

AN INTEGRATED FIRST-ORDER THEORY OF POINTS AND INTERVALS OVER LINEAR ORDERS (PART I)

WILLEM CONRADIE, SALIH DURHAN, AND GUIDO SCIAVICCO

School of Mathematics, University of the Witwatersrand, Johannesburg, South Africa
e-mail address: willem.conradie@wits.ac.za

Nesin Mathematics Village, Sirince, Turkey
e-mail address: salih1@gmail.com

Department of Mathematics and Computer Science, University of Ferrara, Ferrara, Italy
e-mail address: guido.sciavicco@unife.it

ABSTRACT. There are two natural and well-studied approaches to temporal ontology and reasoning: point-based and interval-based. Usually, interval-based temporal reasoning deals with points as a particular case of duration-less intervals. A recent result by Balbiani, Goranko, and Sciavicco presented an explicit two-sorted point-interval temporal framework in which time instants (points) and time periods (intervals) are considered on a par, allowing the perspective to shift between these within the formal discourse. We consider here two-sorted first-order languages based on the same principle, and therefore including relations, as first studied by Reich, among others, between points, between intervals, and inter-sort. We give complete classifications of its sub-languages in terms of relative expressive power, thus determining how many, and which, are the intrinsically different extensions of two-sorted first-order logic with one or more such relations. This approach roots out the classical problem of whether or not points should be included in a interval-based semantics.

1. INTRODUCTION

The relevance of temporal logics in many theoretical and applied areas of computer science and AI, such as theories of action and change, natural language analysis and processing, and constraint satisfaction problems, is widely recognized. While the predominant approach in the study of temporal reasoning and logics has been based on the assumption that time points (instants) are the primary temporal ontological entities, there has also been significant activity in the study of interval-based temporal reasoning and logics over the past two decades. The variety of binary relations between intervals in linear orders was first studied systematically by Allen [AH87, All83, AF94], who explored their use in systems for time management and planning. Allen’s work and much that follows from it is based on the assumption that time can be represented as a dense line, and that points are excluded from the semantics. At

Key words and phrases: Interval-based Temporal Logics; Expressiveness; Allen’s Relations.

We would like to thank Dr. Davide Bresolin, of the University of Padova, for his help. The research of the first author was supported by the National Research Foundation of South Africa grant number 81309..

the modal level, Halpern and Shoham [HS91] introduced the multi-modal logic HS that comprises modal operators for all possible relations (known as Allen’s relations [All83]) between two intervals in a linear order, and it has been followed by a series of publications studying the expressiveness and decidability/undecidability and complexity of the fragments of HS, e.g., [BMM⁺14, BMG⁺14]. Many studies on interval logics have considered the so-called ‘non-strict’ interval semantics, allowing point-intervals (with coinciding endpoints) along with proper ones, and thus encompassing the instant-based approach, too; more recent ones, instead, started to treat pure intervals only. Yet, little has been done so far on the formal treatment of both temporal primitives, points and intervals, in a unified two-sorted framework. A detailed philosophical study of both approaches, point-based and interval-based, can be found in [vB91] (see also [CM00]). A similar mixed approach has been studied in [AH89]. [MH06] contains a study of the two sorts and the relations between them in dense linear orders. More recently, a modal logic that includes different operators for points and interval has been presented in [BGS11].

The present paper provides a systematic treatment of point and interval relations (including equality between points and between intervals treated on the same footing as the other relations) at the first-order level. Our work is motivated, among other observations, by the fact that natural languages incorporate both ontologies on a par, without assuming the primacy of one over the other, and have the capacity to shift the perspective smoothly from instants to intervals and vice versa within the same discourse, e.g.: *when the alarm goes on, it stays on until the code is entered*, which contains two instantaneous events and a non-instantaneous one. Moreover, there are various temporal scenarios which neither of the two ontologies alone can grasp properly since neither the treatment of intervals as the sets of their internal points, nor the treatment of points as ‘instantaneous’ intervals, is really adequate. The technical identification of intervals with sets of their internal points, or of points as instantaneous intervals leads also to conceptual problems like the confusion of events and fluents. Instantaneous events are represented by time intervals and should be distinguished from instantaneous holding of fluents, which are evaluated at time points: therefore, the point a should be distinguished from the interval $[a, a]$, and the truths in these should not necessarily imply each other. Finally, we note that, while differences in expressiveness have been found between the strict and non-strict semantics for some interval logics (see [MGMS11], for example), so far, no distinction in the decidability of the satisfiability has been found. Therefore, we believe that an attempt to systemize the role of points, intervals, and their interaction, would make good sense not only from a purely ontological point of view, but also from algorithmic and computational perspectives.

Previous Work and Motivations. As presented in the early work of van Benthem [vB91] and Allen and Hayes [AH85], interval temporal reasoning can be formalized as an extension of first-order logic with equality with one or more relations, and the properties of the resulting language can be studied; obviously, the same applies when relations between points are considered too. In this paper we ask the question: interpreted over linear orders, how many and which expressively different languages can be obtained by enriching first-order logic with relations between intervals, between points, and between intervals and points? Since, as we shall see, there are 26 different relations (including equality of both sorts) between points, intervals, and points and intervals, 2^{26} is an upper bound on this number. (It is worth noticing that in [MH06] the authors distinguish 30 relations, instead of 26; this is due to the fact that the concepts of the point a starting the interval $[a, b]$ and meeting it are

considered to be different.) However, since certain relations are definable in terms of other ones, the actual number is less and in fact, as we shall show, much less. The answer also depends on our choices of certain semantic parameters, specifically, the class of linear orders over which we construct our interval structures. In this paper, in Part I, we consider the classification problem relative to:

- (i) the class of all linear orders;
- (ii) the class of all weakly discrete linear orders (i.e., orders in which every point with a successor/predecessor has an immediate one).

In Part II of this paper we consider:

- (iii) the class of all dense linear orders;
- (iv) the class of all unbounded linear orders;

Apart from the intrinsic interest and naturalness of this classification problem, its outcome has some important repercussions, principally in the reduction of the number of cases that need to be considered in other problems relating to these languages. For example, it reduces the number of representation theorems that are needed: given the *dual* nature of time intervals (i.e., they can be abstract first-order individuals with specific characteristics, or they can be defined as ordered pairs over a linear order), one of the most important problems that arises is the existence or not of a *representation theorem*. Consider any class of linear orders: given a specific extension of first-order logic with a set of interval relations (such as, for example, *meets* and *during*), does there exist a set of axioms in this language which would constrain (abstract) models of this signature to be isomorphic to concrete ones? Various representation theorems exist in the literature for languages that include interval relations only: van Benthem [vB91], over rationals and with the interval relations *during* and *before*, Allen and Hayes [AH85], for the dense unbounded case without point intervals and for the relation *meets*, Ladkin [Lad78], for point-based structures with a quaternary relation that encodes meeting of two intervals, Venema [Ven91], for structures with the relations *starts* and *finishes*, Goranko, Montanari, and Sciavicco [GMS03], for linear structures with *meets* and *met-by*, Bochman [Boc90], for point-interval structures, and Coetzee [Coe09] for dense structure with *overlaps* and *meets*. Clearly, if two sets of relations give rise to expressively equivalent languages, two separate representations theorems for them are not needed. In which cases are representation theorems still outstanding? Preliminary works that provide similar classifications appeared in [CS11] for first-order languages with equality and only interval-interval relations, and in [CDS12] for points and intervals (with equality between intervals treated on a par with the other relations) but only over the class of all linear orders. Finally, a complete study of first-order interval temporal logics enables a deeper understanding of their modal counterparts based on their shared relational semantics.

Structure of the paper. This paper is structured as follows. Section 2 provides the necessary preliminaries, along with an overview of the general methodology used in this paper. In Section 3 we study the expressive power of the language by analyzing the definability properties of each basic relation in the class *Lin*. Section 4 deals with the ‘other half’ of the picture, that is, undefinability results, and presenting all maximally incomplete sets in this class (i.e., those subsets of relations that do not allow one to define the remaining ones, and are maximal in this sense with respect to the subset relation); we also deal with completeness and incompleteness results for the class of all discrete linear orders in this section. Section 5 presents an account of all our results in a structured way, including the

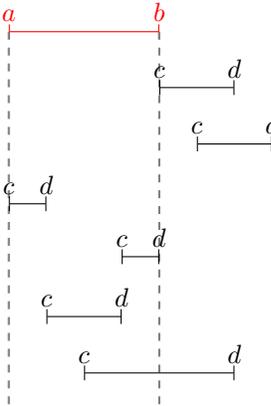
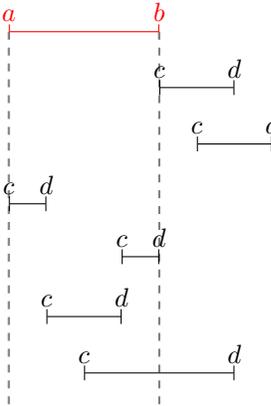
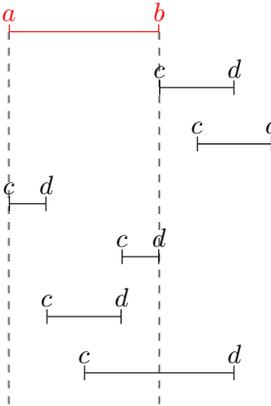
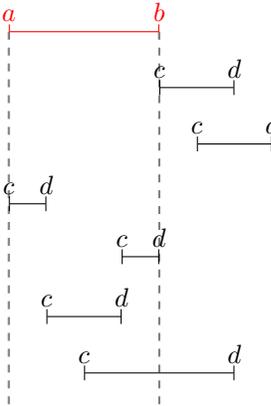
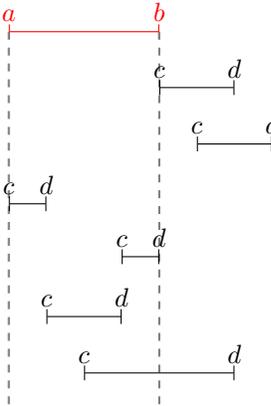
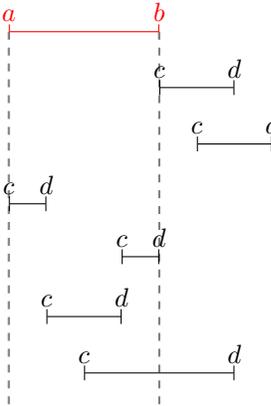
$(meets, m)$	$[a, b]$	34_{ii}	$[c, d] \Leftrightarrow b = c$	
$(before, b)$	$[a, b]$	44_{ii}	$[c, d] \Leftrightarrow b < c$	
$(starts, s)$	$[c, d]$	14_{ii}	$[a, b] \Leftrightarrow a = c, d < b$	
$(finishes, f)$	$[c, d]$	03_{ii}	$[a, b] \Leftrightarrow b = d, a < c$	
$(during, d)$	$[c, d]$	04_{ii}	$[a, b] \Leftrightarrow a < c, d < b$	
$(overlaps, o)$	$[a, b]$	24_{ii}	$[c, d] \Leftrightarrow a < c < b < d$	

Table 1: Interval-interval relations, a.k.a. Allen’s relations. The equality relation is not depicted.

projections of these to natural sub-languages, before concluding. The classes of all dense and the class of all unbounded linearly ordered sets will be treated in Part II (forthcoming).

2. BASICS

2.1. Syntax and semantics. Given a linear order $\mathbb{D} = \langle D, < \rangle$, we call the elements of D *points* (denoted by a, b, \dots) and define an *interval* as an ordered pair $[a, b]$ of points in D , where $a < b$. Abstract intervals will be denoted by I, J, \dots , and so on. Now, as we have mentioned above, there are 13 possible relations, including equality, between any two intervals. From now on, we call these *interval-interval* relations. Besides equality, there are 2 different relations that may hold between any two points (*before* and *after*), called hereafter *point-point* relations, and 5 different relations that may hold between a point and an interval and vice-versa: we call those *interval-point* and *point-interval* relations, respectively, and we use the term *mixed* relations to refer to them indistinctly. Interval-interval relations are exactly Allen’s relations [All83]; point-point relations are the classical relations on a linear order, and mixed relations will be explained below. Traditionally, interval relations are represented by the initial letter of the description of the relation, like m for *meets*. However, when one considers more relations (like point-point and point-interval relations) this notation becomes confusing, and even more so in the presence of more relations, e.g. when one wants to consider interval relations over a *partial order*¹. We introduce the following notation to resolve this issue: an interval $[a, b]$ induces a partition of \mathbb{D} into five regions (see [Lig91]): region 0 which contains all points less than a , region 1 which contains a only, region 2 which contains all the points strictly between a and b , region 3 which contains only b and region 4 which contains the points greater than b . Likewise, a point c induces a partition of \mathbb{D} into 3 pieces: region 0 contains all the points less than c , region 2 contains only c , and region 4 contains all the points greater than c . Interval-interval relations will be denoted by $I\kappa\kappa'_{ii}J$ (where the subscript $_{ii}$ refers to interval-interval relations), where $k, k' \in \{0, 1, 2, 3, 4\}$, and k

¹This paper is focused on linear orders only; nevertheless, it is our intention to complete this study to include the treatment of partial orders also, and, at this stage, we want to make sure that we will be able to keep the notation consistent.

- a *concrete interval structure of signature S* is a relational structure $\mathcal{F} = \langle \mathbb{I}(\mathbb{D}), r_1, r_2, \dots, r_n \rangle$, where $S = \{r_1, \dots, r_n\} \subseteq \mathfrak{J}$, and
- a *concrete point-interval structure of signature S* is a two-sorted relational structure $\mathcal{F} = \langle \mathbb{D}, \mathbb{I}(\mathbb{D}), r_1, r_2, \dots, r_n \rangle$, where $S = \{r_1, \dots, r_n\} \subseteq \mathfrak{R}$.

Since all relations between intervals, points, and all mixed relations are already implicit in $\mathbb{I}(\mathbb{D})$, we shall often simply write $\langle \mathbb{I}(\mathbb{D}) \rangle$ for a concrete interval structure $\langle \mathbb{I}(\mathbb{D}), r_1, r_2, \dots, r_n \rangle$, and $\langle \mathbb{D}, \mathbb{I}(\mathbb{D}) \rangle$ for a concrete point-interval structure $\langle \mathbb{D}, \mathbb{I}(\mathbb{D}), r_1, r_2, \dots, r_n \rangle$; this is in accordance with the standard usage in much of the literature on interval temporal logics. Moreover, we denote by $FO + S$ the language of first-order logic without equality and relation symbols corresponding to the relations in S . Finally, \mathcal{F} is further said to be *of the class C* ($C \in \{\text{Lin}, \text{Den}, \text{Dis}, \text{Unb}, \text{Fin}\}$) when \mathbb{D} belongs to the specific class of linear orders C .

2.2. (Un)definability and Truth Preserving Relations. We describe here the most important tools that we use to classify the expressive power of our (sub-)languages.

Definition 2.4. Let $S \subseteq \mathfrak{R}$, and C a class of linear orders. We say that $FO + S$ *defines* $r \in \mathfrak{R}$ over C , denoted by $FO + S \rightarrow_C r$, if there exists $FO + S$ -formula $\varphi(x, y)$ such that $\varphi(x, y) \leftrightarrow r(x, y)$ is valid on the class of concrete point-interval structures of signature $(S \cup \{r\})$ based on C .

By $FO + S \rightarrow r$ we denote the fact that $FO + S \rightarrow_{\text{Lin}} r$ (and hence $FO + S \rightarrow_C r$ for every $C \in \{\text{Lin}, \text{Den}, \text{Dis}, \text{Unb}, \text{Fin}\}$). Obviously, $FO + S \rightarrow r$ for all $r \in S$.

Definition 2.5. Let $S, S' \subseteq \mathfrak{R}$ and C a class of linear orders. We say that S is:

- *S' -complete over C* (resp., *S' -incomplete over C*) if and only if $FO + S \rightarrow_C r$ for all $r \in S'$ (resp., $FO + S \not\rightarrow_C r$ for some $r \in S'$), and
- *minimally S' -complete over C* (resp., *maximally S' -incomplete over C*) if and only if it is S' -complete (resp., S' -incomplete) over C , and every proper subset (resp., every proper superset) of S is S' -incomplete (resp., S' -complete) over the same class.

The notion of (minimally) r -completeness and (maximally) r -incompleteness over C is immediately deduced from the above one, by taking $S' = \{r\}$ and denoting the latter simply by r . Moreover, one can project the above definitions over some interesting subsets of \mathfrak{R} , such as $\mathfrak{J}, \mathfrak{M}$ or \mathfrak{P} , obtaining relative completeness and incompleteness.

Let $C' \subseteq C$ be two classes of linear orders. Notice that if $FO + S \rightarrow_C r$ then $FO + S \rightarrow_{C'} r$ and, contrapositively, that if $FO + S \not\rightarrow_{C'} r$ then $FO + S \not\rightarrow_C r$. So specifically, if S is S' -complete over C , then it is also S' -complete over C' . Also, if S is S' -incomplete over C' , then it is also S' -incomplete over C . Notice however, that minimality and maximality of complete and incomplete sets does not necessarily transfer between super and subclasses in a similar way. In what follows, in order to prove that $FO + S \not\rightarrow_C r$ for some r and some class C , we shall repeatedly apply the following definition and (rather standard) procedure.

Definition 2.6. Let $\mathcal{F} = \langle \mathbb{D}, \mathbb{I}(\mathbb{D}), S \rangle$ and $\mathcal{F}' = \langle \mathbb{D}', \mathbb{I}(\mathbb{D}'), S \rangle$ be concrete structures where $S \subseteq \mathfrak{R}$. A binary relation $\zeta \subseteq (\mathbb{D} \cup \mathbb{I}(\mathbb{D})) \times (\mathbb{D}' \cup \mathbb{I}(\mathbb{D}'))$ is called a *surjective S -truth preserving relation* if and only if:

- (i) ζ respects sorts, i.e., $\zeta = \zeta_p \cup \zeta_i$, where $\zeta_p \subseteq \mathbb{D} \times \mathbb{D}'$ and $\zeta_i \subseteq \mathbb{I}(\mathbb{D}) \times \mathbb{I}(\mathbb{D}')$;
- (ii) ζ respects the relations in S , i.e., if $(a, a'), (b, b') \in \zeta_p$ and $(I, I'), (J, J') \in \zeta_i$, then:
 - (a) $r(a, b)$ if and only if $r(a', b')$ for every point-point relation $r \in S$;

- (b) $r(I, a)$ if and only if $r(I', a')$ for every interval-point relation $r \in S$;
- (c) $r(I, J)$ if and only if $r(I', J')$ for every interval-interval relation $r \in S$;
- (iii) ζ is total and surjective, i.e.:
 - (a) for every $a \in \mathbb{D}$ (resp., $I \in \mathbb{I}(\mathbb{D})$), there exist $a' \in \mathbb{D}'$ (resp., $I' \in \mathbb{I}(\mathbb{D}')$) such that $(a, a') \in \zeta_p$ (resp., $(I, I') \in \zeta_i$);
 - (b) for every $a' \in \mathbb{D}'$ (resp., $I' \in \mathbb{I}(\mathbb{D}')$), there exist $a \in \mathbb{D}$ (resp., $I \in \mathbb{I}(\mathbb{D})$) such that $(a, a') \in \zeta_p$ (resp., $(I, I') \in \zeta_i$).

If we add to Definition 2.6 the requirement that that ζ should be functional, we obtain nothing but the definition of an isomorphism between two-sorted first-order structures or, equivalently, an isomorphism between single sorted first-order structures with predicates added for ‘point’ and ‘interval’ (see e.g. [Hod93]). As one would expect, surjective S -truth preserving relations preserve the truth of all first-order formulas in signature S . This is stated in Theorem 2.8, below. The reason why we consider only interval-point relations instead of all mixed relations is that, as we shall explain, we can limit ourselves to work without inverse relations, and point-interval relations are the inverse of interval-point ones.

Definition 2.7. If ζ is a surjective S -truth preserving relation, we say that ζ *breaks* $r \notin S$ if and only if there are:

- (i) $(a, a'), (b, b') \in \zeta_p$ such that $r(a, b)$ but $\neg r(a', b')$, if r is point-point, or
- (ii) $(a, a') \in \zeta_p$ and $(I, I') \in \zeta_i$ such that $r(I, a)$ but $\neg r(I', a')$, if r is interval-point, or
- (iii) $(I, I'), (J, J') \in \zeta_i$ such that $r(I, J)$ but $\neg r(I', J')$, if r is interval-interval.

The following result is, as already mentioned, a straightforward generalization of the classical result on the preservation of truth under isomorphism between first-order structures, and it is proved by an easy induction on formulas, using clause (ii) of Definition 2.6 to establish the base case for atomic formulas and clause (iii) for the inductive step for the quantifiers.

Theorem 2.8. *If $\zeta = \zeta_p \cup \zeta_i$ is a surjective S -truth preserving relation between $\mathcal{F} = \langle \mathbb{D}, \mathbb{I}(\mathbb{D}), S \rangle$ and $\mathcal{F}' = \langle \mathbb{D}', \mathbb{I}(\mathbb{D}'), S \rangle$, and $a_1, \dots, a_k \in \mathbb{D}$, $a'_1, \dots, a'_k \in \mathbb{D}$, $I_1, \dots, I_l \in \mathbb{I}(\mathbb{D})$, and $I'_1, \dots, I'_l \in \mathbb{I}(\mathbb{D}')$ are such that $(a_j, a'_j) \in \zeta_p$ for $1 \leq j \leq k$, and $(I_j, I'_j) \in \zeta_i$ for $1 \leq j \leq l$, then for every $FO+S$ formulas $\varphi(x_p^1, \dots, x_p^k, y_i^1, \dots, y_i^l)$ with free variables $x_p^1, \dots, x_p^k, y_i^1, \dots, y_i^l$, we have that*

$$\mathcal{F} \models \varphi(a_1, \dots, a_k, I_1, \dots, I_l) \text{ if and only if } \mathcal{F}' \models \varphi(a'_1, \dots, a'_k, I'_1, \dots, I'_l).$$

Thus, to show that $FO + S \not\models r$ for a given $r \in \mathfrak{R}$, it is sufficient to find two concrete point-interval structures \mathcal{F} and \mathcal{F}' and a surjective S -truth preserving relation ζ between \mathcal{F} and \mathcal{F}' which breaks r . For the readers' convenience, let us refer to surjective S -truth preserving relations as simply S -relations.

Although there are other constructions that could be used to show that relations are not definable in $FO + S$, e.g. elementary embeddings or Ehrenfeucht-Fraïssé games, we have found S -relations sufficient for our purposes in this paper.

2.3. Strategy. The main objective of this paper is to establish all expressively different subsets of \mathfrak{R} (and, then, of $\mathfrak{I}, \mathfrak{M}$ or \mathfrak{P}) over the mentioned classes of linear orders. To this end, for each $r \in \mathfrak{R}$ we compute all expressively different minimally r -complete and all maximally r -incomplete subsets of \mathfrak{R} , from which we can easily deduce all expressively different minimally r -complete and maximally r -incomplete subsets of $\mathfrak{I}, \mathfrak{M}$ and \mathfrak{P} ; minimally \mathfrak{R} - (resp., \mathfrak{I} -, \mathfrak{M} -, \mathfrak{P} -) complete and maximally incomplete subsets are, then, deduced as

a consequence of the above results. The set \mathfrak{R} contains, as we have mentioned, 26 different relations. This means that there are 2^{26} potentially different extensions of first-order logic to be studied. Clearly, unless we design a precise strategy that allows us to reduce the number of results to be proved, the task becomes cumbersome.

As a first simplification principle observe that, since we are working within first-order logic, all inverses of relations are explicitly definable, and hence we only need to assume as primitive a set which contains all relation up to inverses, which implies that point-interval relations can be omitted if we consider all interval-point ones. Accordingly, let \mathfrak{I}^+ be the set of interval-interval relations given in Tab. 1 together with $=_i$, \mathfrak{M}^+ be the set of interval-point relations given in Tab. 2, and let $\mathfrak{P}^+ = \{<, =_p\}$. Lastly let $\mathfrak{R}^+ = \mathfrak{I}^+ \cup \mathfrak{M}^+ \cup \mathfrak{P}^+$.

In order to further reduce the number of results to be presented, consider what follows. The *order dual* of a structure $\mathcal{F} = \langle \mathbb{D}, \mathbb{I}(\mathbb{D}) \rangle$ is the structure $\mathcal{F}^\partial = \langle \mathbb{D}^\partial, \mathbb{I}(\mathbb{D}^\partial) \rangle$ based on the order dual \mathbb{D}^∂ (obtained by reversing the order) of the underlying linear order \mathbb{D} . All classes considered in this paper are closed under taking order duals.

Definition 2.9. The *reversible relations* are exactly the members of the set $\{0_{ip}, 1_{ip}, 3_{ip}, 4_{ip}, 14_{ii}, 03_{ii}\}$. The relations belonging to the complement $\mathfrak{R}^+ \setminus \{0_{ip}, 1_{ip}, 3_{ip}, 4_{ip}, 14_{ii}, 03_{ii}\}$ are called *symmetric*; if, in addition, $r = 2_{ip}$ or $r = 04_{ii}$, then r is said *self-symmetric*. If $r = 0_{ip}$ (resp., $r = 1_{ip}, r = 14_{ii}$), its *reverse* is $r = 4_{ip}$ (resp., $r = 3_{ip}, r = 03_{ii}$), and the other way around. Finally, the *symmetric* S' of a subset $S \subseteq \mathfrak{R}^+$ is obtained by replacing every reversible relation in S with its reverse. We shall use the notation $S \sim S'$ to indicate that sets S and S' are symmetric.

This definition is motivated by the following easily verifiable facts. Let $r \in \mathfrak{R}^+$, \mathcal{F} be a structure, and x and y be elements of \mathcal{F} of the appropriate sorts for r ; then:

- (i) if r is a reversible relation, with reverse r' , then $\mathcal{F} \models r(x, y)$ if and only if $\mathcal{F}^\partial \models r'(x, y)$;
- (ii) if r is self-symmetric, then $\mathcal{F} \models r(x, y)$ if and only if $\mathcal{F}^\partial \models r(x, y)$;
- (iii) if r is a symmetric, but not self-symmetric, relation, then $\mathcal{F} \models r(x, y)$ if and only if $\mathcal{F}^\partial \models r(y, x)$.

The following crucial lemma capitalizes on these facts.

Lemma 2.10. *Let $S, S' \subset \mathfrak{R}^+$ be such that $S \sim S'$. If r is a symmetric relation, then $FO + S \rightarrow r$ if and only if $FO + S' \rightarrow r$. Moreover, if r is a reversible relation with reverse r' , then $FO + S \rightarrow r$ if and only if $FO + S' \rightarrow r'$.*

Proof. Let $S, S' \subset \mathfrak{R}^+$ such that $S \sim S'$. For any $FO + S$ formula φ that defines a given relation (and, therefore, with exactly two free variables), let φ' denote the formula obtained from φ by replacing every occurrence of a reversible relation with its reverse, and by swapping the arguments of every symmetric, but not self-symmetric, relation (occurrences of every self-symmetric relation are left unchanged). Induction on formulas then shows that $\mathcal{F} \models \varphi(x, y)$ (after substituting x, y with elements of the appropriate sorts) if and only if $\mathcal{F}^\partial \models \varphi'(x, y)$, for any structure \mathcal{F} . The base case of the induction is taken care of by the three observations preceding this lemma. Now, suppose that a $FO + S$ formula $\varphi(x, y)$ defines a symmetric relation r . We claim that φ' also defines r . Let \mathcal{F} be an arbitrary structure of signature $S \cup \{r\}$. Then $\mathcal{F}^\partial \models \varphi(x, y) \leftrightarrow r(x, y)$, and hence $\mathcal{F} \models \varphi'(x, y) \leftrightarrow r(y, x)$ if r is not self-symmetric, and $\mathcal{F} \models \varphi'(x, y) \leftrightarrow r(x, y)$ otherwise. Next, suppose that the $FO + S$ formula $\varphi(x, y)$ defines a reversible relation r . We claim that φ' defines its reverse r' . Let \mathcal{F}

<pre> proc Undef ($r \in \mathfrak{R}^+$) { for all $S \subset \mathfrak{R}^+$ { { $S = \text{Closure}(S)$; { if ($(r \notin S)$ and (S is maximal)) { then list S } } } </pre>	<pre> proc Closure ($set\ S, rules\ def_rules$) { while (S changes) { { for all $1 \leq i \leq size(def_rules)$ { { if ($def_rules[i]$ applies) { { then $S = \text{Apply}(S, def_rule[i])$ } } } } } return S </pre>
--	---

Figure 1: Pseudo-code to identify maximally r -incomplete sets.

be an arbitrary structure of signature $S \cup \{r\}$. Then $\mathcal{F}^\partial \models \varphi(x, y) \leftrightarrow r(x, y)$, and, hence, $\mathcal{F} \models \varphi'(x, y) \leftrightarrow r'(x, y)$. \square

In conclusion, we can limit our attention to 14 out of 26 relations by disregarding the inverses of relations in \mathfrak{R}^+ , and we do not need to explicitly analyze complete and incomplete sets for 3_{ip} , 4_{ip} , and 03_{ii} as those correspond exactly to symmetric of complete and incomplete sets for 0_{ip} , 1_{ip} , and 14_{ii} , respectively. This means that only 11 relations are to be analyzed (which we can refer to as *explicit* relations).

Even under the mentioned simplifications, there is a huge number of results to be presented and displayed. Let r be anyone of the explicit relations. In order to correctly identifying all minimally r -complete sets ($\text{mcs}(r)$), we need to know all maximally r -incomplete sets ($\text{MIS}(r)$) over the same class, and the other way around. To this end, we proceed in the following way:

- (1) fixed a class of linearly ordered sets and an explicit relation r , we first guess the r -complete subsets of \mathfrak{R}^+ , obtaining a first approximation of the definability rules for r ;
- (2) then, we apply the algorithm in Fig. 1, which uses the set of r -complete subsets of \mathfrak{R}^+ (the parameter def_rules) to obtain a first approximation of the maximally r -incomplete sets;
- (3) after that, we prove that every R_1, R_2, \dots, R_k listed as a maximally r -incomplete set is actually r -incomplete, and, if not, we repeat from step 1, using the acquired knowledge to update the set of r -complete subsets of \mathfrak{R}^+ ;
- (4) at this point, the sets $S_1, S_2, \dots, S_{k'}$ listed at step 1 are, actually, all minimally r -complete: for each i , S_i is r -complete by definition, and if there was a r -complete set $S \subset S_i$, then for some R_j listed as maximally r -incomplete set we could not prove its r -incompleteness. Therefore, $S_1, S_2, \dots, S_{k'}$ are all minimally r -complete, and, as a consequence, R_1, R_2, \dots, R_k are all maximally r -incomplete.

Once the above process is completed for every relation, we can then easily deduce all minimally \mathfrak{R}^+ -complete and all maximally \mathfrak{R}^+ -incomplete sets, to complete the picture. A similar procedure works for \mathfrak{T}^+ , \mathfrak{M}^+ , and \mathfrak{B}^+ .

The most common notational conventions used in the paper are listed in Tab. 3.

3. COMPLETENESS RESULTS IN THE CLASS Lin

In this section, we start analyzing the inter-definability of relations in \mathfrak{R}^+ , and. In particular, we consider the case in which we do not assume any particular property of the underlying linear order. It is convenient to begin by focusing our attention to the sub-languages induced by \mathfrak{M}^+ and \mathfrak{T}^+ ; notice, in this respect, that while the semantic counterpart of the

$\mathbb{D}, \mathbb{D}', \dots$	(generic) linearly ordered sets
x_p, y_p, \dots	first-order variables for points
x_i, y_i, \dots	first-order variables for intervals
x, y, \dots	first-order variables of any sort
<i>before, ...</i>	relations in text are <i>emphasized</i>
$\mathcal{F}, \mathcal{F}', \dots$	(generic) concrete (point-)interval structures
S, S', \dots	(generic) subsets of \mathfrak{R} -relations
$\zeta (\zeta_p, \zeta_i)$	surjective relation (for points, for intervals)
$Id_p(Id_i)$	‘identity’ relation over points (intervals)
C, C'	(generic) class of linearly ordered sets
$FO + S \rightarrow_C r$	S defines r (w.r.t. the class C)
$S \sim S'$	S and S' are symmetric
$a \in \mathbb{D}$	a is a point of D , where $\mathbb{D} = (D, <)$
<u>S</u>	in the text, a new proof case is <u>underlined</u>
r	generic relation
mcs (mcs (r))	minimally complete set (minimally r -complete set)
MIS (MIS (r))	maximally incomplete set (maximally r -incomplete set)

Table 3: Notational conventions used in this paper.

sub-language $FO + \mathfrak{J}^+$ is essentially single-sorted (it is interpreted on interval structures), in the case of $FO + \mathfrak{M}^+$ (interpreted on point-interval ones) both sorts are necessary. The results for \mathfrak{M}^+ and \mathfrak{J}^+ can also be found in [CDS12].

Throughout our analysis we shall make extensive use of the following schema for the definability part: for every definability equation $r(x, y) \leftrightarrow \varphi(x, y)$, we denote by $\varphi(x, y)$ the right-hand part of the definition, indicating that x, y are the only free variables in it; we then take a generic point-interval structure $\mathcal{F} = \langle \mathbb{D}, \mathbb{I}(\mathbb{D}) \rangle$, and show that $\mathcal{F} \models \varphi(x, y)$ (where x, y have been instantiated with suitable constants of the right type) if and only if $r(x, y)$ (again, after the due instantiation). We shall therefore omit the specification of these symbols and their meaning, as it remains the same in every proof. In order to make the text more readable, we shall present the results for each relation r by means a table with at most four columns under the following headings:

- (1) *Proved*, which contains those r -complete sets for which we give an explicit proof in the corresponding lemma;
- (2) *Symmetric*, which contains, for each r -complete set listed in the *Proved* column, its symmetric one (if r is symmetric);
- (3) *Implied*, which contains all r -complete sets that can be deduced from those in the first two columns plus the definability results presented earlier in the paper;
- (4) *Deduction Chain*, which is not empty if *Implied* is not empty, and it makes the chain of deductions explicit.

When all (explicit) relations have been treated, we shall present the result of applying the algorithm in Fig. 1, and we shall prove the undefinability results, completing the process and consequently proving the minimality of the complete sets.

Relation	Proved	Symmetric	Implied	Deduction Chain
34_{ii}	$\{14_{ii}, 24_{ii}\}$ $\{14_{ii}, 03_{ii}\}$ $\{14_{ii}, 44_{ii}\}$ $\{14_{ii}, 04_{ii}\}$	$\{03_{ii}, 24_{ii}\}$ - $\{03_{ii}, 44_{ii}\}$ $\{03_{ii}, 04_{ii}\}$		
14_{ii}	34_{ii}	-	$\{03_{ii}, 24_{ii}\}$ $\{03_{ii}, 44_{ii}\}$ $\{03_{ii}, 04_{ii}\}$	34_{ii} 34_{ii} 34_{ii}
24_{ii}	34_{ii}	-	$\{14_{ii}, 03_{ii}\}$ $\{14_{ii}, 44_{ii}\}$ $\{14_{ii}, 04_{ii}\}$ $\{03_{ii}, 44_{ii}\}$ $\{03_{ii}, 04_{ii}\}$	34_{ii} 34_{ii} 34_{ii} 34_{ii} 34_{ii}
04_{ii}	34_{ii}	-	$\{14_{ii}, 24_{ii}\}$ $\{14_{ii}, 03_{ii}\}$ $\{14_{ii}, 44_{ii}\}$ $\{03_{ii}, 24_{ii}\}$ $\{03_{ii}, 44_{ii}\}$	34_{ii} 34_{ii} 34_{ii} 34_{ii} 34_{ii}
44_{ii}	34_{ii}	-	$\{14_{ii}, 24_{ii}\}$ $\{14_{ii}, 03_{ii}\}$ $\{14_{ii}, 04_{ii}\}$ $\{03_{ii}, 24_{ii}\}$ $\{03_{ii}, 04_{ii}\}$	34_{ii} 34_{ii} 34_{ii} 34_{ii} 34_{ii}
$=_i$	34_{ii} 14_{ii}	- 03_{ii}		

Table 4: The spectrum of the $\text{mcs}(r)$, for each $r \in \mathcal{J}^+$. - Class: Lin.
(Lemma 3.1)

3.1. Definability in \mathcal{J}^+ and in \mathcal{M}^+ . We now study the minimal definability of \mathcal{J}^+ relations, first, and, then, of \mathcal{M}^+ relations.

Lemma 3.1. *Tab. 4 is correct.*

Proof. First, we prove the \mathcal{J}^+ -completeness of $\{34_{ii}\}$, as well as the fact that every relation in \mathcal{J}^+ is $=_i$ -complete, and, then, we prove that every other subset is 34_{ii} -complete; completeness for the remaining relations is a mere consequence, as it can be seen in the table. As for the first step, we simply express every other interval-interval relation, as follows:

$$\begin{aligned}
x_i 44_{ii} y_i &\leftrightarrow \exists z_i (x_i 34_{ii} z_i \wedge z_i 34_{ii} y_i) \\
x_i 14_{ii} y_i &\leftrightarrow \exists z_i (x_i 34_{ii} z_i \wedge \forall k_i ((z_i 34_{ii} k_i \leftrightarrow y_i 34_{ii} k_i) \wedge (k_i 34_{ii} x_i \leftrightarrow k_i 34_{ii} y_i))) \\
x_i 03_{ii} y_i &\leftrightarrow \exists z_i (z_i 34_{ii} x_i \wedge \forall k_i ((k_i 34_{ii} z_i \leftrightarrow k_i 34_{ii} y_i) \wedge (x_i 34_{ii} k_i \leftrightarrow y_i 34_{ii} k_i))) \\
x_i 04_{ii} y_i &\leftrightarrow \exists z_i (x_i 14_{ii} z_i \wedge z_i 03_{ii} y_i) \\
x_i 24_{ii} y_i &\leftrightarrow \exists z_i (z_i 03_{ii} x_i \wedge z_i 14_{ii} y_i) \\
x_i =_i y_i &\leftrightarrow \forall z_i ((x_i 34_{ii} z_i \leftrightarrow y_i 34_{ii} z_i) \wedge (z_i 34_{ii} x_i \leftrightarrow z_i 34_{ii} y_i)) \\
x_i =_i y_i &\leftrightarrow \forall z_i ((x_i 14_{ii} z_i \leftrightarrow y_i 14_{ii} z_i) \wedge (z_i 14_{ii} x_i \leftrightarrow z_i 14_{ii} y_i))
\end{aligned}$$

All above equations but the last two are very simple, and do not require further explanation. Moreover, the \mathcal{J}^+ -completeness of $\{34_{ii}\}$ is a known result (except for equality between intervals): it has been formally proved in [AH85] assuming density and unboundedness of

the structure, but a closer look shows that those additional hypothesis were not needed. As for the fact that $=_i$ can be expressed with $\{34_{ii}\}$, assume that $\mathcal{F} \models \varphi([a, b], [c, d])$. We wish to show that $a = c$ and $b = d$. Suppose, by way of contradiction, that $a \neq c$. If $a < c$, then the interval $[a, c]$ *meets* the interval $[c, d]$, but it does not *meet* $[a, b]$, contradicting the second conjunct of φ ; if $c < a$, a similar argument shows that the second conjunct fails. Symmetrically, we can prove that $b \neq d$ leads to a contradiction with the first conjunct of φ . Also, it is obvious that if $[a, b] = [c, d]$, they *meet* and are *met by* exactly the same intervals. The case of $\{14_{ii}\}$ is very similar. The remaining cases are more difficult, and require some non-trivial definitions. First, let us observe that having the weaker relation:

$$34_{ii} \cup 44_{ii}$$

is enough to get 34_{ii} , and viceversa: in fact, $[a, b]$ *meets* $[b, c]$ if and only if $[a, b]34_{ii} \cup 44_{ii}[b, c]$ and no other interval in between them has the same property. This is explicitly expressed by the formula $x_i 34_{ii} y_i \leftrightarrow x_i (34_{ii} \cup 44_{ii}) y_i \wedge \neg \exists z_i (x_i (34_{ii} \cup 44_{ii}) z_i \wedge z_i (34_{ii} \cup 44_{ii}) y_i)$. We use this observation in the rest of this proof, as in the remaining cases we are able to define precisely the relation $34_{ii} \cup 44_{ii}$:

$$x_i 34_{ii} \cup 44_{ii} y_i \leftrightarrow \begin{cases} \neg(x_i 24_{ii} y_i \vee y_i 24_{ii} x_i \vee x_i 14_{ii} y_i \vee y_i 14_{ii} x_i) \wedge \{14_{ii}, 24_{ii}\} \\ \exists z_i (x_i 14_{ii} z_i \wedge \neg(y_i 14_{ii} z_i)) \wedge \\ \forall z_i (y_i 14_{ii} z_i \rightarrow \neg(x_i 24_{ii} z_i)) \wedge \\ \forall z_i t_i ((z_i 14_{ii} y_i \wedge x_i 14_{ii} t_i) \rightarrow \neg(z_i 24_{ii} t_i)) \wedge \\ \forall z_i (x_i 14_{ii} z_i \rightarrow \neg(y_i 24_{ii} z_i)) \wedge \\ \forall z_i t_i ((y_i 14_{ii} z_i \wedge x_i 14_{ii} t_i) \rightarrow \neg z_i 24_{ii} t_i) \\ \\ \exists z_i (x_i 14_{ii} z_i \wedge y_i 03_{ii} z_i) \wedge \{14_{ii}, 03_{ii}\} \\ \neg \exists z_i (z_i 03_{ii} x_i \wedge z_i 14_{ii} y_i) \\ \\ \neg \exists z_i (\varphi_1(z_i, y_i) \wedge \varphi_2(z_i, x_i)) \{14_{ii}, 44_{ii}\} \end{cases}$$

where:

$$\begin{aligned} \varphi_1(x_i, y_i) &\leftrightarrow x_i 14_{ii} y_i \vee y_i 14_{ii} x_i \vee \forall z_i ((z_i 14_{ii} x_i \leftrightarrow z_i 14_{ii} y_i) \wedge (x_i 14_{ii} z_i \leftrightarrow y_i 14_{ii} z_i)) \\ \varphi_2(x_i, y_i) &\leftrightarrow \forall z_i (x_i 44_{ii} z_i \leftrightarrow y_i 44_{ii} z_i) \wedge (\neg \exists z_i (x_i 14_{ii} z_i) \leftrightarrow \neg \exists z_i (y_i 14_{ii} z_i)) \end{aligned}$$

As for the case of $\{14_{ii}, 24_{ii}\}$, assume $\mathcal{F} \models \varphi([a, b], [c, d])$; we want to prove that $b \leq c$. It is easy to see that the requirement excludes every other possibility. First, observe that $[a, b]$ and $[c, d]$ cannot *overlap* each other, nor can *start* each other, thanks to the first line. The point b cannot be the last one of the model as there must be an interval *started by* x_i (second line), and since such an interval cannot be *started by* y_i (second line), x_i and y_i cannot be equal. If y_i was *during* x_i , there would be an interval that is *started by* y_i and *overlapped by* x_i , which is a contradiction (third line). If x_i was *during* y_i , there would be an interval z_i that *starts* y_i and an interval t_i *started by* x_i , and z_i would *overlap* t_i , which is, again, a contradiction (fourth line). It is then easy to see that x_i and y_i cannot *finish* each other (third and fifth line); finally we would have a contradiction if $c \leq a$ (fifth line). Thus, the only remaining possibility is the correct one. Conversely, if we assume that $b \leq c$, it is straightforward to see that all requirements are respected. Let us now consider the case $\{14_{ii}, 44_{ii}\}$, which is slightly harder. Consider, first, the definition of φ_1 : it is easy to

Relation	Proved	Implied	Deduction Chain
0_{ip}	$\{1_{ip}, 2_{ip}\}$ $\{1_{ip}, 3_{ip}\}$	$\{1_{ip}, 4_{ip}\}$ $\{2_{ip}, 3_{ip}\}$ $\{2_{ip}, 4_{ip}\}$	$0_{ip} \cup 2_{ip}, 3_{ip}$ 4_{ip} $0_{ip} \cup 3_{ip}, 1_{ip}$
1_{ip}	$\{0_{ip}, 3_{ip}\}$	$\{2_{ip}, 3_{ip}\}$	$0_{ip}, 4_{ip}$
2_{ip}	$\{0_{ip}, 4_{ip}\}$ $\{1_{ip}, 3_{ip}\}$	$\{0_{ip}, 3_{ip}\}$ $\{1_{ip}, 4_{ip}\}$	1_{ip} 0_{ip}

Table 5: The spectrum of the $mcs(r)$, for each $r \in \mathfrak{M}^+$. - Class: Lin.
(Lemma 3.2)

see that if \mathcal{F} is concrete interval structure, then $\mathcal{F} \models \varphi_1([a, b], [c, d])$ if and only if $a = c$. Let us now analyze the definition of φ_2 . Consider $[a, b], [c, d] \in \mathcal{F}$. If $b = d$ then it is clear that $\mathcal{F} \models \varphi_2([a, b], [c, d])$. Suppose that $\mathcal{F} \models \varphi_2([a, b], [c, d])$. We claim that $b = d$. Suppose, by way of contradiction that $b \neq d$. As $\varphi_2(x, y) \leftrightarrow \varphi_2(y, x)$ we may assume that $b < d$. If d is the greatest point of the linear order \mathbb{D} then the last conjunct of φ_2 does not hold. If there is a point e which is greater than d , then $[a, b]44_{ii}[d, e]$ and $\neg[c, d]44_{ii}[d, e]$, which means that the first conjunct of φ_2 does not hold. So we have that $\mathcal{F} \models \varphi_2([a, b], [c, d])$ if and only if $b = d$. Finally, we want to show that $\mathcal{F} \models [a, b]34_{ii} \cup 44_{ii}[c, d]$ if and only if $\mathcal{F} \models \varphi([a, b], [c, d])$ where φ denotes the right-hand part of the last equivalence considered for this set. Assume that $\mathcal{F} \models \varphi([a, b], [c, d])$. If $c < b$ then $z = [c, b]$ witnesses the failure of $\varphi([a, b], [c, d])$. So $b \leq c$ and hence $\mathcal{F} \models [a, b]34_{ii} \vee 44_{ii}[c, d]$. Now assume that $\mathcal{F} \models [a, b]34_{ii} \cup 44_{ii}[c, d]$, i.e., $b \leq c$. If $\mathcal{F} \models \varphi_1(z_i, [c, d])$, then $z_i = [c, e]$ with $c < e$ which implies $b < e$. So $\mathcal{F} \models \neg\varphi_2(z_i, [a, b])$. Therefore we have $\mathcal{F} \models \varphi([a, b], [c, d])$. For the case $\{14_{ii}, 03_{ii}\}$, assume that $\mathcal{F} \models \varphi([a, b], [c, d])$. We wish to show that $\mathcal{F} \models [a, b]34_{ii} \cup 44_{ii}[c, d]$, i.e., that $b \leq c$. Suppose, by way of contradiction, that $c < b$. By assumption, there exists an interval $z_i = [e, f]$ such that $a = e < b < f$ and $e < c < d = f$. Then $a < c < b < d$, hence $[c, b]03_{ii}[a, b]$ and $[c, b]14_{ii}[c, d]$, contradicting $\mathcal{F} \models \neg\exists z_i((z_i 03_{ii}[a, b]) \wedge (z_i 14_{ii}[c, d]))$. Conversely, suppose that $\mathcal{F} \models [a, b]34_{ii} \cup 44_{ii}[c, d]$, i.e., $a < b \leq c < d$. Then the interval $z_i = [a, d]$ witnesses the first conjunct of the definition. Moreover, for any z_i , if $z_i 03_{ii}[a, b]$ then $z_i = [a', b]$. Then $\neg(z_i 14_{ii}[c, d])$, as $b \leq c$, proving that the second conjunct also holds. Finally, as for the set $\{14_{ii}, 04_{ii}\}$, we can easily see that it is 34_{ii} -complete by means of an indirect definition, that is, by defining 24_{ii} :

$$x_i 24_{ii} y_i \leftrightarrow \exists z_i (x_i 14_{ii} z_i \wedge \neg(y_i 04_{ii} z_i) \wedge \exists t_i (t_i 04_{ii} z_i \wedge t_i 14_{ii} y_i)) \wedge \exists t_i (t_i 14_{ii} y_i \wedge \forall w_i (x_i 04_{ii} w_i \rightarrow t_i 04_{ii} w_i))$$

whose correctness is immediate. □

We now focus our attention to \mathfrak{M}^+ . Recall that models here are based on point-interval structures; we are therefore allowed to define interval-interval relations whenever we need them.

Lemma 3.2. *Tab. 5 is correct.*

Proof. Let us focus, first, on 0_{ip} , and consider the following definitions:

$$x_i 0_{ip} y_p \leftrightarrow \begin{cases} \exists z_i w_p (z_i 1_{ip} y_p \wedge x_i 1_{ip} w_p \wedge z_i 2_{ip} w_p) & \{1_{ip}, 2_{ip}\} \\ \exists z_i w_p (x_i 1_{ip} w_p \wedge z_i 3_{ip} w_p \wedge z_i 1_{ip} y_p) & \{1_{ip}, 3_{ip}\} \end{cases}$$

The two cases above, namely $\{1_{ip}, 2_{ip}\}$ and $\{1_{ip}, 3_{ip}\}$ are almost immediate to see. Now, consider the case $\{1_{ip}, 4_{ip}\}$. We can prove that it defines 0_{ip} by exclusion. As a matter of fact, we can see that:

$$x_i 0_{ip} \cup 2_{ip} y_p \leftrightarrow \exists z_i (\forall w_p (z_i 4_{ip} w_p \leftrightarrow x_i 4_{ip} w_p) \wedge z_i 1_{ip} z_p) \wedge \neg(x_i 1_{ip} y_p)$$

In this way, we have that 3_{ip} is definable by difference:

$$x_i 3_{ip} y_p \leftrightarrow \neg(x_i 1_{ip} y_p \vee x_i 4_{ip} y_p \vee x_i 0_{ip} \cup 2_{ip} y_p)$$

and, therefore, the set is 0_{ip} -complete by using $\{1_{ip}, 3_{ip}\}$. Let us now consider the case of $\{2_{ip}, 3_{ip}\}$. To deal with it, we first observe that this set is 4_{ip} -complete, because $\{1_{ip}, 2_{ip}\}$ defines 0_{ip} and we can then use Lemma 2.10. Now, we can directly define 0_{ip} :

$$x_i 0_{ip} y_p \leftrightarrow \begin{aligned} & \neg(x_i 2_{ip} y_p) \wedge \neg(x_i 3_{ip} y_p) \wedge \neg(x_i 4_{ip} y_p) \wedge & \{2_{ip}, 3_{ip}\} \\ & \exists z_i (\forall w_p (z_i 4_{ip} w_p \leftrightarrow x_i 4_{ip} w_p) \wedge \forall w_p (x_i 2_{ip} w_p \rightarrow z_i 2_{ip} w_p) \wedge \\ & \quad \exists w_p (z_i 2_{ip} w_p \wedge \neg(x_i 2_{ip} w_p)) \wedge \neg(z_i 2_{ip} y_p)) \end{aligned}$$

As for the case $\{2_{ip}, 4_{ip}\}$, we reason in a similar way. By slightly modifying the above definition, we obtain a weaker relation:

$$x_i 0_{ip} \cup 3_{ip} y_p \leftrightarrow \begin{aligned} & \neg(x_i 2_{ip} y_p) \wedge \neg(x_i 4_{ip} y_p) \wedge & \{2_{ip}, 4_{ip}\} \\ & \forall z_i ((\forall w_p (z_i 4_{ip} w_p \leftrightarrow x_i 4_{ip} w_p) \wedge \forall w_p (x_i 2_{ip} w_p \rightarrow z_i 2_{ip} w_p) \wedge \\ & \quad \exists w_p (z_i 2_{ip} w_p \wedge \neg(x_i 2_{ip} w_p))) \rightarrow \neg(z_i 2_{ip} y_p)) \end{aligned}$$

Then, 1_{ip} is defined by difference, and 0_{ip} -completeness becomes a consequence of the 0_{ip} -completeness of $\{1_{ip}, 2_{ip}\}$, seen above. Let us consider the 1_{ip} -completeness. First, as for $\{0_{ip}, 3_{ip}\}$ we have that:

$$x_i 1_{ip} y_p \leftrightarrow \begin{aligned} & \neg(x_i 0_{ip} y_p) \wedge \neg(x_i 3_{ip} y_p) \wedge & \{0_{ip}, 3_{ip}\} \\ & \neg \exists z_i (z_i 3_{ip} y_p \wedge \forall w_p (z_i 0_{ip} w_p \leftrightarrow x_i 0_{ip} w_p)) \end{aligned}$$

Then, for the case of $\{2_{ip}, 3_{ip}\}$, we already know that this set is 0_{ip} -complete, and therefore it must be also 1_{ip} -complete thanks to the above argument. We end this proof by analyzing the 2_{ip} -complete sets:

$$x_i 2_{ip} y_p \leftrightarrow \begin{cases} \begin{aligned} & \neg(x_i 0_{ip} y_p) \wedge \neg(x_i 4_{ip} y_p) \wedge & \{0_{ip}, 4_{ip}\} \\ & \exists z_i z_p (\neg(x_i 0_{ip} z_p) \wedge \neg(x_i 4_{ip} z_p) \wedge \neg(z_i 0_{ip} y_p) \wedge \neg(z_i 4_{ip} y_p) \wedge z_i 0_{ip} z_p) \\ & \exists z_i z_p (\neg(x_i 0_{ip} z_p) \wedge \neg(x_i 4_{ip} z_p) \wedge \neg(z_i 0_{ip} y_p) \wedge \neg(z_i 4_{ip} y_p) \wedge z_i 4_{ip} z_p) \end{aligned} \\ \\ \begin{aligned} & \exists z_i z_p (x_i 1_{ip} z_p \wedge z_i 1_{ip} z_p \wedge z_i 3_{ip} y_p) \wedge & \{1_{ip}, 3_{ip}\} \\ & \exists z_i z_p (z_i 1_{ip} y_p \wedge z_i 3_{ip} z_p \wedge x_i 3_{ip} z_p) \end{aligned} \end{cases}$$

Proved	Symmetric
$\{<\}$	-
$\{1_{ip}\}$	$\{3_{ip}\}$

Proved	Symmetric
$\{0_{ip}, 2_{ip}\}$	$\{2_{ip}, 4_{ip}\}$
$\{0_{ip}, 3_{ip}\}$	$\{1_{ip}, 4_{ip}\}$
$\{0_{ip}, 4_{ip}\}$	-
$\{1_{ip}, 2_{ip}\}$	$\{2_{ip}, 3_{ip}\}$
$\{1_{ip}, 3_{ip}\}$	-
$\{14_{ii}\}$	$\{03_{ii}\}$
$\{34_{ii}\}$	-

Table 6: The spectrum of the $\text{mcs}(=_{ip})$ (left) and of the $\text{mcs}(=_{ii})$ (right). - Class: Lin. (Lemma 3.3 and Lemma 3.4)

All of the above are easy to prove. Also, the correctness of the remaining two sets is immediate: from $\{0_{ip}, 3_{ip}\}$ we define 1_{ip} and, then, we use $\{1_{ip}, 3_{ip}\}$, and from $\{1_{ip}, 4_{ip}\}$ we define 0_{ip} , and, then, we use $\{0_{ip}, 4_{ip}\}$. \square

This concludes our preliminary analysis of the expressiveness of our languages when we limit ourselves to relation in \mathfrak{T}^+ and \mathfrak{M}^+ . We shall use these results in the rest of this section, dealing with the expressiveness within \mathfrak{R}^+ .

3.2. Definability in \mathfrak{R}^+ . In the rest of this section, we assume that the set of relations is \mathfrak{R}^+ ; unlike the previous results, we shall treat the relations one by one. We begin our study by considering those sets that define the equality between points; then we move to the equality between intervals, which is no more complicate than the previous one, although there are more ways to define $=_i$ than to define $=_p$.

Lemma 3.3. *Tab. 6 (left) is correct.*

Proof. Consider the following definitions

$$x_p =_p y_p \leftrightarrow \begin{cases} \neg(x_p < y_p) \wedge \neg(y_p < x_p) & \{<\} \\ \forall x_i(x_i 1_{ip} x_p \leftrightarrow x_i 1_{ip} y_p) & \{1_{ip}\} \end{cases}$$

The case of $\{<\}$ is trivial. As for the case of $\{1_{ip}\}$; suppose that $\mathcal{F} \models \varphi(a, b)$. Clearly, it implies that \overline{a} is the starting point of an interval if and only if b is the starting point of that interval. Hence $a = b$. On the other hand it is obvious that if $a = b$, then $\mathcal{F} \models \varphi(a, b)$. \square

Lemma 3.4. *Tab. 6 (right) is correct.*

Proof. Consider the following definition:

$$x_i =_i y_i \leftrightarrow \forall z_p(x_i r_{ip} z_p \leftrightarrow y_i r_{ip} z_p) \wedge \forall z_p(x_i r'_{ip} z_p \leftrightarrow y_i r'_{ip} z_p) \quad \{r_{ip}, r'_{ip}\}$$

where $\{r_{ip}, r'_{ip}\} = S$, for any S in the left-hand part of the table with $S \subseteq \mathfrak{M}^+$. All such cases are based on the same, simple observation: in order to constrain two intervals to be the same

Proved	Symmetric	Implied	Deduction Chain
$\{0_{ip}, 1_{ip}\}$	$\{3_{ip}, 4_{ip}\}$	$\{1_{ip}, 14_{ii}, 24_{ii}\}$	$=_i$ (Section 3.1)
$\{0_{ip}, 3_{ip}\}$	$\{1_{ip}, 4_{ip}\}$	$\{1_{ip}, 14_{ii}, 04_{ii}\}$	$=_i$ (Section 3.1)
$\{1_{ip}, 2_{ip}\}$	$\{2_{ip}, 3_{ip}\}$	$\{1_{ip}, 14_{ii}, 44_{ii}\}$	$=_i$ (Section 3.1)
$\{1_{ip}, 3_{ip}\}$	-	$\{3_{ip}, 24_{ii}, 03_{ii}\}$	$=_i$ (Section 3.1)
$\{1_{ip}, 03_{ii}\}$	$\{3_{ip}, 14_{ii}\}$	$\{3_{ip}, 03_{ii}, 04_{ii}\}$	$=_i$ (Section 3.1)
$\{1_{ip}, 34_{ii}\}$	$\{3_{ip}, 34_{ii}\}$	$\{3_{ip}, 03_{ii}, 44_{ii}\}$	$=_i$ (Section 3.1)
$\{1_{ip}, 04_{ii}, =_i\}$	$\{3_{ip}, 04_{ii}, =_i\}$		
$\{1_{ip}, 24_{ii}, =_i\}$	$\{3_{ip}, 24_{ii}, =_i\}$		
$\{1_{ip}, 44_{ii}, =_i\}$	$\{3_{ip}, 44_{ii}, =_i\}$		

Table 7: The spectrum of the $\text{mcs}(<)$. - Class: Lin.
(Lemma 3.5)

interval, it suffices to fix the two endpoints. This is to say that, for each side of the intervals, we can simply express the fact that they have the same sets of points in a given point-interval relation with it. So, for example, consider $\{0_{ip}, 2_{ip}\}$. Assume that $\mathcal{F} \models \varphi([a, b], [c, d])$: we obtain $a = c$ from the first conjunct of φ , and $b = d$ from the second conjunct. On the other hand, it is immediate to see that if $[a, b] = [c, d]$ then $\mathcal{F} \models \varphi([a, b], [c, d])$. The basic idea is now clear: by means of 0_{ip} we fix the left endpoints, and by means of 2_{ip} we fix the right endpoint (in this particular case, 2_{ip} serves the right side, but, for example, in the case of $\{2_{ip}, 4_{ip}\}$, it would serve the left one). Notice that the only pairs missing from the list (and the list of symmetric sets) are $\{0_{ip}, 1_{ip}\}$ and its symmetric one, for which this idea does not apply (they are, in fact, $=_i$ -incomplete). The remaining definitions are already included in Lemma 3.1. \square

The case of ‘less then’ between point is the first non-trivial case, as there are already many possible different definitions.

Lemma 3.5. *Tab. 7 is correct.*

Proof. Consider, first, the following definitions:

$$x_p < y_p \leftrightarrow \left\{ \begin{array}{ll} \exists x_i(x_i 1_{ip} x_p \wedge x_i 3_{ip} y_p) & \{1_{ip}, 3_{ip}\} \\ (\neg \exists z_i(z_i 1_{ip} y_p) \wedge \exists z_i(z_i 1_{ip} x_p)) \vee \exists z_i(z_i 0_{ip} x_p \wedge z_i 1_{ip} y_p) & \{0_{ip}, 1_{ip}\} \\ (\exists x_i(x_i 1_{ip} x_p) \wedge \neg \exists x_i(x_i 1_{ip} y_p)) \vee \exists x_i(x_i 1_{ip} x_p \wedge x_i 2_{ip} y_p) & \{1_{ip}, 2_{ip}\} \\ \exists z_i(z_i 3_{ip} y_p \wedge \forall k_i(k_i 3_{ip} x_p \rightarrow \neg \forall z_p(z_i 0_{ip} z_p \leftrightarrow k_i 0_{ip} z_p)) \wedge \neg(z_i 3_{ip} x_p)) & \{0_{ip}, 3_{ip}\} \\ (\exists x_i(x_i 1_{ip} x_p) \wedge \neg \exists x_i(x_i 1_{ip} y_p)) \vee \exists x_i y_i(x_i 1_{ip} x_p \wedge y_i 1_{ip} y_p \wedge y_i 03_{ii} x_i) & \{1_{ip}, 03_{ii}\} \end{array} \right.$$

The case of $\{1_{ip}, 3_{ip}\}$ is straightforward. Consider the case of $\{0_{ip}, 1_{ip}\}$. Assume that $\mathcal{F} \models \varphi(a, b)$. If the first disjunct of φ is satisfied then b is the largest point of the linear order \mathbb{D} and a is not the largest point of \mathbb{D} which implies $a < b$ as required. If the second disjunct of φ holds, witnessed by the interval $[c, d]$, then $a < c$ and $b = c$, again leading to $a < b$. On the other hand assume $a < b$. If b is the largest point of \mathbb{D} then a is not, and hence the first

disjunct of φ holds. If b is not the largest point of the the linear order \mathbb{D} then we can pick $c \in \mathbb{D}$ with $b < c$ and the interval $[b, c]$ witnesses the second disjunct of φ . Now consider the case $\{0_{ip}, 3_{ip}\}$, and assume, first, that $\mathcal{F} \models \varphi(a, b)$. So, some interval z_i ends at b ; if, by contradiction, $a = b$, then z_i also ends at a , contradicting the last conjunct, and if $b < a$, then the interval k_i that starts at the beginning point of z_i and ends at a contradicts the second conjunct. If, on the other hand, we assume $a < b$, we can take the interval $z_i = [a, b]$ to satisfy $\varphi(a, b)$, and we make sure that z_i does not end at x_p , nor any interval ending at x_p may possibly start together with z_i . Next, consider the case $\{1_{ip}, 2_{ip}\}$. Assume that $\mathcal{F} \models \varphi(a, b)$. If the first disjunct of φ holds then, b is the largest point of \mathbb{D} and a is not which leads to $a < b$. If the second disjunct of φ holds, witnessed by the interval $x_i = [c, d]$, then $a = c$ and $c < b < d$ which again leads to $a < b$. On the other hand, assume $a < b$, and let us prove that $\mathcal{F} \models \varphi(a, b)$. If b is the greatest point of \mathbb{D} , then the first disjunct of φ holds. If it is not, then, there is a $c \in \mathbb{D}$ such that $c > b > a$ and the second disjunct of φ holds, witnessed by the interval $x_i = [a, c]$. Let us prove the $<$ -completeness of $\{1_{ip}, 03_{ii}\}$. Assume that $\mathcal{F} \models \varphi(a, b)$. If the first disjunct of φ holds then b is the largest point of \mathbb{D} and a is not, which implies that $a < b$. Suppose that the second disjunct of φ holds, witnessed by the intervals $x_i = [c, d]$ and $y_i = [e, f]$. Then $a = c$, $b = e$ and $c < e < f = d$, hence $a < b$. Now assume that $a < b$. If b is the largest point of \mathbb{D} then the first disjunct of φ holds. Otherwise the second disjunct of φ holds witnessed by the intervals $x_i = [a, c]$ and $y_i = [b, c]$ where $b < c$. Now, let us focus on the following group of definitions:

$$x_p < y_p \leftrightarrow \left\{ \begin{array}{l} (\exists x_i(x_i 1_{ip} x_p) \wedge \neg \exists x_i(x_i 1_{ip} y_p)) \vee \\ \exists x_i y_i z_i (x_i 1_{ip} x_p \wedge y_i 1_{ip} y_p \wedge z_i 34_{ii} y_i \wedge \forall t_i (t_i 34_{ii} x_i \leftrightarrow t_i 34_{ii} z_i)) \end{array} \right. \quad \{1_{ip}, 34_{ii}\}$$

$$\left\{ \begin{array}{l} \exists x_i y_i (x_i 1_{ip} x_p \wedge y_i 1_{ip} y_p \wedge y_i 04_{ii} x_i) \vee \\ (\exists x_i(x_i 1_{ip} x_p) \wedge \neg \exists x_i(x_i 1_{ip} y_p)) \vee \\ (\exists x_i y_i (x_i 1_{ip} x_p \wedge y_i 1_{ip} x_p \wedge x_i \neq y_i) \wedge \\ \neg \exists x_i y_i (x_i 1_{ip} y_p \wedge y_i 1_{ip} y_p \wedge x_i \neq y_i)) \end{array} \right. \quad \{1_{ip}, 04_{ii}, =_i\}$$

$$\left\{ \begin{array}{l} \exists x_i y_i (x_i 1_{ip} x_p \wedge y_i 1_{ip} y_p \wedge x_i 24_{ii} y_i) \vee \\ (\exists x_i(x_i 1_{ip} x_p) \wedge \neg \exists x_i(x_i 1_{ip} y_p)) \vee \\ (\exists x_i y_i (x_i 1_{ip} x_p \wedge y_i 1_{ip} x_p \wedge x_i \neq y_i) \wedge \\ \neg \exists x_i y_i (x_i 1_{ip} y_p \wedge y_i 1_{ip} y_p \wedge x_i \neq y_i)) \end{array} \right. \quad \{1_{ip}, 24_{ii}, =_i\}$$

$$\left\{ \begin{array}{l} \exists x_i (x_i 1_{ip} x_p \wedge \forall y_i (y_i 1_{ip} y_p \rightarrow \exists z_i (x_i 44_{ii} z_i \wedge \neg (y_i 44_{ii} z_i)))) \vee \\ (\exists x_i(x_i 1_{ip} x_p) \wedge \neg \exists x_i(x_i 1_{ip} y_p)) \vee \\ (\exists x_i y_i (x_i 1_{ip} x_p \wedge y_i 1_{ip} x_p \wedge x_i \neq y_i) \wedge \\ \neg \exists x_i y_i (x_i 1_{ip} y_p \wedge y_i 1_{ip} y_p \wedge x_i \neq y_i)) \end{array} \right. \quad \{1_{ip}, 44_{ii}, =_i\}$$

The case of $\{1_{ip}, 34_{ii}\}$ is very similar to the previous one. Indeed, if we assume that $\mathcal{F} \models \varphi(a, b)$, and that the second conjunct of φ holds (if the first conjunct holds, then we reason as in the previous case), then, if $x_i = [c, d]$ and $y_i = [e, f]$, we have that $a = c$, $b = e$, and the interval $z_i = [a, b]$ must exist, so that it is met by exactly the same intervals that meet x_i . Now, assume that $a < b$, where b is not the largest point of \mathbb{D} (otherwise, the first disjunct of φ holds). Then, the second disjunct of φ holds witnessed by the intervals $x_i = [a, c]$, $y_i = [b, c]$, and $z_i = [a, b]$ where $b < c$. Now consider the case of $\{1_{ip}, 04_{ii}, =_i\}$. Suppose that $\mathcal{F} \models \varphi(a, b)$. If the first disjunct of φ holds, witnessed by the intervals $x_i = [c, d]$ and

$y_i = [e, f]$, then $a = c$, $b = e$ and $c < e < f < d$ and we get $a < b$. If the second disjunct of φ holds, then b is the largest point of \mathbb{D} and a is not, so $a < b$. If the third disjunct of φ holds, then there are at least two points in \mathbb{D} which are greater than a and there is at most one point in \mathbb{D} greater than b which again leads to $a < b$. Now, assume $a < b$; we want to show that $\mathcal{F} \models \varphi(a, b)$. If b is the largest point of \mathbb{D} , then a is not and therefore the second disjunct of φ holds. If there is exactly one point in \mathbb{D} which is greater than b , then there are at least two points in \mathbb{D} greater than a and therefore the third disjunct of φ holds. Now we may assume that there are points $c, d \in \mathbb{D}$ with $a < b < c < d$. Then the first disjunct of φ holds witnessed by the intervals $x_i = [a, d]$ and $y_i = [b, c]$. The case of $\{1_{ip}, 2_{4ii}, =_i\}$ is similar to the previous one. Lastly, we consider the case of $\{1_{ip}, 4_{4ii}, =_i\}$. Suppose that $\mathcal{F} \models \varphi(a, b)$ and that the first disjunct of φ holds, witnessed by the interval $x_i = [a, c]$. Suppose, towards a contradiction, that $b < c$. Then the interval $y_i = [b, c]$ satisfies $y_i 1_{ip} b$ and therefore there is an interval z_i such that $[a, c] 4_{4ii} z_i$ and $\neg([b, c] 4_{4ii} z_i)$ which is impossible. Therefore $c \leq b$ and we obtain $a < b$. If the second or the third disjunct of φ holds then we again obtain $a < b$ precisely as in the previous case. Now assume $a < b$; we want to show that $\mathcal{F} \models \varphi(a, b)$. If b is the greatest point of \mathbb{D} then the second disjunct of φ holds and if there is exactly one point in \mathbb{D} which is greater than b then the third disjunct of φ holds. So we may assume there are two points $c, d \in \mathbb{D}$ such that $b < c < d$. We claim that the first disjunct of φ holds, witnessed by the interval $x_i = [a, b]$. Let $y_i = [b, e]$. We want to find an interval z_i such that $[a, b] 4_{4ii} z_i$ and $\neg[b, e] 4_{4ii} z_i$. Let f be the minimum of c and e . Then the interval $z_i = [f, d]$ has the desired property, from which we see that the first disjunct of φ holds. \square

We are now moving to mixed relations, starting with 0_{ip} . Recall that 0_{ip} and 1_{ip} are reversible relations: symmetric sets of complete ones are complete for their (respective) reverse, and, thus, they do not appear in their tables.

Lemma 3.6. *Tab. 8 is correct.*

Proof. Consider the following definitions:

$$x_i 0_{ip} y_p \leftrightarrow \begin{cases} \exists z_p (x_i 1_{ip} z_p \wedge y_p < z_p) & \{1_{ip}, <\} \\ \exists z_p (\neg(x_i 2_{ip} z_p) \wedge \neg(x_i 3_{ip} z_p) \wedge \\ \forall z_i \forall k_p ((z_i 3_{ip} z_p \wedge x_i 3_{ip} k_p) \rightarrow \neg(z_i 2_{ip} k_p)) \wedge y_p < z_p) & \{2_{ip}, 3_{ip}\} \end{cases}$$

The case of $\{1_{ip}, <\}$ is straightforward, and needs no explanation. Consider, now, the case $\{2_{ip}, 3_{ip}\}$. The fact that this set is $<$ -complete is proved above. So, assume first $\mathcal{F} \models \varphi([a, b], c)$; there must be some d which, thanks to the first two conjuncts, can only be placed in such a way that $d \leq a$ or $b < d$, and, thanks to the third conjunct, the possibility $b < d$ is eliminated: in fact, if we had $b < d$, we could take $z_i = [a, d]$ and $k_p = b$ to contradict the third conjunct. So, since $d \leq a$ and $c < d$, it must be that $c < a$ as we wanted. Conversely, suppose that $c < a$: we take $z_p = a$, which is not *during* x_i nor does it *end* x_i , and it is such that no interval *ending* at z_p may possibly *contain* b . Moreover, $y_p = c < a$, and so $\varphi([a, b], c)$. Now, focus on the following definitions:

Proved	Implied	Deduction Chain
$\{1_{ip}, <\}$	$\{1_{ip}, 2_{ip}\}$	$<$
$\{2_{ip}, 3_{ip}\}$	$\{1_{ip}, 3_{ip}\}$	$<$
$\{2_{ip}, 4_{ip}\}$	$\{1_{ip}, 4_{ip}\}$	$<$
$\{2_{ip}, 14_{ii}, <\}$	$\{1_{ip}, 14_{ii}, 24_{ii}\}$	$<$
$\{2_{ip}, 03_{ii}, <\}$	$\{1_{ip}, 14_{ii}, 04_{ii}\}$	$<$
$\{3_{ip}, 14_{ii}\}$	$\{1_{ip}, 14_{ii}, 44_{ii}\}$	$<$
$\{4_{ip}, 14_{ii}, <\}$	$\{1_{ip}, 24_{ii}, =_i\}$	$<$
	$\{1_{ip}, 04_{ii}, =_i\}$	$<$
	$\{1_{ip}, 44_{ii}, =_i\}$	$<$
	$\{1_{ip}, 03_{ii}\}$	$<$
	$\{1_{ip}, 34_{ii}\}$	$<$
	$\{2_{ip}, 34_{ii}, <\}$	03_{ii} (Section 3.1)
	$\{3_{ip}, 34_{ii}\}$	14_{ii} (Section 3.1)
	$\{3_{ip}, 24_{ii}, 03_{ii}\}$	34_{ii} (Section 3.1)
	$\{3_{ip}, 03_{ii}, 04_{ii}\}$	34_{ii} (Section 3.1)
	$\{3_{ip}, 03_{ii}, 44_{ii}\}$	34_{ii} (Section 3.1)
	$\{4_{ip}, 34_{ii}, <\}$	14_{ii} (Section 3.1)
	$\{4_{ip}, 24_{ii}, 03_{ii}, <\}$	34_{ii} (Section 3.1)
	$\{4_{ip}, 03_{ii}, 04_{ii}, <\}$	34_{ii} (Section 3.1)
	$\{4_{ip}, 03_{ii}, 44_{ii}, <\}$	34_{ii} (Section 3.1)

Table 8: The spectrum of the $mcs(o_{ip})$. - Class: Lin.
(Lemma 3.6)

$$x_i o_{ip} y_p \leftrightarrow \begin{cases} \neg(x_i 2_{ip} y_p) \wedge \neg(x_i 4_{ip} y_p) \wedge & \{2_{ip}, 4_{ip}\} \\ \exists z_i (\forall k_p (x_i 4_{ip} k_p \leftrightarrow z_i 4_{ip} k_p) \wedge \\ \exists k_p (z_i 2_{ip} k_p \wedge \neg(x_i 2_{ip} k_p)) \wedge \neg(z_i 2_{ip} y_p)) \wedge \\ \neg \forall z_i (\exists k_p (z_i 4_{ip} k_p \wedge \neg(x_i 4_{ip} k_p)) \rightarrow z_i 4_{ip} y_p) \\ \\ \exists z_p k_p (y_p < z_p \wedge z_p < k_p \wedge \neg(x_i 2_{ip} y_p) \wedge & \{2_{ip}, 14_{ii}, <\} \\ \neg(x_i 2_{ip} z_p) \wedge \neg(x_i 2_{ip} k_p) \wedge \\ \forall z_i (z_i 2_{ip} z_p \rightarrow \neg(x_i 14_{ii} z_i))) \end{cases}$$

Let us focus on the the case of $\{2_{ip}, 14_{ii}, <\}$. Suppose $\mathcal{F} \models \varphi([a, b], c)$. So, there are two points d, e such that $c < d < e$. Now $d = z_p$ cannot be *after* a , for if it was we could find an interval $z_i = [a, e]$, where $e > b$ (since e and d cannot be *during* x_i) and such that x_i *starts* z_i , which contradicts the last conjunct. Then, $d \leq a$, which implies $c < a$ as we wanted. Suppose, on the other hand, that $c < a$: to satisfy the requirements, it suffices to take $z_p = a$ and $k_p = b$. The correctness of the case $\{2_{ip}, 4_{ip}\}$ is based on the fact that we can, first, eliminate the possibility that y_p is *during* or *after* x_i ; then, we eliminate the possibility that y_p *starts* x_i by stating that there must be an interval *finished by* x_i that does not have y_p *during* it; finally, we eliminate the possibility that y_p *ends* x_i by stipulating the existence of an interval (the one that *starts* at y_p) which *ends* before the right endpoint of x_i that does not have y_p *after* it. Finally, consider the following definitions:

Proved	Implied	Deduction Chain
$\{0_{ip}, <\}$	$\{0_{ip}, 3_{ip}\}$	$<$
	$\{2_{ip}, 3_{ip}\}$	$<, 0_{ip}$
	$\{2_{ip}, 4_{ip}, <\}$	0_{ip}
	$\{2_{ip}, 14_{ii}, <\}$	0_{ip}
	$\{2_{ip}, 03_{ii}, <\}$	0_{ip}
	$\{2_{ip}, 34_{ii}, <\}$	0_{ip}
	$\{3_{ip}, 14_{ii}\}$	$<, 0_{ip}$
	$\{3_{ip}, 24_{ii}, 03_{ii}\}$	$<, 0_{ip}$
	$\{3_{ip}, 03_{ii}, 04_{ii}\}$	$<, 0_{ip}$
	$\{3_{ip}, 03_{ii}, 44_{ii}\}$	$<, 0_{ip}$
	$\{3_{ip}, 34_{ii}\}$	$<, 0_{ip}$
	$\{4_{ip}, 14_{ii}, <\}$	0_{ip}
	$\{4_{ip}, 24_{ii}, 03_{ii}, <\}$	0_{ip}
	$\{4_{ip}, 03_{ii}, 04_{ii}, <\}$	0_{ip}
	$\{4_{ip}, 03_{ii}, 44_{ii}, <\}$	0_{ip}
	$\{4_{ip}, 34_{ii}, <\}$	0_{ip}

Table 9: The spectrum of the $mcs(1_{ip})$. - Class: Lin.
(Lemma 3.7)

$$x_i 0_{ip} y_p \leftrightarrow \begin{cases} \exists z_p z_i (x_i 03_{ii} z_i \wedge z_i 2_{ip} z_p \wedge \neg(x_i 2_{ip} z_p) \wedge y_p < z_p) & \{2_{ip}, 03_{ii}, <\} \\ \exists z_p (\neg(x_i 3_{ip} z_p) \wedge \neg \exists z_i ((z_i 14_{ii} x_i \vee x_i 14_{ii} z_i) \wedge z_i 3_{ip} z_p) \wedge y_p < z_p) & \{3_{ip}, 14_{ii}\} \\ \exists z_p (\neg \forall k_p (z_p < k_p \leftrightarrow x_i 4_{ip} k_p) \wedge \neg \exists z_i ((z_i 14_{ii} x_i \vee x_i 14_{ii} z_i) \wedge \forall k_p (z_p < k_p \leftrightarrow z_i 4_{ip} k_p)) \wedge y_p < z_p) & \{4_{ip}, 14_{ii}, <\} \end{cases}$$

For the sake of the case $\{2_{ip}, 03_{ii}, <\}$, suppose that $\mathcal{F} \models \varphi([a, b], c)$. So, there is a point z_p such that it is not *during* x_i , and since it must be *during* an interval $z_i = [e, b]$, it can only be $d \leq a$. Since $c < d$ we have $c < a$. Suppose, on the other hand, that $c < a$: to satisfy the requirements, it suffices to take $z_i = [c, b]$ and $z_p = a$. As for the set $\{3_{ip}, 14_{ii}\}$, first, recall its $<$ -completeness; then, we state the existence of a point for which we can eliminate the possibility that z_p *ends* x_i by means of the first conjunct, and, after that, we can eliminate the possibility that z_p is *during* or *after* x_i : if that were the case, there would be an interval z_i *starting* or *started by* x_i such that z_p is its right endpoint. The only possibility left is therefore that z_p is less than or equal to the beginning point of x_i , and therefore the last conjunct guarantees that $x_i 0_{ip} y_p$. Finally, the case $\{4_{ip}, 14_{ii}, <\}$ is identical to the previous one, with the only difference that $x_i 3_{ip} y_p$ can be expressed by asserting that $y_p < k_p$ and $x_i 4_{ip} k_p$ are equivalent. As for the implied definitions, notice that in some cases we exploit the \mathcal{J}^+ -completeness results from Section 3.1. \square

Lemma 3.7. *Tab. 9 is correct.*

Proof. Only one new definition is needed:

Proved	Symmetric	Implied	Deduction Chain
$\{0_{ip}, 4_{ip}\}$	-	$\{0_{ip}, 3_{ip}\}$	1_{ip}
$\{1_{ip}, 3_{ip}\}$	-	$\{0_{ip}, 14_{ii}, 24_{ii}, <\}$	4_{ip}
$\{1_{ip}, 03_{ii}\}$	$\{3_{ip}, 14_{ii}\}$	$\{0_{ip}, 14_{ii}, 04_{ii}, <\}$	4_{ip}
		$\{0_{ip}, 14_{ii}, 44_{ii}, <\}$	4_{ip}
		$\{0_{ip}, 03_{ii}, <\}$	1_{ip}
		$\{0_{ip}, 34_{ii}, <\}$	03_{ii} (Section 3.1)
		$\{1_{ip}, 4_{ip}\}$	3_{ip}
		$\{1_{ip}, 14_{ii}, 24_{ii}\}$	3_{ip}
		$\{1_{ip}, 14_{ii}, 04_{ii}\}$	3_{ip}
		$\{1_{ip}, 14_{ii}, 44_{ii}\}$	3_{ip}
		$\{1_{ip}, 34_{ii}\}$	03_{ii} (Section 3.1)
		$\{3_{ip}, 24_{ii}, 03_{ii}\}$	1_{ip}
		$\{3_{ip}, 03_{ii}, 04_{ii}\}$	1_{ip}
		$\{3_{ip}, 03_{ii}, 44_{ii}\}$	1_{ip}
		$\{3_{ip}, 34_{ii}\}$	14_{ii} (Section 3.1)
		$\{4_{ip}, 14_{ii}, <\}$	3_{ip}
		$\{4_{ip}, 24_{ii}, 03_{ii}, <\}$	1_{ip}
		$\{4_{ip}, 03_{ii}, 04_{ii}, <\}$	1_{ip}
		$\{4_{ip}, 03_{ii}, 44_{ii}, <\}$	1_{ip}

Table 10: The spectrum of the $mcs(2_{ip})$. - Class: Lin.
(Lemma 3.8)

$$x_i 1_{ip} y_p \leftrightarrow \forall z_p (x_i 0_{ip} z_p \leftrightarrow z_p < y_p) \quad \{0_{ip}, <\}$$

This definition is quite straightforward: $[a, b]1_{ip}a$ if, and only if, the points c less than a are exactly the same points c such that $[a, b]0_{ip}c$. \square

We complete this part by analyzing the definability for 2_{ip} . Notice that here we also take advantage of the results of Section 3.1.

Lemma 3.8. *Tab. 10 is correct.*

Proof. In this case, three new definitions are needed:

$$x_i 2_{ip} y_p \leftrightarrow \begin{cases} \exists z_i k_i z_p k_p (z_i 1_{ip} z_p \wedge k_i 3_{ip} k_p \wedge z_i 3_{ip} y_p \wedge k_i 1_{ip} y_p \wedge x_i 1_{ip} z_p \wedge x_i 3_{ip} k_p) & \{1_{ip}, 3_{ip}\} \\ \exists z_i k_i (\forall z_p (z_i 0_{ip} z_p \leftrightarrow x_i 0_{ip} z_p) \wedge \forall k_p (k_i 4_{ip} k_p \leftrightarrow x_i 4_{ip} k_p) \wedge \\ \quad \neg \exists t_i (\forall t_p (t_i 0_{ip} t_p \leftrightarrow k_i 0_{ip} t_p) \wedge \forall t_p (t_i 4_{ip} t_p \leftrightarrow z_i 4_{ip} t_p)) \wedge \\ \quad \neg (z_i 4_{ip} y_p) \wedge \neg (k_i 0_{ip} y_p)) & \{0_{ip}, 4_{ip}\} \\ \exists z_i (z_i 03_{ii} x_i \wedge z_i 1_{ip} y_p) & \{1_{ip}, 03_{ii}\} \end{cases}$$

For the 2_{ip} -completeness of $\{1_{ip}, 3_{ip}\}$, we begin by assuming that $\mathcal{F} \models \varphi([a, b], c)$. So, there must be two intervals z_i, k_i , the former *starting* at a , and the latter *ending* at b ; the point c must, at the same time, *end* z_i (meaning that $a < c$) and *start* k_i (implying that $c < b$), and we deduce $a < c < b$. If we assume $a < c < b$, it is enough to take $z_i = [a, c]$ and $k_i = [c, b]$

Proved	Symmetric	Implied	Deduction Chain
$\{1_{ip}, 3_{ip}\}$	-	$\{24_{ii}, 14_{ii}\}$	Section 3.1
$\{2_{ip}, 14_{ii}\}$	$\{2_{ip}, 03_{ii}\}$	$\{14_{ii}, 44_{ii}\}$	Section 3.1
$\{4_{ip}, 14_{ii}\}$	$\{0_{ip}, 03_{ii}\}$	$\{14_{ii}, 03_{ii}\}$	Section 3.1
$\{0_{ip}, 2_{ip}\}$	$\{2_{ip}, 4_{ip}\}$	$\{14_{ii}, 04_{ii}\}$	Section 3.1
		$\{24_{ii}, 03_{ii}\}$	Section 3.1
		$\{03_{ii}, 04_{ii}\}$	Section 3.1
		$\{03_{ii}, 44_{ii}\}$	Section 3.1
		$\{0_{ip}, 3_{ip}\}$	1_{ip}
		$\{0_{ip}, 4_{ip}\}$	2_{ip}
		$\{1_{ip}, 2_{ip}\}$	0_{ip}
		$\{1_{ip}, 4_{ip}\}$	3_{ip}
		$\{1_{ip}, 03_{ii}\}$	2_{ip}
		$\{2_{ip}, 3_{ip}\}$	4_{ip}
		$\{3_{ip}, 14_{ii}\}$	2_{ip}

Table 11: The spectrum of the $mcs(34_{ii})$. - Class: Lin.
(Lemma 3.9)

to satisfy the conditions. We move now to the 2_{ip} -completeness of $\{0_{ip}, 4_{ip}\}$, and let us assume, again, that $\mathcal{F} \models \varphi([a, b], c)$. The interval z_i must *start* at a , and the interval k_i must *end* at b . Moreover, the ending point of k_i cannot be *after* the beginning point of z_i (third conjunct). Now, in this situation, y_p cannot be a or *before* it (because of the fifth conjunct), and it cannot be b or *after* it (because of the fourth conjunct), and the only possibility left out is $a < c < b$. If, conversely, we assume $a < c < b$, it is enough to take $z_i = [a, c]$ and $k_i = [c, b]$ to satisfy the entire set of conditions. Finally, the 2_{ip} -completeness of $\{1_{ip}, 03_{ii}\}$ is very easy. If $\mathcal{F} \models \varphi([a, b], c)$, then there exists an interval z_i that *finishes* $[a, b]$, and for its *starting* point, that is, $y_p = c$, it must hold that $a < c < b$. On the other hand, if $a < c < b$, we just take $z_i = [c, b]$ to satisfy φ . \square

We now focus on interval-interval relations other than equality, starting with 34_{ii} , which, being \mathcal{J}^+ -complete on its own, plays a very special role. Most of the work has already been taken care of in Section 3.1. As a matter of fact, we already know some subsets of \mathcal{J}^+ that are 34_{ii} -complete, and we certainly have to define those subsets of $\mathfrak{R}^+ \setminus \mathcal{J}^+$ which are. It remains to establish whether other subsets of \mathcal{J}^+ , other than those seen in Section 3.1, become 34_{ii} -complete within the language of $FO + \mathfrak{R}^+$: the fact that this is not the case will be a consequence of the results shown in Section 4.

Lemma 3.9. *Tab. 11 is correct.*

Proof. The fact that the set $\{1_{ip}, 3_{ip}\}$ is 34_{ii} -complete depends on the following easy definition:

$$x_i 34_{ii} y_i \leftrightarrow \exists z_p (x_i 3_{ip} z_p \wedge y_i 1_{ip} z_p) \quad \{1_{ip}, 3_{ip}\}$$

The second two sets in the leftmost column can be proven 34_{ii} -complete by means of defining, as in Section 3.1, the weaker relation $34_{ii} \cup 44_{ii}$:

Proved	Implied	Deduction Chain
$\{0_{ip}, 2_{ip}\}$	$\{0_{ip}, 3_{ip}\}$	2_{ip}
$\{0_{ip}, 4_{ip}\}$	$\{0_{ip}, 03_{ii}, <\}$	2_{ip}
	$\{1_{ip}, 2_{ip}\}$	0_{ip}
	$\{1_{ip}, 3_{ip}\}$	$0_{ip}, 4_{ip}$
	$\{1_{ip}, 4_{ip}\}$	0_{ip}
	$\{1_{ip}, 03_{ii}\}$	34_{ii}
	$\{2_{ip}, 3_{ip}\}$	0_{ip}
	$\{2_{ip}, 4_{ip}\}$	0_{ip}
	$\{34_{ii}\}$	Section 3.1
	$\{2_{ip}, 03_{ii}\}$	34_{ii}
	$\{24_{ii}, 03_{ii}\}$	Section 3.1
	$\{03_{ii}, 04_{ii}\}$	Section 3.1
	$\{03_{ii}, 44_{ii}\}$	Section 3.1

Table 12: The spectrum of the $mcs(14_{ii})$. - Class: Lin.
(Lemma 3.10.)

$$x_i 34_{ii} \cup 44_{ii} y_i \leftrightarrow \begin{cases} \exists z_i (x_i 14_{ii} z_i \wedge \forall k_p (z_i 4_{ip} k_p \leftrightarrow y_i 4_{ip} k_p)) \wedge & \{4_{ip}, 14_{ii}\} \\ \neg \exists z_i (z_i 14_{ii} y_i \wedge \forall k_p (z_i 4_{ip} k_p \leftrightarrow x_i 4_{ip} k_p)) & \\ \\ \exists z_p z_i (x_i 14_{ii} z_i \wedge z_i 2_{ip} z_p \wedge \neg (x_i 2_{ip} z_p)) \wedge & \{2_{ip}, 14_{ii}\} \\ \forall k_i (x_i 14_{ii} k_i \rightarrow k_i 2_{ip} z_p) \wedge & \\ \forall k_i (y_i 14_{ii} k_i \rightarrow \neg (k_i 2_{ip} z_p)) \wedge & \\ \neg (y_i 2_{ip} z_p), & \end{cases}$$

As for $\{4_{ip}, 14_{ii}\}$, assume, first, that $\mathcal{F} \models \varphi([a, b], [c, d])$. We wish to show that $\mathcal{F} \models [a, b] 34_{ii} \cup 44_{ii} [c, d]$, i.e., that $b \leq c$. Suppose, by way of contradiction, that $c < b$. The only possible interval which satisfies the first conjunct is $[a, d]$. Specifically we have $b < d$. Now the interval $[c, b]$ falsifies the second conjunct. Conversely assume that $\mathcal{F} \models [a, b] 34_{ii} \cup 44_{ii} [c, d]$, i.e., $a < b \leq c < d$. Then the interval $z_i = [a, d]$ witnesses the first conjunct of the definition, and, for any z_i , if z_i starts y_i , then it cannot end at b , proving that the second conjunct also holds. As for $\{2_{ip}, 14_{ii}\}$ assume, first, that $\mathcal{F} \models \varphi([a, b], [c, d])$. We wish to show that $\mathcal{F} \models [a, b] 34_{ii} \cup 44_{ii} [c, d]$, i.e., that $b \leq c$. Because of the first four conjuncts, there must be an interval started by x_i , and z_p must be placed as the ending point of $[a, b]$, that is, $z_p = b$; then, y_i must start at z_p or after it, otherwise it would happen that either $y_i 2_{ip} z_p$ or some k_i started by y_i is such that $k_i 2_{ip} z_p$. On the other hand, assume that $\mathcal{F} \models [a, b] 34_{ii} \cup 44_{ii} [c, d]$, i.e., that $b \leq c < d$. In this case, it suffices to take $z_p = b$ and $z_i = [a, d]$ to satisfy all conjuncts. Finally, the 34_{ii} -completeness of $\{0_{ip}, 2_{ip}\}$ can be seen as follows:

$$x_i 14_{ii} y_i \leftrightarrow \forall k_p (x_i 0_{ip} k_p \leftrightarrow y_i 0_{ip} k_p) \wedge \exists k_p (y_i 2_{ip} k_p \wedge \neg (x_i 2_{ip} k_p)) \quad \{0_{ip}, 2_{ip}\}$$

which is immediate to prove, as we stipulate that x_i starts y_i if and only if x_i and y_i have the same points before them and there exists a point (the ending point of x_i) that it is during y_i and not during x_i , and which allows us to obtain the result via the set $\{2_{ip}, 14_{ii}\}$. \square

Proved	Symmetric	Implied	Deduction Chain
$\{2_{ip}, <\}$	-	$\{0_{ip}, 2_{ip}\}$	34_{ii}
$\{0_{ip}, 44_{ii}, 04_{ii}\}$	$\{4_{ip}, 44_{ii}, 04_{ii}\}$	$\{0_{ip}, 3_{ip}\}$	34_{ii}
$\{1_{ip}, 44_{ii}, 04_{ii}\}$	$\{3_{ip}, 44_{ii}, 04_{ii}\}$	$\{0_{ip}, 4_{ip}\}$	34_{ii}
		$\{0_{ip}, 03_{ii}\}$	34_{ii}
		$\{1_{ip}, 2_{ip}\}$	34_{ii}
		$\{1_{ip}, 3_{ip}\}$	34_{ii}
		$\{1_{ip}, 4_{ip}\}$	34_{ii}
		$\{1_{ip}, 03_{ii}\}$	34_{ii}
		$\{2_{ip}, 3_{ip}\}$	34_{ii}
		$\{2_{ip}, 4_{ip}\}$	34_{ii}
		$\{2_{ip}, 14_{ii}\}$	34_{ii}
		$\{2_{ip}, 03_{ii}\}$	34_{ii}
		$\{3_{ip}, 14_{ii}\}$	34_{ii}
		$\{4_{ip}, 14_{ii}\}$	34_{ii}
		$\{14_{ii}, 03_{ii}\}$	34_{ii}
		$\{14_{ii}, 04_{ii}\}$	34_{ii}
		$\{14_{ii}, 44_{ii}\}$	34_{ii}
		$\{03_{ii}, 44_{ii}\}$	34_{ii}
		$\{04_{ii}, 03_{ii}\}$	34_{ii}
		$\{34_{ii}\}$	Section 3.1

Table 13: The spectrum of the $\text{mcs}(24_{ii})$. - Class: Lin.
(Lemma 3.11.)

The only non-symmetric interval-interval relation, 14_{ii} , does not present particularly difficult problems.

Lemma 3.10. *Tab. 12 is correct.*

Proof. The fact that the set $\{0_{ip}, 2_{ip}\}$ is 14_{ii} -complete has already been proved as a necessary step to delineate the $\text{mcs}(34_{ii})$. As for the set $\{0_{ip}, 4_{ip}\}$, it is easily verified that:

$$x_i 14_{ii} y_i \leftrightarrow \forall z_p (x_i 0_{ip} z_p \leftrightarrow y_i 0_{ip} z_p) \wedge \exists z_p (x_i 4_{ip} z_p \wedge \neg (y_i 4_{ip} z_p)), \quad \{0_{ip}, 4_{ip}\} \quad \square$$

Lemma 3.11. *Tab. 13 is correct.*

Proof. Consider the following definitions:

$$x_i 24_{ii} y_i \leftrightarrow \left\{ \begin{array}{l}
\exists z_p k_p (x_i 2_{ip} z_p \wedge \forall t_p (x_i 2_{ip} t_p \rightarrow t_p < k_p) \quad \{2_{ip}, <\} \\
\wedge \forall s_p (\forall t_p (x_i 2_{ip} t_p \rightarrow t_p < s_p) \rightarrow \neg(s_p < k_p)) \\
\wedge y_i 2_{ip} k_p \wedge \neg(y_i 2_{ip} z_p)) \\
\\
\exists z_i t_i w_i (\forall k_p (x_i 0_{ip} k_p \leftrightarrow z_i 0_{ip} k_p) \wedge \forall k_p (y_i 0_{ip} k_p \leftrightarrow w_i 0_{ip} k_p) \wedge \quad \{0_{ip}, 44_{ii}, 04_{ii}\} \\
z_i 44_{ii} t_i \wedge \exists s_i (w_i 04_{ii} s_i \wedge \neg(y_i 04_{ii} s_i)) \wedge \\
\neg \forall k_p (x_i 0_{ip} k_p \leftrightarrow y_i 0_{ip} k_p) \wedge \\
\bigwedge_{\substack{u_i, v_i \in \{w_i, x_i, y_i, z_i, t_i\} \\ \{u_i, v_i\} \neq \{z_i, t_i\}}} (\neg(u_i 04_{ii} v_i) \wedge \neg(u_i 44_{ii} v_i))) \\
\\
\exists z_i t_i w_i (\exists k_p (x_i 1_{ip} k_p \wedge z_i 1_{ip} k_p) \wedge \exists k_p (y_i 1_{ip} k_p \wedge w_i 1_{ip} k_p) \wedge \quad \{1_{ip}, 44_{ii}, 04_{ii}\} \\
z_i 44_{ii} t_i \wedge \exists s_i (w_i 04_{ii} s_i \wedge \neg(y_i 04_{ii} s_i)) \wedge \\
\neg \exists k_p (x_i 1_{ip} k_p \wedge y_i 1_{ip} k_p) \wedge \\
\bigwedge_{\substack{u_i, v_i \in \{w_i, x_i, y_i, z_i, t_i\} \\ \{u_i, v_i\} \neq \{z_i, t_i\}}} (\neg(u_i 04_{ii} v_i) \wedge \neg(u_i 44_{ii} v_i)))
\end{array} \right.$$

Let us start proving the correctness of the case $\{2_{ip}, <\}$. Suppose that $\mathcal{F} \models [a, b]24_{ii}[c, d]$. Then, by definition, $a < c < b < d$; by setting $z_p = c$ and $k_p = b$ we satisfy all requirements. Conversely, suppose that $\mathcal{F} \models \varphi([a, b], [c, d])$. First, observe that $x_i = [a, b]$ must have an internal point z_p . Then, the only way to place k_p is by having that k_p is the smallest point which is greater than every internal point of x_i , that is, k_p must *end* x_i . Now, y_i must *contain* k_p (so, it must *end* after it), but not z_p (so, it must *start* after z_p or on it). Consider, now, $\{1_{ip}, 44_{ii}, 04_{ii}\}$. Suppose that $\mathcal{F} \models [a, b]24_{ii}[c, d]$. Then, by definition, $a < c < b < d$; by setting $z_i = [a, c]$, $w_i = [c, b]$, $t_i = [b, d]$, and $s_i = [a, d]$, we satisfy all requirements. Suppose, now, that $\mathcal{F} \models \varphi([a, b], [c, d])$. We prove that $[a, b]24_{ii}[c, d]$ by eliminating every other possibility. First, observe that they cannot have their *starting* point in common (third line), and they cannot *contain* each other nor can one of them be *after* the other (fourth line). Thus, we eliminated $=_i, 14_{ii}, 04_{ii}, 44_{ii}$ among the possible relations between x_i and y_i . Assume, by way of contradiction, that $y_i 34_{ii} x_i$. The second and fourth conjuncts together imply that $w_i 14_{ii} y_i$, and since w_i (resp., z_i) has the same *starting* point as y_i (resp., x_i), and since $z_i 44_{ii} t_i$, obviously $w_i 44_{ii} t_i$, which is in contradiction with the fourth line. Assume now that $x_i 34_{ii} y_i$. First, observe that z_i cannot be shorter than x_i , otherwise it would be $z_i 44_{ii} w_i$ (contradicting the fourth line); but then, z_i cannot be equal to x_i or longer than it, as in that case $x_i 44_{ii} t_i$ (contradicting, once again, the requirement in the fourth line). Now, suppose that $x_i 03_{ii} y_i$. Let $w_i = [e, f]$, where $e = c$. Clearly, to comply with the fourth line, it cannot be that $f < a$ or that $f > b (= d)$. If, on the other hand, $a \leq f \leq b$, then it is impossible to place $z_i = [g, h]$ ($g = a$), as it must be that $h < f$ (since $w_i 44_{ii} t_i$ leads to a contradiction), but, then, $z_i 04_{ii} y_i$, in contradiction with the fourth line. Therefore, we cannot place w_i , which means that we have a contradiction. Next suppose that $y_i 03_{ii} x_i$. Let $w_i = [e, f]$, where $e = c$; clearly, $f \geq d$, as, otherwise, we would have that x_i *contains* w_i (forbidden by the fourth line); but, then, it is impossible to find any s_i that *contains* w_i and not y_i (second line), leading to a contradiction. Finally, if $y_i 24_{ii} x_i$, it is impossible to place w_i, z_i and t_i in such a way that $z_i 44_{ii} t_i$, $\neg(z_i 04_{ii} w_i)$, and $\neg(w_i 04_{ii} t_i)$. Having eliminated every other possibility, it must be that $x_i 24_{ii} y_i$. Finally, observe that the case $\{0_{ip}, 44_{ii}, 04_{ii}\}$

can be dealt with the very same argument, only having 0_{ip} playing the same role as 1_{ip} . The corresponding definition is therefore almost identical to the one of the previous case. \square

Lemma 3.12. *Tab. 14 is correct.*

Proof. Consider the following definitions:

$$\begin{array}{l}
 x_i 04_{ii} y_i \leftrightarrow \left\{ \begin{array}{l}
 \exists z_i t_i z_p t_p x'_i x''_i (y_i 2_{ip} z_p \wedge y_i 2_{ip} t_p \wedge z_i 24_{ii} t_i \wedge x'_i 44_{ii} x''_i \wedge z_i 2_{ip} z_p \wedge t_i 2_{ip} t_p \wedge \{2_{ip}, 24_{ii}, 44_{ii}\} \\
 \neg(z_i 2_{ip} t_p) \wedge \neg(t_i 2_{ip} z_p) \wedge \\
 \bigwedge_{s_i \in \{x_i, x'_i, x''_i\}} (\neg(s_i 2_{ip} z_p) \wedge \neg(s_i 2_{ip} t_p)) \wedge \\
 \bigwedge_{s_i \in \{y_i, x_i, z_i, t_i\}, r \in \{44_{ii}, 24_{ii}\}} (\neg(x'_i r s_i) \wedge \neg(s_i r x''_i)) \wedge \\
 \bigwedge_{s_i \in \{y_i, x'_i, x''_i, z_i, t_i\}, r \in \{44_{ii}, 24_{ii}\}} (\neg(x_i r s_i) \wedge \neg(s_i r x_i)) \wedge \\
 \bigwedge_{s_i \in \{y_i, x_i, z_i, t_i\}, r \in \{44_{ii}, 24_{ii}\}} (\neg(x''_i r s_i) \wedge \neg(s_i r x''_i))
 \end{array} \right. \\
 \\
 x_i 04_{ii} y_i \leftrightarrow \left\{ \begin{array}{l}
 \exists z_i t_i (\forall k_p (x_i 0_{ip} k_p \leftrightarrow z_i 0_{ip} k_p) \wedge \forall k_p (y_i 0_{ip} k_p \leftrightarrow t_i 0_{ip} k_p) \wedge \{0_{ip}, 24_{ii}, 44_{ii}\} \\
 \forall w_i (y_i 44_{ii} w_i \leftrightarrow z_i 44_{ii} w_i) \wedge t_i 24_{ii} z_i \wedge \\
 \neg \forall k_p (x_i 0_{ip} k_p \leftrightarrow y_i 0_{ip} k_p) \wedge \\
 \bigwedge_{\substack{s_i, w_i \in \{x_i, y_i, z_i, t_i\} \\ \{s_i, w_i\} \neq \{t_i, z_i\}}} (\neg(s_i 24_{ii} w_i) \wedge \neg(s_i 44_{ii} w_i))
 \end{array} \right. \\
 \\
 x_i 04_{ii} y_i \leftrightarrow \left\{ \begin{array}{l}
 \exists z_i t_i (\forall k_p (x_i 1_{ip} k_p \leftrightarrow z_i 1_{ip} k_p) \wedge \forall k_p (y_i 1_{ip} k_p \leftrightarrow t_i 1_{ip} k_p) \wedge \{1_{ip}, 24_{ii}, 44_{ii}\} \\
 \forall w_i (y_i 44_{ii} w_i \leftrightarrow z_i 44_{ii} w_i) \wedge t_i 24_{ii} z_i \wedge \\
 \neg \exists k_p (x_i 1_{ip} k_p \wedge y_i 1_{ip} k_p) \wedge \\
 \bigwedge_{\substack{s_i, w_i \in \{x_i, y_i, z_i, t_i\} \\ \{s_i, w_i\} \neq \{t_i, z_i\}}} (\neg(s_i 24_{ii} w_i) \wedge \neg(s_i 44_{ii} w_i))
 \end{array} \right.
 \end{array}$$

Let us start by $\{2_{ip}, 24_{ii}, 44_{ii}\}$. Suppose that $\mathcal{F} \models [c, d] 04_{ii}[a, b]$. Then, by definition, $a < c < d < b$; to satisfy φ we set: $x'_i = [a, c]$, $x''_i = [d, b]$, $z_i = [a, d]$, $t_i = [c, b]$, $z_p = c$, and $t_p = d$. Suppose, now, that $\mathcal{F} \models \varphi([c, d], [a, b])$ and that $z_i = [b_z, e_z]$, $t_i = [b_t, e_t]$, $x' = [b', e']$ and $x'' = [b'', e'']$. First, observe that $y_i = [a, b]$ must have two internal points z_p and t_p , and that z_p (t_p) must be *contained* in z_i (t_i) and not in t_i (z_i). Since z_i *overlaps* t_i we must have $z_p < t_p$. We deduce that $a < z_p \leq b_t < e_z \leq t_p < b$. The constraints on x' and x'' in lines 1, 3, 4 and 6 of the definition imply that we must have $e' = b_t = z_p$ and $b'' = e_z = t_p$. This gives $a < z_p = b_t = e' < b'' = e_z = t_p < b$. Now, the only way to place $x_i = [c, d]$ without violating the third and fifth line of the definition is that $e' = c < d = b''$. Consequently $a < c < b < d$, as desired. Consider, now, $\{1_{ip}, 24_{ii}, 44_{ii}\}$. Suppose that $\mathcal{F} \models [c, d] 04_{ii}[a, b]$. Then, by definition, $a < c < d < b$; by setting $z_i = [c, b]$ and $t_i = [a, d]$, we satisfy all requirements. Suppose, now, that $\mathcal{F} \models \varphi([c, d], [a, b])$. Once more, we eliminate every possible relation between x_i and y_i , except for $x_i 04_{ii} y_i$. As before, 14_{ii} , 24_{ii} , and 44_{ii} are immediately eliminated (third and fourth line). If $x_i 34_{ii} y_i$, then, since z_i (resp., t_i) and x_i (resp., y_i) have the same *starting* point (first line), it is not possible to have $t_i 24_{ii} z_i$, contradicting the second line. Suppose, now, that $y_i 34_{ii} x_i$. First, observe that z_i (which *starts* at the same point as x_i) cannot *start* or be *equal* to x_i , since in that case $t_i 24_{ii} x_i$ (contradicting the fourth line). But then, if z_i is *started* by x_i , there will be some interval that is *after* y_i which is not *after* z_i itself, contradicting the second line. Thus, 34_{ii} is

Proved	Symmetric	Implied	Deduction Chain
$\{0_{ip}, 24_{ii}, 44_{ii}\}$	$\{4_{ip}, 24_{ii}, 44_{ii}\}$	$\{0_{ip}, 2_{ip}\}$	34_{ii}
$\{1_{ip}, 24_{ii}, 44_{ii}\}$	$\{3_{ip}, 24_{ii}, 44_{ii}\}$	$\{0_{ip}, 3_{ip}\}$	34_{ii}
$\{2_{ip}, 24_{ii}, 44_{ii}\}$	-	$\{0_{ip}, 4_{ip}\}$	34_{ii}
		$\{0_{ip}, 03_{ii}\}$	34_{ii}
		$\{1_{ip}, 2_{ip}\}$	34_{ii}
		$\{1_{ip}, 3_{ip}\}$	34_{ii}
		$\{1_{ip}, 4_{ip}\}$	34_{ii}
		$\{1_{ip}, 03_{ii}\}$	34_{ii}
		$\{2_{ip}, 3_{ip}\}$	34_{ii}
		$\{2_{ip}, 4_{ip}\}$	34_{ii}
		$\{2_{ip}, 14_{ii}\}$	34_{ii}
		$\{2_{ip}, 03_{ii}\}$	34_{ii}
		$\{2_{ip}, 44_{ii}, <\}$	24_{ii}
		$\{3_{ip}, 14_{ii}\}$	34_{ii}
		$\{4_{ip}, 14_{ii}\}$	34_{ii}
		$\{14_{ii}, 24_{ii}\}$	34_{ii}
		$\{14_{ii}, 03_{ii}\}$	34_{ii}
		$\{14_{ii}, 44_{ii}\}$	34_{ii}
		$\{24_{ii}, 03_{ii}\}$	34_{ii}
		$\{03_{ii}, 44_{ii}\}$	34_{ii}
		$\{34_{ii}\}$	Section 3.1

Table 14: The spectrum of the $mcs(04_{ii})$. - Class: Lin.
(Lemma 3.12.)

eliminated. Assume that $x_i 03_{ii} y_i$. As before, z_i cannot *start* or be *equal* to x_i , since in that case $t_i 24_{ii} x_i$ (contradicting the fourth line). Moreover, if z_i is *started by* x_i , it would imply that $y_i 24_{ii} z_i$, again, a contradiction with the fourth line. If $y_i 03_{ii} x_i$, $y_i =_i x_i$, or $y_i 04_{ii} x_i$, then it is impossible to correctly place z_i and t_i in such a way that $t_i 24_{ii} z_i$. Having eliminated in this case all other possibilities, we conclude that $x_i 04_{ii} y_i$. Once again, we can deal with $\{0_{ip}, 24_{ii}, 44_{ii}\}$ with the exact same argument, where 0_{ip} plays the role of 1_{ip} . \square

Lemma 3.13. *Tab. 15 is correct.*

Proof. As always, we start with the definition:

$$\begin{aligned}
x_i 44_{ii} y_i \leftrightarrow & \exists z_i t_i (\exists k_p (x_i 1_{ip} k_p \wedge z_i 1_{ip} k_p) \wedge \neg \exists k_p (x_i 1_{ip} k_p \wedge y_i 1_{ip} k_p) \wedge \{1_{ip}, 24_{ii}, 04_{ii}\} \\
& \forall w_i (\forall k_p (y_i 1_{ip} k_p \rightarrow w_i 1_{ip} k_p) \rightarrow \neg (z_i 04_{ii} w_i)) \wedge \\
& \forall w_i (\forall k_p (t_i 1_{ip} k_p \rightarrow w_i 1_{ip} k_p) \rightarrow \neg (w_i 04_{ii} x_i)) \wedge \\
& \forall w_i (\forall k_p (y_i 1_{ip} k_p \rightarrow w_i 1_{ip} k_p) \rightarrow \neg (z_i 24_{ii} w_i)) \wedge \\
& \forall w_i (\forall k_p (y_i 1_{ip} k_p \rightarrow w_i 1_{ip} k_p) \rightarrow \neg (x_i 04_{ii} w_i)) \wedge \\
& \forall w_i (\forall k_p (y_i 1_{ip} k_p \rightarrow w_i 1_{ip} k_p) \rightarrow \neg (w_i 24_{ii} z_i)) \wedge \\
& z_i 24_{ii} t_i \wedge \bigwedge_{\substack{s_i, w_i \in \{x_i, y_i, z_i, t_i\} \\ \{s_i, w_i\} \neq \{z_i, t_i\}}} (\neg (s_i 24_{ii} w_i) \wedge \neg (s_i 04_{ii} w_i))
\end{aligned}$$

Following the same schema of the previous two relations, consider the case of $\{1_{ip}, 24_{ii}, 04_{ii}\}$, and suppose that $\mathcal{F} \models [a, b] 44_{ii} [c, d]$. Then, by definition, $a < b < c < d$; by setting $z_i = [a, c]$

Proved	Symmetric	Implied	Deduction Chain
$\{0_{ip}, 24_{ii}, 04_{ii}\}$	$\{4_{ip}, 24_{ii}, 04_{ii}\}$	$\{0_{ip}, 2_{ip}\}$	34_{ii}
$\{1_{ip}, 24_{ii}, 04_{ii}\}$	$\{3_{ip}, 24_{ii}, 04_{ii}\}$	$\{0_{ip}, 3_{ip}\}$	34_{ii}
		$\{0_{ip}, 4_{ip}\}$	34_{ii}
		$\{0_{ip}, 03_{ii}\}$	34_{ii}
		$\{1_{ip}, 2_{ip}\}$	34_{ii}
		$\{1_{ip}, 3_{ip}\}$	34_{ii}
		$\{1_{ip}, 4_{ip}\}$	34_{ii}
		$\{1_{ip}, 03_{ii}\}$	34_{ii}
		$\{2_{ip}, 3_{ip}\}$	34_{ii}
		$\{2_{ip}, 4_{ip}\}$	34_{ii}
		$\{2_{ip}, 14_{ii}\}$	34_{ii}
		$\{2_{ip}, 03_{ii}\}$	34_{ii}
		$\{3_{ip}, 14_{ii}\}$	34_{ii}
		$\{4_{ip}, 14_{ii}\}$	34_{ii}
		$\{14_{ii}, 24_{ii}\}$	34_{ii}
		$\{14_{ii}, 03_{ii}\}$	34_{ii}
		$\{14_{ii}, 04_{ii}\}$	34_{ii}
		$\{24_{ii}, 03_{ii}\}$	34_{ii}
		$\{03_{ii}, 04_{ii}\}$	34_{ii}
		$\{34_{ii}\}$	Section 3.1

Table 15: The spectrum of the $mcs(44_{ii})$. - Class: Lin.
(Lemma 3.13.)

and $t_i = [b, d]$, all requirements are satisfied. Suppose, now, that $\mathcal{F} \models \varphi([a, b], [c, d])$. We prove that $[a, b]44_{ii}[c, d]$ by eliminating every other possible relation that may hold between x_i and y_i . The relations $=_i$, 14_{ii} , 24_{ii} and 04_{ii} are eliminated immediately thanks to the first and last lines of the definition. Suppose, by way of contradiction, that $x_i34_{ii}y_i$. The interval z_i , which must *start* at a (first line), cannot end between c and d in order to comply with the fact that it cannot *overlap* y_i (last line). If z_i *ends* at d then, since $z_i24_{ii}t_i$, we must have (i) t_i beginning between a and b which causes it to contain y_i contradicting the last line, or (ii) t_i beginning at $b = c$ which causes a contradiction with the fourth line, or (iii) t_i beginning between at c and d causing y_i to *overlap* t_i , contradicting the last line. If z_i *ends* after d , then y_i is *contained* in z_i contradicting the last line. If z_i *ends* at c then t_i *overlaps* x_i contradicting the last line. If z_i *ends* between a and b the overlap between z_i and t_i begins with t_i and is contained in x_i , contradicting the third line. This eliminates all possibilities for the placement of z_i and hence the case $x_i34_{ii}y_i$. Suppose, now, that $y_i34_{ii}x_i$. Then there exists an interval starting with y_i which *overlaps* z_i , contradicting the sixth line. Thus, 34_{ii} is eliminated. Similarly, if $x_i03_{ii}y_i$, then there exists an interval starting with y_i which *overlaps* z_i , contradicting the sixth line. Suppose next that $y_i03_{ii}x_i$. Note that z_i must end after x_i as the overlap between z_i and t_i cannot be *contained* in x_i (third line) and x_i cannot overlap t_i (last line). This means that z_i *contains* y_i , contradicting once more the last line. Thus, 03_{ii} is eliminated as well. Finally, if $y_i44_{ii}x_i$, it is impossible to correctly place z_i and t_i . Having eliminated every other possibility, we conclude that $x_i44_{ii}y_i$. As in the proofs of the previous two relations, the same argument solves the case of $\{0_{ip}, 24_{ii}, 04_{ii}\}$:

$$\begin{aligned}
x_i 44_{ii} y_i \leftrightarrow & \exists z_i t_i (\forall k_p (x_i 0_{ip} k_p \leftrightarrow z_i 0_{ip} k_p) \wedge \neg \forall k_p (x_i 0_{ip} k_p \leftrightarrow y_i 0_{ip} k_p) \wedge \{0_{ip}, 24_{ii}, 04_{ii}\} \\
& \forall w_i (\forall k_p (y_i 0_{ip} k_p \rightarrow w_i 0_{ip} k_p) \rightarrow \neg (z_i 04_{ii} w_i)) \wedge \\
& \forall w_i (\forall k_p (t_i 0_{ip} k_p \rightarrow w_i 0_{ip} k_p) \rightarrow \neg (w_i 04_{ii} x_i)) \wedge \\
& \forall w_i (\forall k_p (y_i 0_{ip} k_p \rightarrow w_i 0_{ip} k_p) \rightarrow \neg (z_i 24_{ii} w_i)) \wedge \\
& \forall w_i (\forall k_p (y_i 0_{ip} k_p \rightarrow w_i 0_{ip} k_p) \rightarrow \neg (x_i 04_{ii} w_i)) \wedge \\
& \forall w_i (\forall k_p (y_i 0_{ip} k_p \rightarrow w_i 0_{ip} k_p) \rightarrow \neg (w_i 24_{ii} z_i)) \wedge \\
& z_i 24_{ii} t_i \wedge \bigwedge_{\substack{s_i, w_i \in \{x_i, y_i, z_i, t_i\} \\ \{s_i, w_i\} \neq \{z_i, t_i\}}} (\neg (s_i 24_{ii} w_i) \wedge \neg (s_i 04_{ii} w_i))
\end{aligned} \quad \square$$

4. INCOMPLETENESS RESULTS IN THE CLASS Lin AND THE CLASS Dis

We now describe and prove the ‘other half’ of the picture, by identifying all maximally incomplete sets for each relation $r \in \mathfrak{R}^+$. To treat incompleteness, since most incomplete set appears as MIS for more than one relation, we present these results as follows. In the table we list, in the leftmost column the maximal incomplete sets, and in the topmost row all \mathfrak{R}^+ -relations. Whenever the crossing point of a column and a row is marked, the set corresponding to its row is a MIS for the relation corresponding to its column, and that fact will be justified in the proof. The section at the top contains those sets for which we give an explicit construction, while the section at the bottom contains all symmetric results. Finally, the sets in the topmost part which are symmetric to themselves will be proved r -incomplete up to reverse of relations (i.e., their r' -incompleteness relative to the reverse of a given r is implied, but not mentioned). The results for the sub-languages induced by \mathfrak{J}^+ and \mathfrak{M}^+ , which turn out to be much simpler than those for \mathfrak{R}^+ , are included in the next section. As already mentioned, we shall prove specific undefinability results via surjective truth-preserving relations between models, and we shall be therefore forced to choose a specific class of linearly ordered sets. At the end of this analysis, it shall turn out that the expressive power in the classes Lin and Dis is identical (every counterexample for the class Lin is based on a discrete structure) while Den and Unb are different from the former two, and (slightly) different from each other. The class Fin (treated in Part II, along with Den and Unb), shall require, as we shall see, a deeper analysis. Throughout the following proofs, let \mathcal{F} and \mathcal{F}' be two concrete structures, and, for every given case, let S be a set of relations that we claim to be r -incomplete for a given r . Case-by-case, we shall define two domains \mathbb{D}, \mathbb{D}' and a surjective S -relation between \mathcal{F} and \mathcal{F}' that breaks r . Notation- and terminology-wise, most of the constructions are based on the *same* domain (in terms of their elements and relative ordering): we distinguish them by means of the superscript $'$, and, with an abuse of terminology, we use the term *identity* relation (over points or over intervals), denoted by Id_p or Id_i to indicate the relation that respects the element’s name (i.e., it relates a with a' , $[a, b]$ with $[a', b']$, and so on). Notice that, all together, the results presented here imply those reported in Section 5, which will therefore require no formal proof. Notice also that each set S listed as r -incomplete for some r in this section has been obtained by means of the technique shown in Fig. 1. This means that it must be maximal by definition: if that was not the case, then $S \cup \{r'\}$ would be r -incomplete for some $r' \notin S$, but $S \cup \{r'\}$ contains some r -complete set, which is a contradiction.

<i>Proved</i>	$=_i$	34_{ii}	14_{ii}	03_{ii}	24_{ii}	04_{ii}	44_{ii}
$\{44_{ii}, 04_{ii}, 24_{ii}\}$	•						
$\{44_{ii}, 04_{ii}, 24_{ii}, =_i\}$		•	•	•			
$\{03_{ii}, =_i\}$		•	•		•	•	•
<i>Symmetric</i>	$=_i$	34_{ii}	14_{ii}	03_{ii}	24_{ii}	04_{ii}	44_{ii}
$\{14_{ii}, =_i\}$		•		•	•	•	•

Table 16: $\text{MIS}(r)$, for each $r \in \mathcal{J}^+$; upper part: sets for which we give an explicit construction; lower part: symmetric ones. - Classes: Lin, Dis.

(Lemma 4.2.)

In what follows, two cases are particularly difficult, and they require an additional concept defined here.

Definition 4.1. We say that the structure $\langle \mathbb{D}^*, \prec \rangle$ is *iterated discrete* if and only if:

- (i) $\mathbb{D}^* = \mathbb{Z} \times \mathbb{Q}$;
- (ii) \prec is defined as follow: $(n, q) \prec (m, r)$ if and only if (1) $q < r$, or (2) $q = r$ and $n < m$, that is, it defines a linear order as the reverse lexicographic order between pairs.

Notice that an iterated discrete structure is, in fact, discrete: the direct predecessor (resp., successor) or (n, q) is $(n - 1, q)$ (resp., $(n + 1, q)$). The two cases that require a construction based on an iterated discrete structure (one in the sub-language induced by \mathcal{J}^+ , the other one in the general case of \mathfrak{R}^+) could be sorted out by a similar (and simpler) construction based on \mathbb{Q} ; however, basing it on a discrete structure yields the stronger incompleteness result we want, that applies to both Dis and Lin.

4.1. (Maximal) \mathcal{J}^+ - and \mathfrak{M}^+ -Incompleteness. As we have done while analyzing the definability of relations, we focus our attention, first, on the sub-languages induced by \mathcal{J}^+ and by \mathfrak{M}^+ .

Lemma 4.2. *Tab. 16 is correct and complete.*

Proof. The fact that $\{44_{ii}, 04_{ii}, 24_{ii}\}$ is maximally incomplete for $=_i$ is justified as follows. Let $\mathbb{D} = \mathbb{D}' = \{a < b < c\}$, and $\zeta = \zeta_i = \{([a, c], [a, b])\} \cup Id_i$. We have that ζ is total and the interval-interval relations in S are trivially respected. But $[a, b]$ is the image of both $[a, b]$ and $[a, c]$, proving that $=_i$ is broken. Now, we prove that $\{44_{ii}, 04_{ii}, 24_{ii}, =_i\}$ is 34_{ii} -, 14_{ii} -, and 03_{ii} -incomplete. To this end, we take $\mathbb{D} = \mathbb{D}' = \{a < b < c\}$ and $\zeta = \zeta_i = \{([a, c], [a', c']), ([a, b], [b', c']), ([b, c], [a', b'])\}$. The relation $\zeta : \mathcal{F} \rightarrow \mathcal{F}'$ is clearly surjective (actually, it is a bijection, so equality between intervals is preserved) with respect to interval, and every relations in S is trivially respected. However, it clearly brakes all intended relations. Consider, now, the case of $\{03_{ii}, =_i\}$, which we have to prove that it is 34_{ii} -, 14_{ii} -, 24_{ii} -, 04_{ii} -, 44_{ii} -incomplete. We proceed as follows: first, we give a construction that proves this set to be 34_{ii} -, 14_{ii} -, 24_{ii} -, and 44_{ii} -incomplete, and, then, give a second construction that proves it to be 04_{ii} -incomplete. As for the first step, define \mathcal{F} and \mathcal{F}' both based on an iterated discrete structure \mathbb{D}^* (see Def. 4.1). Define a relation $\zeta = \zeta_i$ between them as follows: ζ_i is the union of $\{((n, q), (m, r)), [(n', q'), (m', r')]\} \mid q = r \text{ and } n, m \in \mathbb{Z}$ and $\{((n, q), (m, r)), [(n', q' - |r' - q'|), (m', r')]\} \mid q < r \text{ and } n, m \in \mathbb{Z}$. In this way, finite intervals

<i>Proved</i>	0_{ip}	1_{ip}	2_{ip}	3_{ip}	4_{ip}
$\{3_{ip}, 4_{ip}\}$	•	•	•		
$\{0_{ip}, 2_{ip}, 4_{ip}\}$		•		•	
<i>Symmetric</i>	0_{ip}	1_{ip}	2_{ip}	3_{ip}	4_{ip}
$\{0_{ip}, 1_{ip}\}$			•	•	•

Table 17: $\text{MIS}(r)$, for each $r \in \mathfrak{M}^+$; upper part: sets for which we give an explicit construction; lower part: symmetric ones. - Classes: Lin, Dis.

(Lemma 4.3.)

(i.e., those containing a finite number of points) are related to themselves, while infinite ones are related to intervals with the same *ending* point but twice the length (with respect the rational coordinate). We want to prove that ζ is an S -relation. Obviously, the only interesting cases are those that involve at least one infinite interval. Consider two intervals $[(n, q), (m, r)]$ and $[(l, s), (m, r)]$ with $(l, s) \prec (n, q)$; they are 03_{ii} -related, so either both are infinite or $[(n, q), (m, r)]$ is finite and $[(l, s), (m, r)]$ is infinite, so under ζ the endpoints are kept constant and either both are doubled in length or $[(n, q), (m, r)]$ is kept fixed and only $[(l, s), (m, r)]$ is doubled. Either way, 03_{ii} is respected. Since ζ is a bijection by definition, equality between intervals is respected too, and therefore ζ is a proper S -relation. It is clear, on the other hand, that 34_{ii} , $14_{,ii}24_{ii}$ and 44_{ii} are broken. Now, as for the second step, consider once again \mathcal{F} and \mathcal{F}' both based on an iterated discrete structure \mathbb{D}^* and define a new S -relation $\zeta = \zeta_i$ between them as follows: ζ_i is the union of $\{([(n, q), (m, r)], [(n', q'), (m', r')]) \mid q = r \text{ and } n, m \in \mathbb{Z}\}$ and $\{([(n, q), (m, r)], [(n', q' + (|r' - q'|/2)), (m', r')]) \mid q < r \text{ and } n, m \in \mathbb{Z}\}$. In this way, finite intervals are related to themselves, while infinite ones are related to intervals with the same *ending* point but *half* the length (with respect to the rational coordinate). It is straightforward to check that this is a bijective S -relation which breaks 04_{ii} and respects both equality between intervals and the relation 03_{ii} . The fact that the table is complete is a consequence of the following observation: every set not listed is either contained in some of the listed ones (and, thus, it is not maximal with respect to incompleteness) or \mathfrak{I}^+ -complete (from the results of Section 3). \square

Lemma 4.3. *Tab. 17 is correct and complete.*

Proof. Let us start by proving that $\{3_{ip}, 4_{ip}\}$ is $0_{ip^-}, 1_{ip^-}$, and 2_{ip^-} -incomplete. Take $\mathbb{D} = \mathbb{D}' = \{a < b < c\}$, $\zeta_p = Id_p$, and $\zeta_i = \{([b, c], [a', c']), ([a, c], [b', c']), ([a, b], [a', b'])\}$. It is easy to verify that 3_{ip} , 4_{ip} are respected, but none of the other mixed relations are. Then, to prove that $\{0_{ip}, 2_{ip}, 4_{ip}\}$ is 1_{ip^-} and 3_{ip^-} -incomplete, we simply take $\mathbb{D} = \mathbb{D}' = \{a < b\}$, $\zeta_p = \{(a, b'), (b, a')\}$, and $\zeta_i = Id_i$. \square

4.2. (Maximal) \mathfrak{R}^+ -Incompleteness. To conclude this section, we analyze the results shown in Tab. 18. Notice that some of the constructions presented here can be actually considered as generalizations of those seen above.

Lemma 4.4. *Tab. 18 is correct and complete.*

<i>Proved</i>	$=_p$	$=_i$	$<$	0_{ip}	1_{ip}	2_{ip}	3_{ip}	4_{ip}	34_{ii}	14_{ii}	03_{ii}	24_{ii}	04_{ii}	44_{ii}
$\{0_{ip}, 2_{ip}, 4_{ip}\} \cup \mathcal{J}^+$	•													
$\{=_{p, <}, 0_{ip}, 1_{ip}, 44_{ii}, 04_{ii}, 24_{ii}\}$		•												
$\{=_{p, <}, 2_{ip}, 44_{ii}, 04_{ii}, 24_{ii}\}$		•												
$\{=_{p, =_i}, 1_{ip}, 14_{ii}\}$			•	•										
$\{=_{p, =_i}, 1_{ip}, 44_{ii}, 04_{ii}, 24_{ii}\}$			•	•										
$\{=_{p, =_i}, 0_{ip}, 2_{ip}, 4_{ip}\} \cup \mathcal{J}^+$			•		•		•							
$\{=_{p, =_i}, 2_{ip}\} \cup \mathcal{J}^+$				•				•						
$\{=_{p, =_i, <}, 2_{ip}, 44_{ii}, 04_{ii}, 24_{ii}\}$				•	•		•	•	•	•				
$\{=_{p, =_i, <}, 3_{ip}, 4_{ip}, 44_{ii}, 04_{ii}, 24_{ii}\}$				•	•	•			•	•	•			
$\{=_{p, =_i, <}, 3_{ip}, 4_{ip}, 03_{ii}\}$				•	•	•			•	•		•	•	•
$\{=_{p, =_i}, 4_{ip}\} \cup \mathcal{J}^+$				•		•								
$\{=_{p, <}\} \cup \mathcal{J}^+$				•	•	•	•	•						
$\{=_{p, =_i, <}, 0_{ip}, 1_{ip}, 04_{ii}\}$												•		•
$\{=_{p, =_i, <}, 0_{ip}, 1_{ip}, 44_{ii}\}$												•	•	
$\{=_{p, =_i}, 2_{ip}, 04_{ii}, 44_{ii}\}$												•		
$\{=_{p, =_i, <}, 04_{ii}, 44_{ii}\}$												•		
$\{=_{p, =_i, <}, 2_{ip}, 24_{ii}, 44_{ii}\}$													•	
$\{=_{p, =_i, <}, 0_{ip}, 1_{ip}, 24_{ii}\}$													•	•
<i>Symmetric</i>	$=_p$	$=_i$	$<$	0_{ip}	1_{ip}	2_{ip}	3_{ip}	4_{ip}	34_{ii}	14_{ii}	03_{ii}	24_{ii}	04_{ii}	44_{ii}
$\{=_{p, <}, 3_{ip}, 4_{ip}, 44_{ii}, 04_{ii}, 24_{ii}\}$		•												
$\{=_{p, =_i}, 3_{ip}, 03_{ii}\}$			•					•						
$\{=_{p, =_i}, 3_{ip}, 44_{ii}, 04_{ii}, 24_{ii}\}$			•					•						
$\{=_{p, =_i, <}, 0_{ip}, 1_{ip}, 44_{ii}, 04_{ii}, 24_{ii}\}$						•	•	•	•	•	•			
$\{=_{p, =_i, <}, 0_{ip}, 1_{ip}, 14_{ii}\}$						•	•	•	•		•	•	•	•
$\{=_{p, =_i}, 0_{ip}\} \cup \mathcal{J}^+$						•		•						
$\{=_{p, =_i, <}, 3_{ip}, 4_{ip}, 04_{ii}\}$												•		•
$\{=_{p, =_i, <}, 3_{ip}, 4_{ip}, 44_{ii}\}$												•	•	
$\{=_{p, =_i, <}, 3_{ip}, 4_{ip}, 24_{ii}\}$													•	•

Table 18: $\text{MIS}(r)$, for each $r \in \mathfrak{R}^+$; upper part: sets for which we give an explicit construction; lower part: symmetric ones. - Classes: Lin, Dis.

(Lemma 4.4.)

Proof. We need to give a construction for 18 different sets. Most of these constructions are similar to each other, but no part of any of them can be actually be re-used. Notice that every construction is based on a discrete set: this is why definability results over the classes Dis and Lin coincide. Observe, also, that most of the constructions are actually based on finite sets, which suggests that the behaviour of these languages on Fin is very similar to that of Lin and Dis; nevertheless, as we have mentioned, in Part II a deeper analysis is required to complete the case Fin.

Let S be $\{0_{ip}, 2_{ip}, 4_{ip}\} \cup \mathcal{J}^+$; we have to prove that it is $=_p$ -incomplete. To this end, we simply take $\mathbb{D} = \mathbb{D}' = \{a < b\}$ and $\zeta = (\zeta_p, \zeta_i)$, where $\zeta_i = Id_i$, $\zeta_p = \{(a, a'), (a, b'), (b, a')\}$. The relation $\zeta : \mathcal{F} \rightarrow \mathcal{F}'$ is clearly total and surjective with respect to both interval and points, and every relations in S is trivially respected. However, the pairs (a, a') and (a, b') show that equality between points is not respected. Let S be $\{=_{p, <}, 0_{ip}, 1_{ip}, 44_{ii}, 04_{ii}, 24_{ii}\}$. We

have to prove here that it is $=_i$ -incomplete. In this case, let $\mathbb{D} = \mathbb{D}' = \{a < b < c\}$, $\zeta_p = Id_p$ and $\zeta_i = \{([a, c], [a, b])\} \cup Id_i$. Again, S is total and surjective. Moreover, interval-interval relations are trivially respected, and point-interval relations are respected thanks to the fact that the *beginning* points of intervals are maintained through ζ . But $[a, b]$ is the image of both $[a, b]$ and $[a, c]$, proving that $=_i$ is broken. A very similar construction, based on the same structures, with $\zeta_p = Id_p$ and $\zeta_i = \{([b, c], [a, b])\} \cup Id_i$, covers the $=_i$ -incompleteness of $\{=_{ip}, <, 2_{ip}, 4_{ii}, 0_{4ii}, 2_{4ii}\}$. Let S be $\{=_{ip}, =_i, 1_{ip}, 1_{4ii}\}$; we shall prove that it is 0_{ip} - and $<$ -incomplete. In this case, let $\mathbb{D} = \mathbb{D}' = \mathbb{Z}$, and define $\zeta = (\zeta_p, \zeta_i)$ as follows: $(a, -a) \in \zeta_p$ for every $a \in \mathbb{Z}$, and $([a, b], [-a, -a + |b - a|]) \in \zeta_i$ for every $[a, b] \in \mathbb{I}(\mathbb{D})$, so that the length of every interval is preserved. Now, the interval-interval relation 1_{4ii} must be respected by ζ , as intervals are mapped along with their beginning point, while the ending point does not move relative to the beginning point; for the same reason, 1_{ip} is also respected. Finally, ζ is a bijection, so $=_i$ and $=_p$ are respected too; as it breaks both $<$ and 0_{ip} , these cannot be expressed in this language. Let S be $\{=_{ip}, 1_{ip}, 4_{ii}, 0_{4ii}, 2_{4ii}\}$; we will prove that it is $<$ - and 0_{ip} -incomplete. We take $\mathbb{D} = \mathbb{D}' = \{a < b < c\}$ and define $\zeta_p = \{(a, b'), (b, a'), (c, c')\}$ and $\zeta_i = \{([a, b], [b', c']), ([b, c], [a', b']), ([b, c], [a', c']), ([a, c], [b', c'])\}$. All interval-interval relations are (vacuously) respected, 1_{ip} is respected as well (as *beginning* points of intervals are preserved), and equality between points is respected too (as ζ_p is a bijection), but $<$ and 0_{ip} are broken. If S be $\{=_{ip}, 0_{ip}, 2_{ip}, 4_{ip}\} \cup \mathcal{J}^+$, then take $\mathbb{D} = \mathbb{D}' = \{a < b\}$, $\zeta_i = \{([a, b], [a', b'])\}$ and $\zeta_p = \{(a, b'), (b, a')\}$, which respects S and breaks $<$, 1_{ip} , and 3_{ip} , as it can be immediately verified. Now, let S be $\{=_{ip}, 2_{ip}\} \cup \mathcal{J}^+$. To prove that S is 0_{ip} - and 4_{ip} -incomplete we take $\mathbb{D} = \mathbb{D}' = \{a < b < c\}$, and $\zeta = (\zeta_p, \zeta_i)$, where $\zeta_i = Id_i$, $\zeta_p = \{(a, c'), (c, a'), (b, b')\}$. The relation $\zeta : \mathcal{F} \rightarrow \mathcal{F}'$ is clearly surjective respectively to both interval and points, and every relations in S is trivially preserved; on the other hand, both 0_{ip} and 4_{ip} are broken. Let S be $\{=_{ip}, =_i, <, 2_{ip}, 4_{ii}, 0_{4ii}, 2_{4ii}\}$; we prove that it is 0_{ip} -, 1_{ip} -, 3_{ip} -, 4_{ip} -, 1_{4ii} -, $0_{ip}3$ -, and 3_{4ii} -incomplete. It is enough to take $\mathbb{D} = \mathbb{D}' = \{a < b < c\}$, with $\zeta = (\zeta_p, \zeta_i)$ defined as $\zeta_p = Id_p$ and $\zeta_i = \{([a, c], [a', c']), ([a, b], [b', c']), ([b, c], [a', b'])\}$, to have a relation ζ which is surjective respectively to both intervals and points, preserves every relations in S , and breaks all the indicated relations. If S is $\{=_{ip}, =_i, <, 3_{ip}, 4_{ip}, 4_{ii}, 0_{4ii}, 2_{4ii}\}$, we have to prove that it is 0_{ip} -, 1_{ip} -, 2_{ip} -, 1_{4ii} -, 0_{3ii} -, and 3_{4ii} -incomplete. Take $\mathbb{D} = \mathbb{D}' = \{a < b < c\}$, $\zeta_p = Id_p$, and $\zeta_i = \{([b, c], [a', c']), ([a, c], [b', c']), ([a, b], [a', b'])\}$. It is easy to verify that all 9 relations in S are respected, but none of the indicated ones are. Now, let S be $\{=_{ip}, =_i, <, 3_{ip}, 4_{ip}, 0_{3ii}\}$; we prove that it is 0_{ip} -, 1_{ip} -, and 2_{ip} -incomplete, and, also, r -incomplete for every $r \in \mathcal{J}^+ \setminus \{0_{3ii}, =_i\}$. As in Lemma 4.2, we proceed in two steps, first proving the incompleteness for all indicated relations but 0_{4ii} , and, then, dealing with 0_{4ii} on its own. As for the first step, define \mathcal{F} and \mathcal{F}' both based on the iterated discrete structure \mathbb{D}^* , as in Def. 4.1. Define a relation ζ between them as follows:

(i) $\zeta_p = Id_p$;

(ii) ζ_i is the union of

$$\{([(n, q), (m, r)], [(n', q'), (m', r')]) \mid q = r \text{ and } n, m \in \mathbb{Z}\}$$

and

$$\{([(n, q), (m, r)], [(n', q' - |r' - q'|), (m', r')]) \mid q < r \text{ and } n, m \in \mathbb{Z}\}.$$

In this construction, which generalizes to the case of \mathfrak{R}^+ the one already seen in Lemma 4.2 by adding the ζ_p component, finite intervals (i.e., those containing a finite number of points)

are related to themselves, while infinite ones are related to intervals with the same *ending* point but twice the length (w.r.t. the rational coordinate). We want to prove that ζ is an S -relation. Obviously, the only interesting cases are those that involve at least one infinite interval. Suppose, first, that $[(n, q), (m, r)]$ is 3_{ip} - or 4_{ip} -related to some point; under ζ , these relations are clearly respected. So, consider two intervals $[(n, q), (m, r)]$ and $[(l, s), (m, r)]$ with $(l, s) \prec (n, q)$; they are 03_{ii} -related, so either both are infinite or $[(n, q), (m, r)]$ is finite and $[(l, s), (m, r)]$ is infinite, so under ζ the endpoints are kept constant and either both are ‘doubled’ in length or $[(n, q), (m, r)]$ is kept fixed and only $[(l, s), (m, r)]$ is ‘doubled’. Either way, 03_{ii} is respected. Since ζ is a bijection, equalities of both sort is respected too, and therefore ζ is a proper S -relation. It is clear, on the other hand, that 0_{ip} , 1_{ip} , 2_{ip} and 34_{ii} , 14_{ii} , 24_{ii} and 44_{ii} are broken. As for the second step, define, again, \mathcal{F} and \mathcal{F}' both based on an iterated discrete structure \mathbb{D}^* , and define a new relation ζ as follows: (i) $\zeta_p = Id_p$; (ii) ζ_i is the union of $\{([(n, q), (m, r)], [(n', q'), (m', r')]) \mid q = r \text{ and } n, m \in \mathbb{Z}\}$ and $\{([(n, q), (m, r)], [(n', q' + (|r' - q'|/2)), (m', r')]) \mid q < r \text{ and } n, m \in \mathbb{Z}\}$. In this way, finite intervals are related to themselves, while infinite ones are related to intervals with the same *ending* point but *half* the length (with respect to the rational coordinate). It is straightforward to check that this relation preserves both equalities and 03_{ii} , and that it breaks 04_{ii} as we wanted. Let S be $\{=_{p, 4_{ip}}\} \cup \mathcal{I}^+$; we prove that it is 0_{ip} - and 2_{ip} -incomplete. For this, we take $\mathbb{D} = \mathbb{D}' = \{a < b < c\}$; we define $\zeta_p = \{(a, b'), (b, a'), (c, c')\}$ and $\zeta_i = Id_i$, and we have that all interval-interval relations are respected, 4_{ip} is respected as well ($\zeta_p(c, c')$ holds, and c and c' are the only points in relation 4_{ip} with some interval), and equality between points is respected too (as ζ_p is a bijection), but 0_{ip} and 2_{ip} are broken. Let S be $\{=_{p, <}\} \cup \mathcal{I}^+$; we prove here that this set is \mathfrak{M}^+ -incomplete. To this aim we can simply take two copies of the integers, $\zeta_p = \{(n, n' + 1)\}$ for every $n \in \mathbb{Z}$, and $\zeta_i = Id_i$; clearly, ζ respects S , as the only broken relations are the mixed ones. If S is $\{=_{p, =i, <}, 0_{ip}, 1_{ip}, 04_{ii}\}$, for which we have to prove its 44_{ii} - and 24_{ii} -incompleteness, we take $\mathbb{D} = \mathbb{D}' = \{a < b < c < d\}$, $\zeta_p = Id_p$, and $\zeta_i = \{([a, b], [a', c']), ([a, c], [a', b'])\}$ plus the identity relation on every other interval; relations 0_{ip} and 1_{ip} are preserved, since the *starting* point of every interval does not change, the relation 04_{ii} is preserved as well, as the only intervals that are not identically related have no *contained* intervals (neither they are *contained* by any other), and since ζ is a bijection, equalities are preserved too. Nevertheless, the two indicated relations are broken. Let S be $\{=_{p, =i, <}, 0_{ip}, 1_{ip}, 44_{ii}\}$; we have to prove that it is 04_{ii} - and 24_{ii} -incomplete. We proceed as before, by taking $\mathbb{D} = \mathbb{D}' = \{a < b < c < d\}$, $\zeta_p = Id_p$, and $\zeta_i = \{([a, c], [a', d']), ([a, d], [a', c'])\}$ plus the identity relation on every other interval, and, as it is immediate to check, all interval-interval and point-interval relations in S are respected, but 04_{ii} and 24_{ii} are not respected. When S is $\{=_{i, =p, 2_{ip}, 04_{ii}, 44_{ii}}\}$, we need to prove that it is 24_{ii} -incomplete. To this end, we take $\mathbb{D} = \mathbb{D}' = \{a < b < c < d\}$, $\zeta_p(b) = c'$, $\zeta_p(c) = b'$ plus the identity relation over the other points, and $\zeta_i = \{([a, c], [b', d']), ([b, d], [a', c'])\}$ plus the identity relation on every other interval. The relation 2_{ip} is preserved, since the internal points of every interval involved in ζ are moved along the interval that *contains* them, the relations 04_{ii} and 44_{ii} are respected, as the only intervals that are not identically related via ζ are not *contained* in, or *before* any other interval, and since ζ is a bijection, equalities are respected as well. Conversely, 24_{ii} is broken. An identical construction, where we simply take $\zeta_p = Id_p$, proves the 24_{ii} -incompleteness of $\{=_{i, =p, <}, 04_{ii}, 44_{ii}\}$. When we have to prove that $\{=_{i, =p, <}, 2_{ip}, 24_{ii}, 44_{ii}\}$ is 04_{ii} -incomplete, we simply take $\mathbb{D} = \mathbb{D}' = \{a < b < c < d\}$, $\zeta_p = Id_p$, and $\zeta_i = \{([b, c], [c', d']), ([c, d], [b', c'])\}$ plus the identity relation on every other

<pre> proc MinDef_MaxUndef () for all $S \subseteq \mathfrak{R}^+$ { if $\forall r \in \mathfrak{R}^+ (r \in \text{Closure}(S))$ and { S is minimal, list S as $\text{mcs}(\mathfrak{R}^+)$ } if $\exists r \in \mathfrak{R}^+ (r \notin \text{Closure}(S))$ and { S is maximal, list S as $\text{MIS}(\mathfrak{R}^+)$ } return; } </pre>	<pre> proc IsExpressiveAs (set S, S') { $S = \text{Closure}(S)$; $S' = \text{Closure}(S')$; if $S = S'$ return Yes; return No; } </pre>
--	--

Figure 2: Pseudo-code to compare the expressive power of subsets of \mathfrak{R}^+ .

interval, and, finally, to prove that $\{=_{p, =i}, <, 0_{ip}, 1_{ip}, 24_{ii}\}$ is 44_{ii} - and 04_{ii} -incomplete, we take $\mathbb{D} = \mathbb{D}' = \{a < b < c < d\}$, $\zeta_p = Id_p$, and $\zeta_i = \{([a, b], [a', d']), ([a, d], [a', b'])\}$ plus the identity relation on every other interval. \square

5. HARVEST: THE COMPLETE PICTURE FOR Lin AND Dis

We are now capable to identify all expressively different subsets of \mathfrak{R}^+ , \mathfrak{T}^+ , and \mathfrak{M}^+ . One can easily establish the expressive power of a particular subset S with respect to another subset S' by simply comparing, containment-wise, the closure (by definability) of S with the closure of S' (see Fig. 2). Clearly, it is very difficult to display the resulting Hasse-diagram; we then limit ourselves to list all maximally incomplete sets for each case. The following theorem is a consequence of all results seen so far.

Theorem 5.1. *If a set of relations is listed:*

- as $\text{mcs}(\mathfrak{R}^+)$ (resp., $\text{mcs}(\mathfrak{T}^+)$, $\text{mcs}(\mathfrak{M}^+)$) in Tab. 19 left (resp., right-top, right-bottom) side, left column, then it is minimally \mathfrak{R}^+ -complete (resp., minimally \mathfrak{T}^+ -complete, minimally \mathfrak{M}^+ -complete) in the class of all linearly ordered sets and in the class of all discrete linearly ordered sets.
- as $\text{MIS}(\mathfrak{R}^+)$ (resp., $\text{MIS}(\mathfrak{T}^+)$, $\text{MIS}(\mathfrak{M}^+)$) in Tab. 19 left (resp., right-top, right-bottom) side, right column, then it is maximally \mathfrak{R}^+ -incomplete (resp., maximally \mathfrak{T}^+ -incomplete, maximally \mathfrak{M}^+ -incomplete) in the class of all linearly ordered sets and in the class of all discrete linearly ordered sets.

6. CONCLUSIONS

We considered here the two-sorted first-order temporal language that includes relations between intervals, points, and inter-sort, and we treated equality between points and between intervals as any other relation, with no special role. Under four different assumptions on the underlying structure, namely, linearity only, linearity+discreteness, linearity+density, and linearity+unboundedness, we asked the question: which relation can be first-order defined by which subset of all relations? As a result, we identified all possible inter-definability between relations, all minimally complete, and all maximally incomplete subsets of relations. These inter-definability results allow one to effectively compute all expressively different subsets of relations, and, with minimal effort, also all expressively different subsets of relations for the interesting sub-languages of interval relations only or mixed relations only. Two out of four interesting classes of linearly ordered sets are treated in this Part I, and the remaining two

\mathfrak{R}^+	
mcs	MIS
$\{0_{ip}, 2_{ip}, <\}$	$\{=i, =p, <, 0_{ip}, 1_{ip}, 14_{ii}\}$
$\{0_{ip}, 3_{ip}\}$	$\{=i, =p, <, 0_{ip}, 1_{ip}, 24_{ii}, 04_{ii}, 44_{ii}\}$
$\{0_{ip}, 4_{ip}, <\}$	$\{=p, 0_{ip}, 2_{ip}, 4_{ip}\} \cup \mathfrak{J}^+$
$\{0_{ip}, 14_{ii}, 24_{ii}, <\}$	$\{=i, =p, <, 2_{ip}, 24_{ii}, 04_{ii}, 44_{ii}\}$
$\{0_{ip}, 14_{ii}, 04_{ii}, <\}$	$\{=p, <\} \cup \mathfrak{J}^+$
$\{0_{ip}, 14_{ii}, 44_{ii}, <\}$	$\{=i, =p, <, 3_{ip}, 4_{ip}, 03_{ii}\}$
$\{0_{ip}, 03_{ii}, <\}$	$\{=i, =p, <, 3_{ip}, 4_{ip}, 24_{ii}, 04_{ii}, 44_{ii}\}$
$\{0_{ip}, 34_{ii}, <\}$	
$\{1_{ip}, 2_{ip}\}$	
$\{1_{ip}, 3_{ip}\}$	
$\{1_{ip}, 4_{ip}\}$	
$\{1_{ip}, 14_{ii}, 24_{ii}\}$	
$\{1_{ip}, 14_{ii}, 04_{ii}\}$	
$\{1_{ip}, 14_{ii}, 44_{ii}\}$	
$\{1_{ip}, 03_{ii}\}$	
$\{1_{ip}, 34_{ii}\}$	
$\{2_{ip}, 3_{ip}\}$	
$\{2_{ip}, 4_{ip}, <\}$	
$\{2_{ip}, 14_{ii}, <\}$	
$\{2_{ip}, 03_{ii}, <\}$	
$\{2_{ip}, 34_{ii}, <\}$	
$\{3_{ip}, 14_{ii}\}$	
$\{3_{ip}, 24_{ii}, 03_{ii}\}$	
$\{3_{ip}, 03_{ii}, 04_{ii}\}$	
$\{3_{ip}, 03_{ii}, 44_{ii}\}$	
$\{3_{ip}, 34_{ii}\}$	
$\{4_{ip}, 14_{ii}, <\}$	
$\{4_{ip}, 24_{ii}, 03_{ii}, <\}$	
$\{4_{ip}, 03_{ii}, 04_{ii}, <\}$	
$\{4_{ip}, 03_{ii}, 44_{ii}, <\}$	
$\{4_{ip}, 34_{ii}, <\}$	

\mathfrak{J}^+	
mcs	MIS
$\{34_{ii}\}$	$\{=i, 14_{ii}\}$
$\{14_{ii}, 24_{ii}\}$	$\{=i, 24_{ii}, 04_{ii}, 44_{ii}\}$
$\{14_{ii}, 04_{ii}\}$	$\{=i, 03_{ii}\}$
$\{14_{ii}, 03_{ii}\}$	
$\{14_{ii}, 44_{ii}\}$	
$\{03_{ii}, 24_{ii}\}$	
$\{03_{ii}, 04_{ii}\}$	
$\{03_{ii}, 44_{ii}\}$	

\mathfrak{M}^+	
mcs	MIS
$\{0_{ip}, 3_{ip}\}$	$\{0_{ip}, 1_{ip}\}$
$\{1_{ip}, 2_{ip}\}$	$\{0_{ip}, 2_{ip}, 4_{ip}\}$
$\{1_{ip}, 3_{ip}\}$	$\{3_{ip}, 4_{ip}\}$
$\{1_{ip}, 4_{ip}\}$	
$\{2_{ip}, 3_{ip}\}$	

Table 19: Left: minimally \mathfrak{R}^+ -complete and maximally \mathfrak{R}^+ -incomplete sets. Right, top: minimally \mathfrak{J}^+ -complete and maximally \mathfrak{J}^+ -incomplete sets. Right, bottom: minimally \mathfrak{M}^+ -complete and maximally \mathfrak{M}^+ -incomplete sets. - Classes: Lin, Dis.

are dealt with in Part II (forthcoming). There are several aspects of temporal reasoning in computer science to which this extensive study can be related:

- first-order logic over linear orders extended with temporal relations between points, intervals and mixed, are the very foundation of modal logics for temporal reasoning, and it is necessary to have a complete understanding of the former in order to deal with the latter;
- automated reasoning techniques for interval-based modal logics are at their first stages; an uncommon, but promising approach is to treat them as pure modal logics over particular Kripke-frames, whose first-order properties are, in fact, representation theorems such as

- those (indirectly) treated in this paper. As a future work, we also plan to systematically study the area of representation theorems;
- the decidability of pure first-order theories extended with interval relations is well-known [Lad]; nevertheless, these results hinge on the decidability of $\text{MFO}[\langle]$, while we believe that they could be refined both algorithmically and computationally;
 - the study of other related languages, important in artificial intelligence, can benefit from our results, such as first-order and modal logics for spatial reasoning where basic objects are, for example, rectangles.

REFERENCES

- [AF94] J.F. Allen and G. Ferguson. Actions and events in interval temporal logic. *Journal of Logic and Computation*, 4(5):531–579, 1994.
- [AH85] J.F. Allen and P. Hayes. A common-sense theory of time. In *Proc. of the 9th International Joint Conference on Artificial Intelligence*, pages 528–531. Morgan Kaufmann, 1985.
- [AH87] J.F. Allen and P. J. Hayes. Short time periods. In *Proc. of the 10th International Joint Conference on Artificial Intelligence*, pages 981–983, 1987.
- [AH89] J.F. Allen and P.J. Hayes. Moments and points in an interval-based temporal logic. *Computational Intelligence*, 5:225–238, 1989.
- [All83] J.F. Allen. Maintaining knowledge about temporal intervals. *Communications of the ACM*, 26(11):832–843, 1983.
- [BGS11] P. Balbiani, V. Goranko, and G. Sciavicco. Two-sorted point-interval temporal logics. *Electronic Notes on Theoretical Computer Science*, 278:31 – 45, 2011.
- [BMG⁺14] D. Bresolin, D. Della Monica, V. Goranko, A. Montanari, and G. Sciavicco. The dark side of interval temporal logic: Marking the undecidability border. *Annals of Mathematics and Artificial Intelligence*, 71(1-3):41 – 83, 2014.
- [BMM⁺14] D. Bresolin, D. Della Monica, A. Montanari, P. Sala, and G. Sciavicco. Interval temporal logics over strongly discrete linear orders: Expressiveness and complexity;. *Theoretical Computer Science*, 560:269 – 291, 2014.
- [Boc90] A. Bochman. Concerted instant-interval temporal semantics I: Temporal ontologies. *Notre Dame Journal of Formal Logic*, 31(3):403–414, 1990.
- [CDS12] W. Conradie, S. Durhan, and G. Sciavicco. An integrated first-order theory of points and intervals: Expressive power in the class of all linear orders. In *Proc. of the 19th International Symposium on Temporal Representation and Reasoning (TIME)*, pages 47 – 51. IEEE, 2012.
- [CM00] L. Chittaro and A. Montanari. Temporal representation and reasoning in artificial intelligence: Issues and approaches. *Annals of Mathematics and Artificial Intelligence*, 28(1-4):47–106, 2000.
- [Coe09] C. J. Coetsee. Representation theorems for classes of interval structures. Master’s thesis, Department of Mathematics, University of Johannesburg, 2009.
- [CS11] W. Conradie and G. Sciavicco. On the expressive power of first order-logic extended with Allen’s relations in the strict case. In *Proc. of the 2011 Conferencia Española para la Inteligencia Artificial (CAEPIA)*, volume 7023 of *Lecture Notes in Computer Science*, pages 173–182. Springer, 2011.
- [GMS03] V. Goranko, A. Montanari, and G. Sciavicco. Propositional interval neighborhood temporal logics. *Journal of Universal Computer Science*, 9(9):1137 – 1167, 2003.
- [Hod93] W. Hodges. *Model Theory*. Encyclopedia of Mathematics and its Applications. Cambridge University Press, 1993.
- [HS91] J. Halpern and Y. Shoham. A propositional modal logic of time intervals. *Journal of the ACM*, 38(4):935–962, 1991.
- [Lad] P.B. Ladkin. Deciding first-order statements about time intervals. Kestrel Institute Technical Report.
- [Lad78] P.B. Ladkin. *The Logic of Time Representation*. PhD thesis, University of California, Berkeley, 1978.
- [Lig91] G. Ligozat. On generalized interval calculi. In *Proc. of the 9th National Conference on Artificial Intelligence (AAAI 91)*, pages 234–240, 1991.

- [MGMS11] D. Della Monica, V. Goranko, A. Montanari, and G. Sciavicco. Expressiveness of the Interval Logics of Allen's Relations on the Class of all Linear Orders: Complete Classification. In *Proc. of the 22nd International Joint Conference on Artificial Intelligence (IJCAI)*, pages 845–850, 2011.
- [MH06] J. Ma and P.J. Hayes. Primitive intervals versus point-based intervals: Rivals or allies? *Comput. J.*, 49(1):32–41, 2006.
- [vB91] J. van Benthem. *The Logic of Time (2nd Edition)*. Kluwer Academic Press, 1991.
- [Ven91] Y. Venema. A modal logic for chopping intervals. *Journal of Logic and Computation*, 1(4):453–476, 1991.

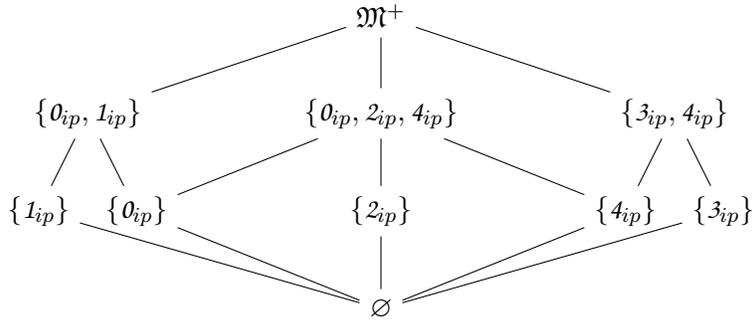


Figure 3: The lattice of expressively different fragments of $FO + \mathfrak{M}^+$

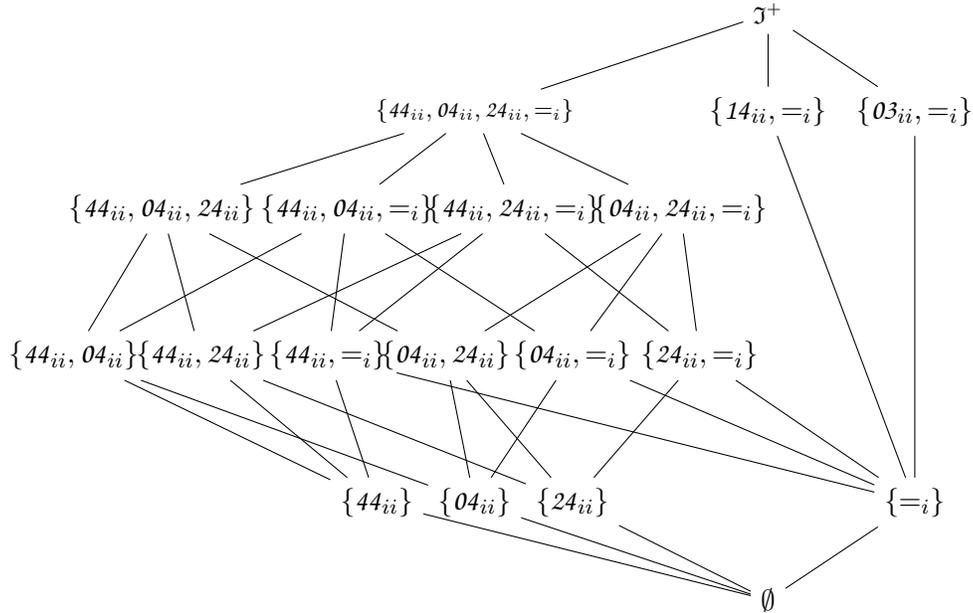


Figure 4: The lattice of expressively different fragments of $FO + \mathfrak{J}^+$

APPENDIX: HASSE DIAGRAMS

A pictorial representation of the results of this paper, in terms of relative expressive power of subsets of relations is shown here.

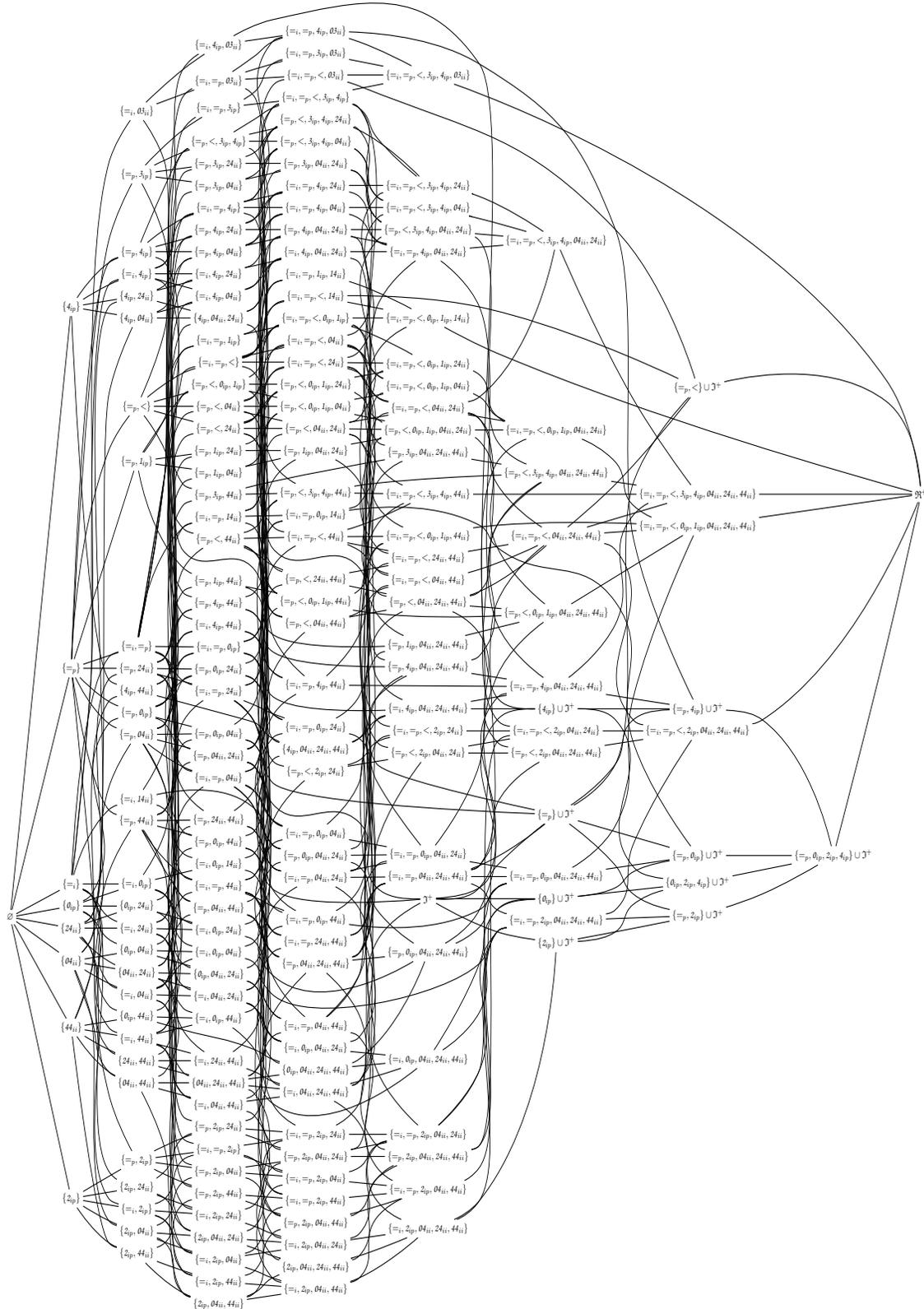


Figure 5: The lattice of expressively different fragments of $FO + \mathfrak{R}^+$