K_6 MINORS IN LARGE 6-CONNECTED GRAPHS

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ABSTRACT

Jørgensen conjectured that every 6-connected graph G with no K_6 minor has a vertex whose deletion makes the graph planar. We prove the conjecture for all sufficiently large graphs.

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1 Introduction

Graphs in this paper are allowed to have loops and multiple edges. A graph is a *minor* of another if the first can be obtained from a subgraph of the second by contracting edges. An H *minor* is a minor isomorphic to H. A graph G is *apex* if it has a vertex v such that $G\backslash v$ is planar. (We use $\backslash$ for deletion.) Jørgensen [4] made the following beautiful conjecture.

Conjecture 1.1 *Every 6-connected graph with no K_6 minor is apex.*

This is related to Hadwiger's conjecture [3], the following.

Conjecture 1.2 *For every integer $t \geq 1$, if a loopless graph has no K_t minor, then it is $(t-1)$-colorable.*

Hadwiger's conjecture is known for $t \leq 6$. For $t = 6$ it has been proven in [13] by showing that a minimal counterexample to Hadwiger's conjecture for $t = 6$ is apex. The proof uses an earlier result of Mader [6] that every minimal counterexample to Conjecture 1.2 is 6-connected. Thus Conjecture 1.1, if true, would give more structural information. Furthermore, the structure of all graphs with no K_6 minor is not known, and appears complicated and difficult. On the other hand, Conjecture 1.1 provides a nice and clean statement for 6-connected graphs. Unfortunately, it, too, appears to be a difficult problem. In this paper we prove Conjecture 1.1 for all sufficiently large graphs, as follows.

Theorem 1.3 *There exists an absolute constant N such that every 6-connected graph on at least N vertices with no K_6 minor is apex.*

The second and third author recently announced a generalization [7] of Theorem 1.3, where 6 is replaced by an arbitrary integer t. The result states that for every integer t there exists an integer N_t such that every t-connected graph on at least N_t vertices with no K_t minor has a set of at most $t-5$ vertices whose deletion makes the graph planar. The proof follows a different strategy, but makes use of several ideas developed in this paper and its companion [5].

We use a number of results from the Graph Minor series of Robertson and Seymour, and also three results of our own that are proved in [5]. The first of those is a version of Theorem 1.3 for graphs of bounded tree-width, the following. (We will not define tree-width here, because it is sufficiently well-known, and because we do not need the concept *per se*, only several theorems that use it.)

Theorem 1.4 *For every integer w there exists an integer N such that every 6-connected graph of tree-width at most w on at least N vertices and with no K_6 minor is apex.*

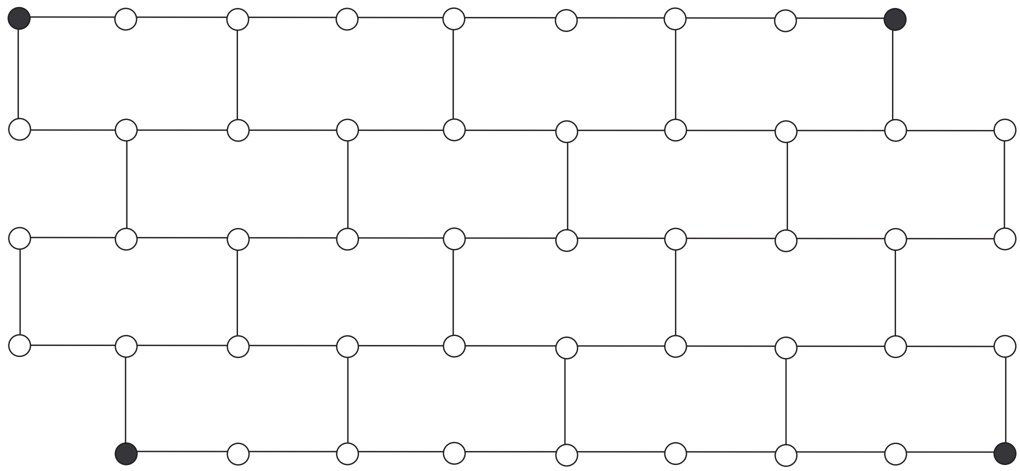

Figure 1: An elementary wall of height 4.

Theorem 1.4 reduces the proof of Theorem 1.3 to graphs of large tree-width. By a result of Robertson and Seymour [9] those graphs have a large grid minor. However, for our purposes it is more convenient to work with walls instead. Let $h \geq 2$ be even. An *elementary wall of height h* has vertex-set

$$\{(x,y) : 0 \leq x \leq 2h+1, 0 \leq y \leq h\} - \{(0,0),(2h+1,h)\}$$

and an edge between any vertices (x,y) and (x',y') if either

- $|x - x'| = 1$ and $y = y'$, or
- $x = x'$, $|y - y'| = 1$ and x and $\max\{y, y'\}$ have the same parity.

Figure 1 shows an elementary wall of height 4. A *wall of height h* is a subdivision of an elementary wall of height h. The result of [9] (see also [2, 8, 14]) can be restated as follows.

Theorem 1.5 *For every even integer $h \geq 2$ there exists an integer w such that every graph of tree-width at least w has a subgraph isomorphic to a wall of height h.*

The *perimeter* of a wall is the cycle that bounds the infinite face when the wall is drawn as in Figure 1. Now let C be the perimeter of a wall H in a graph G. The *compass of H in G* is the restriction of G to X, where X is the union of $V(C)$ and the vertex-set of the unique component of $G \backslash V(C)$ that contains a vertex of H. Thus H is a subgraph of its compass, and the compass is connected. A wall H with perimeter C in a graph G is *planar* if its compass can be drawn in the plane with C bounding the infinite face. In Section 2 we prove the following.

Theorem 1.6 *For every even integer $t \geq 2$ there exists an even integer $h \geq 2$ such that if a 5-connected graph G with no K_6 minor has a wall of height at least h, then either it is apex, or has a planar wall of height t.*

Actually, in the proof of Theorem 1.6 we need Lemma 2.4 that is proved in [5]. The lemma says that if a 5-connected graph with no K_6 minor has a subgraph isomorphic to subdivision of a pinwheel with sufficiently many vanes (see Figure 3), then it is apex.

By Theorem 1.6 we may assume that our graph G has an arbitrarily large planar wall H. Let C be the perimeter of H, and let K be the compass of H. Then C separates G into K and another graph, say J, such that $K \cup J = G$, $V(K) \cap V(J) = V(C)$ and $E(K) \cap E(J) = \emptyset$. Next we study the graph J. Since the order of the vertices on C is important, we are lead to the notion of a "society", introduced by Robertson and Seymour in [10].

Let Ω be a cyclic permutation of the elements of some set; we denote this set by $V(\Omega)$. A *society* is a pair (G, Ω), where G is a graph, and Ω is a cyclic permutation with $V(\Omega) \subseteq V(G)$. Now let J be as above, and let Ω be one of the cyclic permutations of $V(C)$ determined by the order of vertices on C. Then (J, Ω) is a society that is of primary interest to us. We call it the *anticompass society of H in G*.

We say that (G, Ω, Ω_0) is a *neighborhood* if G is a graph and Ω, Ω_0 are cyclic permutations, where both $V(\Omega)$ and $V(\Omega_0)$ are subsets of $V(G)$. Let Σ be a plane, with some orientation called "clockwise." We say that a neighborhood (G, Ω, Ω_0) is *rural* if G has a drawing Γ in Σ without crossings (so G is planar) and there are closed discs $\Delta_0 \subseteq \Delta \subseteq \Sigma$, such that

(i) the drawing Γ uses no point of Σ outside Δ, and none in the interior of Δ_0, and

(ii) for $v \in V(G)$, the point of Σ representing v in the drawing Γ lies in $bd(\Delta)$ (respectively, $bd(\Delta_0)$) if and only if $v \in V(\Omega)$ (respectively, $v \in V(\Omega_0)$), and the cyclic permutation of $V(\Omega)$ (respectively, $V(\Omega_0)$) obtained from the clockwise orientation of $bd(\Delta)$ (respectively, $bd(\Delta_0)$) coincides (in the natural sense) with Ω (respectively, Ω_0).

We call $(\Sigma, \Gamma, \Delta, \Delta_0)$ a *presentation* of (G, Ω, Ω_0).

Let (G_1, Ω, Ω_0) be a neighborhood, let (G_0, Ω_0) be a society with $V(G_0) \cap V(G_1) = V(\Omega_0)$, and let $G = G_0 \cup G_1$. Then (G, Ω) is a society, and we say that (G, Ω) is the *composition* of the society (G_0, Ω_0) with the neighborhood (G_1, Ω, Ω_0). If the neighborhood (G_1, Ω, Ω_0) is rural, then we say that (G_0, Ω_0) is a *planar truncation* of (G, Ω). We say that a society (G, Ω) is *k-cosmopolitan*, where $k \geq 0$ is an integer, if for every planar truncation (G_0, Ω_0) of (G, Ω) at least k vertices in $V(\Omega_0)$ have at least two neighbors in $V(G_0)$. At the end of Section 2 we deduce

Theorem 1.7 *For every integer $k \geq 1$ there exists an even integer $t \geq 2$ such that if G is a simple graph of minimum degree at least six and H is a planar wall of height t in G, then the anticompass society of H in G is k-cosmopolitan.*

For a fixed presentation $(\Sigma, \Gamma, \Delta, \Delta_0)$ of a neighborhood (G, Ω, Ω_0) and an integer $s \geq 0$ we define an *s-nest* for $(\Sigma, \Gamma, \Delta, \Delta_0)$ to be a sequence $(C_1, C_2, \ldots, C_s)$ of pairwise disjoint cycles of G such that $\Delta_0 \subseteq \Delta_1 \subseteq \cdots \subseteq \Delta_s \subseteq \Delta$, where Δ_i denotes the closed disk in Σ bounded by the image under Γ of C_i. We say that a society (G, Ω) is *s-nested* if it is the

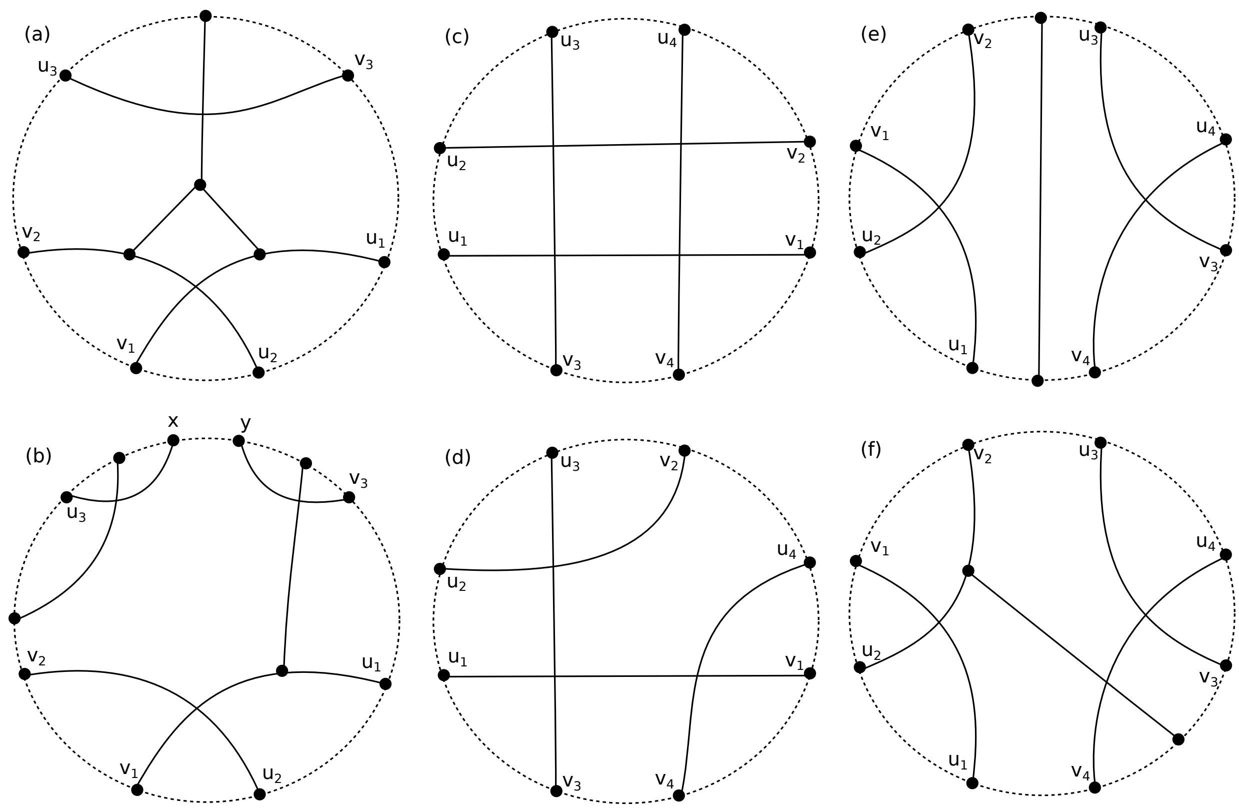

Figure 2: (a),(b) A turtle. (c),(d) A gridlet. (e),(f) A separated doublecross.

composition of a society (G_1, Ω_0) with a rural neighborhood (G_2, Ω, Ω_0) that has an s-nest for some presentation of (G_2, Ω, Ω_0).

Let Ω be a cyclic permutation. For $x \in V(\Omega)$ we denote the image of x under Ω by $\Omega(x)$. If $X \subseteq V(\Omega)$, then we denote by $\Omega|X$ the restriction of Ω to X. That is, $\Omega|X$ is the permutation Ω' defined by saying that $V(\Omega') = X$ and $\Omega'(x)$ is the first term of the sequence $\Omega(x), \Omega(\Omega(x)), \ldots$ which belongs to X. Let $v_1, v_2, \ldots, v_k \in V(\Omega)$ be distinct. We say that $(v_1, v_2, \ldots, v_k)$ is *clockwise* in Ω (or simply *clockwise* when Ω is understood from context) if $\Omega'(v_{i-1}) = v_i$ for all $i = 1, 2, \ldots, k$, where v_0 means v_k and $\Omega' = \Omega|\{v_1, v_2, \ldots, v_k\}$. For $u, v \in V(\Omega)$ we define $u\Omega v$ as the set of all $x \in V(\Omega)$ such that either $x = u$ or $x = v$ or (u, x, v) is clockwise in Ω.

A separation of a graph is a pair (A, B) such that $A \cup B = V(G)$ and there is no edge with one end in $A - B$ and the other end in $B - A$. The order of (A, B) is $|A \cap B|$. We say that a society (G, Ω) is *k-connected* if there is no separation (A, B) of G of order at most $k - 1$ with $V(\Omega) \subseteq A$ and $B - A \neq \emptyset$. A *bump* in (G, Ω) is a path in G with at least one edge, both ends in $V(\Omega)$ and otherwise disjoint from $V(\Omega)$.

Let (G, Ω) be a society and let $(u_1, u_2, v_1, v_2, u_3, v_3)$ be clockwise in Ω. For $i = 1, 2$ let P_i be a bump in G with ends u_i and v_i, and let L be either a bump with ends u_3 and v_3, or the union of two internally disjoint bumps, one with ends u_3 and $x \in u_3\Omega v_3$ and the other with ends v_3 and $y \in u_3\Omega v_3$. In the former case let $Z = \emptyset$, and in the latter case let Z be the subinterval of $u_3\Omega v_3$ with ends x and y, including its ends. Assume that P_1, P_2, L are pairwise disjoint. Let $q_1, q_2 \in V(P_1) \cup V(P_2) \cup v_3\Omega u_3 - \{u_3, v_3\}$ be distinct such that neither of the sets $V(P_1) \cup v_3\Omega u_1$, $V(P_2) \cup v_2\Omega u_3$ includes both q_1 and q_2. Let Q_1 and Q_2 be two not necessarily disjoint paths with one end in $u_3\Omega v_3 - Z - \{u_3, v_3\}$ and the other end q_1 and q_2, respectively, both internally disjoint from $V(P_1 \cup P_2 \cup L) \cup V(\Omega)$. In those circumstances we say that $P_1 \cup P_2 \cup L \cup Q_1 \cup Q_2$ is a *turtle* in (G, Ω). We say that P_1, P_2 are the *legs*, L is the *neck*, and $Q_1 \cup Q_2$ is the *body* of the turtle. (See Figure 2(a),(b).)

Let (G, Ω) be a society, let $(u_1, u_2, u_3, v_1, v_2, v_3)$ be clockwise in Ω, and let P_1, P_2, P_3 be disjoint bumps such that P_i has ends u_i and v_i. In those circumstances we say that P_1, P_2, P_3 are *three crossed paths* in (G, Ω).

Let (G, Ω) be a society, and let $u_1, u_2, u_3, u_4, v_1, v_2, v_3, v_4 \in V(\Omega)$ be such that either $(u_1, u_2, u_3, v_2, u_4, v_1, v_4, v_3)$ or $(u_1, u_2, u_3, u_4, v_2, v_1, v_4, v_3)$ or $(u_1, u_2, u_3, v_2 = u_4, v_1, v_4, v_3)$ is clockwise. For $i = 1, 2, 3, 4$ let P_i be a bump with ends u_i and v_i such that these bumps are pairwise disjoint, except possibly for $v_2 = u_4$. In those circumstances we say that P_1, P_2, P_3, P_4 is a *gridlet*. (See Figure 2(c),(d).)

Let (G, Ω) be a society and let $(u_1, u_2, v_1, v_2, u_3, u_4, v_3, v_4)$ be clockwise or counter-clockwise in Ω. For $i = 1, 2, 3, 4$ let P_i be a bump with ends u_i and v_i such that these bumps are pairwise disjoint, and let P_5 be a path with one end in $V(P_1) \cup v_4\Omega u_2 - \{u_2, v_1, v_4\}$, the other end in $V(P_3) \cup v_2\Omega u_4 - \{v_2, v_3, u_4\}$, and otherwise disjoint from $P_1 \cup P_2 \cup P_3 \cup P_4$. In those circumstances we say that $P_1, P_2, \ldots, P_5$ is a *separated doublecross.*(See Figure 2(e),(f).)

A society (G, Ω) is *rural* if G can be drawn in a disk with $V(\Omega)$ drawn on the boundary of the disk in the order given by Ω. A society (G, Ω) is *nearly rural* if there exists a vertex $v \in V(G)$ such that the society $(G\backslash v, \Omega\backslash v)$ obtained from (G, Ω) by deleting v is rural.

In Sections 4–9 we prove the following. The proof strategy is explained in Section 5. It uses a couple of theorems from [10] and Theorem 4.1 that we prove in Section 4.

Theorem 1.8 *There exists an integer $k \geq 1$ such that for every integer $s \geq 0$ and every 6-connected s-nested k-cosmopolitan society (G, Ω) either (G, Ω) is nearly rural, or G has a triangle C such that $(G\backslash E(C), \Omega)$ is rural, or (G, Ω) has an s-nested planar truncation that has a turtle, three crossed paths, a gridlet, or a separated doublecross.*

Finally, we need to convert a turtle, three crossed paths, gridlet and a separated doublecross into a K_6 minor. Let G be a 6-connected graph, let H be a sufficiently high planar wall in G, and let (J, Ω) be the anticompass society of H in G. We wish to apply to Theorem 1.8 to (J, Ω). We can, in fact, assume that H is a subgraph of a larger planar wall H' that includes s concentric cycles $C_1, C_2, \ldots, C_s$ surrounding H and disjoint from H, for some suitable integer s, and hence (J, Ω) is s-nested. Theorem 1.8 guarantees a turtle or paths in (J, Ω) forming three crossed paths, a gridlet, or a separated double-cross, but it does not say how the turtle or paths might intersect the cycles C_i. In Section 10 we prove a theorem that says that the cycles and the turtle (or paths) can be changed such that after possibly sacrificing a lot of the cycles, the remaining cycles and the new turtle (or paths) intersect nicely. Using that information it is then easy to find a K_6 minor in G. We complete the proof of Theorem 1.3 in Section 11.

2 Finding a planar wall

Let a *pinwheel with four vanes* be the graph pictured in Figure 3. We define a pinwheel with k vanes analogously. A graph G is *internally* 4*-connected* if it is simple, 3-connected, has at least five vertices, and for every separation (A, B) of G of order three, one of A, B induces a graph with at most three edges.

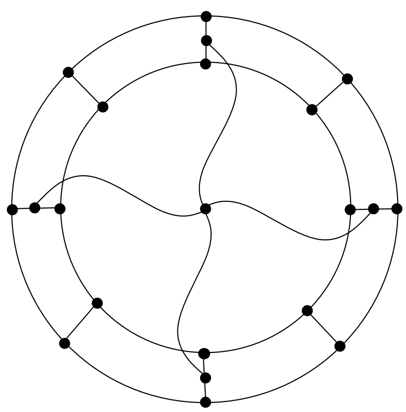

Figure 3: A pinwheel with four vanes.

The objective of this section is to prove the following theorem.

Theorem 2.1 *For every even integer $t \geq 2$ there exists an even integer h such that if H is a wall of height at least h in an internally* 4*-connected graph G, then either*

(1) *G has a K_6 minor, or*

(2) *G has a subgraph isomorphic to a subdivision of a pinwheel with t vanes, or*

(3) *G has a planar wall of height t.*

In the proof we will be using several results from [11]. Their statements require the following terminology: distance function, (l, m)-star over H, external (l, m)-star over H, subwall, dividing subwall, flat subwall, cross over a wall. We refer to [11] for precise definitions, but we offer the following informal descriptions. The distance of two distinct vertices s, t of a wall is the minimum number of times a curve in the plane joining s and t intersects the drawing of the wall, when the wall is drawn as in Figure 1. An (l, m)-star over a wall H in G is a subdivision of a star with l leaves such that only the leaves and possibly the center belong to H, and the leaves are pairwise at distance at least m. The star is external if the center does not belong to H. A subwall of a wall is dividing if its perimeter separates the subwall from the rest of the wall. A cross over a wall is a set of two disjoint paths joining the diagonally opposite pairs of "corners" of the wall, the vertices represented by solid circles in Figure 1. A subwall H is flat in G if there is no cross P, Q over H such that $P \cup Q$ is a subgraph of the compass of H in G.

We begin with the following easy lemma. We leave the proof to the reader.

Lemma 2.2 *For every integer t there exist integers l and m such that if a graph G has a wall H with an external (l, m)-star, then it has a subgraph isomorphic to a subdivision of a pinwheel with t vanes.*

We need one more lemma, which follows immediately from [11, Theorem 8.6].

Lemma 2.3 *Every flat wall in an internally 4-connected graph is planar.*

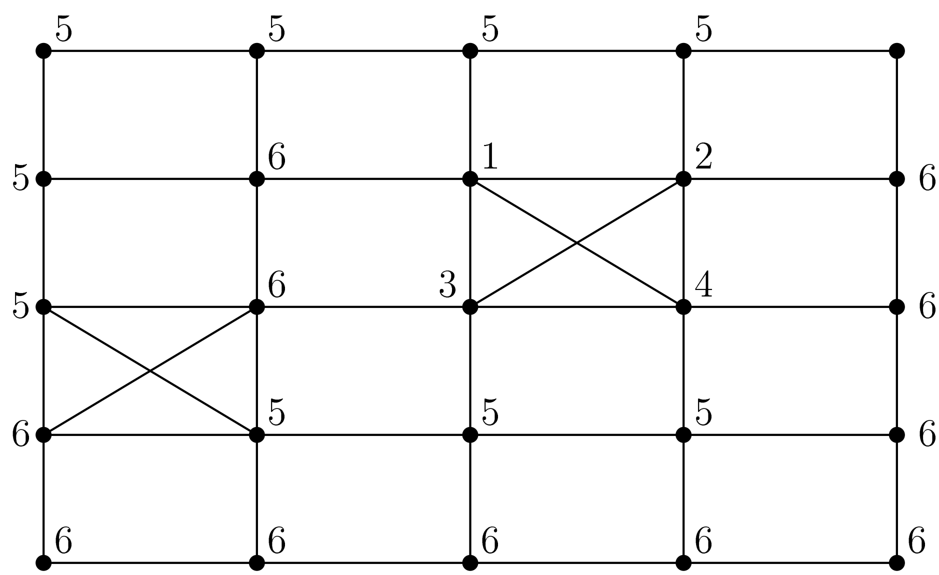

Figure 4: A K_6 minor in a grid with two crosses.

Proof of Theorem 2.1. Let $t \geq 1$ be given, let l, m be as in Lemma 2.2, let $p = 6$, and let k, r be as in [11, Theorem 9.2]. If h is sufficiently large, then H has $k + 1$ subwalls of height at least t, pairwise at distance at least r. If at least k of these subwalls are non-dividing, then by [11, Theorem 9.2] G either has a K_6 minor, or an (l, m)-star over H, in which case it has a subgraph isomorphic to a pinwheel with t vanes by Lemma 2.2. In either case the theorem holds, and so we may assume that at least two of the subwalls, say H_1 and H_2, are dividing. We may assume that H_1 and H_2 are not planar, for otherwise the theorem holds. Let $i \in \{1, 2\}$. By Lemma 2.3 the wall H_i is not flat, and hence its compass has a cross

$P_i \cup Q_i$. Since the subwalls H_1 and H_2 are dividing, it follows that the paths P_1, Q_1, P_2, Q_2 are pairwise disjoint. Thus G has a minor isomorphic to the graph shown in Figure 4, but that graph has a minor isomorphic to a minor of K_6, as indicated by the numbers in the figure. Thus G has a K_6 minor, and the theorem holds. $\square$

To deduce Theorem 1.6 we need the following lemma, proved in [5, Lemma 5.3].

Lemma 2.4 *If a 5-connected graph G with no K_6 minor has a subdivision isomorphic to a pinwheel with 20 vanes, then G is apex.*

Proof of Theorem 1.6. Let $t \geq 2$ be an even integer. We may assume that $t \geq 20$. Let h be as in Theorem 2.1, and let G be a 5-connected graph with no K_6 minor. From Theorem 2.1 we deduce that G either satisfies the conclusion of Theorem 1.6, or has a subdivision isomorphic to a pinwheel with t vanes. In the latter case the theorem follows from Lemma 2.4. $\square$

We need the following theorem of DeVos and Seymour [1].

Theorem 2.5 *Let (G, Ω) be a rural society such that G is a simple graph and every vertex of G not in $V(\Omega)$ has degree at least six. Then $|V(G)| \leq |V(\Omega)|^2/12 + |V(\Omega)|/2 + 1$.*

Proof of Theorem 1.7. Let $k \geq 1$ be an integer, and let t be an even integer such that if W is the elementary wall of height t and $|V(W)| \leq \ell^2/12 + \ell/2 + 1$, then $\ell > 6k - 6$. Let K be the compass of H in G, let (J, Ω) be the anticompass society of H in G, let (G_0, Ω_0) be a planar truncation of (J, Ω), and let $\ell = |V(\Omega_0)|$. Thus (J, Ω) is the composition of (G_0, Ω_0) with a rural neighborhood (G', Ω, Ω_0). Then $|V(H)| \leq \ell^2/12 + \ell/2 + 1$ by Theorem 2.5 applied to the society $(K \cup G', \Omega_0)$, and hence $\ell > 6k - 6$. Let L be the graph obtained from $K \cup G'$ by adding a new vertex v and joining it to every vertex of $V(\Omega_0)$ and by adding an edge joining every pair of nonadjacent vertices of $V(\Omega_0)$ that are consecutive in Ω_0. Then L is planar. Let s be the number of vertices of $V(\Omega_0)$ with at least two neighbors in G_0. Then all but s vertices of $K \cup G'$ have degree in L at least six. Thus the sum of the degrees of vertices of L is at least $6|V(K \cup G')| - 6s + \ell$. On the other hand, the sum of the degrees is at most $6|V(L)| - 12$, because L is planar, and hence $s \geq k$, as desired. $\square$

3 Rural societies

If P is a path and $x, y \in V(P)$, we denote by xPy the unique subpath of P with ends x and y. Let (G, Ω) be a society. An *orderly transaction* in (G, Ω) is a sequence of k pairwise disjoint bumps $\mathcal{T} = (P_1, \ldots, P_k)$ such that P_i has ends u_i and v_i and $u_1, u_2, \ldots, u_k, v_k, v_{k-1}, \ldots, v_1$ is clockwise. Let M be the graph obtained from $P_1 \cup P_2 \cup \cdots \cup P_k$ by adding the vertices of $V(\Omega)$ as isolated vertices. We say that M is the *frame* of $\mathcal{T}$. We say that a path Q in G is *$\mathcal{T}$-coterminal* if Q has both ends in $V(\Omega)$ and is otherwise disjoint from it and for every $i = 1, 2, \dots, k$ the following holds: if Q intersects P_i, then their intersection is a path whose one end is a common end of Q and P_i.

Let (G, Ω) be a society, and let M and $\mathcal{T}$ be as in the previous paragraph. Let $i \in \{1, 2, \dots, k\}$ and let Q be a $\mathcal{T}$-coterminal path in $G \backslash V(P_i)$ with one end in $v_i \Omega u_i$ and the other end in $u_i \Omega v_i$. In those circumstances we say that Q is a *$\mathcal{T}$-jump over P_i*, or simply a *$\mathcal{T}$-jump*.

Now let $i \in \{0, 1, \dots, k\}$ and let Q_1, Q_2 be two disjoint $\mathcal{T}$-coterminal paths such that Q_j has ends x_j, y_j and $(u_i, x_1, x_2, u_{i+1}, v_{i+1}, y_1, y_2, v_i)$ is clockwise in Ω, where possibly $u_i = x_1$, $x_2 = u_{i+1}$, $v_{i+1} = y_1$, or $y_2 = v_i$, and u_0 means x_1, u_{k+1} means x_2, v_{k+1} means y_1, and v_0 means y_2. In those circumstances we say that (Q_1, Q_2) is a *$\mathcal{T}$-cross in region i*, or simply a *$\mathcal{T}$-cross*.

Finally, let $i \in \{1, 2, \dots, k\}$ and let Q_0, Q_1, Q_2 be three paths such that Q_j has ends x_j, y_j and is otherwise disjoint from all members of $\mathcal{T}$, $x_0, y_0 \in V(P_i)$, the vertices x_1, x_2 are internal vertices of $x_0 P_i y_0$, $y_1, y_2 \notin V(P_i)$, $y_1 \in u_{i-1} \Omega u_i \cup v_i \Omega v_{i-1}$, $y_2 \in u_i \Omega u_{i+1} \cup v_{i+1} \Omega v_i$, and the paths Q_0, Q_1, Q_2 are pairwise disjoint, except possibly $x_1 = x_2$. In those circumstances we say that (Q_0, Q_1, Q_2) is a *$\mathcal{T}$-tunnel under P_i*, or simply a *$\mathcal{T}$-tunnel*.

Intuitively, if we think of the paths in $\mathcal{T}$ as dividing the society into "regions", then a $\mathcal{T}$-jump arises from a $\mathcal{T}$-path whose ends do not belong to the same region. A $\mathcal{T}$-cross arises from two $\mathcal{T}$-paths with ends in the same region that cross inside that region, and furthermore, each path in $\mathcal{T}$ includes at most two ends of those crossing paths. Finally, a $\mathcal{T}$-tunnel can be converted into a $\mathcal{T}$-jump by rerouting P_i along Q_0. However, in some applications such rerouting will be undesirable, and therefore we need to list $\mathcal{T}$-tunnels as outcomes.

Let M be a subgraph of a graph G. An *M-bridge* in G is a connected subgraph B of G such that $E(B) \cap E(M) = \emptyset$ and either $E(B)$ consists of a unique edge with both ends in M, or for some component C of $G \backslash V(M)$ the set $E(B)$ consists of all edges of G with at least one end in $V(C)$. The vertices in $V(B) \cap V(M)$ are called the *attachments* of B. Now let M be such that no block of M is a cycle. By a *segment of M* we mean a maximal subpath P of M such that every internal vertex of P has degree two in M. It follows that the segments of M are uniquely determined. Now if B is an M-bridge of G, then we say that B is *unstable* if some segment of M includes all the attachments of B, and otherwise we say that B is *stable*.

A society (G, Ω) is *rurally 4-connected* if for every separation (A, B) of order at most three with $V(\Omega) \subseteq A$ the graph $G[B]$ can be drawn in a disk with the vertices of $A \cap B$ drawn on the boundary of the disk. A society is *cross-free* if it has no cross. The following, a close relative of Lemma 2.3, follows from [10, Theorem 2.4].

Theorem 3.1 *Every cross-free rurally 4-connected society is rural.*

Lemma 3.2 *Let (G,Ω) be a rurally 4-connected society, let $\mathcal{T} = (P_1,\ldots,P_k)$ be an orderly transaction in (G,Ω), and let M be the frame of $\mathcal{T}$. If every M-bridge of G is stable and (G,Ω) is not rural, then (G,Ω) has a $\mathcal{T}$-jump, a $\mathcal{T}$-cross, or a $\mathcal{T}$-tunnel.*

Proof. For $i = 1,2,\ldots,k$ let u_i and v_i be the ends of P_i numbered as in the definition of orderly transaction, and for convenience let P_0 and P_{k+1} be null graphs. We define $k+1$ cyclic permutations $\Omega_0,\Omega_1,\ldots,\Omega_k$ as follows. For $i = 1,2,\ldots,k-1$ let $V(\Omega_i) := V(P_i) \cup V(P_{i+1}) \cup u_i\Omega u_{i+1} \cup v_{i+1}\Omega v_i$ with the cyclic order defined by saying that $u_i\Omega u_{i+1}$ is followed by $V(P_{i+1})$ in order from u_{i+1} to v_{i+1}, followed by $v_{i+1}\Omega v_i$ followed by $V(P_i)$ in order from v_i to u_i. The cyclic permutation Ω_0 is defined by letting $v_1\Omega u_1$ be followed by $V(P_1)$ in order from u_1 to v_1, and Ω_k is defined by letting $u_k\Omega v_k$ be followed by $V(P_k)$ in order from v_k to u_k.

Now if for some M-bridge B of G there is no index $i \in \{0,1,\ldots,k\}$ such that all attachments of B belong to $V(\Omega_i)$, then (G,Ω) has a $\mathcal{T}$-jump. Thus we may assume that such index exists for every M-bridge B, and since B is stable that index is unique. Let us denote it by $i(B)$. For $i = 0,1,\ldots,k$ let G_i be the subgraph of G consisting of $P_i \cup P_{i+1}$, the vertex-set $V(\Omega_i)$ and all M-bridges B of G with $i(B) = i$. The society (G_i,Ω_i) is rurally 4-connected. If each (G_i,Ω_i) is cross-free, then each of them is rural by Theorem 3.1 and it follows that (G,Ω) is rural. Thus we may assume that for some $i = 0,1,\ldots,k$ the society (G_i,Ω_i) has a cross (Q_1,Q_2). If neither P_i nor P_{i+1} includes three or four ends of the paths Q_1 and Q_2, then (G,Ω) has a $\mathcal{T}$-cross. Thus we may assume that P_i includes both ends of Q_1 and at least one end of Q_2. Let x_j, y_j be the ends of Q_j. Since the M-bridge of G containing Q_2 is stable, it has an attachment outside P_i, and so if needed, we may replace Q_2 by a path with an end outside P_i (or conclude that (G,Ω) has a $\mathcal{T}$-jump). Thus we may assume that u_i, x_1, x_2, y_1, v_i occur on P_i in the order listed, and $y_2 \notin V(P_i)$.

The M-bridge of G containing Q_1 has an attachment outside P_i. If it does not include Q_2 and has an attachment outside $V(P_i) \cup \{y_2\}$, then (G,Ω) has a $\mathcal{T}$-jump or $\mathcal{T}$-cross, and so we may assume not. Thus there exists a path Q_3 with one end x_3 in the interior of Q_1 and the other end $y_3 \in V(Q_2) - \{x_2\}$ with no internal vertex in $M \cup Q_1 \cup Q_2$. We call the triple (Q_1,Q_2,Q_3) a *tripod*, and the path $y_3Q_2y_2$ the *leg* of the tripod. If v is an internal vertex of $x_1P_iy_1$, then we say that v is *sheltered* by the tripod (Q_1,Q_2,Q_3). Let L be a path that is the leg of some tripod, and subject to that L is minimal. From now on we fix L and will consider different tripods with leg L; thus the vertices x_1, y_1, x_2, x_3 may change, but y_2 and y_3 will remain fixed as the ends of L.

Let $x_1', y_1' \in V(P_i)$ be such that they are sheltered by no tripod with leg L, but every internal vertex of $x_1'P_iy_1'$ is sheltered by some tripod with leg L. Let X' be the union of $x_1'P_iy_1'$ and all tripods with leg L that shelter some internal vertex of $x_1'P_iy_1'$, let $X :=$

$X'\backslash V(L)\backslash\{x'_1, y'_1\}$ and let $Y := V(M \cup L) - x'_1 P_i y'_1 - \{y_3\}$. Since (G, Ω) is rurally 4-connected we deduce that the set $\{x'_1, y'_1, y_3\}$ does not separate X from Y in G. It follows that there exists a path P in $G\backslash\{x'_1, y'_1, y_3\}$ with ends $x \in X$ and $y \in Y$. We may assume that P has no internal vertex in $X \cup Y$. Let (Q_1, Q_2, Q_3) be a tripod with leg L such that either x is sheltered by it, or $x \in V(Q_1 \cup Q_2 \cup Q_3)$. If $y \notin V(L \cup P_i)$, then by considering the paths P, Q_1, Q_2, Q_3 it follows that either (G, Ω) has a $\mathcal{T}$-jump or $\mathcal{T}$-tunnel. If $y \in V(L)$, then there is a tripod whose leg is a proper subpath of L, contrary to the choice of L. Thus we may assume that $y \in V(P_i)$, and that $y \in V(P_i)$ for every choice of the path P as above. If $x \in V(Q_1 \cup Q_2 \cup Q_3)$ then there is a tripod with leg L that shelters x'_1 or y'_1, a contradiction. Thus $x \in V(P_i)$. Let B be the M-bridge containing P. Since $y \in V(P_i)$ for all choices of P it follows that the attachments of B are a subset of $V(P_i) \cup \{y_2\}$. But B is stable, and hence y_2 is an attachment of B. The minimality of L implies that B includes a path from y to y_3, internally disjoint from L. Using that path and the paths P, Q_1, Q_2, Q_3 it is now easy to construct a tripod that shelters either x'_1 or y'_1, a contradiction. $\square$

4 Leap of length five

A *leap of length* k in a society (G, Ω) is a sequence of $k+1$ pairwise disjoint bumps $P_0, P_1, \ldots, P_k$ such that P_i has ends u_i and v_i and $u_0, u_1, u_2, \ldots, u_k, v_0, v_k, v_{k-1}, \ldots, v_1$, is clockwise. In this section we prove the following.

Theorem 4.1 *Let (G, Ω) be a 6-connected society with a leap of length five. Then (G, Ω) is nearly rural, or G has a triangle C such that $(G\backslash E(C), \Omega)$ is rural, or (G, Ω) has three crossed paths, a gridlet, a separated doublecross, or a turtle.*

The following is a hypothesis that will be common to several lemmas of this section, and so we state it separately to avoid repetition.

Hypothesis 4.2 Let (G, Ω) be a society with no three crossed paths, a gridlet, a separated doublecross, or a turtle, let $k \geq 1$ be an integer, let

$$(u_0, u_1, u_2, \ldots, u_k, v_0, v_k, v_{k-1}, \ldots, v_1)$$

be clockwise, and let $P_0, P_1, \ldots, P_k$ be pairwise disjoint bumps such that P_i has ends u_i and v_i. Let $\mathcal{T}$ be the orderly transaction $(P_1, P_2, \ldots, P_k)$, let M be the frame of $\mathcal{T}$ and let

$$Z = u_1 \Omega u_k \cup v_k \Omega v_1 \cup V(P_2) \cup V(P_3) \cup \cdots \cup V(P_{k-1}) - \{u_1, u_k, v_1, v_k\}.$$

Let $Z_1 = v_1 \Omega u_1 - \{u_0, u_1, v_1\}$ and $Z_2 = u_k \Omega v_k - \{v_0, u_k, v_k\}$.

If H is a subgraph of G, then an *H-path* is a (possibly trivial) path with both ends in $V(H)$ and otherwise disjoint from H. This is somewhat non-standard, typically an H-path is required to have at least one edge, but we use our definition for convenience. We say that a vertex v of P_0 is *exposed* if there exists an $(M \cup P_0)$-path P with one end v and the other in Z.

Lemma 4.3 *Assume Hypothesis 4.2 and let $k \geq 3$. Let R_1, R_2 be two disjoint $(M \cup P_0)$-paths in G such that R_i has ends $x_i \in V(P_0)$ and $y_i \in V(M) - \{u_0, v_0\}$, and assume that u_0, x_1, x_2, v_0 occur on P_0 in the order listed, where possibly $u_0 = x_1$, or $v_0 = x_2$, or both. Then either $y_1 \in V(P_1) \cup v_1\Omega u_1$, or $y_2 \in V(P_k) \cup u_k\Omega v_k$, or both. In particular, there do not exist two disjoint $(M \cup P_0)$-paths from $V(P_0)$ to Z.*

Proof. The second statement follows immediately from the first, and so it suffices to prove the first statement. Suppose for a contradiction that there exist paths R_1, R_2 satisfying the hypotheses but not the conclusion of the lemma. By using the paths $P_2, P_3, \ldots, P_{k-1}$ we conclude that there exist two disjoint paths Q_1, Q_2 in G such that Q_i has ends $x_i \in V(P_0)$ and $z_i \in V(\Omega)$, and is otherwise disjoint from $V(P_0) \cup V(\Omega)$, and if Q_i intersects some P_j for $j \in \{1, 2, \ldots, k\}$, then $j \in \{2, \ldots, k-1\}$ and $Q_i \cap P_j$ is a path one of whose ends is a common end of Q_i and P_j. Furthermore, $z_1 \in u_1\Omega v_1 - \{u_1, v_1\}$ and $z_2 \in v_k\Omega u_k - \{u_k, v_k\}$. From the symmetry we may assume that either (u_0, v_0, z_2, z_1), or (u_0, z_1, v_0, z_2) or (u_0, v_0, z_1, z_2) is clockwise. In the first two cases (G, Ω) has a separated doublecross (the two pairs of crossing bumps are P_1 and $Q_1 \cup u_0P_0x_1$, and P_k and $Q_2 \cup v_0P_0x_2$, and the fifth path is a subpath of P_2), unless the second case holds and $z_1 \in u_k\Omega v_0$ or $z_2 \in v_1\Omega u_0$, or both. By symmetry we may assume that $z_1 \in u_k\Omega v_0$. Then, if $z_2 \in v_{k-2}\Omega u_0$, (G, Ω) has a gridlet formed by the paths $P_k, P_{k-1}, u_0P_0x_1 \cup Q_1$ and $v_0P_0x_2 \cup Q_2$. Otherwise, $z_2 \in v_k\Omega v_{k-2} - \{v_k, v_{k-2}\}$ and (G, Ω) has a turtle with legs P_k and $v_0P_0x_2 \cup Q_2$, neck P_1 and body $u_0P_0x_2 \cup Q_1$.

Finally, in the third case (G, Ω) has a turtle or three crossed paths. More precisely, if $z_2 \in v_0\Omega v_1 - \{v_1\}$, then (G, Ω) has a turtle described in the paragraph above. Otherwise, by symmetry, we may assume that $z_2 \in v_1\Omega u_0$ and $z_1 \in v_0\Omega v_k$, in which case $v_0P_0x_2 \cup Q_2$, $u_0P_0x_1 \cup Q_1$ and P_2 are the three crossed paths. $\square$

Lemma 4.4 *Assume Hypothesis 4.2 and let $k \geq 2$. Then $(G\backslash V(P_0), \Omega\backslash V(P_0))$ has no $\mathcal{T}$-jump.*

Proof. Suppose for a contradiction that $(G\backslash V(P_0), \Omega\backslash V(P_0))$ has a $\mathcal{T}$-jump. Thus there is an index $i \in \{1, 2, \ldots, k\}$ and a $\mathcal{T}$-coterminal path P in $G\backslash V(P_0 \cup P_i)$ with ends $x \in v_i\Omega u_i$ and $y \in u_i\Omega v_i$. Let $j \in \{1, 2, \ldots, k\} - \{i\}$. Then using the paths P_0, P_i, P_j and P we deduce that (G, Ω) has either three crossed paths or a gridlet, in either case a contradiction. $\square$

Lemma 4.5 *Assume Hypothesis 4.2 and let $k \geq 2$. Let $v \in V(P_0)$ be such that there is no $(M \cup P_0)$-path in $G \backslash v$ from vP_0v_0 to $vP_0u_0 \cup V(P_1 \cup P_2 \cup \cdots \cup P_{k-1}) \cup v_k\Omega u_k - \{v_k, u_k\}$ and none from vP_0u_0 to $V(P_2 \cup P_3 \cup \cdots \cup P_k) \cup u_1\Omega v_1 - \{u_1, v_1\}$. Then $(G\backslash v, \Omega\backslash v)$ has no $\mathcal{T}$-jump.*

Proof. The hypotheses of the lemma imply that every $\mathcal{T}$-jump in $(G\backslash v, \Omega\backslash v)$ is disjoint from P_0. Thus the lemma follows from Lemma 4.4. □

Lemma 4.6 *Assume Hypothesis 4.2, let $k \geq 3$, and let $v \in V(P_0)$ be such that no vertex in $V(P_0) - \{v\}$ is exposed. Let $i \in \{0, 1, \ldots, k\}$ be such that $(G\backslash v, \Omega\backslash v)$ has a $\mathcal{T}$-cross (Q_1, Q_2) in region i. Then $i \in \{0, k\}$ and v is not exposed. Furthermore, assume that $i = 0$, and that there exists an $(M \cup P_0)$-path Q with one end v and the other end in $P_1 \cup v_1\Omega u_1 - \{u_0\}$, and that v_0P_0v is disjoint from $Q_1 \cup Q_2$. Then for some $j \in \{1, 2\}$ there exist $p \in V(Q_j \cap u_0P_0v)$ and $q \in V(Q_j \cap Q)$ such that pP_0v and qQv are internally disjoint from $Q_1 \cup Q_2$.*

Proof. If $i \notin \{0, k\}$, then the $\mathcal{T}$-cross is disjoint from P_0 by the choice of v, and hence the $\mathcal{T}$-cross and P_0 give rise to three crossed paths. To complete the proof of the first assertion we may assume that $i = 0$ and that v is exposed. Subject to these assumptions we choose Q_1 and Q_2 so that $Q_1 \cup Q_2 \cup P_0$ is minimal. Since v is exposed there exists a $\mathcal{T}$-coterminal path Q' from v to $y \in Z \cap V(\Omega)$ disjoint from $P_0 \cup P_1 \cup P_k\backslash v$. Let $Q'' = Q' \cup vP_0v_0$. If $Q'' \cap (Q_1 \cup Q_2) = \emptyset$ then (G, Ω) has a separated doublecross, where one pair of crossed paths is obtained from the $\mathcal{T}$-cross, the other pair is P_k and Q'', and the fifth path is a subpath of P_2. Thus we may assume that there exists $x \in V(Q'') \cap V(Q_j)$ for some $j \in \{1, 2\}$ and that x is chosen so that $xQ''y$ is internally disjoint from $Q_1 \cup Q_2$. For $r = 1, 2$ let $z_r \in v_1\Omega u_1 - \{v_1, u_1\}$ be an end of Q_r such that Q_{3-r} has one end in $z_r\Omega v_0$ and another in $v_0\Omega z_r$. If $x \in V(Q')$, then Q_j is disjoint from P_0, because v is the only exposed vertex and $v \notin V(Q_1) \cup V(Q_2)$. Thus $z_jQ_jx \cup xQ'y$ is a $\mathcal{T}$-jump disjoint from P_0, contrary to Lemma 4.4. It follows that $x \in V(v_0P_0v)$, and Q' is disjoint from $Q_1 \cup Q_2$.

Let $x' \in V(P_0) \cap (V(Q_1) \cup V(Q_2))$ be chosen so that $x'P_0v_0$ is internally disjoint from $Q_1 \cup Q_2$. Without loss of generality, we assume that $x' \in V(Q_1)$. Define $P_0' = v_0P_0x' \cup x'Q_1z_1$. Let $x'' \in vP_0u_0 \cap (V(P_0') \cup V(Q_2) \cup \{u_0\})$ be chosen so that vP_0x'' is internally disjoint from $P_0' \cup Q_2$. If $x'' \notin V(P_0')$ then the path $Q' \cup vP_0u_0$, if $x'' = u_0$, or the path $Q' \cup vP_0x'' \cup x''Q_2z_2$, if $x'' \in V(Q_2)$ is a $\mathcal{T}$-jump, disjoint from P_0', contradicting Lemma 4.4. (See Figure 5(a).) If $x'' \in V(P_0')$ then $x''P_0v \cup Q'$ and $Q_1\backslash(V(P_0) - \{x'\})$ are paths with one end in $V(P_0')$ and another in $V(\Omega)$, contradicting Lemma 4.3, after we replace P_0 by P_0' and P_1 by Q_2 in M. (See Figure 5(b).) This proves the first assertion of the lemma.

To prove the second statement of the lemma we assume that $i = 0$ and that Q is a path from v to $v' \in v_1\Omega u_1 - \{u_0\}$, disjoint from $M \cup P_0\backslash v$, except that $P_1 \cap Q$ may be a path with one end v'. Let the ends of Q_1, Q_2 be labeled as in the definition of $\mathcal{T}$-cross. If P_0

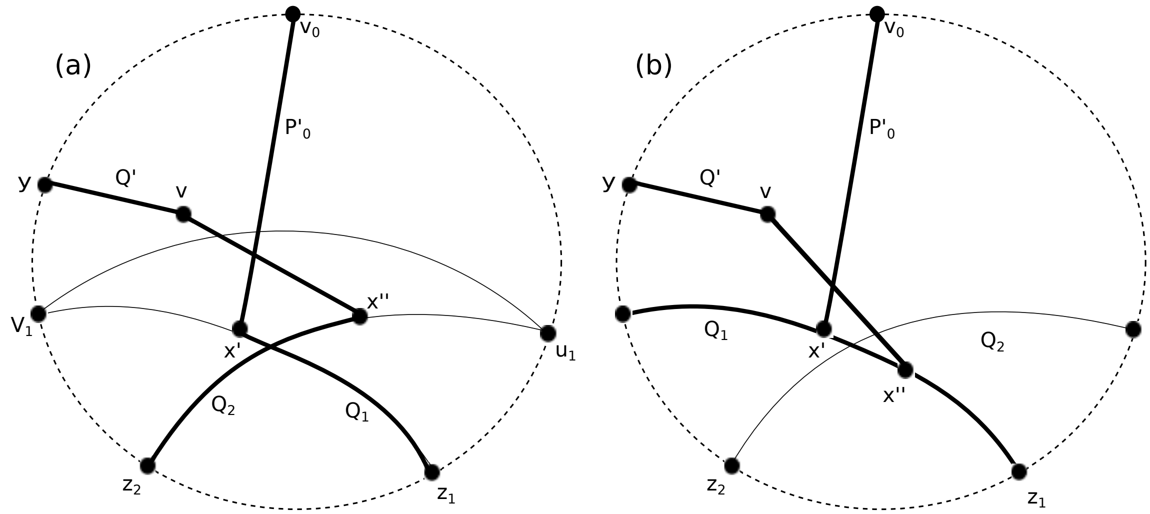

Figure 5: Configurations considered in the proof of the first assertion of Lemma 4.6.

is disjoint from $Q_1 \cup Q_2$, then (G, Ω) has three crossed paths (if (y_2, u_0, x_1) is clockwise) or a gridlet with paths Q_1, Q_2, P_0, P_2 (if (x_1, u_0, x_2) or (y_1, u_0, y_2) is clockwise), or a separated doublecross with paths Q_1, Q_2, P_0, P_2, P_k (if (v_1, u_0, y_1) or (x_2, u_0, u_1) is clockwise). Thus we may assume that P_0 intersects $Q_1 \cup Q_2$. (Please note that v_0P_0v is disjoint from $Q_1 \cup Q_2$ by hypothesis.) Similarly we may assume that Q intersects $Q_1 \cup Q_2$, for otherwise we apply the previous argument with P_0 replaced by $Q \cup vP_0v_0$. Let $p \in V(Q_1 \cup Q_2) \cap u_0P_0v$ and $q \in V(Q_1 \cup Q_2) \cap V(Q)$ be chosen to minimize pP_0v and qQv. If p and q belong to different paths Q_1, Q_2, then (G, Ω) has a turtle with legs Q_1, Q_2, neck P_k and body $pP_0v_0 \cup qQv$. Thus p and q belong to the same Q_j and the lemma holds. □

In the proof of the following lemma we will be applying Lemma 3.2. To guarantee that the conditions of Lemma 3.2 are satisfied, we will need a result from [5]. We need to precede the statement of this result by a few definitions.

Let M be a subgraph of a graph G, such that no block of M is a cycle. Let P be a segment of M of length at least two, and let Q be a path in G with ends $x, y \in V(P)$ and otherwise disjoint from M. Let M' be obtained from M by replacing the path xPy by Q; then we say that M' was obtained from M by *rerouting* P along Q, or simply that M' was obtained from M by *rerouting*. Please note that P is required to have length at least two, and hence this relation is not symmetric. We say that the rerouting is *proper* if all the attachments of the M-bridge that contains Q belong to P. The following is proved in [5, Lemma 2.1].

Lemma 4.7 *Let G be a graph, and let M be a subgraph of G such that no block of M is a cycle. Then there exists a subgraph M' of G obtained from M by a sequence of proper reroutings such that if an M'-bridge B of G is unstable, say all its attachments belong to* *a segment P of M', then there exist vertices $x, y \in V(P)$ such that some component of $G \backslash \{x, y\}$ includes a vertex of B and is disjoint from $M \backslash V(P)$.*

Lemma 4.8 *Assume Hypothesis 4.2, and let $k \geq 4$. If every leap of length $k-1$ has at most one exposed vertex, (G, Ω) is 4-connected and $(G\backslash v, \Omega\backslash v)$ is rurally 4-connected for every $v \in V(P_0)$, then (G, Ω) is nearly rural.*

Proof. Since (G, Ω) has no separated doublecross it follows that it does not have a $\mathcal{T}$-cross both in region 0 and region k. Thus we may assume that it has no $\mathcal{T}$-cross in region k. Similarly, it follows that it does not have a $\mathcal{T}$-tunnel under both P_1 and P_k, or a $\mathcal{T}$-cross in region 0 and a $\mathcal{T}$-tunnel under P_k. Thus we may also assume that (G, Ω) has no $\mathcal{T}$-tunnel under P_k. If some leap of length k in (G, Ω) has an exposed vertex, then we may assume that v is an exposed vertex. Otherwise, let the leap $(P_0, P_1, \ldots, P_k)$ and $v \in V(P_0)$ be chosen such that either $v = u_0$ or there exists an $(M \cup P_0)$-path with one end v and the other end in $P_1 \cup v_1\Omega u_1 - \{u_0\}$, and, subject to that, vP_0v_0 is as short as possible.

By Lemma 4.7 we may assume, by properly rerouting M if necessary, that every M-bridge of $G\backslash v$ is stable. Since the reroutings are proper the new paths P_i will still be disjoint from P_0, and the property that defines v will continue to hold. Similarly, the facts that there is no $\mathcal{T}$-cross in region k and no $\mathcal{T}$-tunnel under P_k remain unaffected. We claim that $(G\backslash v, \Omega\backslash v)$ is rural.

We apply Lemma 3.2 to the society $(G\backslash v, \Omega\backslash v)$ and orderly transaction $\mathcal{T}$. We may assume that $(G\backslash v, \Omega\backslash v)$ is not rural, and hence by Lemma 3.2 the society $(G\backslash v, \Omega\backslash v)$ has a $\mathcal{T}$-jump, a $\mathcal{T}$-cross or a $\mathcal{T}$-tunnel. By the choice of v there exists a path Q from v to $v' \in v_k\Omega u_k - \{v_k, u_k\}$ such that Q does not intersect $P_k \cup P_0\backslash v$ and intersects at most one of $P_1, P_2, \ldots, P_{k-1}$. Furthermore, if it intersects P_i for some $i \in \{1, 2, \ldots, k-1\}$ then $P_i \cap Q$ is a path with one end a common end of both. (If $v = u_0$ then we can choose Q to be a one vertex path.)

We claim that v satisfies the hypotheses of Lemma 4.5. To prove this claim suppose for a contradiction that P is an $(M \cup P_0)$-path violating that hypothesis. Suppose first that P and Q are disjoint. Then P joins different components of $P_0\backslash v$ by Lemma 4.3. But then changing P_0 to the unique path in $P_0 \cup P$ that does not use v either produces a leap with at least two exposed vertices, or contradicts the minimality of vP_0v_0. Thus P and Q intersect. Since no leap of length k has two or more exposed vertices, it follows that v is not exposed. Thus P has one end in u_0P_0v by the minimality of vP_0v_0, and the other end in $P_k \cup u_k\Omega v_k$, because v is not exposed. But then $P \cup Q$ includes a $\mathcal{T}$-jump disjoint from P_0, contrary to Lemma 4.4. This proves our claim that v satisfies the hypotheses of Lemma 4.5. We conclude that $(G\backslash v, \Omega\backslash v)$ has no $\mathcal{T}$-jump.

Assume now that $(G\backslash v, \Omega\backslash v)$ has a $\mathcal{T}$-cross (Q_1, Q_2) in region i for some integer $i \in \{0, 1, \ldots, k\}$. By the first part of Lemma 4.6 and the fact that there is no $\mathcal{T}$-cross in region

k it follows that $i = 0$ and v is not exposed. We have $v \neq u_0$, for otherwise $V(P_0) \cap (V(Q_1) \cup V(Q_2)) = \emptyset$ and either Q_1, Q_2, P_0 are three crossed paths, or Q_1, Q_2, P_0, P_3, P_2 is a separated double cross in (G, Ω). Since v is not exposed we deduce that Q satisfies the requirements of Lemma 4.6. By the first part of Lemma 4.6 and the assumption made earlier it follows that $i = 0$ and v is not exposed. But the existence of Q and the second statement of Lemma 4.6 imply that some leap of length k has at least two exposed vertices, a contradiction. (To see that let j, p, q be as in Lemma 4.6. Replace P_1 by Q_{3-j} and replace P_0 by a suitable subpath of $Q_j \cup pP_0v_0 \cup qQv$.)

We may therefore assume that $(G\backslash v, \Omega\backslash v)$ has a $\mathcal{T}$-tunnel (Q_0, Q_1, Q_2) under P_i for some $i \in \{1, 2, \ldots, k\}$. Then the leap $L' = (P_0, P_1, \ldots, P_{i-1}, P_{i+1}, \ldots, P_k)$ of length $k - 1 \geq 3$ has a $\mathcal{T}'$-cross, where $\mathcal{T}'$ is the corresponding orderly society, and the result follows in the same way as above. □

Lemma 4.9 *Assume Hypothesis 4.2 and let $k \geq 3$. If there exist at least two exposed vertices, then there exists a cycle C and three disjoint $(M \cup C)$-paths R_1, R_2, R_3 such that R_i has ends $x_i \in V(C)$ and $y_i \in V(M)$, $C\backslash\{x_1, x_2, x_3\}$ is disjoint from M, $y_1 = u_0$, $y_2 = v_0$ and $y_3 \in Z$.*

Proof. Let x_1 be the closest exposed vertex to u_0 on P_0, and let x_2 be the closest exposed vertex to v_0. Let $R_1 = x_1P_0u_0$ and let $R_2 = x_2P_0v_0$. For $i = 1, 2$ let S_i be an $(M \cup P_0)$-path with one end x_i and the other end in Z. By Lemma 4.3 S_1 and S_2 intersect, and so we may assume that $S_1 \cap S_2$ is a path R_3 containing an end of both S_1 and S_2, say y_3. Let x_3 be the other end of R_3. Then $P_0 \cup S_1 \cup S_2$ includes a unique cycle C. The cycle C and paths R_1, R_2, R_3 are as desired for the lemma. □

If the cycle C in Lemma 4.9 can be chosen to have at least four vertices, then we say that the leap $(P_0, P_1, \ldots, P_k)$ is *diverse.*

Lemma 4.10 *Assume Hypothesis 4.2, let $k \geq 4$, and let there be no diverse leap of length k. If C is as in Lemma 4.9 and $(G\backslash E(C), \Omega)$ is rurally 4-connected, then $(G\backslash E(C), \Omega)$ is rural.*

Proof. Since the leap $(P_0, P_1, \ldots, P_k)$ is not diverse, it follows that C is a triangle. Let R_1, R_2, R_3 and their ends be numbered as in Lemma 4.9. We may assume that $P_0 = R_1 \cup R_2 + x_1x_2$. Since there is no diverse leap, Lemma 4.3 implies that there is no path in $G\backslash E(C)\backslash V(P_k)$ from x_2 to $v_k\Omega u_k$, and none in $G\backslash E(C)\backslash V(P_1)$ from x_1 to $u_1\Omega v_1$. It also implies that no vertex on P_0 is exposed in $G\backslash x_1x_3\backslash x_2x_3$.

As in Lemma 4.8, we can apply Lemma 4.7 and assume, by properly rerouting M if necessary, that the conditions of Lemma 3.2 are satisfied. We assume that the society $(G\backslash E(C), \Omega)$ has a $\mathcal{T}$-jump, a $\mathcal{T}$-cross, or a $\mathcal{T}$-tunnel, as otherwise by Lemma 3.2 $(G\backslash E(C), \Omega)$ is rural.

By the observation at the end of the previous paragraph this $\mathcal{T}$-jump, $\mathcal{T}$-cross, or $\mathcal{T}$-tunnel cannot use both x_1 and x_2; say it does not use x_2. But that contradicts Lemma 4.5 or the first part of Lemma 4.6, applied to $v = x_2$ and the graph $G\backslash x_1x_3$, in case of a $\mathcal{T}$-jump or a $\mathcal{T}$-cross.

Thus we may assume that $(G\backslash E(C)\backslash x_2, \Omega\backslash x_2)$ has a $\mathcal{T}$-tunnel (Q_0, Q_1, Q_2) under P_i for some $i \in \{1, 2, \ldots, k\}$. But then the leap $L' = (P_0, P_1, \ldots, P_{i-1}, P_{i+1}, \ldots, P_k)$ of length $k - 1 \geq 3$ has a $\mathcal{T}'$-cross (Q'_1, Q'_2), where $\mathcal{T}'$ is the corresponding orderly transaction, Q'_1 is obtained from P_i by rerouting along Q_0 and Q'_2 is the union of $Q_1 \cup Q_2$ with the subpath of P_i joining the ends of Q_1 and Q_2. By the first half of Lemma 4.6 applied to the graph $G\backslash x_1x_3$, the leap L', $v := x_2$ and the $\mathcal{T}'$-cross (Q'_1, Q'_2) we may assume that $i = 1$ and that $y_3 \in v_2\Omega u_2 - \{u_0\}$. By the second half of Lemma 4.6 applied to the same entities and $Q := R_3 + x_3x_2$ there exist $j \in \{1, 2\}$, $p \in V(Q'_j \cap R_1)$ and $q \in V(Q'_j \cap Q)$ such that pP_0x_2 and qQx_2 are internally disjoint from $Q'_1 \cup Q'_2$. If $j = 1$, then p, q belong to the interior of Q_0, and the leap $(P_0, P_1, \ldots, P_k)$ is diverse, as a subpath of Q_0 joins a vertex of R_1 to a vertex of Q in $G\backslash x_1x_3$. If $j = 2$ then we obtain a diverse leap from $(P_0, P_1, \ldots, P_k)$ by replacing P_1 by Q'_1 and replacing P_0 by a suitable subpath of $Q \cup v_0P_0p \cup Q'_2$. $\square$

Lemma 4.11 *Assume Hypothesis 4.2, let $k \geq 3$, let (G, Ω) be 4-connected, let C, R_1, R_2, R_3 be as in Lemma 4.9, and assume that C is not a triangle. Then there exist four disjoint $(M \cup C)$-paths, each with one end in $V(C)$ and the other end respectively in the sets $\{u_0\}$, $\{v_0\}$, Z and $V(P_1 \cup P_k)$.*

Proof. By an application of the proof of the max-flow min-cut theorem there exist four disjoint $(M \cup C)$-paths, each with one end in $V(C)$ and the other end respectively in the sets $\{u_0\}$, $\{v_0\}$, Z and $V(M)$. By Lemma 4.3 the fourth path does not end in $V(M) - V(P_1) - V(P_k)$. The result follows. $\square$

Lemma 4.12 *Assume Hypothesis 4.2, let $k \geq 3$, let C, R_1, R_2, R_3 be as in Lemma 4.9, let $D := M \cup C \cup R_1 \cup R_2 \cup R_3$, and let R_4 be a D-path with ends $x_4 \in V(C) - \{x_1, x_2, x_3\}$ and $y_4 \in V(P_1)$. Then x_1, x_2, x_3, x_4 occur on C in the order listed. Furthermore, if R is a D-path with ends $x \in V(C) - \{x_1, x_2, x_3\}$ and $y \in V(M)$, then x_1, x_2, x_3, x occur on C in the order listed and $y \in V(P_1)$.*

Proof. The vertices x_1, x_2, x_3, x_4 occur on C in the order listed by Lemma 4.3. Now let R be as stated. By Lemma 4.3 we have $y \in V(P_1 \cup P_k)$, and so by the first part of the lemma we may assume that $y \in V(P_k)$. By the symmetric statement to the first half of the lemma it follows that x_1, x_2, x, x_3 occur on C in the order listed. We may assume that P_0 is the unique path from u_0 to v_0 in $R_1 \cup R_2 \cup C\backslash x_3$. Then $R_4 \cup R \cup C\backslash V(P_0)$ includes a $\mathcal{T}$-jump disjoint from P_0, contrary to Lemma 4.4. $\square$

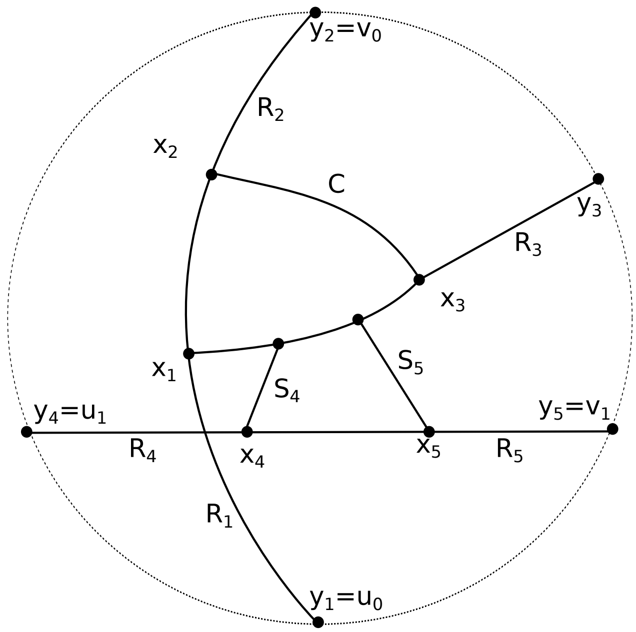

Figure 6: Hypothesis 4.13.

We need to further upgrade the assumptions of Hypothesis 4.2, as follows.

Hypothesis 4.13 Assume Hypothesis 4.2. Let C be a cycle with distinct vertices x_1, x_2, x_3 such that $C \backslash \{x_1, x_2, x_3\}$ is disjoint from M. Let R_1, R_2, R_3 be pairwise disjoint $(M \cup C)$-paths such that R_i has ends x_i and y_i, where $y_1 = u_0$, $y_2 = v_0$, and $y_3 \in Z$. By a *ray* we mean an $(M \cup C)$-path from C to M, disjoint from $R_1 \cup R_2 \cup R_3$. We say that a vertex $v \in V(P_1)$ is *illuminated* if there is a ray with end v. Let $x_4, x_5 \in V(P_1)$ be illuminated vertices such that either $x_4 = x_5$, or u_1, x_4, x_5, v_1 occur on P_1 in the order listed, and $x_4 P_1 x_5$ includes all illuminated vertices. Let $R_4 := u_1 P_1 x_4$ and $R_5 := v_1 P_1 x_5$, and let $y_4 := u_1$ and $y_5 := v_1$. Let S_4 and S_5 be rays with ends x_4 and x_5, respectively, and let $A_0 := V(M) - V(P_1)$ and $B_0 := V(C \cup S_4 \cup S_5 \cup x_4 P_1 x_5)$. (See Figure 6.)

Lemma 4.14 *Assume Hypothesis 4.13, let $k \geq 3$, and let (G, Ω) be 6-connected. Then $x_4 \neq x_5$, and the path $x_4 P_1 x_5$ has at least one internal vertex.*

Proof. If $x_4 = x_5$ or $x_4 P_1 x_5$ has no internal vertex, then by Lemma 4.12 the set $\{x_1, x_2, \ldots, x_5\}$ is a cutset separating C from $M \backslash V(P_1)$, contrary to the 6-connectivity of (G, Ω). Note that $V(C) - \{x_1, x_2, \ldots, x_5\}$ is non-empty as it includes an end of a ray. $\square$

Assume Hypothesis 4.13. By Lemma 4.14 the paths $R_1, R_2, \ldots, R_5$ are disjoint paths from A_0 to B_0. The following lemma follows by a standard "augmenting path" argument.

Lemma 4.15 *Assume Hypothesis 4.13, and let $k \geq 2$. If there is no separation (A, B) of order at most five with $A_0 \subseteq A$ and $B_0 \subseteq B$, then there exist an integer n and internally* *disjoint paths $Q_1, Q_2, \ldots, Q_n$ in G, where Q_i has distinct ends a_i and b_i such that*

(i) *$a_1 \in A_0 - \{y_1, y_2, \ldots, y_5\}$ and $b_n \in B_0 - \{x_1, x_2, \ldots, x_5\}$,*

(ii) *for all $i = 1, 2, \ldots, n-1$, $a_{i+1}, b_i \in V(R_t)$ for some $t \in \{1, 2, \ldots, 5\}$, and y_t, a_{i+1}, b_i, x_t are pairwise distinct and occur on R_t in the order listed,*

(iii) *if $a_i, b_j \in V(R_t)$ for some $t \in \{1, 2, \ldots, 5\}$ and $i, j \in \{1, 2, \ldots, 5\}$ with $i > j + 1$, then either $a_i = b_j$, or y_t, b_j, a_i, x_t occur on R_t in the order listed, and*

(iv) *for $i = 1, 2, \ldots, n$, if a vertex of Q_i belongs to $A_0 \cup B_0 \cup V(R_1 \cup R_2 \cup \cdots \cup R_5)$, then it is an end of Q_i.*

The sequence of paths $(Q_1, Q_2, \ldots, Q_n)$ as in Lemma 4.15 will be called an *augmenting sequence.*

Lemma 4.16 *Assume Hypothesis 4.13, and let $k \geq 3$. Then there is no augmenting sequence $(Q_1, Q_2, \ldots, Q_n)$, where Q_1 is disjoint from P_2.*

Proof. Suppose for a contradiction that there is an augmenting sequence $(Q_1, Q_2, \ldots, Q_n)$, where Q_1 is disjoint from P_2, and let us assume that the leap $(P_0, P_1, \ldots, P_k)$, cycle C, paths R_1, R_2, R_3, S_4, S_5 and augmenting sequence $(Q_1, Q_2, \ldots, Q_n)$ are chosen with n minimum. Let the ends of the paths Q_i be labeled as in Lemma 4.15. We may assume that P_0 is the unique path from u_0 to v_0 in $R_1 \cup R_2 \cup C \backslash x_3$. We proceed in a series of claims.

(1) *The vertex b_n belongs to the interior of $x_4 P_1 x_5$.*

To prove (1) suppose for a contradiction that $b_n \in V(C \cup S_4 \cup S_5)$. By Lemma 4.12, the choice of x_4, x_5 and the fact that $a_n \neq x_4, x_5$ by Lemma 4.15(ii) we deduce that $a_n \in V(R_i)$ for some $i \in \{1, 2, 3\}$. Then we can use Q_n to modify C to include $a_n R_i x_i$ (and modify R_1, R_2, R_3 accordingly), in which case $(Q_1, Q_2, \ldots, Q_{n-1})$ is an augmentation contradicting the choice of n. This proves (1).

(2) *$a_i, b_i \in V(R_j)$ for no $i \in \{1, 2, \ldots, n\}$ and no $j \in \{1, 2, \ldots, 5\}$.*

To prove (2) suppose to the contrary that $a_i, b_i \in V(R_j)$. Then $1 < i < n$ and by rerouting R_j along Q_i we obtain an augmentation $(Q_1, Q_2, \ldots, Q_{i-2}, Q_{i-1} \cup b_{i-1} R_j a_{i+1} \cup Q_{i+1}, Q_{i+2}, \ldots, Q_n)$, contrary to the minimality of n. This proves (2).

(3) *$a_i, b_i \in V(R_1 \cup R_2 \cup R_3)$ for no $i \in \{1, 2, \ldots, n\}$.*

Using (2) the proof of (3) is analogous to the argument at the end of the proof of Claim (1).

(4) *$a_i, b_i \in V(R_4 \cup R_5)$ for no $i \in \{1, 2, \ldots, n\}$.*

By (2) one of a_i, b_i belongs to R_4 and the other to R_5. We can reroute P_1 along Q_i, and then $(Q_1, Q_2, \ldots, Q_{i-1})$ becomes an augmentation, contrary to the minimality of n.

(5) *For $i = 1, 2, \ldots, n-1$, the graph $Q_i \cup R_1 \cup R_2 \cup R_3$ includes no $\mathcal{T}$-jump.*

This claim follows from (3), Lemma 4.3 and Lemma 4.4 applied to P_0.

(6) $a_1 \notin v_1 \Omega u_1$.

To prove (6) suppose for a contradiction that $a_1 \in v_1 \Omega u_1$. Since $a_1 \neq y_1$, we may assume from the symmetry that $a_1 \in v_1 \Omega y_1 - \{y_1\}$. Then $b_1 \in V(P_1 \cup R_1)$ by (5). But if $b_1 \in V(R_i)$, where $i = 1$ or $i = 5$, then by rerouting R_i along Q_1 we obtain an augmenting sequence $(Q_2 \cup x_1 R_i a_2, Q_3, Q_4, \ldots, Q_n)$, contrary to the choice of n. Thus $b_1 \in u_1 P_1 x_5$. By replacing P_1 by the path $Q_1 \cup u_1 P_1 b_1$ and considering the paths R_3 and $S_5 \cup R_5$ we obtain contradiction to Lemma 4.3. This proves (6).

(7) $a_1 \notin u_k \Omega v_k$.

Similarly as in the proof of (6), if $a_1 \in u_k \Omega v_k$, then $b_1 \in V(R_2)$ by (5), and we reroute R_2 along Q_1 to obtain a contradiction to the minimality of n. This proves (7).

(8) $a_1 \in V(P_k)$.

To prove (8) we may assume by (6) and (7) that $a_1 \in Z$. Then $b_1 \in V(R_3 \cup P_1)$ by (5). If $b_1 \in V(R_3)$, then we reroute R_3 along Q_1 as before. Thus $b_1 \in V(P_1)$. It follows from (5) and the hypothesis $V(P_2) \cap V(Q_1) = \emptyset$ that $a_1 \in u_1 \Omega u_2 - \{u_1, u_2\}$ or $a_1 \in v_2 \Omega v_1 - \{v_1, v_2\}$, and so from the symmetry we may assume the latter.

Let us assume for a moment that $y_3 \in a_1 \Omega v_1$. We reroute P_1 along $Q_1 \cup b_1 P_1 v_1$. The union of R_3, R_2 and a path in C between x_2 and x_3, avoiding x_1, x_4, x_5, will play the role of P_0 after rerouting. If $b_1 \in x_4 P_1 v_1 - \{x_4\}$, then $R_1 \cup C \cup S_4 \cup R_4$ includes two disjoint paths that contradict Lemma 4.3 applied to the new frame and new path P_0. Therefore $b_1 \in V(R_4)$, and hence $(u_1 P_1 a_2 \cup Q_2, Q_3, \ldots, Q_n)$ is an augmenting sequence after the rerouting, contrary to the choice of n.

It follows that $y_3 \notin a_1 \Omega v_1$. If $b_1 \in V(R_5)$, we replace P_1 by $Q_1 \cup u_1 P_1 b_1$; then $(v_1 P_1 a_2 \cup Q_2, Q_3, \ldots, Q_n)$ is an augmenting sequence that contradicts the choice of n. So it follows that $b_1 \in u_1 P_1 x_5$. But now (G, Ω) has a gridlet using the paths P_0, P_k, $Q_1 \cup u_1 P_1 b_1$ and a subpath of $R_5 \cup S_5 \cup R_3 \cup C \backslash V(P_0)$. This proves (8).

(9) $n > 1$.

To prove (9) suppose for a contradiction that $n = 1$. Thus b_1 belongs to the interior of $x_4 P x_5$ by (1), and $a_1 \in V(P_k)$ by (8). But then Q_1 is a $\mathcal{T}$-jump, contrary to (5).

(10) $b_1 \in V(R_3)$.

To prove (10) we first notice that $b_1 \in V(R_2 \cup R_3)$ by (5), (9) and (1). Suppose for a contradiction that $b_1 \in V(R_2)$. Then $a_2 \in V(R_2)$, but $b_2 \notin V(R_1 \cup R_2 \cup R_3)$ by (3) and $b_2 \notin V(P_1)$ by (5), a contradiction. This proves (10).

Let P_{12} and P_{34} be two disjoint subpaths of C, where the first has ends x_1, x_2, and the second has ends x_3, x_4. By (8) and (10) the path $Q_1 \cup b_1 R_3 x_3 \cup P_{34} \cup S_4$ is a $\mathcal{T}$-jump disjoint from $R_1 \cup P_{12} \cup R_2$, contrary to Lemma 4.4. □

We are now ready to prove Theorem 4.1.

Proof of Theorem 4.1. Let (G, Ω) be a 6-connected society with a leap of length five. Thus we may assume that Hypothesis 4.2 holds for $k = 5$. By Lemma 4.8 either (G, Ω) is nearly rural, in which case the theorem holds, or there exists a leap of length at least four with at least two exposed vertices. Thus we may assume that there exists a leap of length four with at least two exposed vertices. Let C be a cycle as in Lemma 4.9. If there is no diverse leap, then C is a triangle, $(G \backslash E(C), \Omega)$ is rurally 4-connected and hence rural by Lemma 4.10, and the theorem holds. Thus we may assume that the cycle C is not a triangle, and so by Lemma 4.11 we may assume that Hypothesis 4.13 for $k = 4$ holds. By Lemma 4.14 and the 6-connectivity of G there is no separation (A, B) as described in Lemma 4.15, and hence by that lemma there exists an augmenting sequence $(Q_1, Q_2, \ldots, Q_n)$. By Lemma 4.16 the path Q_1 intersects P_2, and hence Q_1 is disjoint from P_3, contrary to Lemma 4.16 applied to the leap (P_0, P_1, P_3, P_4) of length three and an augmenting sequence $(Q_1', Q_2, \ldots, Q_n)$, where Q_1' is the union of Q_1 and $a_1 P_2 u_2$ or $a_1 P_2 v_2$. □

5 Societies of bounded depth

Let (G, Ω) be a society. A *linear decomposition* of (G, Ω) is an enumeration $\{t_1, \ldots, t_n\}$ of $V(\Omega)$ where $(t_1, \ldots, t_n)$ is clockwise, together with a family $(X_i : 1 \leq i \leq n)$ of subsets of $V(G)$, with the following properties:

(i) $\bigcup(G[X_i] : 1 \leq i \leq n) = G$,

(ii) for $1 \leq i \leq n$, $t_i \in X_i$, and

(iii) for $1 \leq i \leq i' \leq i'' \leq n$, $X_i \cap X_{i''} \subseteq X_{i'}$.

The *depth* of such a linear decomposition is

$$\max(|X_i \cap X_{i'}| : 1 \leq i < i' \leq n),$$

and the depth of (G, Ω) is the minimum depth of a linear decomposition of (G, Ω). Theorems (6.1), (7.1) and (8.1) of [10] imply the following.

Theorem 5.1 *There exists an integer d such that every 4-connected society (G, Ω) either has a separated doublecross, three crossed paths or a leap of length five, or some planar truncation of (G, Ω) has depth at most d.*

In light of Theorems 4.1 and 5.1, in the remainder of the paper we concentrate on societies of bounded depth. We need a few definitions. Let (G, Ω) be a society, let $u_1, u_2, \ldots, u_{4t}$ be clockwise in Ω, and let $P_1, P_2, \ldots, P_{2t}$ be disjoint bumps in G such that for $i = 1, 2, \ldots, 2t$ the path P_{2i-1} has ends u_{4i-3} and u_{4i-1}, and the path P_{2i} has ends u_{4i-2} and u_{4i}. In those circumstances we say that (G, Ω) has t *disjoint consecutive crosses*.

Now let $u_1, v_1, w_1, u_2, v_2, w_2, \ldots, u_t, v_t, w_t$ be clockwise in Ω, let $x \in V(G) - \{u_1, v_1, w_1, \ldots, u_t, v_t, w_t\}$, for $i = 1, 2, \ldots, t$ let P_i be a path in $G \backslash x$ with ends u_i and w_i and otherwise disjoint from $V(\Omega)$, let Q_i be a path with ends x and v_i and otherwise disjoint from $V(\Omega)$, and assume that the paths P_i and Q_i are pairwise disjoint, except that the paths Q_i meet at x. Let W be the union of all the paths P_i and Q_i. We say that W is a *windmill with t vanes*, and that the graph $P_i \cup Q_i$ is a *vane* of the windmill.

Finally, let $u_1, u_2, \ldots, u_t$ and $v_1, v_2, \ldots, v_t$ be vertices of $V(\Omega)$ such that for all $x_i \in \{u_i, v_i\}$ the sequence $x_1, x_2, \ldots, x_t$ is clockwise in Ω. Let $z_1, z_2 \in V(G) - \{u_1, v_1, \ldots, u_t, v_t\}$ be distinct, for $i = 1, 2, \ldots, t$ let P_i be a path in $G \backslash z_2$ with ends z_1 and u_i and otherwise disjoint from $V(\Omega)$, and let Q_i be a path in $G \backslash z_1$ with ends z_2 and v_i and otherwise disjoint from $V(\Omega)$. Assume that the paths P_i and Q_j are disjoint, except that the P_i share z_1, the Q_i share z_2 and P_i and Q_i are allowed to intersect. Let F be the union of all the paths P_i and Q_i. Then we say that F is a *fan with t blades*, and we say that $P_i \cup Q_i$ is a *blade* of the fan. The vertices z_1 and z_2 will be called the *hubs* of the fan. In Section 8 we prove the following theorem.

Theorem 5.2 *For every two integers d and t there exists an integer k such that every* 6-*connected k-cosmopolitan society (G, Ω) of depth at most d contains one of the following:*

(1) *t disjoint consecutive crosses, or*

(2) *a windmill with t vanes, or*

(3) *a fan with t blades.*

Unfortunately, windmills and fans are nearly rural, and so for our application we need to improve Theorem 5.2. We need more definitions.

Let $x, u_i, v_i, w_i, P_i, Q_i$ be as in the definition of a windmill W with t vanes, let $a, b, c, d \in V(G)$ be such that $u_1, v_1, w_1, \ldots, u_t, v_t, w_t, a, b, c, d$ is clockwise in Ω, and let (P, Q) be a cross disjoint from W whose paths have ends in $\{a, b, c, d\}$. In those circumstances we say that $W \cup P \cup Q$ is a *windmill with t vanes and a cross*.

Now let u_i, v_i, P_i, Q_i be as in the definition of a fan F with t blades, and let $a, b, c, d \in V(\Omega)$ be such that all $x_i \in \{u_i, v_i\}$ the sequence $x_1, x_2, \ldots, x_t, a, b, c, d$ is clockwise in Ω. Let (P, Q) be a cross disjoint from F whose paths have ends in $\{a, b, c, d\}$. In those circumstances we say that $W \cup P \cup Q$ is a *fan with t blades and a cross*.

Let $z_1, z_2, u_i, v_i, P_i, Q_i$ be as in the definition of a fan F with t blades, and let $a_1, b_1, c_1, a_2, b_2, c_2 \in V(G)$ be such that all $x_i \in \{u_i, v_i\}$ the sequence $x_1, x_2, \ldots, x_t, a_1, b_1, c_1, a_2, b_2, c_2$ is clockwise in Ω, except that we permit $c_1 = a_2$. For $i = 1, 2$ let L_i be a path in $G \backslash V(F)$ with ends a_i and c_i and otherwise disjoint from $V(\Omega)$, and let S_i be a path with ends z_i and b_i and otherwise disjoint from $V(F) \cup V(\Omega)$. If the paths L_1, L_2, S_1, S_2 are pairwise disjoint, except possibly for L_1 intersecting L_2 at $c_1 = a_2$, then we say that $F \cup L_1 \cup L_2 \cup S_1 \cup S_2$ is a *fan with t blades and two jumps*.

Now let u_i, v_i, P_i, Q_i be as in the definition of a fan F with $t+1$ blades, and let $a, b \in V(\Omega)$ be such that all $x_i \in \{u_i, v_i\}$ the sequence $x_1, x_2, \ldots, x_t, a, x_{t+1}, b$ is clockwise in Ω. Let P be a path in $G \backslash V(F)$ with ends a and b, and otherwise disjoint from $V(F)$. We say that $F \cup P$ is a *fan with t blades and a jump*. In Section 9 we improve Theorem 5.2 as follows.

Theorem 5.3 *For every two integers d and t there exists an integer k such that every 6-connected k-cosmopolitan society (G, Ω) of depth at most d is either nearly rural, or contains one of the following:*

(1) *t disjoint consecutive crosses, or*

(2) *a windmill with t vanes and a cross, or*

(3) *a fan with t blades and a cross, or*

(4) *a fan with t blades and a jump, or*

(5) *a fan with t blades and two jumps.*

For $t = 4$ each of the above outcomes gives a turtle, and hence we have the following immediate corollary.

Corollary 5.4 *For every integer d there exists an integer k such that every 6-connected k-cosmopolitan society (G, Ω) of depth at most d is either nearly rural, or has a turtle.*

The next four sections are devoted to proofs of Theorems 5.2 and 5.3. The proof of Theorem 5.2 will be completed in Section 8 and the proof of Theorem 5.3 will be completed in Section 9. At that time we will be able to deduce Theorem 1.8.

6 Crosses and goose bumps

In this section we prove that a society (G, Ω) either satisfies Theorem 5.2, or it has many disjoint bumps. If X is a set and Ω is a cyclic permutation, we define $\Omega \backslash X$ to be $\Omega|(V(\Omega) - X)$. Let $P_1, P_2, \ldots, P_k$ be a set of pairwise disjoint bumps in (G, Ω), where P_i has ends u_i and v_i and $u_1, v_1, u_2, v_2, \ldots, u_k, v_k$ is clockwise in Ω. In those circumstances we say that $P_1, P_2, \ldots, P_k$ is a *goose bump in (G, Ω) of strength k.*

Lemma 6.1 *Let b, d and t be positive integers, and let (G, Ω) be a society of depth at most d. Then either (G, Ω) has a goose bump of strength b, or there is a set $X \subseteq V(G)$ of size at most $(b-1)d$ such that the society $(G \backslash X, \Omega \backslash X)$ has no bump.*

Proof. Let $(t_1, t_2, \ldots, t_n)$ and $(X_1, X_2, \ldots, X_n)$ be a linear decomposition of (G, Ω) of depth at most d, and for $i = 1, 2, \ldots, n-1$ let $Y_i = X_i \cap X_{i+1}$. If P is a bump in (G, Ω), then the axioms of a linear decomposition imply that

$$I_P := \{i \in \{1, 2, \ldots, n-1\} : Y_i \cap V(P) \neq \emptyset\}$$

is a nonempty subinterval of $\{1, 2, \ldots, n-1\}$. It follows that either there exist bumps $P_1, P_2, \ldots, P_b$ such that $I_{P_1}, I_{P_2}, \ldots, I_{P_b}$ are pairwise disjoint, or there exists a set $I \subseteq \{1, 2, \ldots, n-1\}$ of size at most $b-1$ such that $I \cap I_P \neq \emptyset$ for every bump P. In the former case $P_1, P_2, \ldots, P_b$ is a desired goose bump, and in the latter case the set $X := \bigcup_{i \in I} Y_i$ is as desired. $\square$

The proof of the following lemma is similar and is omitted.

Lemma 6.2 *Let t and d be positive integers, and let (G, Ω) be a society of depth at most d. Then either (G, Ω) has t disjoint consecutive crosses, or there is a set $X \subseteq V(G)$ of size at most $(t-1)d$ such that the society $(G \backslash X, \Omega \backslash X)$ is cross-free.*

Lemma 6.3 *Let d, b, t be positive integers, let $k \geq (b-1)d + (t-1)\binom{(b-1)d}{2} + 1$ and let (G, Ω) be a 3-connected society of depth at most d such that at least k vertices in $V(\Omega)$ have at least two neighbors in $V(G)$. Then (G, Ω) has either a fan with t blades, or a goose bump of strength b.*

Proof. By Lemma 6.1 we may assume that there exists a set $X \subseteq V(G)$ of size at most $(b-1)d$ such that $(G \backslash X, \Omega \backslash X)$ has no bump. There are at least $(t-1)\binom{(b-1)d}{2} + 1$ vertices in $V(\Omega) - X$ with at least two neighbors in $V(G)$. Let v be one such vertex, and let H be the component of $G \backslash X$ containing v. Since $(G \backslash X, \Omega \backslash X)$ has no bumps it follows that $V(H) \cap V(\Omega) = \{v\}$. By the fact that v has at least two neighbors in G (if $V(H) = \{v\}$) or the 3-connectivity of (G, Ω) (if $V(H) \neq \{v\}$) it follows that H has at least two neighbors in X. Thus there exist distinct vertices z_1, z_2 such that for at least t vertices of $v \in V(\Omega) - X$ the component of $G \backslash X$ containing v has z_1 and z_2 as neighbors. It follows that (G, Ω) has a fan with t blades, as desired. $\square$

7 Intrusions, invasions and wars

Let Ω be a cyclic permutation. A *base* in Ω is a pair (X, Y) of subsets of $V(\Omega)$ such that $|X \cap Y| = 2$, $X \cup Y = V(\Omega)$ and for distinct elements $x_1, x_2 \in X$ and $y_1, y_2 \in Y$ the sequence (x_1, y_1, x_2, y_2) is not clockwise. Now let (G, Ω) be a society. A separation (A, B) of G is called an *intrusion* in (G, Ω) if there exists a base (X, Y) in Ω such that $X \subseteq A$, $Y \subseteq B$ and there exist disjoint paths $(P_v)_{v \in A \cap B}$, each with one end in X, the other end in Y and with

$v \in V(P_v)$. The intrusion (A, B) is *minimal* if there is no intrusion (A', B') of order $|A \cap B|$ with base (X, Y) such that A' is a proper subset of A. The paths P_v will be called *longitudes* for the intrusion (A, B). We say that (A, B) is *based* at (X, Y), and that (X, Y) is a base for (A, B). An intrusion (A, B) in (G, Ω) is an *invasion* if $|A \cap B \cap V(\Omega)| = 2$.

Lemma 7.1 *Let d be a positive integer, and let (G, Ω) be a society of depth at most $d - 1$. Then for every base (X, Y) in Ω there exists an intrusion of order at most $2d$ based at (X, Y).*

Proof. Let $(t_1, t_2, \ldots, t_n)$ and $(X_1, X_2, \ldots, X_n)$ be a linear decomposition of (G, Ω) of depth at most $d - 1$, and let $X \cap Y = \{t_i, t_j\}$. Let $i', j' \in \{1, 2, \ldots, n\}$ be such that $|i - i'| = |j - j'| = 1$, and let $Z := (X_i \cap X_{i'}) \cup (X_j \cap X_{j'}) \cup \{t_i, t_j\}$. It follows from the axioms of a linear decomposition that $|Z| \leq 2d$ and that Z separates X from Y in G. Thus there exists a separation (A, B) of G of order at most $2d$ with $X \subseteq A$ and $Y \subseteq B$. Any such separation (A, B) with $|A \cap B|$ minimum is as desired by Menger's theorem. □

An intrusion (A, B) in a society (G, Ω) is *t-separating* if (G, Ω) has goose bumps $P_1, P_2, \ldots, P_t$ and $Q_1, Q_2, \ldots, Q_t$ such that $V(P_i) \subseteq A - B$ and $V(Q_i) \subseteq B - A$ for all $i = 1, 2, \ldots, t$.

Lemma 7.2 *Let d, s, t be positive integers, and let (G, Ω) be a society of depth at most $d-1$ with a goose bump of strength $t(s + 2d)$. Then there exist s-separating minimal intrusions $(A_1, B_1), (A_2, B_2), \ldots, (A_t, B_t)$ of order at most $2d$ such that $A_i \cap A_j \subseteq B_i \cap B_j$ for all pairs of distinct indices $i, j = 1, 2, \ldots, t$.*

Proof. Let $\mathcal{P}$ be the set of paths comprising a goose bump of strength $t(s+2d)$. Thus there exist bases $(X_1, Y_1), (X_2, Y_2), \ldots (X_t, Y_t)$ such that the sets X_i are pairwise disjoint and for each $i = 1, 2, \ldots, t$ exactly $s+2d$ of the paths in $\mathcal{P}$ have both ends in X_i. By Lemma 7.1 there exists, for each $i = 1, 2, \ldots, t$, an intrusion (A_i, B_i) of order at most $2d$ based at (X_i, Y_i).

Let us choose, for each $i = 1, 2, \ldots, t$, an intrusion (A_i, B_i) of order at most $2d$ based at (X_i, Y_i) in such a way that

$$\sum_{i=1}^{t} |A_i| \text{ is minimum.} \tag{1}$$

We claim that $A_i \cap A_j \subseteq B_i \cap B_j$. To prove the claim suppose to the contrary that say $x \in A_1 \cap A_2 - B_1 \cap B_2$. Let

$$\begin{aligned} A'_1 &= A_1 \cap B_2, \\ B'_1 &= A_2 \cup B_1, \\ A'_2 &= A_2 \cap B_1, \\ B'_2 &= A_1 \cup B_2. \end{aligned}$$

Then (A'_1, B'_1) and (A'_2, B'_2) are separations of G with $X_1 \subseteq A'_1$, $Y_1 \subseteq B'_1$, $X_2 \subseteq A'_2$ and $Y_2 \subseteq B'_2$. We have

$$|A_1 \cap B_1| + |A_2 \cap B_2| = |A'_1 \cap B'_1| + |A'_2 \cap B'_2|.$$

Furthermore, since each longitude for (A_1, B_1) intersects $A_1' \cap B_1'$ we deduce that $|A_1' \cap B_1'| \geq |A_1 \cap B_1|$, and similarly $|A_2' \cap B_2'| \geq |A_2 \cap B_2|$. Thus the last two inequalities hold with equality, and hence the longitudes for (A_1, B_1) are also longitudes for (A_1', B_1'), and the longitudes for (A_2, B_2) are longitudes for (A_2', B_2'). It follows that for $i = 1, 2$ the separation (A_i', B_i') is an intrusion in (G, Ω) based at (X_i, Y_i) of order $|A_i \cap B_i|$. Since $A_1 \cap A_2 - (B_1 \cap B_2) = (A_1 \cap A_2 - B_1) \cup (A_1 \cap A_2 - B_2)$ we may assume that $x \in A_1 - B_2$. But then replacing (A_1, B_1) by (A_1', B_1') produces a set of intrusions that contradict (1). This proves our claim that $A_i \cap A_j \subseteq B_i \cap B_j$ for all distinct integers $i, j = 1, 2, \ldots, t$.

Since at most $2d$ of the paths in $\mathcal{P}$ with ends in X_i can intersect $A_i \cap B_i$, we deduce that each intrusion (A_i, B_i) is s-separating. Moreover, each (A_i, B_i) is clearly minimal by (1). $\square$

We need a lemma about subsets of a set.

Lemma 7.3 *Let d and t be positive integers, and let $\mathcal{F}$ be a family of $2^{\binom{d+1}{2}} t^d$ distinct subsets of a set S, where each member of $\mathcal{F}$ has size at most d. Then there exist a set $X \subset S$ of size at most $\binom{d+1}{2}$ and a family $\mathcal{F}' \subseteq \mathcal{F}$ of size at least t such that $F \cap F' \subseteq X$ for every two distinct sets $F, F' \in \mathcal{F}'$.*

Proof. We proceed by induction on $d + t$. If $d = 1$ or $t = 1$, then the lemma clearly holds, and so we may assume that $d, t > 1$. Let $F_0 \in \mathcal{F}$ be minimal with respect to inclusion. If $\mathcal{F}$ has a subfamily $\mathcal{F}_1$ of at least $2^{\binom{d+1}{2}}(t-1)^d$ sets disjoint from F_0, then the result follows from the induction hypothesis applied to $\mathcal{F}_1$ and by adding F_0 to the family thus obtained. If the family $\mathcal{F}_2 = \{F - F_0 : F \in \mathcal{F}, F \cap F_0 \neq \emptyset\}$ includes at least $2^{\binom{d}{2}} t^{d-1}$ distinct sets, then the result follows from the induction hypothesis applied to $\mathcal{F}_2$ by adding F_0 to the set thus obtained. Thus we may assume neither of the two cases holds. Thus

$$|\mathcal{F}| \leq 2^{\binom{d+1}{2}}(t-1)^d - 1 + 2^d 2^{\binom{d}{2}} t^{d-1} - 1 + 1 < 2^{\binom{d+1}{2}} t^d,$$

a contradiction. $\square$

Lemma 7.4 *Let d, s, t be positive integers, and let (G, Ω) be a society of depth at most $d - 1$ with a goose bump of strength $2^{\binom{2d+1}{2}} t^{2d}(s + 2d)$. Then there exist a set $X \subseteq V(G)$ of size at most $\binom{2d+1}{2}$ and s-separating intrusions $(A_1, B_1), (A_2, B_2), \ldots, (A_t, B_t)$ in $(G \backslash X, \Omega \backslash X)$ such that $A_i \cap A_j = \emptyset$ for all pairs of distinct indices $i, j = 1, 2, \ldots, t$.*

Proof. Let $T = 2^{\binom{2d+1}{2}} t^{2d}$. By Lemma 7.2 there exist s-separating minimal intrusions $(A_1, B_1), (A_2, B_2), \ldots, (A_T, B_T)$ of order at most $2d$ such that $A_i \cap A_j \subseteq B_i \cap B_j$ for all pairs of distinct indices $i, j = 1, 2, \ldots, t$. By Lemma 7.3 applied to the sets $A_i \cap B_i$ there exist a set $X \subseteq \bigcup_{i=1}^{T}(A_i \cap B_i)$ of size at most $\binom{2d+1}{2}$ and a subset of t of those intrusions, say $(A_1, B_1), (A_2, B_2), \ldots, (A_t, B_t)$, such that $A_i \cap B_i \cap A_j \cap B_j \subseteq X$ for all distinct integers $i, j = 1, 2, \ldots, t$. It follows that $(A_i - X, B_i - X)$ are as required for $(G \backslash X, \Omega \backslash X)$. $\square$

Our next objective is to prove, albeit with weaker bounds, that the conclusion of Lemma 7.4 can be strengthened to assert that the intrusions (A_i, B_i) therein are actually invasions.

Let (A, B) be an intrusion in a society (G, Ω) based at (X, Y). A path P in $G[A]$ is a *meridian* for (A, B) if its ends are the two vertices of $X \cap Y$. If P is a meridian for (A, B) and $(L_v)_{v \in A \cap B}$ are longitudes for (A, B), then the graph $(P \cup \bigcup_{v \in A \cap B} L_v) \backslash (B - A)$ is called a *frame* for (A, B).

Lemma 7.5 *Let λ and s be positive integers, let $s' = (s-1)(\lambda-1)+1$, let (G, Ω) be a cross-free society, and let (A, B) be an s'-separating minimal intrusion in (G, Ω) of order at most λ. Then there exists an s-separating minimal invasion (C, D) in (G, Ω) of order at most λ with a frame F such that $V(F) - V(\Omega) \subseteq A$.*

Proof. We may assume that

(1) *there is no integer $\lambda' \leq \lambda$ and an $((s-1)(\lambda'-1)+1)$-separating minimal intrusion (A', B') in (G, Ω) of order at most λ' with A' a proper subset of A,*

for if (A', B') exists, and it satisfies the conclusion of the lemma, then so does (A, B). We first show that (A, B) has a meridian. Indeed, suppose not. Let (X, Y) be a base of (A, B) and let $X \cap Y = \{u, v\}$; then $G[A]$ has no u-v path. Since (G, Ω) is cross-free it follows that $G[A]$ has a separation (A_1, A_2) of order zero such that both $X_1 = X \cap A_1$ and $X_2 = X \cap A_2$ are intervals in Ω. It follows that there exist Y_1, Y_2 such that (X_1, Y_1) and (X_2, Y_2) are bases. Thus $(A_1, A_2 \cup B \cup (X_1 \cap Y_1))$ and $(A_2, A_1 \cup B \cup (X_2 \cap Y_2))$ are minimal intrusions, and one of them violates (1). This proves that (A, B) has a meridian.

Let M be a meridian in (A, B), let $(L_v)_{v \in A \cap B}$ be a collection of longitudes for (A, B) and let $F = M \cup \bigcup_{v \in A \cap B} (L_v \backslash (B - A))$. By the same argument that justifies (1) we may assume that

(2) *there is no integer $\lambda' < \lambda$ and an $((s-1)(\lambda'-1)+1)$-separating minimal intrusion (A', B') in (G, Ω) of order at most λ' with frame F' such that $F' \backslash V(\Omega)$ is a subgraph of F.*

We claim that $|A \cap B \cap V(\Omega)| = 2$. We first prove that $A \cap B \cap X = \{u, v\}$. To this end suppose for a contradiction that $w \in A \cap B \cap X - \{u, v\}$; then w divides X into two cyclic intervals X_1 and X_2 with ends u, w and w, v, respectively. Let Y_1 and Y_2 be the complementary cyclic intervals so that (X_1, Y_1) and (X_2, Y_2) are bases.

For $i = 1, 2$ let A_i consist of w and all vertices $a \in A$ such that there exists a path in $G[A] \backslash w$ with one end a and the other end in $X_i - \{w\}$, and let $A_3 = A - A_1 - A_2$. It follows that $A_1 \cap A_2 = \{w\}$, for if P is a path in $G[A] \backslash w$ with one end in X_1 and the other end in X_2, then (P, P_w) is a cross in (G, Ω), a contradiction. Thus $(A_1, A_2 \cup A_3 \cup B)$ and $(A_2, A_1 \cup A_3 \cup B)$ are minimal intrusions based on (X_1, Y_1) and (X_2, Y_2), respectively, with $A_1, A_2 \subseteq A$. Thus one of them violates (2).

Next we show that $|A \cap B \cap Y| = 2$, and so we suppose for a contradiction that there exists $z \in A \cap B \cap Y - \{u, v\}$. We define $B_1, B_2, B_3, X_1, Y_1, X_2, Y_2$ analogously as in the previous paragraph, but with the roles of A and B reversed. Similarly we find that one of $(A \cup B_1 \cup B_3, B_2)$ and $(A \cup B_2 \cup B_3, B_1)$ is an $((s-1)(\lambda'-1)+1)$-separating minimal intrusion in (G, Ω) of order at most λ', for some $\lambda' < \lambda$, and so from the symmetry we may assume that $(A \cup B_1 \cup B_3, B_2)$ has this property. Since (M, P_z) is not a cross in (G, Ω) it follows that M and P_z intersect. Thus $M \cup P_z$ includes a meridian for $(A \cup B_1 \cup B_3, B_2)$. Finally, since $Z = B_2 \cap (A \cup B_1 \cup B_3) \subseteq A \cap B$, the paths $(L_v)_{v \in Z}$ form longitudes for $(A \cup B_1 \cup B_3, B_2)$, contrary to (2).

Thus we have shown that $A \cap B \cap V(\Omega) = \{u, v\}$. Let Z be the set of all vertices $z \in A$ such that there is no path in $G[A]$ with one end z and the other end in X, let $C = A - Z$ and $D = B \cup Z$. Then (C, D) is an intrusion with $C \cap D = A \cap B$ and F is a frame for (C, D) with $V(F) - V(\Omega) \subseteq C$. Since the order of (C, D) is at least two, it satisfies the conclusion of the lemma. $\square$

We are ready to deduce the main result of this section. By a *war* in a society (G, Ω) we mean a set $\mathcal{W}$ of minimal invasions such that each invasion in $\mathcal{W}$ has a meridian, and $A \cap A' = \emptyset$ for every two distinct invasions $(A, B), (A', B') \in \mathcal{W}$. We say that the war $\mathcal{W}$ is *s-separating* if each invasion in $\mathcal{W}$ is s-separating, we say $\mathcal{W}$ has *order at most* λ if each member of $\mathcal{W}$ has order at most λ, and we say that $\mathcal{W}$ is a *war of intensity* $|\mathcal{W}|$.

Lemma 7.6 *Let s, t and d be positive integers, and let $b = 2^{\binom{2d+1}{2}}(2dt)^{2d}(s(2d-1)+2)$. Then if a cross-free society (G, Ω) of depth at most $d-1$ has a goose bump of strength b, then it has a set X of at most $\binom{2d+1}{2}$ vertices such that the society $(G \backslash X, \Omega \backslash X)$ has an s-separating war of intensity t and order order at most $2d$.*

Proof. Let $s' = (2d-1)(s-1)+1$. By Lemma 7.4 there exist a set $X \subseteq V(G)$ with at most $\binom{2d+1}{2}$ elements and s'-separating intrusions $(A_1, B_1), (A_2, B_2), \ldots, (A_{2dt}, B_{2dt})$ in $(G \backslash X, \Omega \backslash X)$ of order at most $2d$ such that $A_i \cap A_j = \emptyset$ for every pair $i, j = 1, 2, \ldots, 2dt$ of distinct integers. By $2dt$ applications of Lemma 7.5 there exist, for each $i = 1, 2, \ldots, 2dt$, and s-separating minimal invasion (C_i, D_i) in $(G \backslash X, \Omega \backslash X)$ of order at most $2d$ with a frame F_i such that $V(F_i) - V(\Omega) \subseteq V(A_i)$. Let M_i be a meridian for (C_i, D_i), and let (X_i, Y_i) be the base for (C_i, D_i). Since (G, Ω) has depth at most d there exists a set $I \subseteq \{1, 2, \ldots, 2dt\}$ of size t such that the sets $\{X_i\}_{i \in I}$ are pairwise disjoint. By symmetry we may assume that $I = \{1, 2, \ldots, t\}$. We claim that $(C_1, D_1), (C_2, D_2), \ldots, (C_t, D_t)$ are as desired. To prove the claim suppose for a contradiction that say $x \in C_i \cap C_j$. Since (C_i, D_i) is an invasion there exists a path in $G[C_i]$ from x to $X_i \subseteq Y_j$; therefore this path intersects $C_j \cap D_j$. Thus there exists a vertex $v \in C_j \cap D_j \cap C_i$; let L be the longitude of F_j that includes v. But L connects $v \in C_i$ to a vertex of $X_j \subseteq Y_i \subseteq D_i$, and hence intersects $C_i \cap D_i \subseteq V(F_i)$. Thus F_i and F_j

intersect. But $V(F_i)\cap V(F_j)-V(\Omega)\subseteq A_i\cap A_j=\emptyset$ and $V(F_i)\cap V(F_j)\cap V(\Omega)\subseteq X_i\cap X_j=\emptyset$, a contradiction. Thus $(C_1,D_1),(C_2,D_2),\ldots,(C_t,D_t)$ satisfy the conclusion of the lemma. $\square$

8 Using wars

Lemma 8.1 *Let l,t,r be positive integers such that $r\geq (t-1)\binom{l}{2}+1$, let (G,Ω) be a connected society, and let $Z\subseteq V(G)$ be a set of size at most l such that the society $(G\backslash Z,\Omega\backslash Z)$ has a war $\mathcal{W}$ of intensity r such that for every $(A,B)\in\mathcal{W}$ at least two distinct members of Z have at least one neighbor in A. Then (G,Ω) has a fan with t blades.*

Proof. There exist distinct vertices $z_1,z_2\in Z$ and a subset $\mathcal{W}'$ of $\mathcal{W}$ of size t such that for every $(A,B)\in\mathcal{W}'$ both z_1 and z_2 have a neighbor in A. Furthermore, since (A,B) is a minimal intrusion, it follows that for every vertex $a\in A$ there exists a path in $G[A]$ from a to $V(\Omega)$. It follows that (G,Ω) has a fan with t blades, as desired. $\square$

Let (A,B) be an invasion in a cross-free society (G,Ω), based at (X,Y), and let $(L_v)_{v\in A\cap B}$ be longitudes for (A,B). Let Ω' be a cyclic permutation in A defined as follows: for each $u\in Y$, if u is an end of L_v, then we replace u by v, and otherwise we delete u. Then $(G[A],\Omega')$ is a society, and we will call it the *society induced by* (A,B). Since (G,Ω) is cross-free the definition does not depend on the choice of longitudes for (A,B).

Assume now that $(G[A],\Omega')$ is rural. A path P in $G[A]$ is called a *perimeter path* in $(G[A],\Omega')$ if $A\cap B\subseteq V(P)$ and $G[A]$ has a drawing in a disk with vertices of Ω' appearing on the boundary of the disk in the order specified by Ω' and with every edge of P drawn in the boundary of the disk.

The next lemma is easy and we omit its proof.

Lemma 8.2 *Let (A,B) be an invasion with longitudes $\{P_v\}_{v\in A\cap B}$ in a cross-free society (G,Ω). Then the society induced by (A,B) is cross-free.*

Lemma 8.3 *Let (G,Ω) be a 5-connected society, let $Z\subseteq V(G)$ be such that $(G\backslash Z,\Omega\backslash Z)$ is cross-free, and let (A,B) be an invasion in $(G\backslash Z,\Omega\backslash Z)$. If at most one vertex of Z has a neighbor in A, then the society induced in $(G\backslash Z,\Omega\backslash Z)$ by (A,B) is rural and has a perimeter path.*

Proof. Let $(G[A],\Omega')$ be the society induced in $(G\backslash Z,\Omega\backslash Z)$ by (A,B). By Lemma 8.2 it is cross-free and by Theorem 3.1 it is rural. Thus it has a drawing in a disk Δ with $V(\Omega')$ drawn on the boundary of Δ in the order specified by Ω'. When Δ is regarded as a subset of the plane, the unbounded face of $G[A]$ is bounded by a walk W. Let P be a subwalk of W containing $A\cap B$. If P is not a path, then it has a repeated vertex, say x, and $G[A]$ has a separation (C,D) with $C\cap D=\{x\}$ and $A\cap B\cap V(\Omega)\subseteq C$. Since $(G[A],\Omega')$ is cross-free, the latter inclusion implies that $D - C$ is disjoint from $V(\Omega)$ or from $A \cap B$. However, the latter is impossible, which can be seen by considering the drawing of $G[A]$ in Δ. Thus $(D-C) \cap V(\Omega) = \emptyset$, and since (A, B) has longitudes we deduce that $|(D-C) \cap A \cap B| \leq 1$. Let $z \in Z$ be such that no vertex of $Z - \{z\}$ has a neighbor in A. Since (G, Ω) is 4-connected, the fact that $((D-C) \cap A \cap B) \cup \{x, z\}$ does not separate G implies that $D - C$ consists of a unique vertex, say d, and $d \in A \cap B$. Furthermore, the only neighbor of d in A is x. But then $(A - \{d\}, B \cup \{x\})$ contradicts the minimality of (A, B). This proves that P is a path, and it follows that it is a perimeter path for $(G[A], \Omega')$. $\square$

Let (G, Ω) be a society. A set $\mathcal{T}$ of bumps in (G, Ω) is called a *transaction in* (G, Ω) if there exist elements $u, v \in V(\Omega)$ such that each member of $\mathcal{T}$ has one end in $u\Omega v$ and the other end in $V(\Omega) - u\Omega v$. The first part of the next lemma is easy, and the second part is proved in [10, Theorem (8.1)].

Lemma 8.4 *Let (G, Ω) be a society, and let $d \geq 1$ be an integer. If (G, Ω) has depth d, then it has no transaction of cardinality exceeding $2d$. Conversely, if (G, Ω) has no transaction of cardinality exceeding d, then it has depth at most d.*

Lemma 8.5 *Let (G, Ω) be a society of depth d, and let $X \subseteq V(G)$. Then the society $(G \backslash X, \Omega \backslash X)$ has depth at most $2d$.*

Proof. By Lemma 8.4 the society (G, Ω) has no transaction of cardinality exceeding $2d$. Then clearly $(G \backslash X, \Omega \backslash X)$ has no transaction of cardinality exceeding $2d$, and hence has depth at most $2d$ by another application of Lemma 8.4. $\square$

We need one last lemma before we can prove Theorem 5.2. The lemma we need is concerned with the situation when a society of bounded depth "almost" has a windmill with t vanes, except that the paths P_i are not necessarily disjoint and their ends do not necessarily appear in the right order. We begin with a special case when the ends of the paths P_i do appear in the right order.

Lemma 8.6 *Let $t \geq 1$ be an integer, and let $\rho = d(t-1)(t'-1)+1$, where $t' = d(t-1)^2+t$. Let (G, Ω) be a society of depth d, let $(u_1, z_1, v_1, u_2, z_2, v_2, \ldots, u_\rho, z_\rho, v_\rho)$ be clockwise, let $z \in V(G)$, for $i = 1, 2, \ldots, \rho$ let P_i be a bump with ends u_i and v_i, and let Q_i be a path of length at least one with ends z and z_i disjoint from $V(\Omega) - \{z, z_i\}$. Assume that the paths Q_i are pairwise disjoint except for z, and that each is disjoint from every P_j. Then (G, Ω) has either a windmill with t vanes, or a fan with t blades.*

Proof. By the proof of Lemma 6.1 applied to the paths P_i either some t of those paths are vertex-disjoint, in which case (G, Ω) has a windmill with t vanes, or there exists a set $X \subseteq V(G)$ of size at most $(t-1)d$ such that each P_i uses at least one vertex of X. We may therefore assume the latter. For $i = 1, 2, \ldots, \rho$ the path P_i has a subpath P_i' with one end u_i, the other end $x_i \in X$ and no internal vertex in X. Thus there exist $x \in X$ and a set $I \subseteq \{1, 2, \ldots, \rho\}$ of size t' such that $x = x_i$ for all $i \in I$. Let H be the union of all P_i' over $i \in I$. By an application of Lemma 6.1 to the graph $H \backslash x$ we deduce that either $H \backslash x$ has a goose bump of strength t, in which case (G, Ω) has a windmill with t vanes, or H has a set Y of size at most $(t-1)d$ such that $H \backslash Y \backslash x$ has no bumps. In the latter case for each $i \in I$ there is a path P_i'' in H with one end u_i, the other end $y_i \in Y \cup \{x\}$ and otherwise disjoint from $Y \cup \{x\}$. Thus there is a vertex $y \in Y \cup \{x\}$ and a set $J \subseteq I$ of size t such that $y_i = y$ for every $i \in J$. Since $H \backslash Y \backslash x$ has no bumps it follows that P_j'' and $P_{j'}''$ share only y for distinct $j, j' \in J$. Thus (G, Ω) has a fan with t blades, as desired. $\square$

Now we are ready to prove the last lemma in full generality.

Lemma 8.7 *Let $t \geq 1$ be an integer, and let $\xi = (d+1)\rho$, where ρ is as in Lemma 8.6. Let (G, Ω) be a society of depth d, let $z \in V(G)$, for $i = 1, 2, \ldots, \xi$ let (u_i, z_i, v_i) be clockwise, and let $(u_1, z_1, u_2, z_2, \ldots, u_\xi, z_\xi)$ be clockwise. Let P_i be a bump with ends u_i and v_i, and let Q_i be a path of length at least one with ends z and z_i disjoint from $V(\Omega) - \{z, z_i\}$. Assume that the paths Q_i are pairwise disjoint except for z, and that each is disjoint from every P_j. Then (G, Ω) has either a windmill with t vanes, or a fan with t blades.*

Proof. Let $(t_1, t_2, \ldots, t_n)$ be a clockwise enumeration of $V(\Omega)$, and let $(X_1, X_2, \ldots, X_n)$ be a corresponding linear decomposition of (G, Ω) of depth d. Let us fix an integer $i = 1, 2, \ldots, \rho$, and let $I = \{(i-1)(d+1)+1, (i-1)(d+1)+2, \ldots, i(d+1)\}$. For each such i we will construct paths P_i^* and Q_i^* satisfying the hypothesis of Lemma 8.6. In the construction we will make use of the paths P_j and Q_j for $j \in I$.

If $(u_j, z_j, v_j, u_{i(d+1)+1})$ is clockwise for some $j \in I$, then we put $P_i^* = P_j$ and $Q_i^* = Q_j$. Otherwise, letting s be such that $t_s = u_{i(d+1)}$, we deduce that P_j intersects $X_{t_s} \cap X_{t_{s+1}}$ for all $j \in I$. Since $|I| > |X_{t_s} \cap X_{t_{s+1}}|$ it follows that there exist $j < j' \in I$ such that P_j and $P_{j'}$ intersect. Let P_i^* be a subpath of $P_j \cup P_{j'}$ with ends u_j and $u_{j'}$, and let $Q_i^* = Q_j$.

This completes the construction. The lemma follows from Lemma 8.6. $\square$

Proof of Theorem 5.2. Let the integers d and t be given, let ξ be as in Lemma 8.7, let $\ell = 2(t-1)d + \binom{4d+2}{2}$, let $\tau = (t-1)\binom{\ell}{2} + \left(2(t-1)d + \binom{8d+2}{2}\right)(6\xi - 1) + 1$, let b be as in Lemma 7.6 with $s = 1$, $t = \tau$ and d replaced by $4d+1$, and let k be as in Lemma 6.3 applied to b, t, and $4d$. We will prove that k satisfies the conclusion of the theorem.

To that end let (G, Ω) be a k-cosmopolitan society of depth at most d, and let (G_0, Ω_0) be a planar truncation of (G, Ω). Let $S \subseteq V(\Omega_0)$. We say that S is *sparse* if whenever $u_1, u_2 \in S$ are such that there does not exist $w \in S$ such that (u_1, w, u_2) is clockwise, then there exist two disjoint bumps P_1, P_2 in (G_0, Ω_0) such that u_i is an end of P_i. The reader should notice that if H is one of the graphs listed as outcomes (1)-(3) of Theorem 5.2, then $V(H) \cap V(\Omega_0)$

is sparse. We say that (G_0,Ω_0) is *weakly linked* if for every sparse set $S \subseteq V(\Omega_0)$ there exist $|S|$ disjoint paths from S to $V(\Omega)$ with no internal vertex in $V(G_0)$. Thus if the conclusion of the theorem holds for some weakly linked truncation of (G_0,Ω_0), then it holds for (G,Ω) as well. Thus we may assume that (G_0,Ω_0) is a weakly linked truncation of (G,Ω) with $|V(G_0)|$ minimum. We will prove that (G_0,Ω_0) satisfies the conclusion of Theorem 5.2. Since (G_0,Ω_0) is weakly linked, Lemma 8.4 implies that (G_0,Ω_0) has no transaction of cardinality exceeding $2d$, and hence has depth at most $2d$ by Lemma 8.4.

By Lemma 6.2 there exists a set $Z_1 \subseteq V(G_0)$ such that $|Z_1| \leq 2(t-1)d$ and the society $(G_0\backslash Z_1, \Omega_1\backslash Z_1)$ is cross-free. By Lemma 8.5 the society $(G_0\backslash Z_1, \Omega_0\backslash Z_1)$ has depth at most $4d$. By Lemma 6.3 we may assume that $(G_0\backslash Z_1, \Omega_0\backslash Z_1)$ has a goose bump of strength b. By Lemma 7.6 there exists a set $Z_2 \subseteq V(G) - Z_1$ such that $|Z_2| \leq \binom{4d+2}{2}$ and in the society $(G_0\backslash Z, \Omega_0\backslash Z)$ there exists a 1-separating war $\mathcal{W}$ of intensity τ and order at most $8d+2$, where $Z = Z_1 \cup Z_2$. If there exist at least $(t-1)\binom{\ell}{2}+1$ invasions $(A,B) \in \mathcal{W}$ such that at least two distinct vertices of Z have a neighbor in A, then the theorem holds by Lemma 8.1. We may therefore assume that this is not the case, and hence $\mathcal{W}$ has a subset $\mathcal{W}'$ of size at least $|Z|(6\xi-1)+1$ such that for every $(A,B) \in \mathcal{W}'$ at most one vertex of Z has a neighbor in A.

Let $(A,B) \in \mathcal{W}'$ and let $z \in Z$ be such that no vertex in $Z - \{z\}$ has a neighbor in A. By Lemma 8.3 the society $(G_0[A], \Omega')$ induced in $(G_0\backslash Z, \Omega_0\backslash Z)$ by (A,B) is rural and has a perimeter path P. It follows that $(A \cup \{z\}, B \cup \{z\})$ is a separation of G_0. Let $A \cap B = \{w_0, w_1, \ldots, w_s\}$, and let L_i be the longitude containing w_i. Let the ends of L_i be $u_i \in A$ and $v_i \in B$. We may assume that $(u_0, u_1, \ldots, u_s)$ is clockwise. The vertices w_i divide P into paths $P_0, P_1, \ldots, P_s$, where P_i has ends w_{i-1} and w_i. We claim that no P_i includes all neighbors of z. Suppose for a contradiction that P_i does. Let (G,Ω) be the composition of (G_0,Ω_0) with a rural neighborhood (G_1,Ω,Ω_0). Let $G_1' = G_1 \cup G[A \cup \{z\}]$, let $G_0' = G_0\backslash(A-B)$ and let Ω_0' consist of $w_s\Omega w_0$ followed by $w_{s-1}, w_{s-2}, \ldots, w_i$ followed by z followed by $w_{i-1}, w_{i-2}, \ldots, w_1$. Since $(G[A], \Omega')$ is rural and all neighbors of z belong to P_i, it follows that $(G_1', \Omega, \Omega_0')$ is a rural neighborhood and (G,Ω) is the composition of (G_0', Ω_0') with this neighborhood. Thus (G_0', Ω_0') is a planar truncation of (G,Ω). We claim that (G_0', Ω_0') is weakly linked. To prove that let $S' \subseteq V(\Omega_0')$ be sparse. Since (A,B) is a minimal intrusion there exists a set $\mathcal{P}'$ of $|S'|$ disjoint paths from S' to $V(\Omega_0)$ with no internal vertex in G_0'; let S be the set of their ends in $V(\Omega_0)$. Since S' is sparse in (G_0', Ω_0'), it follows that S is sparse in (G_0,Ω_0). Since (G_0,Ω_0) is weakly linked there exists a set $\mathcal{P}$ of $|S|$ disjoint paths in G from S to $V(\Omega)$ with no internal vertex in G_0. By taking unions of members of $\mathcal{P}$ and $\mathcal{P}'$ we obtain a set of paths proving that (G_0', Ω_0') is weakly linked, as desired. Since $\mathcal{W}$ is 1-separating this contradicts the minimality of G_0, proving our claim that no P_i includes all neighbors of z. The same argument, but with $G_1' = G_1 \cup G[A]$ and Ω_0' not including z shows that z has a neighbor in $A - B$.

We have shown, in particular, that exactly one vertex of Z has a neighbor in $A-B$. Thus there exists a subset $\mathcal{W}''$ of $\mathcal{W}'$ of size 6ξ and a vertex $z \in Z$ such that for every $(A, B) \in \mathcal{W}''$ the vertex z has a neighbor in $A - B$. Now let $w = (A, B) \in \mathcal{W}''$, and let the notation be as before. We will construct paths P_w, Q_w such that the hypotheses of Lemma 8.7 will be satisfied for at least half the members $w \in \mathcal{W}''$.

The facts that (A, B) is a minimal intrusion and that z has a neighbor in $A - B$ imply that there exists a path Q_w in $G[A \cup \{z\}]$ from z to $z_w \in V(\Omega_0) \cap A$ and a choice of longitudes $(L_v : v \in A \cap B)$ for (A, B) such that Q_w is disjoint from all L_v. Referring to the subpaths P_i of the perimeter path P defined above, since no P_i includes all neighbors of z it follows that there exists $v \in A \cap B - V(\Omega_0)$. We define P_w to be a path obtained from L_v by suitably modifying L_v inside B such that P_w intersects A' for at most one $(A', B') \in \mathcal{W}'' - \{(A, B)\}$. Such modification is easy to make, using the perimeter path of (A', B'). Let $u_w \in A$ and $v_w \in B$ be the ends of P_w.

The set $\mathcal{W}''$ has a subset $\mathcal{W}'''$ of size ξ such that, using to the notation of the previous paragraph, either (u_w, z_w, v_w) is clockwise for every $w \in \mathcal{W}'''$ or (v_w, z_w, u_w) is clockwise for every $w \in \mathcal{W}'''$, and for every $w \in \mathcal{W}'''$ the path P_w is disjoint from A' for every $(A', B') \in \mathcal{W}''' - \{w\}$. The theorem now follows from Lemma 8.7. □

9 Using lack of near-planarity

In this section we prove Theorems 5.3 and 1.8. The first follows immediately from Theorem 5.2 and the two lemmas below.

Lemma 9.1 *Let (G, Ω) be a rurally 5-connected society that is not nearly rural, and let t be a positive integer. If (G, Ω) has a windmill with $4t + 1$ vanes, then it has a windmill with t vanes and a cross.*

Proof. Let $x, u_i, v_i, w_i, P_i, Q_i$ be as in the definition of a windmill W with $4t+1$ vanes. Since $(G \backslash x, \Omega \backslash \{x\})$ is rurally 4-connected and not rural, it has a cross (P, Q) by Theorem 3.1. We may choose the windmill W and cross (P, Q) in $(G \backslash x, \Omega \backslash \{x\})$ such that $W \cup P \cup Q$ is minimal with respect to inclusion. If the cross does not intersect the windmill, then the lemma clearly holds, and so we may assume that a vane $P_i \cup Q_i$ intersects $P \cup Q$. Let v be a vertex that belongs to both $P_i \cup Q_i$ and $P \cup Q$ such that some subpath R of $P_i \cup Q_i$ with one end v and the other end in $V(\Omega)$ has no vertex in $(P \cup Q) \backslash v$. If R has at least one edge, then $P \cup Q \cup R$ has a proper subgraph that is a cross, contrary to the minimality of $W \cup P \cup Q$. Thus v is an end of P or Q. Since P and Q have a total of four ends, it follows that $P \cup Q$ intersects at most four vanes of W. By ignoring those vanes we obtain a windmill with $4(t - 1) + 1$ vanes, and a cross (P, Q) disjoint from it. The lemma follows. □

Lemma 9.2 *Let (G, Ω) be a rurally 6-connected society that is not nearly rural, and let t be a positive integer. If (G, Ω) has a fan with $16t + 5$ blades, then it has a fan with t blades and a cross, or a fan with t blades and a jump, or a fan with t blades and two jumps.*

Proof. Let z_1, z_2 be the hubs of a fan F_2 with $16t + 5$ blades. If $(G \backslash \{z_1, z_2\}, \Omega \backslash \{z_1, z_2\})$ has a cross, then the lemma follows in the same way as Lemma 9.1, and so we may assume not. Since $(G \backslash z_1, \Omega \backslash \{z_1\})$ has a cross, an argument analogous to the proof of Lemma 9.1 shows that there exists a subfan F_1 of F_2 with $4t + 1$ blades (that is, F_1 is obtained by ignoring a set of $12t + 4$ blades), and two paths L_2, S_2 with ends a_2, c_2 and b_2, z_2, respectively, such that $x_1, x_2, \ldots, x_{4t+1}$, a_2, b_2, c_2 is clockwise in Ω for every choice of $x_1, x_2, \ldots, x_{4t+1}$ as in the definition of a fan, and the graphs $L_2, S_2 \backslash z_2, F_1$ are pairwise disjoint. By using the same argument and the fact that $(G \backslash z_2, \Omega \backslash \{z_2\})$ has a cross we arrive at a subfan F of F_1 with t blades and paths L_1, S_1 satisfying the same properties, but with the index 2 replaced by 1. We may assume that F, L_1, L_2, S_1, S_2 are chosen so that $F \cup L_1 \cup L_2 \cup S_1 \cup S_2$ is minimal with respect to inclusion. This will be referred to as "minimality."

If the paths L_1, L_2, S_1, S_2 are pairwise disjoint, except possibly for shared ends and possibly S_1 and S_2 intersecting, then it is easy to see that the lemma holds, and so we may assume that an internal vertex of L_1 belongs to $L_2 \cup S_2$. Let v be the first vertex on L_1 (in either direction) that belongs to $L_2 \cup S_2$, and suppose for a contradiction that v is not an end of L_1. Let L_1' be a subpath of L_1 with one end v, the other end in $V(\Omega)$ and no internal vertex in $L_2 \cup S_2$. Then by replacing a subpath of L_2 or S_2 by L_1' we obtain either a contradiction to minimality, or a cross that is a subgraph of $L_1 \cup L_2 \cup S_1 \cup S_2 \backslash \{z_1, z_2\}$, also a contradiction. This proves that v is an end of L_1, and hence both ends of L_1 are also ends of L_2 or S_2. In particular, L_1 and L_2 share at least one end.

Suppose first that one end of L_1 is an end of S_2. Thus from the symmetry we may assume that a_1 is an end of L_2 and $c_1 = b_2$; thus $a_2 = a_1$, because a_2, b_2, c_2 is clockwise. But now c_2 is not an end of L_1 or S_1, and so the argument of the previous paragraph implies that no internal vertex of L_2 belongs to $S_1 \cup L_1$. The paths S_1, S_2, L_2 now show that (G, Ω) has a fan with t blades and a jump.

We may therefore assume that $a_1 = a_2$ and $c_1 = c_2$. Let H be the union of $L_1, L_2, S_1 \backslash z_1$, $S_2 \backslash z_2$, and $V(\Omega)$. Then the society (H, Ω) is rural, as otherwise $(G \backslash \{z_1, z_2\}, \Omega)$ has a cross. Let Γ be a drawing of (H, Ω) in a disk Δ such that the vertices of $V(\Omega)$ are drawn on the boundary of Δ in the clockwise order specified by Ω. Let $\Delta' \subseteq \Delta$ be a disk such that Δ' includes every path in Γ with ends a_1 and c_1, and the boundary of Δ' includes $a_1 \Omega c_1$ and a path P of Γ from a_1 to c_1. Then L_1 and L_2 lie in Δ', and since L_i is disjoint from $S_i \backslash z_i$ it follows that $S_1 \backslash z_1$ and $S_2 \backslash z_2$ are inside Δ' and, in particular, are disjoint from P. By considering P, S_1 and S_2 we obtain a fan with t blades and a jump. $\square$

Proof of Theorem 5.3. Let d and t be integers, let k be an integer such that Theorem 5.2 holds for d and $16t+5$, and let (G,Ω) be a 6-connected k-cosmopolitan society of depth at most d. We may assume that (G,Ω) is not nearly rural, for otherwise the theorem holds. By Theorem 5.2 the society (G,Ω) has t disjoint consecutive crosses, or a windmill with $4t+1$ vanes, or a fan with $16t+5$ blades. In the first case the theorem holds, and in the second and third case the theorem follows from Lemma 9.1 and Lemma 9.2, respectively. □

For the proof of Theorem 1.8 we need one more lemma. Let us recall that presentation of a neighborhood was defined prior to Theorem 1.7.

Lemma 9.3 *Let d and s be integers, let (G,Ω) be an s-nested society, and let (G',Ω') be a planar truncation of (G,Ω) of depth at most d. Then (G,Ω) has an s-nested planar truncation of depth at most $2(d+2s)$.*

Proof. By a vortical decomposition of a society (G,Ω) we mean a collection $(Z_v : v \in V(\Omega))$ of sets such that

(i) $\bigcup(Z_v : v \in V(\Omega)) = V(G)$ and every edge of G has both ends in Z_v for some $v \in V(\Omega)$,

(ii) for $v \in V(\Omega)$, $v \in Z_v$, and

(iii) if (v_1,v_2,v_3,v_4) is clockwise in Ω, then $Z_{v_1} \cap Z_{v_3} \subseteq Z_{v_2} \cup Z_{v_4}$.

The *depth* of such a vortical decomposition is $\max |Z_u \cap Z_v|$, taken over all pairs of distinct vertices $u,v \in V(\Omega)$ that are consecutive in Ω, and the depth of (G,Ω) is the minimum depth of a vortical decomposition of (G,Ω). Thus if (G,Ω) has depth at most d, then the corresponding linear decomposition also serves as a vortical decomposition of depth at most d.

Let (G,Ω) be an s-nested society, and let it be the composition of a society (G_0,Ω_0) with a rural neighborhood (G_1,Ω,Ω_0), where the neighborhood has a presentation $(\Sigma,\Gamma_1,\Delta,\Delta_0)$ with an s-nest $C_1,C_2,\ldots,C_s$. Let $\Delta_0,\Delta_1,\ldots,\Delta_s$ be as in the definition of s-nest. Let (G',Ω') be a planar truncation of (G,Ω) of depth at most d. Then (G,Ω) is the composition of (G',Ω') with a rural neighborhood (G_2,Ω,Ω'), and we may assume that (G_2,Ω,Ω') has a presentation $(\Sigma,\Gamma_2,\Delta,\Delta')$, where $\Delta_0 \subseteq \Delta'$. We may assume that the s-nest $C_1,C_2,\ldots,C_s$ is chosen as follows: first we select C_1 such that $\Delta_0 \subseteq \Delta_1$ and the disk Δ_1 is as small as possible, subject to that we select C_2 such that $\Delta_1 \subseteq \Delta_2$ and the disk Δ_2 is as small as possible, subject to that we select C_3, and so on.

Let Δ^* be a closed disk with $\Delta' \subseteq \Delta^* \subseteq \Delta$. We say that Δ^* is *normal* if whenever an interior point of an edge $e \in E(\Gamma_1)$ belongs to the boundary of Δ^*, then e is a subset of the boundary of Δ^*. A normal disk Δ^* defines a planar truncation (G^*,Ω^*) in a natural way as follows: G^* is consists of all vertices and edges that of G either belong to G', or their image under Γ_1 belongs to Δ^*, and Ω^* consists of vertices of G whose image under Γ_1 belongs to the boundary Δ^* in the order determined by the boundary of Δ^*.

Given a normal disk Δ^* and two vertices $u, v \in V(G)$ we define $\xi_{\Delta^*}(u, v)$, or simply $\xi(u, v)$ as follows. If u is adjacent to v, and the image e under Γ_1 of the edge uv is a subset of the boundary of Δ^*, and for every internal point x on e there exists an open neighborhood U of x such that $U \cap \Delta^* = U \cap \Delta_i$, then we let $\xi(u, v) = i$. Otherwise we define $\xi(u, v) = 0$. A short explanation may be in order. If the image e of uv is a subset of the boundary of Δ^*, then this can happen in two ways: if we think of e as having two sides, either Δ^* and Δ_i appear on the same side, or on opposite sides of e. In the definition of ξ it is only edges with Δ^* and Δ_i on the same side that count.

We may assume, by shrinking Δ' slightly, that the boundary of Δ' does not include an interior point of any edge of Γ_2. Then Δ' is normal, and the corresponding planar truncation is (G', Ω'). Since a linear decomposition of (G', Ω') of depth at most d may be regarded as a vortical decomposition of (G', Ω') of depth at most d, we may select a normal disk Δ^* that gives rise to a planar truncation (G^*, Ω^*) of (G, Ω), and we may select a vortical decomposition $(Z_v : v \in V(\Omega^*))$ of (G^*, Ω^*) such that $|Z_u \cap Z_v| \leq d + 2\xi(u, v)$ for every pair of consecutive vertices of Ω^*. Furthermore, subject to this, we may choose Δ^* such that the number of unordered pairs u, v of distinct vertices of G with $\xi(u, v) = s$ is maximum, subject to that the number of unordered pairs u, v of distinct vertices of G with $\xi(u, v) = s - 1$ is maximum, subject to that the number of unordered pairs u, v of distinct vertices of G with $\xi(u, v) = s - 2$ is maximum, and so on.

We will show that (G^*, Ω^*) satisfies the conclusion of the theorem. Let $(t_1, t_2, \ldots, t_n)$ be an arbitrary clockwise enumeration of $V(\Omega^*)$, and let $X_i := Z_{t_i} \cup (Z_{t_1} \cap Z_{t_n})$. Then $(X_1, X_2, \ldots, X_n)$ is a linear decomposition of (G^*, Ω^*) of depth at most $2(d + 2s)$.

To complete the proof we must show that (G^*, Ω^*) is s-nested, and we will do that by showing that each C_i is a subgraph of G^*. To this end we suppose for a contradiction that it is not the case, and let $i_0 \in \{1, 2, \ldots, s\}$ be the minimum integer such that C_{i_0} is not a subgraph of G^*.

If C_{i_0} has no edge in G^*, then we can construct a new society (G_3, Ω_3), where Ω_3 consists of the vertices of C_{i_0} in order, and obtain a contradiction to the choice of (G^*, Ω^*). Since the construction is very similar but slightly easier than the one we are about to exhibit, we omit the details. Instead, we assume that C_{i_0} includes edges of both G^* and $G \backslash E(G^*)$. Thus there exist vertices $x, y \in V(C_{i_0}) \cap V(\Omega^*)$ such that some subpath P of C_{i_0} with ends x and y has no internal vertex in $V(\Omega^*)$. Let B denote the boundary of Δ^*. There are three closed disks with boundaries contained in $B \cup P$. One of them is Δ^*; let D be the one that is disjoint from Δ_0. If the interior of D is a subset of Δ_{i_0} and includes no edge of C_{i_0}, then we say that P is a *good segment.* It follows by a standard elementary argument that there is a good segment.

Thus we may assume that P is a good segment, and that the notation is as in the previous paragraph. There are two cases: either D is a subset of Δ^*, or the interiors of D

and Δ^* are disjoint. Since the former case is handled by a similar, but easier construction, we leave it to the reader and assume the latter case. Let $(s_0, s_1, \ldots, s_{t+1})$ be clockwise in Ω^* such that $s_0, s_1, \ldots, s_{t+1}$ are all the vertices that belong to $D \cap \Delta^*$. Thus $\{s_0, s_{t+1}\} = \{x, y\}$. Let $r_0 = s_0, r_1, \ldots, r_k, r_{k+1} = s_{t+1}$ be all the vertices of P, in order, let H be the subgraph of G^* consisting of all vertices and edges whose images under Γ_1 belong to D, and let $X := \{s_0, s_1, \ldots, s_{t+1}, r_0, r_1, \ldots, r_{k+1}\}$. We can regard H as drawn in a disk with the vertices $s_0, s_1, \ldots, s_{t+1}, r_k, r_{k-1}, \ldots, r_1$ drawn on the boundary of the disk in order. We may assume that every component of H intersects X. The way we chose the cycles C_{i_0} implies that every path in $H \backslash \{s_1, s_2, \ldots, s_k\}$ that joins two vertices of P is a subpath of P. We will refer to this property as the convexity of H. For $i = 0, 1, \ldots, k+1$ let b_i be the maximum index j such that the vertex s_j can be reached from $\{r_0, r_1, \ldots, r_i\}$ by a path in H with no internal vertex in X. We define $b_{-1} := -1$, and let R_i be the set of all vertices of H that can be reached from $\{r_i, s_{b_{i-1}+1}, s_{b_{i-1}+2}, \ldots, s_{b_i}\}$ by a path with no internal vertex in X. The convexity of H implies that for $i < j$ the only possible member of $R_i \cap R_j$ is s_{b_i}. We now define a new society (G^{**}, Ω^{**}) as follows. The graph G^{**} will be the union of G^* and H, and the cyclic permutation is defined by replacing the subsequence $s_0, s_1, \ldots, s_{t+1}$ of Ω^* by the sequence $r_0, r_1, \ldots, r_k, r_{k+1}$. We define the sets Z_v^{**} as follows. For $v \in V(\Omega^*) - V(\Omega^{**})$ we let $Z_v^{**} := Z_v$. If $v = r_i$ and $b_i > b_{i-1}$ we define Z_v^{**} to be the union of $R_i \cup \{s_{b_i}, r_{i-1}\}$ and all Z_{s_j} for $j = b_{i-1}+1, b_{i-1}+2, \ldots, b_i$. If $v = r_i$ and $b_i = b_{i-1}$ we define $Z_v^{**} := R_i \cup \{s_{b_i}, r_{i-1}\} \cup (Z_{s_{b_i}} \cap Z_{s_{b_i+1}})$. It is straightforward to verify that (G^{**}, Ω^{**}) is a planar truncation of (G, Ω) and that $(Z_v^{**} : v \in V(\Omega^{**}))$ is a vortical decomposition of (G^{**}, Ω^{**}). We claim that $\xi_{\Delta^*}(s_j, s_{j+1}) < i_0$ for all $j = 0, 1, \ldots, t$. To prove this we may assume that s_j is adjacent to s_{j+1}, and let e be the image under Γ_1 of the edge $s_j s_{j+1}$. It follows that e is a subset of Δ_{i_0}, and hence if $s_j s_{j+1} \in E(C_k)$ for some k, then $k \le i_0$. Furthermore, if equality holds, then Δ_{i_0} and Δ^* lie on opposite sides of e, and hence $\xi_{\Delta^*}(s_j, s_{j+1}) = 0$. This proves our claim that $\xi_{\Delta^*}(s_j, s_{j+1}) < i_0$. Since for $i = 0, 1, \ldots, k$ we have $Z_{r_i}^{**} \cap Z_{r_{i+1}}^{**} \subseteq (Z_{s_{b_i}} \cap Z_{s_{b_i+1}}) \cup \{r_i, s_{b_i}\}$, and $\xi_{\Delta^{**}}(r_i, r_{i+1}) = i_0$, we deduce that

$$|Z_{r_i}^{**} \cap Z_{r_{i+1}}^{**}| \le |Z_{s_{b_i}} \cap Z_{s_{b_i+1}}| + 2 \le d + \xi_{\Delta^*}(s_{b_i}, s_{b_i+1}) \le d + 2\xi_{\Delta^{**}}(r_i, r_{i+1}).$$

Thus the existence of (G^{**}, Ω^{**}) contradicts the choice of (G^*, Ω^*). This completes our proof that $C_1, C_2, \ldots, C_s$ are subgraphs of G^*, and hence (G^*, Ω^*) is s-nested, as desired. $\square$

Proof of Theorem 1.8. Let d be as in Theorem 5.1, and let k be as in Corollary 5.4 applied to $2(d + 2s)$ in place of d. We claim that k satisfies Theorem 1.8. To prove that let (G, Ω) be a 6-connected s-nested k-cosmopolitan society that is not nearly rural. Since (G, Ω) is an s-nested planar truncation of itself, by Theorem 5.1 we may assume that (G, Ω) has either a leap of length five, in which case it satisfies Theorem 1.8 by Theorem 4.1, or it has a planar truncation of depth at most d. In the latter case it has an s-nested planar truncation (G', Ω') of depth at most $2(d+2s)$ by Lemma 9.3, and the theorem follows from Corollary 5.4 applied to the society (G', Ω'). $\square$

10 Finding a planar nest

In this section we prove a technical result that applies in the following situation. We will be able to guarantee that some societies (G, Ω) contain certain configurations consisting of disjoint trees connecting specified vertices in $V(\Omega)$. The main result of this section, Theorem 10.3 below, states that if the society is sufficiently nested, then we can make sure that the cycles in some reasonably big nest and the trees of the configuration intersect nicely.

A *target* in a society (G, Ω) is a subgraph F of G such that

(i) F is a forest and every leaf of F belongs to $V(\Omega)$, and

(ii) if $u, v \in V(\Omega)$ belong to a component T of F, then there exists a component $T' \neq T$ of F and $w \in V(T') \cap V(\Omega)$ such that (u, w, v) is clockwise.

We say that a vertex $v \in V(G)$ is F-*special* if either v has degree at least three in F, or v has degree at least two in F and $v \in V(\Omega)$.

Now let F be a target in (G, Ω) and let T be a component of F. Let P be a path in $G \backslash V(\Omega)$ with ends u, v such that $u, v \in V(T)$ and P is otherwise disjoint from F. Let C be the unique cycle in $T \cup P$, and assume that C has at most one F-special vertex. If $C \backslash u \backslash v$ has no F-special vertex, then let P' be the subpath of C that is complementary to P, and if $C \backslash u \backslash v$ has an F-special vertex, say w, then let P' be either the subpath of $C \backslash u$ with ends v and w, or the subpath of $C \backslash v$ with ends u and w. Finally, let F' be obtained from $F \cup P$ by deleting all edges and internal vertices of P'. In those circumstances we say that F' was obtained from F by *rerouting*.

A subgraph F of a rural neighborhood (G, Ω, Ω_0) is *perpendicular* to an s-nest $(C_1, C_2, \ldots, C_s)$ if for every component P of F

(i) P is a path with one end in $V(\Omega)$ and the other in $V(\Omega_0)$, and

(ii) $P \cap C_i$ is a path for all $i = 1, 2, \ldots, s$.

The complexity of a forest F in a society (G, Ω) is

$$\sum (\deg_F(v) - 2)^+ + \sum_{v \in V(\Omega)} (\deg_F(v) - 1)^+,$$

where the first summation is over all $v \in V(G) - V(\Omega)$ and x^+ denotes $\max(x, 0)$.

The following is a preliminary version of the main result of this section.

Theorem 10.1 *Let w, s, k be positive integers, and let $s' = 2w(k+1) + s$. Then for every s'-nested society (G, Ω) such that G has tree-width at most w and for every target F_0 in (G, Ω) of complexity at most k there exists a target F in (G, Ω) obtained from F_0 by repeated* *rerouting such that (G, Ω) can be expressed as a composition of some society with a rural neighborhood (G', Ω, Ω') that has a presentation with an s-nest $(C_1, C_2, \ldots, C_s)$ such that $G' \cap F$ is perpendicular to $(C_1, C_2, \ldots, C_s)$.*

Proof. Suppose that the theorem is false for some integers w, s, k, a society (G, Ω) and target F_0, and choose these entities with $|V(G)| + |E(G)|$ minimum. Let (G, Ω) be the composition of a society (G_0, Ω_0) with a rural neighborhood (G_1, Ω, Ω_0). Let κ be the complexity of $F \cap G_1$ in the society (G_1, Ω), and let $s'' = 2w(\kappa + 1) + s$. Since (G, Ω) is s'-nested and $s'' \leq s'$ we may choose a presentation $(\Sigma, \Gamma, \Delta, \Delta_0)$ of (G_1, Ω, Ω_0) and an s''-nest $(C_1, C_2, \ldots, C_{s''})$ for it. We may assume that $G_0, \Omega_0, G_1, F, \Sigma, \Gamma, \Delta, \Delta_0, C_1, C_2, \ldots, C_{s''}$ are chosen to minimize κ. The minimality of G implies that $G = C_1 \cup C_2 \cup \cdots \cup C_{s'} \cup F$. Likewise, $C_1 \cup C_2 \cup \cdots \cup C_{s'}$ is edge-disjoint from F, for otherwise contracting an edge belonging to the intersection of the two graphs contradicts the minimality of G.

By a *dive* we mean a subpath of $F \cap G_1$ with both ends in $V(\Omega_0)$ and otherwise disjoint from $V(\Omega_0)$. Let P be a dive with ends u, v, and let P' be the corresponding path in Γ. Then $\Delta_0 \cup P'$ separates Σ; let $\Delta(P')$ denote the component of $\Sigma - \Delta_0 - P'$ that is contained in Δ, and let $H(P)$ denote the subgraph of G_1 consisting of all vertices and edges that correspond to vertices or edges of Γ that belong to the closure of $\Delta(P')$. Thus P is a subgraph of $H(P)$. We say that a dive P is *clean* if $H(P) \backslash V(\Omega_0)$ includes at most one F-special vertex, and if it includes one, say v, then $v \in V(P)$, and no edge of $E(F) - E(P)$ incident with v belongs to $H(P)$. The *depth* of a dive P is the maximum integer $d \in \{1, 2, \ldots, s'\}$ such that $V(P) \cap V(C_d) \neq \emptyset$, or 0 if no such integer exists. It follows from planarity that $|V(P) \cap V(C_i)| \geq 2$ for all $i = 1, 2, \ldots, d - 1$.

(1) *Every clean dive has depth at most $2w$.*

To prove the claim suppose for a contradiction that P_1 is a clean dive of depth $d \geq 2w+1$. Thus $V(P_1) \cap V(C_d) \neq \emptyset$. Assume that we have already constructed dives $P_1, P_2, \ldots, P_t$ for some $t \leq w$ such that $V(P_i) \cap V(C_{d-i+1}) \neq \emptyset$ for all $i = 1, 2, \ldots, t$ and $H(P_t) \subseteq H(P_{t-1}) \subseteq \cdots \subseteq H(P_1)$. Since $V(P_t) \cap V(C_{d-t+1}) \neq \emptyset$, there exist distinct vertices $x, y \in V(P_t) \cap V(C_{d-t})$. Furthermore, it is possible to select x, y such that one of subpaths of C_{d-t} with ends x, y, say Q, is a subgraph of $H(P_t)$ and no internal vertex of Q belongs to P_t.

We claim that some internal vertex of Q belongs to F. Indeed, if not, then we can reroute xP_ty along Q to produce a target F' and delete an edge of xP_ty; since P_1 is clean and $H(P_t)$ is a subgraph of $H(P_1)$ this is indeed a valid rerouting as defined above. But this contradicts the minimality of G, and hence some internal vertex of Q, say q, belongs to F. Since P_1 is clean and $H(P_t)$ is a subgraph of $H(P_1)$ it follows that q belongs to a dive P_{t+1} that is a subgraph of $H(P_t) \backslash V(P_t)$. It follows that $H(P_{t+1})$ is a subgraph of $H(P_t)$, thus completing the construction. (See Figure 7.)

The dives $P_1, P_2, \ldots, P_{w+1}$ just constructed are pairwise disjoint and all intersect C_{d-w}.

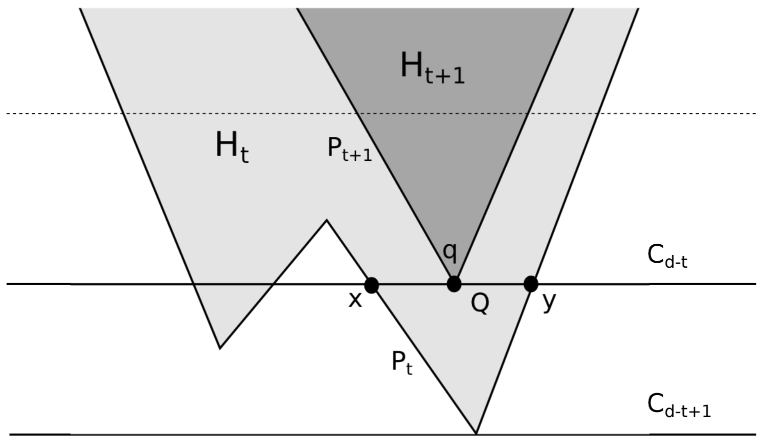

Figure 7: Construction of $H(P_{t+1})$.

Since $d \geq 2w+1$ this implies that $P_1, P_2, \ldots, P_{w+1}$ all intersect each of $C_1, C_2, \ldots, C_{w+1}$, and hence $C_1 \cup P_1, C_2 \cup P_2, \ldots, C_{w+1} \cup P_{w+1}$ is a "screen" in G of "thickness" at least $w+1$. By [16, Theorem (1.4)] the graph G has tree-width at least w, a contradiction. This proves (1).

Our next objective is to prove that $\kappa = 0$. That will take several steps. To that end let us define a dive P to be *special* if $P \backslash V(\Omega_0)$ contains exactly one F-special vertex. By a *bridge* we mean a subgraph B of $G_1 \cap F$ consisting of a component C of $G_1 \backslash V(\Omega_0)$ together with all edges from $V(C)$ to $V(\Omega_0)$ and all ends of these edges.

(2) *If a bridge B includes an F-special vertex not in $V(\Omega_0)$, then B includes a special dive.*

To prove Claim (2) let B be a bridge containing an F-special vertex not in $V(\Omega_0)$. For an F-special vertex $b \in V(B) - V(\Omega_0)$ and an edge $e \in E(B)$ incident with b let P_e be the maximal subpath of B containing e such that one end of P_e is b and no internal vertex of P_e is F-special or belongs to $V(\Omega_0)$. Let u_e be the other end of P_e. The second axiom in the definition of target implies that at most one vertex of F belongs to $V(\Omega)$. Since every F-special vertex in $V(G_1) - V(\Omega)$ has degree at least three, it follows that there exists an F-special vertex $b \in V(B) - V(\Omega_0)$ such that $u_{e_1}, u_{e_2} \in V(\Omega_0)$ for two distinct edges $e_1, e_2 \in E(B)$ incident with b. Then $P_{e_1} \cup P_{e_2}$ is as desired. This proves (2).

By (2) we may select a special dive P with $H(P)$ minimal. We claim that P is clean. For let $v \in V(P) - V(\Omega_0)$ be F-special. If some edge $e \in E(F) - E(P)$ incident with v belongs to $H(P)$, then there exists a subpath P' of F containing e with one end v and the other end in $V(\Omega_0) \cup V(\Omega)$. But P' is a subgraph of $H(P)$, and hence the other end of P' belongs to $V(\Omega_0)$ by planarity. It follows that $P \cup P'$ includes a dive that contradicts the minimality of $H(P)$. This proves that the edge e as above does not exist.

It remains to show that no vertex of $H(P)\backslash V(\Omega_0)$ except v is F-special. So suppose for a contradiction that such vertex, say v', exists. Then $v' \notin V(P)$, because P is special, and hence v' belongs to a bridge $B' \neq B$. But B' includes a special dive by (2), contrary to the choice of P. This proves our claim that P is clean.

By (1) P has depth at most $2w$. In particular, the image under Γ of some F-special vertex belongs to the open disk Δ_{2w+1} bounded by the image under Γ of C_{2w+1}. Let G_0' consist of G_0 and all vertices and edges of G whose images under Γ belong to the closure of Δ_{2w+1}, let G_1' consist of all vertices and edges whose images under Γ belong to the complement of Δ_{2w+1}, and let Ω_0' be defined by $V(\Omega_0') = V(C_{2w+1})$ and let the cyclic order of Ω_0' be determined by the order of $V(C_{2w+1})$. Then (G, Ω) can be regarded as a composition of (G_0', Ω_0') with the rural neighborhood $(G_1', \Omega, \Omega_0')$. This rural neighborhood has a presentation with a σ-nest, where $\sigma = 2w\kappa + s$. On the other hand, the complexity of $F \cap G_1'$ is at most $\kappa - 1$, contrary to the minimality of κ. This proves our claim that $\kappa = 0$.

By repeating the argument of the previous paragraph and sacrificing $2w$ of the cycles C_i we may assume that (G_1, Ω, Ω_0) has a presentation with an s-nest $C_1, C_2, \dots, C_s$ and that there are no dives. It follows that every component P of $F \cap G_1$ is a path with one end in $V(\Omega)$ and the other in $V(\Omega_0)$. To complete the proof of the theorem we must show that $P \cap C_i$ is a path for all $i = 1, 2, \dots, s$. Suppose for a contradiction that that is not the case. Thus for some $i \in \{1, 2, \dots, s\}$ and some component P of $F \cap G_1$ the intersection $P \cap C_i$ is not a path. Thus there exist distinct vertices $x, y \in V(P \cap C_i)$ such that xPy is a path with no edge or internal vertex in C_i. Let us choose P, i, x, y such that, subject to the conditions stated, i is maximum. If $i < s$ and xPy intersects C_{i+1}, then $P \cap C_{i+1}$ is not a path, contrary to the choice of i. If $i = 1$ or xPy does not intersect C_{i-1}, then by rerouting one of the subpaths of C_i with ends x, y along xPy we obtain contradiction to the minimality of G. Thus we may assume that $i > 1$ and that xPy intersects C_{i-1}.

Exactly one of the subpaths of C_i with ends x, y, say Q, has the property that the image under Γ of $xPy \cup Q$ bounds a disk contained in Δ and disjoint from Δ_0. If no component of $F \cap G_1$ other than P intersects Q, then by rerouting F along Q we obtain a contradiction to the minimality of G. Thus there exists a component P' of $F \cap G_1$ other that P that intersects Q, say in a vertex u. The vertex u divides P' into two subpaths P_1' and P_2'. If both P_1' and P_2' intersect C_{i+1}, then P' contradicts the choice of i. Thus we may assume that say P_1' does not intersect C_{i+1}. But P_1' includes a subpath P'' with both ends on C_i and otherwise disjoint from $C_1 \cup C_2 \cup \cdots \cup C_s$, and hence by rerouting C_i along P'' we obtain a contradiction to the minimality of G. This completes the proof of the theorem. $\square$

Before we state the main result of this section we need the following deep result from [12]. A *linkage* in a graph G is a subgraph of G, every component of which is a path. A linkage L in a graph G is *vital* if $V(L) = V(G)$ and there is no linkage $L' \neq L$ in G such that for every two vertices $u, v \in V(G)$, the vertices u, v are the ends of a component of L if and only if they are the ends of a component of L'.

Theorem 10.2 *For every integer $p \geq 0$ there exists an integer w such that every graph that has a vital linkage with p components has tree-width at most w.*

Now we are ready to state and prove the main theorem of this section. If F is a target in a society (G, Ω) we say that a vertex $v \in V(G)$ is *critical* for F if v is either F-special or a leaf of F. We say that two targets F, F' are *hypomorphic* if they have the same set of critical vertices, say X, and $u, v \in X$ are joined by a path in F with no internal vertices in X if and only if they are so joined in F'.

Theorem 10.3 *For every two positive integers s, k there exists an integer s' such that for every s'-nested society (G, Ω) and for every target F in (G, Ω) of complexity at most k there exists a target F in (G, Ω) obtained from a target hypomorphic to F_0 by repeated rerouting such that (G, Ω) can be expressed as a composition of some society with a rural neighborhood (G', Ω, Ω') that has a presentation with an s-nest $(C_1, C_2, \ldots, C_s)$ such that $G' \cap F$ is perpendicular to $(C_1, C_2, \ldots, C_s)$.*

Proof. We proceed by induction on $|V(G)| + |E(G)|$. Let $p = k + 2$, and let w be the bound guaranteed by Theorem 10.2. By hypothesis (G, Ω) is the composition of a society (G_0, Ω_0) with a rural neighborhood (G_1, Ω, Ω_0), where (G_1, Ω, Ω_0) has a presentation $(\Sigma, \Gamma, \Delta, \Delta_0)$ and an s'-nest $(C_1, C_2, \ldots, C_{s'})$. Let X be the set of all vertices critical for F, and let $L = F \backslash X$. Then L is a linkage in $G \backslash X$. If it is vital, then G has tree-width at most $|X| + w \leq 2k + 1 + w$, and hence the theorem follows from Theorem 10.1.

Thus we may assume that L is not vital. Assume first that there exists a vertex $v \in V(G) - V(L)$. If $v \in V(C_i)$ for some $i \in \{1, 2, \ldots, s'\}$, then the theorem follows by induction applied to the graph obtained from G by contracting one of the edges of C_i incident with v; otherwise, the theorem follows by induction applied to the graph $G \backslash v$.

Thus we may assume that $V(L) = V(G)$, and hence there exists a linkage $L' \neq L$ linking the same pairs of terminals. Thus there exists an edge $e \in E(L) - E(L')$. If $e \in E(C_i)$ for some $i \in \{1, 2, \ldots, s'\}$, then the theorem follows by induction by contracting the edge e; otherwise it follows by induction by deleting e, because the linkage L' guarantees that $G \backslash e$ has a target hypomorphic to F. $\square$

11 Chasing a turtle

In this section we prove Theorem 1.3, but first we need the following two theorems.

Theorem 11.1 *There is an integer s such that if an s-nested society (G, Ω) has a turtle, then G has a K_6 minor.*

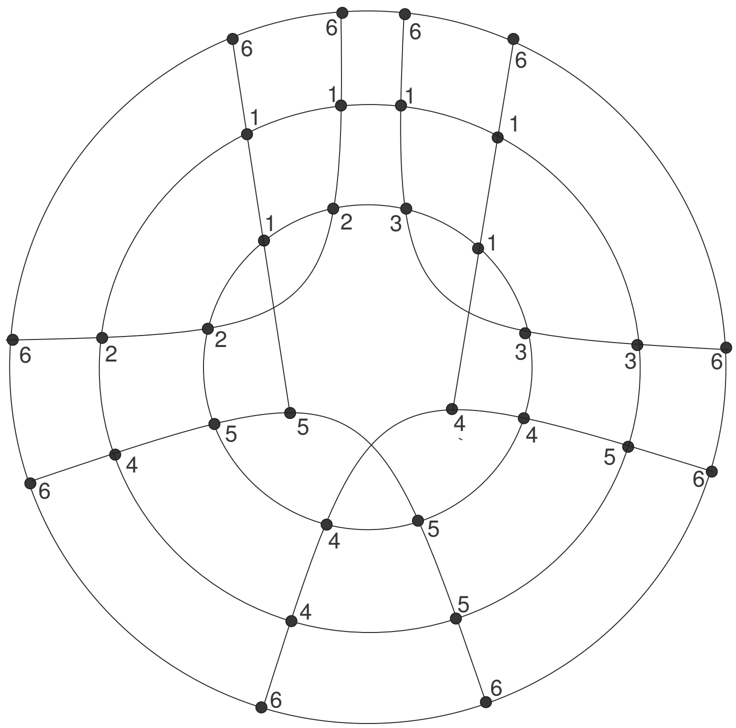

Figure 8: A turtle giving rise to a K_6 minor.

Proof. Let k be the maximum complexity of a turtle, let $s = 3$, and let s' be as in Theorem 10.3. We claim that s' satisfies the theorem. Indeed, let (G, Ω) be an s'-nested society that has a turtle. Since every turtle is a target, and every target obtained from a target hypomorphic to a turtle is again a turtle, we deduce from Theorem 10.3 that (G, Ω) has a turtle F and can be expressed as a composition of a society with a rural neighborhood (G', Ω, Ω') that has a presentation with a 3-nest (C_1, C_2, C_3) such that $G' \cap F$ is perpendicular to (C_1, C_2, C_3). It is now fairly straightforward to deduce that G has a K_6 minor. The argument is illustrated in Figure 8. □

Theorem 11.2 *There is an integer s such that if an s-nested society (G, Ω) has three crossed paths, a separated doublecross or a gridlet, then G has a K_6 minor.*

Proof. The argument is analogous to the proof of the previous theorem, using Figures 9, 10 and 11 instead. We omit the details. □

Proof of Theorem 1.3. Let s be an integer large enough that both Theorem 11.1 and Theorem 11.2 hold for s. Let k be an integer such that Theorem 1.8 holds for this integer. Let t be such that Theorem 1.7 holds for t and the integer k just defined. Let h be an integer such that Theorem 1.6 holds with t replaced by $t + 2s$. Let w be an integer such that Theorem 1.5 holds for the integer h just defined. Finally, let N be as in Theorem 1.4.

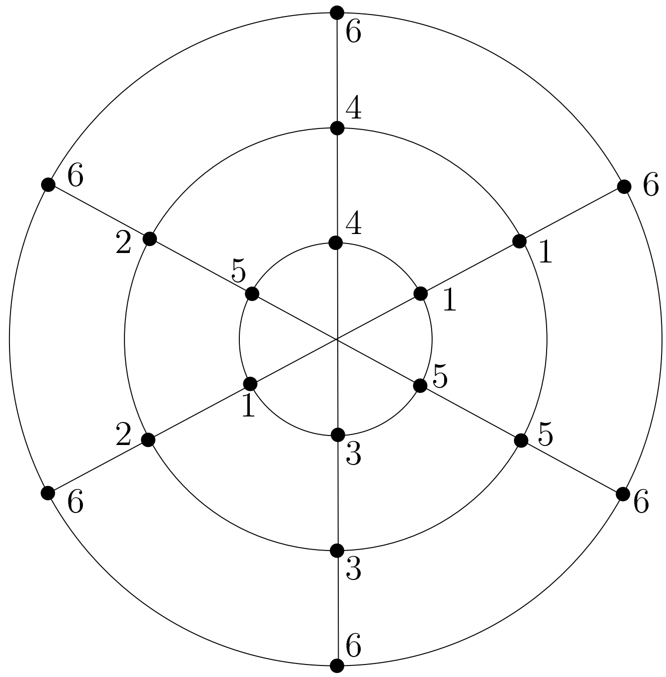

Figure 9: Three crossed paths giving rise to a K_6 minor.

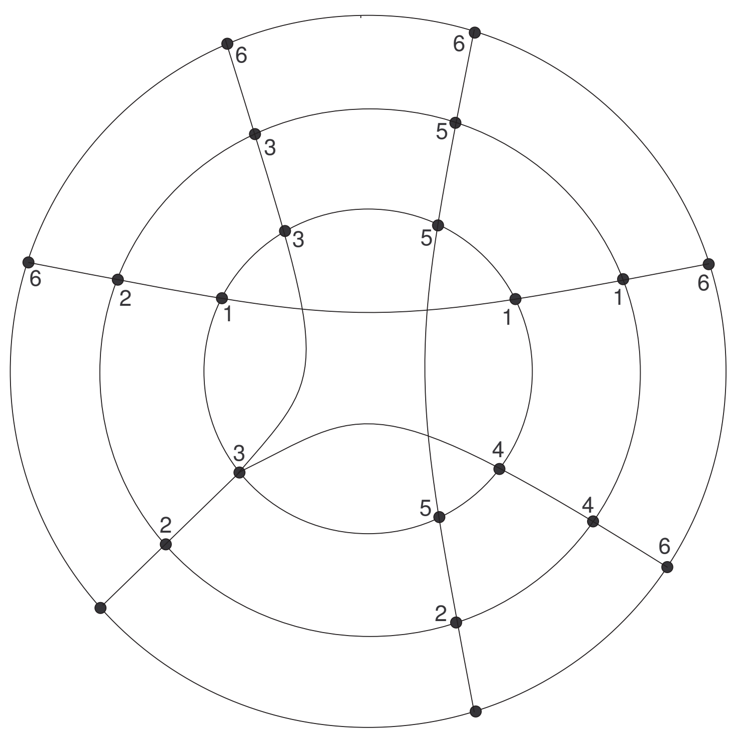

Figure 10: A gridlet giving rise to a K_6 minor.

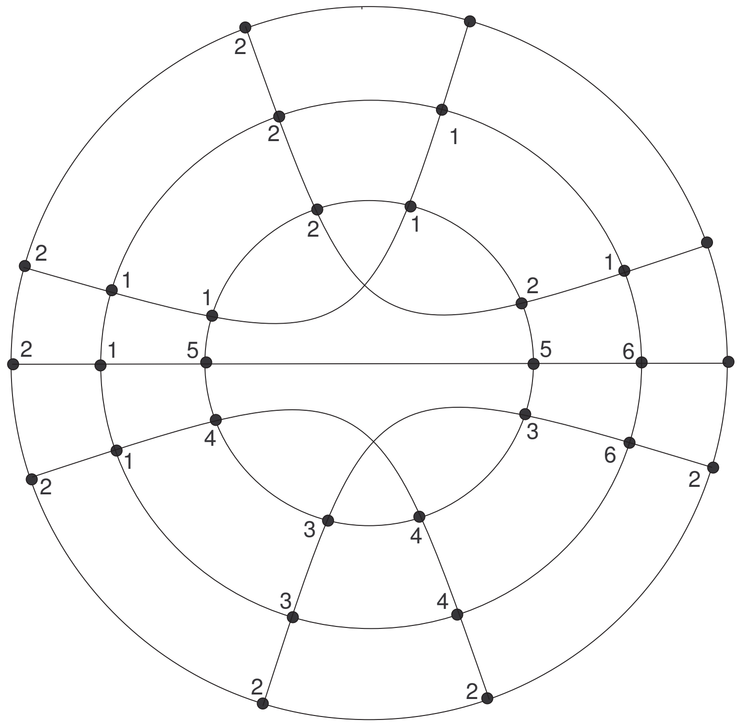

Figure 11: A separated doublecross giving rise to a K_6 minor.

Suppose for a contradiction that G is a 6-connected graph on at least N vertices that is not apex. By Theorem 1.4 G has tree-width exceeding w. By Theorem 1.5 G has a wall of height h. By Theorem 1.6 G has a planar wall H_0 of height $t+2s$. By considering a subwall H of H_0 of height t and s cycles of $H_0 \backslash V(H)$ we find, by Theorem 1.7, that the anticompass society (K, Ω) of H in G is s-nested and k-cosmopolitan. By Theorem 1.8 the society (K, Ω) has a turtle, three crossed paths, a separated doublecross, or a gridlet. By Theorems 11.1 and 11.2 the graph G has a K_6 minor, a contradiction. $\square$

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