \cap Y$-reduction of G[Y] can be drawn in a disk with the vertices s_1, s_2, t_1, t_2 drawn on the boundary of the disk in order. This proves that the sets X_i are pairwise disjoint.

The sets X_i can be modified, using the vertical and horizontal paths of W_1 that are not part of W, to give model of a K_t minor grasped by W', and hence grasped by W_0. The only thing that is missing are edges between the sets $X_{t-3}, X_{t-2}, X_{t-1}, X_t$, and those can be supplied by enlarging these sets using horizontal and vertical paths of W' that are disjoint from all the cycles C_i. We omit the details, which are easy. □

Let G be a graph, let C be a cycle in G, and let J be a C-reduction of G obtained by successively performing elementary C-reductions determined by separations $(A_1, B_1), (A_2, B_2), \ldots, (A_k, B_k)$ in the order listed. More precisely, let $G_0 := G$, for $i = 1, 2, \ldots, k$ let G_i be obtained from G_{i-1} by the elementary C-reduction determined by (A_i, B_i), and let $J = G_k$. If J can be drawn in the plane with C bounding a face, then we say that $(B_1, B_2, \ldots, B_k)$ is a *C-reduction sequence* for G. Given a C-reduction sequence $(B_1, B_2, \ldots, B_k)$, the original separations may be recovered by letting A_i be the set of all vertices $v \in V(G_{i-1})$ such that either $v \notin B_i$, or $v \in B_i$ and v has a neighbor in $V(G_{i-1}) \setminus B_i$.

We now prove Theorem 1.8, which we restate.

Theorem 6.3 *Let $r \geq 2$ and $t \geq 5$ and be integers, let $n = 12288t^{24}$ and $R_0 = 49152t^{24}(40t^2 + (rt)^{2^n})$ and let G be a graph with no K_t minor. If G has an R_0-wall, then there exist a set $A \subseteq V(G)$ of size at most $t-5$ and an r-wall W in G such that $V(W) \cap A = \emptyset$ and W is a flat wall in $G - A$. Furthermore, if (X, Y) is a separation as in the definition of flat wall, then the graph $G[Y]$ has no (R_0+1)-wall.*

Proof. There exists a separation (X_0, Y_0) of G of order at most $t-2$ such that the graph $G[Y_0]$ has an R_0-wall, because the separation $(\emptyset, V(G))$ has said property. We may choose such a separation such that Y_0 is minimal with respect to inclusion. Let G_0 denote the graph $G[Y_0]$. By Theorem 6.2 applied to the graph G_0, an R_0-wall in G_0 and the integer rt in place of r we may assume that there exist a set $A \subseteq V(G_0)$ of size at most $t-5$ and an rt-wall W in G_0 such that $V(W) \cap A = \emptyset$ and W is a flat wall in $G_0 - A$. Let (X_0', Y_0') be a separation as in the definition of flat wall.

Let us select W, X_0', Y_0' as stated in the previous paragraph, and subject to that in such a way that Y_0' is minimal with respect to inclusion. Let $W_1, W_2, \ldots, W_t$ be disjoint r-subwalls of W such that each is disjoint from the outer cycle of W and every two of them are separated by a vertical or horizontal path of W. Let $i = 1, 2, \ldots, t$. By Lemma 6.1 the wall W_i is flat in $G_0 - A$; let (A_i, B_i) be the corresponding separation. We claim that the sets $B_1, B_2, \ldots, B_t$ are pairwise disjoint. Indeed, otherwise there exists a W-path in $G_0 - A$ with ends in different subwalls W_i, a contradiction similarly as in the proof of Theorem 5.2 or Theorem 6.2. Since $|X_0 \cap Y_0| \leq t-2$ we may assume that B_1 is disjoint from $X_0 \cap Y_0$. It follows that $(A_1 \cup X_0, B_1)$ is a separation of $G - A$, and hence the wall W_1 is flat in $G - A$.

It remains to show that $G[B_1]$ has no (R_0+1)-wall. To that end suppose for a contradiction that W_0 is an (R_0+1)-wall in $G[B_1]$, let $Y_1, Y_2, \ldots, Y_k$ be an $A_1 \cap B_1$-reduction sequence for $G[B_1]$, and let (X_i, Y_i) be the corresponding separations. Thus $Y_i \subseteq B_1$ for every $i = 1, 2, \ldots, k$. We may assume that the $A_1 \cap B_1$-reduction sequence is chosen with k maximum.

For each $i = 1, 2, \ldots, k$ either every vertex of W_0 of degree three except possibly one belongs to X_i, or every vertex of W_0 of degree three except possibly one belongs to Y_i. Let us assume first that the latter holds for some index $i \in \{1, 2, \ldots, k\}$. Then $G[Y_i]$ has an R_0-wall (a subwall of W_0) and $(V(G) \setminus (Y_i \setminus X_i), Y_i \cup A)$ is a separation of G of order at most $t - 2$ that contradicts the choice of (X_0, Y_0), because $A \subseteq Y_0$, $Y_i \subseteq B_1 \subseteq Y_0$ and at least one corner of W_1 belongs to $Y_0 \setminus Y_i$. It follows that for each $i = 1, 2, \ldots, k$ every vertex of W_0 of degree three except possibly one belongs to X_i.

Let J be the $A_1 \cap B_1$-reduction of $G[B_1]$ that arises by applying the $A_1 \cap B_1$-reduction sequence $Y_1, Y_2, \ldots, Y_k$. We may assume that J is drawn in a disk Δ in such a way that the vertices of $A_1 \cap B_1$ are drawn on the boundary of Δ in the order determined by the outer cycle of W_1. Since the graph $G[B_1]$ has the R_0-wall W_0, it has an $R_0 \times R_0$-grid minor such that no branch set of the minor that corresponds to a vertex of degree four of the grid minor is a subset of $Y_i \setminus X_i$. With the possible exception of vertices of the outer cycle such a grid minor is preserved under the $A_1 \cap B_1$-reductions, which implies that J has an $(R_0 - 2) \times (R_0 - 2)$-grid minor, and hence an $(R_0/2 - 1)$-wall. Thus J has an rt-wall W' such that the face bounded by the outer cycle of W' includes the boundary of Δ. Let D' be the outer cycle of W'. It follows from the maximality of k that there exist four internally disjoint paths in J from $W' - V(D')$ to $A_1 \cap B_1$ with distinct ends in $A_1 \cap B_1$. By changing the paths if necessary we may assume that each of these four paths intersects D' in a path. Let W'' be an rt-wall in $G[B_1]$ obtained by converting the wall W' into one in $G[B_1]$. This is done mostly by replacing edges of W' that do not belong to G by corresponding paths in $G[Y_i]$ for some $i \in \{1, 2, \ldots, k\}$. Likewise, the four internally disjoint paths in J can be converted to paths in $G[B_1]$. By Lemma 5.1 the wall W'' is flat in $G[B_1]$, and hence in $G_0 - A$; thus the corresponding separation contradicts the choice of (X_0', Y_0'). Thus $G[B_1]$ has no $(R_0 + 1)$-wall, as desired. □

7 An Algorithm

We need algorithmic versions of Lemmas 2.1 and 2.2. In order for those algorithms to run efficiently we need to make some assumptions about the computability of the relation R. It seems best to do so in the context of our application, namely when M is a mesh in the graph G and $(x, y) \in R$ if and only if $d(x, y) < l$ for some integer l, where d is a distance function on M. Let us recall that the notion of a distance function was defined at the beginning of Section 3 by saying that $d(x, y)$ is the distance of $f(x)$ and $f(y)$ in H, where H is a grid minor of M and $f : V(M) \to V(H)$ describes the contraction. We will refer to $f : V(M) \to V(H)$ as a *grid contraction function*. It is clear that given a grid contraction function f, the value $d(x, y)$ can be computed in constant time for any $x, y \in V(M)$. Thus we will use a grid contraction function to represent the distance function on M. We assume that for each $x \in V(M)$ we store the value $f(x)$, and that for each $u \in V(H)$ we store $f^{-1}(u)$ as a list.

Let an integer $l \geq 0$ be fixed, and let $(x, y) \in R$ if and only if $d(x, y) < l$. We need to clarify one issue about the sets $R(x)$. Let us recall that $R(x)$ denotes the set of all $y \in X$ such that $(x, y) \in R$. If $x \in V(M)$, then $R(x)$ can be written as $\bigcup_{v \in V_1 \cup V_2} f^{-1}(v)$ for some sets $V_1, V_2 \subseteq V(H)$, where $|V_1| \leq (2l - 1)^2$ and V_2 is the union of the vertex-sets of at most $2l - 1$

vertical and at most $2l-1$ horizontal paths of H. To see this let V_1 be the set of all vertices $v \in V(H)$ such that there is a curve in the plane connecting v and $f(x)$ that intersects H at most l times and does not use the outer face of H, and V_2 is defined analogously using curves that use the outer face of H.

The following is an algorithmic version of Lemma 2.1. The conclusion is slightly weaker in order to save on running time.

Lemma 7.1 *There exists an algorithm with the following specifications.*
Input: *A graph G on n vertices and m edges, integers $k, l \geq 1$, and a mesh M in G with grid contraction function $f : V(M) \to V(H)$ giving rise to a distance function d on M. For $x, y \in V(M)$ let $(x, y) \in R$ if and only if $d(x, y) < l$.*
Output: *Either k disjoint R-semi-dispersed M-paths, or sets $A \subseteq V(G)$ and $Z \subseteq V(M)$ with $|A| \leq k-1$ and $|Z| \leq 3k-3$ such that every M-path P in $G - A$ with ends x and y either satisfies $d(x, y) \leq 2l-2$ or both $x, y \in \bigcup_{z \in Z} R(z)$.*
Running time: $O(\min\{n, k\}m + n)$.

Proof. We may assume that G has no isolated vertices (by deleting them). If l is at least the number of vertical or horizontal paths in M, then $A := \emptyset$ and any one-element set $Z \subseteq V(M)$ (or $Z = \emptyset$ if $k = 1$ and no M-path with ends far apart exists) satisfy the second condition of the output requirement. Thus we may assume that $l^2 = O(m)$.

The algorithm will proceed in at most $3k$ iterations. At the beginning of each iteration there will be M-paths $P_1, P_2, \ldots, P_s$ and $Q_1, Q_2, \ldots, Q_p$ as in the proof of Lemma 2.1 with ends denoted in the same way. Let A, Z, W be defined as in the proof of Lemma 2.1. At the start of the first iteration we have $s = p = 0$; thus $A = Z = W = \emptyset$. Throughout the algorithm the set W will be of the form $\bigcup_{v \in V} f^{-1}(v)$ for some $V \subseteq V(H)$, and will be presented by marking the elements of V.

For the purpose of this paragraph and the next let us say that a *good path* is an M-path S in $G - A$ with ends x, y, where $x \in V(M) \setminus W$ and $(x, y) \notin R$. We say that S is *very good* if it is good and $d(x, y) \geq 2l - 1$. At the beginning of each iteration we either find a good path, or establish that no very good path exists. We do so by running the following subroutine for every M-bridge B of the graph $G - A$. In the subroutine we first test whether B has an attachment $x \in V(M) \setminus W$. If not, then B does not include a good path and we return that information. Otherwise we test whether B has an attachment y at distance at least l from x; if we find one, then a path in B from x to y is a good path, and we return it. On the other hand, if all attachments of B belong to $R(x)$, then B includes no very good path, and we return that information. This completes the description of the subroutine. It is clear that each call takes time $O(|E(B)|)$, and that if no call to the subroutine returns a good path, then no very good path exists. Thus we either find a good path, or establish that no very good path exists in time $O(m)$.

If no very good path exists, then the sets A and Z satisfy the specifications of the algorithm. We output those sets and terminate the algorithm. If we find a good path S, then we modify the paths P_i and Q_i as in the proof of Lemma 2.1 by either adding a new path P_{s+1} and keeping all but one of the old paths Q_i, or by adding two new paths P_{s+1}, P_{s+2} and discarding one old path P_i and one old path Q_i, or by adding a new path Q_{p+1}. In each case the quantity $2s+p$ increases by one. We update the sets A, Z and W. The set W will be updated by marking $f(v)$ for every vertex v that is being added to W. For every vertex that is being added to Z this involves marking at most $(2l-1)^2$ vertices of H and the vertex-sets of at most $2(l-1)$ vertical and at most $2(l-1)$ horizontal paths of H. Similarly, we unmark vertices that are being deleted from W. The marking of vertical and horizontal paths will be done implicitly, so that the total time spent on marking during each iteration will be $O(l^2)$. If $s \geq k$ we output the paths $P_1, P_2, \ldots, P_k$ and terminate the algorithm; otherwise we go to the next iteration. The second step of the iteration described in this paragraph takes time $O(l^2+n) = O(m)$.

Since the quantity $|Z| = 2s+p$ increases during each iteration and $p \leq s$, the algorithm will terminate after at most $\min\{n, 3k\}$ iterations. Thus the running time is as claimed. □

Likewise there is a version of Lemma 2.2 with a similar proof, which we omit.

Lemma 7.2 *There exists an algorithm with the following specifications.*
Input: *A graph G on n vertices and m edges, integers $k, l \geq 0$, and a mesh M in G with grid contraction function $f : V(M) \to V(H)$ giving rise to a distance function d on M. For $x, y \in V(M)$ let $(x, y) \in R$ if and only if $d(x, y) < l$.*
Output: *Either k disjoint R-dispersed M-paths, or sets $A \subseteq V(G)$ and $Z \subseteq V(M)$ with $|A| \leq k-1$ and $|Z| \leq 3k-3$ such that for every M-path P in $G - A$ its ends can be denoted by x and y such that either $d(x, y) \leq 2l-2$ or $x \in \bigcup_{z \in Z} R(z)$.*
Running time: $O(\min\{n, k\}m + n)$.

Lemma 7.3 *There is an algorithm with the following specifications.*
Input: *A graph G on n vertices and m edges, an integer $t \geq 2$, a mesh in G with grid contraction function $f : V(M) \to V(H)$ giving rise to a distance function d on M, a set $X \subseteq V(M)$ with $|X| = 64(t(t-1))^6$ such that $d(x, y) \geq 2t(t-1)$ for all $x, y \in X$, and a matching $F \subseteq E(G) \setminus E(M)$ in G of size $32(t(t-1))^6$ with vertex-set X.*
Output: *A model of K_t grasped by M.*
Running time: $O(m+n)$.

Proof. This follows from the proof of Lemma 4.3, because it is easy to convert the standard proofs of Lemmas 4.1 and 4.2 into algorithms with running times $O(k^2) = O(m)$, where k is as in those lemmas. □

Lemma 7.4 *There exists an algorithm with the following specifications.*
Input: *A graph G on n vertices and m edges, an integer $t \geq 2$, and a mesh in G with grid contraction function $f : V(M) \to V(H)$ giving rise to a distance function d on M.*
Output: *For $k_0 := 12288(t(t-1))^{12}$ either a model of K_t in G grasped by M, or sets $A \subseteq V(G)$ and $Z \subseteq V(M)$ such that $|A| \leq k_0 - 1$, $|Z| \leq 3k_0 - 2$, and if x, y are the ends of a M-path in $G - A$, then either $d(x, y) < 20t(t-1)$, or each of x, y lies at distance at most $10t(t-1) - 1$ from some vertex of Z.*
Running time: $O(t^{24}m + n)$

Proof. The algorithm follows the proof of Lemma 4.6. We first apply the algorithm of Lemma 7.2 to the graph G, mesh M and integers $l = 2t(t-1)$ and $k = 32(t(t-1))^6$. If the algorithm returns k disjoint dispersed M-paths, then we use the algorithm of Lemma 7.3 to output a model of K_t grasped by M and stop. We may therefore assume that the algorithm of Lemma 7.2 returns sets $A \subseteq V(G)$ and $Z \subseteq V(M)$ such that $|A| \leq k-1$, $|Z| \leq 3k-3$, and for every M-path P in $G - A$ its ends may be denoted by x and y such that either $d(x,y) \leq 4t(t-1)-2$ or $d(x,z) \leq 2t(t-1)-1$ for some $z \in Z$. Next we apply the algorithm of Lemma 7.1 to the graph G, mesh M and integers $l = 10t(t-1)$ and k_0. If the algorithm returns sets A and Z, then we return those sets and stop. We may therefore assume that the algorithm of Lemma 7.1 returns a set of k_0 pairwise disjoint semi-dispersed M-paths. We use the argument of the proof of Lemma 4.6 to use the paths to construct a matching to which we can apply the algorithm of Lemma 7.3 to output a model of K_t grasped by M. $\square$

The following is an algorithm of Kawarabayashi, Li and Reed [8] stated using our terminology.

Theorem 7.5 *There is a polynomial-time algorithm with the following specifications.*
Input: *A graph G with n vertices and m edges and a cycle C in G.*
Output: *Either a C-cross in G, or a C-rendition.*
Running time: *$O(n+m)$.*

Let us remark that the algorithm of Kawarabayashi, Li and Reed [8] is formulated in terms of C-reductions, which is equivalent to C-renditions by Theorem 1.4.

Our last lemma is an algorithmic version of Lemma 5.1.

Lemma 7.6 *There exists an algorithm with the following specifications.*
Input: *A graph G on n vertices and m edges, a subgraph W of G, a cycle C in G, a cycle D in W such that $W - V(D)$ is connected, four internally disjoint paths from $V(W) \setminus V(D)$ to $V(C)$ with distinct ends in C such that each intersects D in a path, and a C-rendition of G.*
Output: *A separation (A,B) in G satisfying (1)–(4) of Lemma 5.1 and an Ω-rendition of $G[B]$, where Ω is a cyclic ordering of $A \cap B$ and the cyclic order is determined by the order on D.*
Running time: *$O(n+m)$.*

Proof. Let (Γ, σ, π) be a C-rendition of G. We construct a track of D as in the proof of Lemma 5.1. Using the track we construct the separation (A, B'), and then modify it to the separation (A,B), as in the proof of Lemma 5.1. Finding the original separation takes time $O(n+m)$, and the modifications take time $\sum_{i=1}^{n} O(|E(\sigma(c_i))|)$. Thus the total running time is $O(n+m)$. $\square$

We are finally ready to describe our main algorithm.

Theorem 7.7 *There is an algorithm with the following specifications.*
Input: *A graph G on n vertices and m edges, integers $r,t \geq 1$, and an R-wall W in G, where $R = 49152t^{24}(60t^2+r)$.*

Output: *Either a model of a K_t minor in G grasped by W, or a set $A \subseteq V(G)$ of size at most $12288t^{24}$ and an r-subwall W' of W such that $V(W') \cap A = \emptyset$ and W' is a flat wall in $G - A$. In the second alternative the algorithm also returns a separation (A, B) as in the definition of flat wall, and an Ω-rendition of $G[B]$, where Ω is a cyclic ordering of $A \cap B$ and the cyclic order is determined by the order on the outer cycle of W'.*
Running time: $O(t^{24}m + n)$.

Proof. We compute a grid contraction function $f : V(W) \to V(H)$ and apply the algorithm of Lemma 7.4 to the graph G, mesh W, function f, and integer t. If the algorithm returns a model of K_t grasped by W, then we return that model and stop. We may therefore assume that the algorithm of Lemma 7.4 returned sets $A \subseteq V(G)$ and $Z \subseteq V(M)$ such that $|A| \leq 12288(t(t-1))^{12}$, $|Z| \leq 3 \cdot 12288(t(t-1))^{12}$, and if x, y are the ends of an M-path in $G - A$, then either $d(x, y) < 20t(t-1)$, or each of x, y lies at distance at most $10t(t-1) - 1$ from some vertex of Z. We define strips similarly as in the proof of Theorem 5.2, except that strips will now consist of $60t(t-1) + r$ consecutive paths. We construct walls $W_1, W_2, \ldots, W_{t(t-1)}$, but this time each will be a $(40t(t-1) + r)$-wall, they will be pairwise at distance at least $20t(t-1)$, and each will be disjoint from the first and last $10t(t-1)$ paths of the strip. We construct the graphs H_i and cycles C_i as in the proof of Theorem 5.2, and apply the algorithm of Theorem 7.5 to each. If each of them has a C_i-cross, then we use those crosses to construct a model of K_t grasped by W, as in the proof of Theorem 5.2. On the other hand if some H_i has a C_i-rendition, then we apply the algorithm of Lemma 7.6 to H_i, wall W_i, its outer cycle and the C_i-rendition to produce a separation (X', Y) satisfying (1)–(4) of Lemma 5.1 and an Ω-rendition of $G[Y]$, where Ω is a cyclic ordering of $X' \cap Y$ and the cyclic order is determined by the order on the outer cycle of W_i. Finally, we convert (X', Y) to a required separation of G as in the proof of Theorem 5.2. □

Acknowledgment

We would like to thank Ann-Kathrin Elm from the University of Hamburg for pointing out a couple of inaccuracies in an early draft of this article. We would also like to acknowledge that since the first version of this article has been posted, Chuzhoy [1] dramatically improved the bound in Theorem 1.7 to $R_0 = \Omega(t(t + r))$.

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8 Appendix: Characterizing graphs with no C-cross

In this section, we present a proof of Theorem 1.3 which characterizes when a given graph G containing a cycle C has a C-cross. The proof is due to Robertson and Seymour [11].

Let G be a graph and C a cycle in G. We first prove the easy "if" implication. We have noted earlier that if H is an elementary C-reduction of G, then H contains a C-cross if and only if G does as well. Let G' be any C-reduction of G. If G' can be drawn in the plane with C bounding the infinite face, then by planarity, there does not exist a C-cross in G'. Consequently, there does not exist a C-cross in G as well.

We now prove the "only if" implication by induction on $|V(G)|+|E(G)|$. If $G = C$, then the theorem clearly holds, and so we may assume that $G \neq C$ and that G has no C-cross. We may assume that G is simple, because deleting loops and parallel edges does not change the validity of either of the statements in the theorem. If G has an elementary C-reduction, then the theorem follows by induction applied to that C-reduction. Thus we may assume that G has no elementary C-reduction. Therefore

(1) *G has no separation (A, B) of order at most three with $V(C) \subseteq A$ and $B \setminus A \neq \emptyset$,*

because if such a separation exists, then choosing one with $|A \cap B|$ minimum gives a separation that determines an elementary C-reduction of G, a contradiction.

Define a *tripod* as a union of paths $P_1, P_2, P_3, Q_1, Q_2, Q_3$ satisfying the following. The paths P_1, P_2, P_3 have a common end $v \in V(G) \setminus V(C)$ and are otherwise pairwise disjoint. Each P_i, $1 \leq i \leq 3$ has exactly one vertex in $V(C)$, call it x_i, and x_i is an end of P_i. The paths Q_1, Q_2, Q_3 have a common end $u \in (V(G) \setminus V(C))$, $u \neq v$, and are otherwise pairwise disjoint. For every $1 \leq i \leq 3$, Q_i has an end $y_i \in V(P_i) - \{v\}$ and Q_i is otherwise disjoint from $P_1 \cup P_2 \cup P_3$.

(2) *The graph G does not contain a tripod.*

To prove (2) assume there exists a tripod T and let the paths $P_1, P_2, P_3, Q_1, Q_2, Q_3$ and the vertices $x_1, x_2, x_3, u, v, y_1, y_2, y_3$ be labeled as in the definition of a tripod. For $i = 1, 2, 3$ let L_i be the subpath of P_i with ends x_i and y_i, and let R_i be the subpath of P_i with ends v and y_i. Let $X = V(R_1 \cup R_2 \cup R_3 \cup Q_1 \cup Q_2 \cup Q_3)$. By (1) there exist four disjoint paths from X to $V(C)$, and by a standard "augmenting path" argument (cf. [4, Section 3]) those paths can be chosen such that three of them have ends in $\{y_1, y_2, y_3\}$ and (possibly different) three of those paths have ends in $\{x_1, x_2, x_3\}$. Thus by possibly replacing the paths L_1, L_2, L_3 by a different set of disjoint paths we may assume that there exists a path Q with one end in $X \setminus \{y_1, y_2, y_3\}$ and the other end in $V(C) - \{x_1, x_2, x_3\}$ that is disjoint from T except for one of its ends. It follows that $T \cup Q$ includes a C-cross, a contradiction, which proves (2).

Let us recall that H-bridges were defined prior to Theorem 5.2 and H-paths were defined at the beginning of Section 2. If P is a C-path, then a $C \cup P$-bridge is *unstable* if all its attachments belong to $V(P)$, and *stable* otherwise.

(3) *There exists a C-path P in G such that every $C \cup P$-bridge is stable.*

To prove (3) we first note that since $G \neq C$, it follows from (1) that G has a C-path. Let P be a C-path chosen such that the number of vertices of $G - V(C \cup P)$ that belong to stable $C \cup P$-bridges is maximum. We claim that P is as desired. To prove the claim we may assume for a contradiction that there exists at least one unstable bridge.

A vertex v of P is *straddled* if it is an internal vertex of P and there exists an unstable bridge with attachments in both components of $P - v$. We claim that there exists at least one straddled vertex in P. Let B be an unstable bridge. If B has at least three vertices, then it has at least three attachments by (1), and therefore a middle attachment is straddled. Otherwise, if B is has only two vertices, then its vertices are not adjacent in P because G is simple, and consequently, there exists a straddled vertex between the vertices of B.

Let R be a maximal subpath of P such that every internal vertex of R is straddled. Note that R has length at least two. As by (1) the ends of R do not form a vertex cut of size two separating the internal vertices of R from C, we see there exists a $C \cup P$-bridge B' with an attachment x that is an internal vertex of R and an attachment which is not contained in R. If B' were unstable, then it must straddle one of the ends of R, violating the maximality of R. We conclude that B' is stable.

The vertex x is straddled by some unstable bridge D. Let u, v be attachments of D such that u, x, v are distinct and appear on P in the order listed. Let P' be obtained from P by replacing the subpath from u to v by a subpath of D from u to v. It follows that every stable $C \cup P$-bridge is a subgraph of a stable $C \cup P'$-bridge, and the vertex x belongs to a stable $C \cup P'$-bridge containing B'. Thus the path P' contradicts the choice of P. This proves (3).

Let P be a C-path in G such that every $C \cup P$-bridge is stable.

(4) *No $C \cup P$-bridge has attachments in different components of $C - V(P)$.*

To prove (4) we note that if such a bridge existed, then it would include a path Q with ends in different components of $C - V(P)$. But then the paths P and Q form a C-cross, a contradiction, which proves (4).

Let C_1, C_2 be the two cycles of $C \cup P$ other than C. It follows from (3) and (4) that every $C \cup P$-bridge is either a C_1-bridge, or a C_2-bridge, and not both. For $i = 1, 2$ let G_i be the union of C_i and all $C \cup P$-bridges of G that are C_i-bridges. Then $G_1 \cup G_2 = G$, $G_1 \cap G_2 = P$, and $|V(G_i)| + |E(G_i)| < |V(G)| + |E(G)|$ for $i = 1, 2$.

(5) *For $i = 1, 2$ the graph G_i has no elementary C_i-reduction.*

To prove (5) let $i \in \{1, , 2\}$. If G_i has an elementary C_i-reduction, then it has a separation (A, B) of order at most three with C_i contained in A and $B \setminus A \neq \emptyset$. Then $(A \cup V(G_{3-i}), B)$ is a separation of G contradicting (1). This proves (5).

By induction and (5), for $i = 1, 2$ the graph G_i either has a C_i-cross, or can be drawn in the plane with C_i bounding a face. If the latter alternative holds for both $i = 1$ and $i = 2$, then the two drawings may be combined to produce a drawing of G in the plane with C bounding a face, as desired. Thus we may assume without loss of generality that G_1 contains a C_1-cross Q_1, Q_2. Let the ends of Q_i be s_i and t_i. If P contains at most two of the vertices s_1, s_2, t_1, t_2, we see that the cross Q_1, Q_2 readily extends to a C-cross in G by possibly using subpaths of P, a contradiction.

We claim that we may assume that $\{s_1, s_2, t_1, t_2\} \not\subseteq V(P)$. To prove this claim we may assume that Q_1 and Q_2 each have their both ends contained in $V(P)$. Since the $C \cup P$-bridge containing Q_1 is stable by (3), it follows that Q_1 has an internal vertex, and there exists a path R from an internal vertex of Q_1 or Q_2 to $V(C_1) \setminus V(P)$ and otherwise disjoint from $P \cup Q_1 \cup Q_2 \cup C_1$. We deduce that $R \cup Q_1 \cup Q_2$ contains a C_1-cross with at least one end not in $V(P)$, as desired. This proves our claim that we may assume that $\{s_1, s_2, t_1, t_2\} \not\subseteq V(P)$.

It now follows that Q_1 and Q_2 have a total of exactly three ends in $V(P)$. Without loss of generality, assume that s_1, s_2, t_1 are contained in $V(P)$ and occur in that order when traversing P. Since the $C \cup P$-bridge containing Q_1 is stable by (3), it follows that Q_1 has an internal vertex, and there exists a path R from an internal vertex of Q_1 to $V(C_1) \setminus V(P)$ that is otherwise disjoint from $P \cup Q_1 \cup C_1$. If R is disjoint from Q_2, then $Q_1 \cup Q_2 \cup R$ includes a C_1-cross with exactly two ends in P, a case already handled. Thus we may assume that R has a subpath S with one end in $Q_1 - \{s_1, t_1\}$, the other end in $Q_2 - s_2$, and otherwise disjoint from $Q_1 \cup Q_2$. Now $S \cup Q_1 \cup Q_2 \cup P$ is a tripod in G, contradicting (2). This final contradiction completes the proof of the theorem. $\square$

The proof of Theorem 1.3 is constructive and readily implies the existence of a polynomial time algorithm for the problem of Theorem 7.5. However, it does not seem to achieve as good a bound on the running time as the algorithm of Kawarabayashi, Li and Reed [8].

This material is based upon work supported by the National Science Foundation. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the National Science Foundation.