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Tiered objects

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Abstract. We investigate the foundations of reasoning over infinite data structures by means of set-theoretical structures arising in the sheaf-theoretic semantics of higher-order intuitionistic logic. Our approach focuses on a natural notion of *tiering* represented by the operation of restriction in the formalization of (pre)sheaves. We relate these tiered objects to final coalgebras and initial algebras of a wide class of endofunctors of the category of sets, and study their order and convergence properties. As a sample application, we derive a general proof principle for tiered objects.

Keywords: complete Heyting algebras, sheaves, initial algebras, final coalgebras, infinite data structures, approximation lemma

1. Introduction

Among the many frameworks that have been proposed in order to deal with infinite objects in a computational context, an important place is taken by structures whose elements are cumulatively arranged in layers according to a general notion of rank taking values in the set of *extended natural numbers* $0, 1, 2, \ldots, \infty$. Structures of this kind have appeared in many guises in the literature. Without aiming at an exhaustive list, we mention the *rank-ordered sets* introduced in [8], motivated by the semantics of recursive types and later extended in several ways, see [2, 26]. Closely related structures are the *complete projection spaces*, [19], that arose independently from work on the projective model of process algebra [5, 25]. The families of equivalence relations studied in [29] and (complete) ultrametric spaces [31, §§8.1-.2] are further ways of formalizing the same basic intuitions.

Central to all these structures is a notion of *tiering*. The starting point of the present paper is the observation that, in many interesting cases, tiered objects can be regarded as a special case of (pre)sheaves over a complete Heyting algebra as formulated by Fourman and Scott [14] in their work on the sheaf-theoretic semantics of intuitionistic higher-order logic. This provides an elegant framework for discussing general properties of infinite objects as made up of their initial segments of finite length, which is the theme underlying all the structures mentioned above. We have construed this theme as an instance of the passage from local to global which is typical of sheaves, an analogy also noticed, for example, in [15] and, in a far more general context albeit less explicitly, in [33].

Important examples of tiered objects arise from final coalgebras. For example, the set $X^\infty =_{\text{def}} X^* \cup X^\omega$ of finite and infinite sequences, or words, over a non-empty alphabet X, is the final coalgebra of the polynomial endofunctor of $\mathbf{Set}$ which assigns to a set Y the set $\{\varepsilon\} \cup (X \times Y)$. It is known that such coalgebras can be regarded both as complete ultrametric spaces [4] and as complete partial orders [1]. However, by making explicit the tiering structure, inductive arguments can be carried out in the place of coinductive proofs that exploit bisimulations in order to prove equality of lazy data structures [16]. In this respect, our work bears on research described, for example, in [28] where the role of tiering is played by sequences of equivalence relations indexed over a well-founded set, allowing recursive definitions over non-inductive types.

After defining tiered objects and proving their basic properties in Section 2, in Section 3 we display examples demonstrating the extent to which tiered objects subsume common approaches to spaces of infinite structures. These examples include finite and infinite words, recursively defined functions, and final coalgebras of ω-continuous set functors. In Section 4, we discuss completeness of tiered objects and show the equivalence of order-theoretic completeness with Cauchy completeness, where Cauchy sequences are defined by exploiting a natural ultrametric or, equivalently, the tiering structure. The latter leads very naturally to a notion of projective completion for tiered objects, which is studied in Section 5. In the last Section we investigate sufficient conditions on endofunctors of $\mathbf{Set}$ so that their final coalgebras are projective completions of initial algebras, applying general results developed in Section 6 and carrying over into the tiered framework results of Adámek [1]. Finally, we set up a proof principle for tiered objects that abstracts the form of reasoning that lies at the basis of some approaches to proving properties of infinite objects, in particular the coinductive proof system for subtyping recursive types described in [7], or the guarded induction principle of [9].

2. Tiered objects

In order to define tiered objects we first notice the existence of a familiar algebraic structure on the extended natural numbers (the *tiers*) $0, 1, 2, \ldots, \infty$.

Definition 2.1. A *complete Heyting algebra* (cHa) is a complete lattice H satisfying the distributive law:

$$a \wedge \bigvee X = \bigvee_{x \in X} (a \wedge x)$$

for any $X \subseteq H$.

Proposition 2.1. $\Omega =_{\text{def}} \{0 < 1 < \cdots < \infty\}$ is a complete Heyting algebra.

The cHa Ω is not a Boolean algebra. Its logical properties are studied in [20]. In the sequel, by Ω we shall always mean the cHa of tiers. Tiered objects, to be introduced presently, are in fact presheaves over

Ω presented, as in [14], by exploiting Heyting-valued restriction and extent operations (see also [17, 30]). We shall not commit to any notion of morphism between tiered objects: this explains the choice of a new name for these structures.

Definition 2.2. (Tiered objects)
A structure

$$\mathbf{A} = \langle A, \cdot \restriction \cdot : A \times \Omega \to A, \mathsf{E} : A \to \Omega \rangle$$

is a *tiered object* if the following equations hold for all $p, q \in \Omega, a \in A$:

$$a \restriction \mathsf{E}\, a = a \tag{1}$$

$$\begin{aligned}(a \restriction p) \restriction q &= a \restriction (p \wedge q) \\ &= (a \restriction q) \restriction p \end{aligned} \tag{2}$$

$$\mathsf{E}(a \restriction p) = \mathsf{E}\, a \wedge p. \tag{3}$$

The operation $\restriction$ and E are called *restriction* and *extent* respectively.

Proposition 2.2. For any $a \in A$, $\mathsf{E}\, a$ is the least $p \in \Omega$ such that $a \restriction p = a$.

Proof:
By equation (1), $\mathsf{E}\, a = a \restriction \mathsf{E}\, a$. If $a = a \restriction p$, then $\mathsf{E}\, a = \mathsf{E}(a \restriction p) = \mathsf{E}\, a \wedge p$ by equation (3), therefore $\mathsf{E}\, a \leq p$. □

We say that two tiered object $\mathbf{A}$ and $\mathbf{B}$ are *isomorphic* if there exists a bijection $f : A \to B$ that respects restriction and extent: for any $a \in A$, $p \in \Omega$,

- $f(a \restriction_{\mathbf{A}} p) = f(a) \restriction_{\mathbf{B}} p$;
- $\mathsf{E}_{\mathbf{B}}\, f(a) = \mathsf{E}_{\mathbf{A}}\, a$.

Remark 2.1. The axioms for tiered objects are almost the same as those of rank-ordered sets [8] (and also those of projection spaces [19]). Structures of both kinds satisfy equations (1) and (2). What is missing from rank-ordered sets is a notion of extent satisfying equation (3). As an example of a rank-ordered set that is not a tiered object, consider "partial streams," namely functions $\sigma : \omega \to A \cup \{*\}$, where $\sigma(i) = *$ means intuitively that position i in stream σ is empty. For a partial stream σ, define

$$\mathsf{E}\, \sigma = \begin{cases} \text{the largest } i \text{ such that } \sigma(i) \neq * & \text{if such an } i \text{ exists} \\ \infty & \text{otherwise} \end{cases}$$

Restriction is defined by the clause:

$$(\sigma(i) \restriction p)\,(i) = \begin{cases} \sigma(i) & \text{if } i < p \\ * & \text{otherwise} \end{cases}$$

It is easy to check that (1) and (2) hold for partial streams, but (3) fails. It is precisely the interaction of extent and restriction in tiered objects that affords a straightforward way of defining a partial order relation on their elements.

2.1. Tiered objects as partial orders

Definition 2.3. For $\mathbf{A}$ a tiered object and $a, b \in A$, define

$$a \sqsubseteq b \Leftrightarrow a = b \upharpoonright \mathsf{E}\, a.$$

Define $a, b \in A$ to be *compatible*, written $a \between b$, if $a \upharpoonright \mathsf{E}\, b = b \upharpoonright \mathsf{E}\, a$. $B \subseteq A$ is *compatible* if its elements are pairwise compatible.

We collect below the basic order properties of tiered objects (see also [14]): the straightforward proofs, demonstrating in particular how natural is the axiomatization based on extent and restriction, are omitted.

Proposition 2.3. Assume that $\mathbf{A}$ is a tiered object. Then, for all $a, b, c \in A$:

1. $a \sqsubseteq b \implies \mathsf{E}\, a \leq \mathsf{E}\, b$.
2. $a \sqsubseteq b, b \sqsubseteq c \implies a \sqsubseteq c$.
3. $a \sqsubseteq b, b \sqsubseteq a \implies a = b$.
4. For any $p, q \in \Omega$, $p \leq q \implies a \upharpoonright p \sqsubseteq a \upharpoonright q$.
5. For any $p \in \Omega$, $a \sqsubseteq b \implies a \upharpoonright p \sqsubseteq b \upharpoonright p$.
6. If $X \subseteq A$ is directed, then it is a countable chain. In particular, every compatible subset of a tiered object is a countable chain.
7. $a \upharpoonright p = a \upharpoonright p + 1 \implies \mathsf{E}\, a \leq p$.
8. $a = b \upharpoonright p \implies a \sqsubseteq b$.
9. $a \sqsubseteq b, \mathsf{E}\, a = \mathsf{E}\, b \implies a = b$.
10. If B is a infinite chain, then $\bigvee_{b \in B} \mathsf{E}\, b = \infty$.
11. $a = b \upharpoonright p, \mathsf{E}\, a \neq p \implies a = b$.
12. $p \geq \mathsf{E}\, a \implies a = a \upharpoonright p$.

As in the Heyting-valued approach to (pre)sheaves, completeness and separation properties play an important role. For defining them we just need to use countable chains, since by Proposition 2.3(6), in a tiered object compatible or directed subsets turn out to be chains.

Definition 2.4. [Separation and completeness] Let $\mathbf{A} = \langle A, \cdot \upharpoonright \cdot : A \times \Omega \to A, \mathsf{E} : A \to \Omega \rangle$ be a tiered object, and let $B \subseteq A$ be a non-empty chain.

1. B is *bounded* if it has some $\sqsubseteq$-upper bound.
2. A *join* for a bounded B is a $\sqsubseteq$-minimal upper bound.
3. $\mathbf{A}$ is *separated* if every infinite bounded B has a unique join.

4. $\mathbf{A}$ is *complete* if every bounded B has a unique join.

5. $\mathbf{A}$ is *pointed* if there exists an element $\bot \in A$ such that $\bot = a \restriction 0$, for all $a \in A$.

Definition 2.4(3) above does not need to mention finite chains since their behaviour is trivial: in any partial order any finite chain has a join which coincides with its greatest element.
Observe that any complete tiered object $\mathbf{A}$ is pointed (because $\emptyset \subseteq A$ is compatible) and separated (by definition). The following Proposition characterizes joins. Points (2) and (3) reproduce Lemmas 4.5(iii) and (iv) from [14]. They entail, in particular, that separated tiered objects have least upper bounds of compatible subsets.

Proposition 2.4. Let $\mathbf{A} = \langle A, \cdot \restriction \cdot : A \times \Omega \to A, \mathsf{E} : A \to \Omega \rangle$ be a tiered object, and $a \in A$:

1. Let $B \subseteq A$ be a bounded infinite chain. Then any upper bound of B is a join.

2. a is a join for a bounded chain $B \subseteq A$ if and only if $\mathsf{E}\, a = \bigvee_{b \in B} \mathsf{E}\, b$ and $b \sqsubseteq a$ for all $b \in B$.

3. $a \restriction \bigvee_{i \in I} p_i$ is a join for $\{a \restriction p_i \mid i \in I\}$ for all $a \in A$ and $\{p_i \mid i \in I\} \subseteq \Omega$.

Proof:
(1) Let B be bounded and a, c be upper bounds for B. Then for any $b \in B$, $b \sqsubseteq a, c$. By Proposition 2.3(10) and 2.3(1) $\mathsf{E}\, a = \mathsf{E}\, c = \infty$. Suppose now $a \sqsubseteq c$. Then by 2.3(9) we derive $a = c$, hence c is minimal, that is c is a join. Since c is arbitrary, we have the thesis.
(2) ($\Rightarrow$) Let B be bounded and a be a join for B. If B is finite, then $a = \max B$, hence a satisfies the thesis. Otherwise $\bigvee_{b \in B} \mathsf{E}\, b = \infty$ by Proposition 2.3(10), hence by Proposition 2.3(1) we have $\mathsf{E}\, a = \infty$, since $b \sqsubseteq a$ for any $b \in B$, and the thesis follows.
($\Leftarrow$) If B is finite, then $\mathsf{E}\, a = \bigvee_{b \in B} \mathsf{E}\, b$ implies that there is $b' \in B$ such that $\mathsf{E}\, a = \mathsf{E}\, b'$. Since $b' \sqsubseteq a$, $a = b'$ by Proposition 2.3(9), hence $a \in B$ is the least upper bound of B, being the greatest element in B. If B is infinite, then the thesis follows immediately by the previous point (1), since by hypothesis a is an upper bound of B.
(3) Let $p' = \bigvee_{i \in I} p_i$. We have, for any $i \in I$, $a \restriction p_i \sqsubseteq a \restriction p'$ by Proposition 2.3(4). If $p' \in \omega$, the thesis is immediate, since $p' = p_j$ for some $j \in I$, and $a \restriction p_j$, as the greatest element of the chain $\{a \restriction p_i \mid i \in I\}$, is the least upper bound. If $p' = \infty$, then observe $a \restriction p' = a$. Let $c \sqsubseteq a$ be another upper bound for $\{a \restriction p_i \mid i \in I\}$. Then we have, for any $i \in I$, $\mathsf{E}(a \restriction p_i) \leq \mathsf{E}\, c$, that is $\mathsf{E}\, a \wedge p_i \leq \mathsf{E}\, c$. Since the set $\{p_i \mid i \in I\}$ is unbounded, this implies $\mathsf{E}\, c = \infty$. Then $a = c$ is a consequence of 2.3(9). Therefore a is minimal and the proof is complete. $\square$

Points (2), (3), and (4) of the next lemma show that extent and restriction are Scott continuous operations. The last point shows that approximation is cumulative.

Lemma 2.1. Let $\mathbf{A}$ be a separated tiered object, and let $B \subseteq A$ be a bounded chain. Then

1. B has a least upper bound, denoted $\bigsqcup B$.

2. $\mathsf{E}(\bigsqcup B) = \bigvee_{b \in B} \mathsf{E}\, b$.

3. For any $p \in \Omega$, $(\bigsqcup B) \restriction p = \bigsqcup_{b \in B}(b \restriction p)$.

4. For any $a \in A$, $\{p_i \mid i \in I\} \subseteq \Omega$, $a \upharpoonright (\bigvee_{i \in I} p_i) = \bigsqcup_{i \in I}(a \upharpoonright p_i)$.

5. For any $a \in A$, $a = \bigsqcup_{n \in \omega}(a \upharpoonright n)$.

Proof:
(1) If B is finite the thesis is trivial, since the greater element of B is its least upper bound. If B is infinite, we have that any upper bound of B is a join for B, by Proposition 2.4(1). On the other hand by definition of separated tiered object, B has a unique join.
(2) $b \sqsubseteq \bigsqcup B$ implies $\mathsf{E}\, b \leq \mathsf{E}(\bigsqcup B)$ by Proposition 2.3(1), hence $\bigvee_{b \in B} \mathsf{E}\, b \leq \mathsf{E}(\bigsqcup B)$. Conversely, if B is finite, we have $\mathsf{E}(\bigsqcup B) \leq \bigvee_{b \in B} \mathsf{E}\, b$, since $\bigsqcup B \in B$. If B is infinite, then $\bigvee_{b \in B} \mathsf{E}\, b = \infty$ by Proposition 2.3(10), hence $\mathsf{E}(\bigsqcup B) \leq \bigvee_{b \in B} \mathsf{E}\, b$ by Proposition 2.3(9).
(3) First note that the chain $\{b \upharpoonright p \mid b \in B\}$ is bounded, hence by the previous point (1) it has a unique join. We have $\bigsqcup_{b \in B}(b \upharpoonright p) \sqsubseteq (\bigsqcup B) \upharpoonright p$, since by Proposition 2.3(5), for any $b \in B$, $b \upharpoonright p \sqsubseteq (\bigsqcup B) \upharpoonright p$. Moreover

$$
\begin{aligned}
\mathsf{E}\left(\bigsqcup_{b \in B}(b \upharpoonright p)\right) &= \bigvee_{b \in B} \mathsf{E}(b \upharpoonright p) && \text{by point (2)} \\
&= \bigvee_{b \in B} ((\mathsf{E}\, b) \wedge p) \\
&= \left(\bigvee_{b \in B} \mathsf{E}\, b\right) \wedge p \\
&= \mathsf{E}\left(\bigsqcup B\right) \wedge p && \text{by point (2)} \\
&= \mathsf{E}\left(\left(\bigsqcup B\right) \upharpoonright p\right)
\end{aligned}
$$

hence we conclude using Proposition 2.3(9).
(4) This point is just Proposition 2.4(3), rephrased with the notation for least upper bounds.
(5)

$$
\begin{aligned}
\bigsqcup_{n \in \omega}(a \upharpoonright n) &= a \upharpoonright (\bigvee_{n \in \omega} n) && \text{by point (3)} \\
&= a \upharpoonright \infty \\
&= a.
\end{aligned}
$$

□

Corollary 2.1. Let $\mathbf{A}$ be a complete tiered object. Then any chain has a least upper bound. Moreover $\mathbf{A}$ is a cpo.

Proof:
The first statement is an immediate consequence of Lemma 2.1(1) and definition of complete tiered object. The second statement follows from the first, since by Proposition 2.3(6) any directed set is a chain. □

The following immediate consequence of Lemma 2.1(5) has an independent interest: it can be seen as a proof principle for complete tiered objects akin to the Approximation Lemma used in functional programming [6, 21].

Lemma 2.2. (The approximation lemma)
If $\mathbf{A}$ is separated,

$$(\forall n \in \omega. \quad a \upharpoonright n = b \upharpoonright n) \implies a = b.$$

3. Examples

3.1. Finite and infinite words

As a preview of the interpretation of tiering just introduced, we sketch how it arises in the case of the set X^∞ of finite and infinite words.

Given any non-empty set X (of "letters"), a finite or infinite word u over X is a mapping $u : \gamma \to X$ for some (von Neumann) ordinal $\gamma \leq \omega$. The empty word ε is the empty map $\emptyset \to X$. Therefore X^∞ can be identified with $\bigcup_{n\in\omega} X^n \cup X^\omega$. Every element $u \in X^\infty$ can be given extent $\mathsf{E}(u) =_{\text{def}} \text{dom}\,(u) \leq \omega$. Besides, for any γ, consider the restriction $u \upharpoonright \gamma$ defined as $u_{|\text{dom}(u)\cap\gamma}$, and observe that for any $\gamma, \delta \leq \omega$:

$$\begin{aligned}
(u \upharpoonright \gamma) \upharpoonright \delta &= u \upharpoonright \gamma \cap \delta && \text{(by properties of restriction)}\\
&= u \upharpoonright \min(\gamma, \delta) && \text{(by properties of von Neumann ordinals)}\\
&= (u \upharpoonright \delta) \upharpoonright \gamma.
\end{aligned}$$

The equations defining X^∞ as a tiered object are satisfied

$$u \upharpoonright 0 = \varepsilon, \tag{4}$$

$$u \upharpoonright \mathsf{E}(u) = u, \tag{5}$$

$$\mathsf{E}(u \upharpoonright \gamma) = \mathsf{E}(u) \cap \gamma \tag{6}$$

and follow immediately from the definition of restriction and extent. For words $u, v \in X^\infty$ we have

$$u \between v \quad \text{iff} \quad u \upharpoonright \text{dom}\,(v) = v \upharpoonright \text{dom}\,(u)\,.$$

If $W \subseteq X^\infty$ is coherent, let the join of W be the function

$$\left(\bigsqcup W\right)(\alpha) = u(\alpha)$$

for any u such that $\alpha \in \mathsf{E}(u)$, where, by definition,

$$\mathsf{E}\left(\bigsqcup W\right) = \bigcup_{w\in W} \mathsf{E}(w).$$

Coherence of W guarantees that this is indeed a function. In fact, if $\alpha \in \mathsf{E}(u) \cap \mathsf{E}(v)$:

$$\begin{aligned}
u(\alpha) &= (u \upharpoonright \mathsf{E}(v)) && (\text{as } \alpha \in \mathsf{E}(u) \cap \mathsf{E}(v))\\
&= (v \upharpoonright \mathsf{E}(u)) && (\text{as } u \between v)\\
&= v(\alpha).
\end{aligned}$$

If we define, for $u, v \in X^\infty$

$$u \sqsubseteq v \quad \text{iff} \quad u = v \restriction \mathbf{E}(u),$$

then $u \subseteq v$ as functions. Furthermore, for coherent W, $\bigsqcup W$ is just union, so $\bigsqcup W$ is precisely the least upper bound of W w.r.t. $\sqsubseteq$. Therefore X^∞ is a complete tiered object.

3.2. Recursion

While tiering is especially useful in situations involving structures of infinite depth, it also shows up in a context which is intimately related to induction over the natural numbers, namely the justification of the iterative definition of functions contained in the classical work of Dedekind [10]. Following [18], we briefly summarize this landmark result by highlighting its relations to tiering. Given a structure $\mathcal{A} = \langle A, a \in A, f : A \to A \rangle$, in order to show that there exists a unique function $\omega \to A$ such that:

$$\begin{array}{rcl} h(0) & = & a \\ h(n+1) & = & f(h(n)) \end{array}$$

we build a set of approximations of h having as domains *segments* $H \subseteq \omega$ such that

- $0 \in H$;
- if $n + 1 \in H$, then $n \in H$.

Clearly, the collection of segments of ω ordered by inclusion is isomorphic to Ω. A *partial function* is any single-valued relation $\varphi \subseteq \omega \times A$ whose domain is a segment and that satisfies

$$\begin{array}{rcl} \varphi(0) & = & a \\ \varphi(n+1) & = & f(\varphi(x)) \end{array}$$

whenever $n + 1 \in \mathrm{dom}\,(\varphi)$. The construction of h proceeds by proving that

1. for every $n \in \omega$ there exists a partial function φ such that $n \in \mathrm{dom}\,(\varphi)$;
2. if $n \in \mathrm{dom}\,(\varphi_1) \cap \mathrm{dom}\,(\varphi_2)$, then $\varphi_1(n) = \varphi_2(n)$.

Finally, let

$$h(n) =_{\mathrm{def}} \varphi(n) \quad \text{where } n \in \mathrm{dom}\,(\varphi).$$

It is easy to see that, by defining $\mathbf{E}\,\varphi =_{\mathrm{def}} \mathrm{dom}\,(\varphi)$ and taking restriction as restriction of a function to a segment, the collection of partial functions is a tiered object. Observe that compatibility of partial functions defined by

$$\varphi_1 \between \varphi_2 \Leftrightarrow (\forall n \in \mathrm{dom}\,(\varphi_1) \cap \mathrm{dom}\,(\varphi_2) \Rightarrow \varphi_1(n) = \varphi_2(n))$$

coincides with their compatibility as elements of a tiered object. The construction of h exploits precisely the completeness of the set of partial functions as a tiered object.

3.3. Final coalgebras as tiered objects

There is an intimate connection between final coalgebras of ω-continuous endofunctors of $\mathbf{Set}$ and complete tiered objects. Assume that $F : \mathbf{Set} \longrightarrow \mathbf{Set}$ is an ω-continuous functor and build the sequence:

$$T_0 \xleftarrow{j_0} T_1 \xleftarrow{j_1} T_2 \xleftarrow{j_2} \cdots \xleftarrow{j_{n-1}} T_n \xleftarrow{j_n} T_{n+1} \xleftarrow{j_{n+2}} \cdots \qquad (7)$$

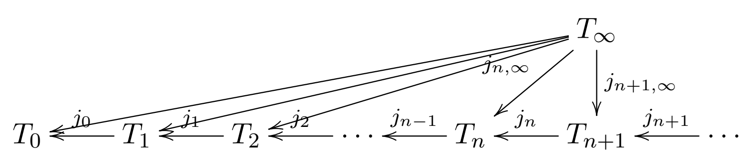
where:

$$\begin{aligned} T_0 &= \mathbf{1} \\ T_{n+1} &= F(T_n) \end{aligned}$$

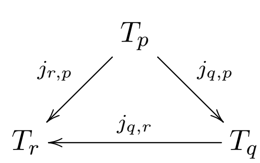
and the pairs j_n, for $n > 0$, are defined inductively by

$$j_n = F(j_{n-1}) : F(T_n) \to F(T_{n-1}).$$

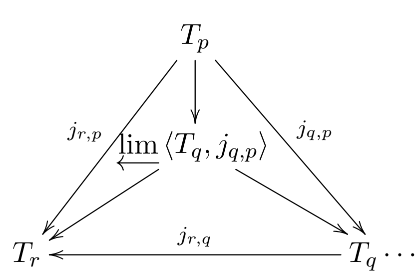
Observe that a cone for diagram (7) of the form:



amounts to a presheaf $T : \Omega^{\mathrm{op}} \longrightarrow \mathbf{Set}$ where $T(p) = T_p$ and, for $p < q \in \omega$, the mapping $T(q) \longrightarrow T(p)$ is the composition $j_{p,q} =_{\mathrm{def}} j_p \circ \cdots \circ j_{q-1}$. As in [14, 4.14(iii)], a *crible* K of an element p of Ω is a set $K \subseteq \Omega$ of elements $q \leq p$ such that $r \leq q \in K$ implies $r \in K$. A crible K of p *covers* p if $p = \bigvee K$. In general, if K is a covering crible of $p \in \Omega$ there is a cone



with vertex $T(p) = T_p$: T is a sheaf exactly when, for every p, the universal mapping towards the limit



is a bijection. In the case of Ω, this condition is non-trivial only when $K = \omega$ and $p = \infty$, therefore we can conclude

Proposition 3.1. For an ω-continuous $F : \mathbf{Set} \longrightarrow \mathbf{Set}$, the presheaf $T : \Omega^{\mathrm{op}} \longrightarrow \mathbf{Set}$ associated with the limiting cone

$$T_\infty =_{\mathrm{def}} \varprojlim \langle T_i, j_i \rangle \quad (j_{n,\infty},\ j_{n+1,\infty})$$

$$T_0 \xleftarrow{j_0} T_1 \xleftarrow{j_1} T_2 \xleftarrow{j_2} \cdots \xleftarrow{j_{n-1}} T_n \xleftarrow{j_n} T_{n+1} \xleftarrow{j_{n+2}} \cdots$$

is a sheaf.

The construction of a complete tiered object $\mathbf{T} = \langle A, \cdot \upharpoonright \cdot : A \times \Omega \to A, \mathsf{E} : A \to \Omega \rangle$ out of the sheaf T can then be carried out as in [14, loc. cit.]:

- $A =_{\mathrm{def}} \coprod_{p \in \Omega} T_p$;
- for $a = \langle p, t \rangle \in A$, $\mathsf{E}(a) = p$;
- for $a = \langle p, t \rangle \in A$ and $q \in \Omega$: $a \upharpoonright q = \langle p \wedge q, j_{p \wedge q, p}(t) \rangle$.

Thus, the complete tiered object $\mathbf{T}$ summarizes the construction of the final F-coalgebra, which is carved out of $\mathbf{T}$ as the subset of maximal elements (i.e., those of extent ∞). Observe that the complete tiered object $\mathbf{A}$ just defined is, in particular, a complete partial order (cpo, [31]); it is also easily seen that such a cpo is algebraic, the compact elements being those of finite extent. This suggests that our association of a complete tiered object to the final coalgebra of an ω-continuous endofunctor of $\mathbf{Set}$ can be exploited in defining *computational models* for final coalgebras, in analogy to the computational models of ultrametric spaces described in [13]. However, we shall not pursue this matter any further in the present paper.

4. Cauchy completeness

Let $\mathbf{A} = \langle A, \upharpoonright, \mathsf{E} \rangle$ be a tiered object. We introduce a notion of (discrete) pseudometric

$$d : A \times A \to \{0\} \cup \{2^{-n} \mid n \in \omega\}$$

by setting $d(x, y) = 2^{-\delta(x,y)}$, where $\delta : A \times A \to \Omega$ is defined by

$$\delta(x, y) = \min\{p \mid x \upharpoonright p \neq y \upharpoonright p\}$$

($\delta(x, y) = \infty$ if the set in the right-hand side is empty). In the separated case d is an ultrametric, but in the general case the first axiom of metric fails: consider for instance $\{n \mid n \in \omega\} \cup \{a, b\}$, with $\mathsf{E}\, n = n$, $\mathsf{E}\, a = \mathsf{E}\, b = \infty$, and for any $m \in \Omega$, $n \upharpoonright m = \min\{n, m\}$, $a \upharpoonright m = b \upharpoonright m = m$. Then $d(a, b) = 0$, since $\delta(a, b) = \infty$, nevertheless $a \neq b$. A Cauchy sequence in the pseudo-ultrametric space $\langle A, d \rangle$ is a sequence $\langle a^{(n)} \rangle_{n \in \omega}$ such that

$$\forall \varepsilon > 0. \exists j(\varepsilon) \in \omega. \forall k \in \omega. d(a^{(j(\varepsilon))}, a^{(j(\varepsilon)+k)}) < \varepsilon$$

which is equivalent to

$$\forall i \in \omega. \exists j(i) \in \omega. \forall k \in \omega. d(a^{(j(i))}, a^{(j(i)+k)}) < 2^{-i}$$

and also (see for example [12]) to

$$\forall i \in \omega . \exists j(i) \in \omega . \forall k \in \omega . \left(a^{(j(i))} \upharpoonright i = a^{(j(i)+k)} \upharpoonright i \right) . \tag{8}$$

Convergence of a Cauchy sequence $\langle a^{(i)} \rangle_{i \in \omega}$ to a limit a is defined by the condition:

$$\forall i \in \omega . \exists j(i) \in \omega . \forall k \in \omega . \left(a \upharpoonright i = a^{(j(i)+k)} \upharpoonright i \right) . \tag{9}$$

Definition 4.1. A tiered object $\mathbf{A}$ is *Cauchy complete* if for any Cauchy sequence there exists a unique limit.

We introduce an equivalence relation on Cauchy sequences: let $s = \langle a^{(i)} \rangle_{i \in \omega}$, $s' = \langle b^{(i)} \rangle_{i \in \omega}$

$$s \ \simeq \ s' \ \Leftrightarrow \ \forall i \in \omega . \exists j(i) \in \omega . \forall k, p \in \omega . \left(a^{(j(i)+k)} \upharpoonright i = b^{(j(i)+p)} \upharpoonright i \right) .$$

We have the following easy lemma.

Lemma 4.1. Given two sequence s, s' in a tiered object $\mathbf{A}$, if $s \ \simeq \ s'$, then s has limit a if and only a is a limit for s'.

Actually $s \ \simeq \ s'$ if and only if the distance $d(a^{(i)}, b^{(i)}) \to 0$ when $i \to \infty$. The following simple notion of projective sequence (used, among others, by [8, 12]) will turn out to be equivalent to that of Cauchy sequence, simplifying later proofs:

Definition 4.2. (Projective completeness)
A sequence $s = \langle a^{(i)} \rangle_{i \in \omega}$ of elements from a tiered object $\mathbf{A}$ is a *projective sequence* if

$$a^{(i)} = a^{(i+1)} \upharpoonright i \quad \text{for all } i \in \omega. \tag{10}$$

We say that s is E-*unbounded* if $\lim_{i \to \infty} \mathsf{E}\, a^{(i)} = \infty$.

A *projective limit* for s is any $a \in A$ such that, for all $n \in \omega$:

$$a^{(n)} = a \upharpoonright n. \tag{11}$$

$\mathbf{A}$ is *projectively complete* if, for every projective sequence $\langle a^{(i)} \rangle_{i \in \omega}$, there is a *unique* projective limit, denoted by $a^{(\infty)}$.

Proposition 4.1. Let $s = \langle a^{(i)} \rangle_{i \in \omega}$ be a projective sequence. Then

1. $\forall i, k \in \omega . a^{(i)} = a^{(i+k)} \upharpoonright i$. In particular, s is a Cauchy sequence.
2. s is a $\sqsubseteq$-chain $a^{(0)} \sqsubseteq a^{(1)} \sqsubseteq \cdots \sqsubseteq a^{(n)} \sqsubseteq a^{(n+1)} \ldots$.
3. $\forall i \in \omega .\ \mathsf{E}\, a^{(i)} \leq i$.
4. $\mathsf{E}\, a^{(i)} < i \implies \forall j \geq i . a^{(j)} = a^{(i)}$.
5. If s is E-unbounded, then $\forall i \in \omega .\ \mathsf{E}\, a^{(i)} = i$.

6. a is a limit for s if and only if a is a projective limit for s.

Proof:
(1) easy by induction.
(2) by Proposition 2.3(8).
(3) $a^{(i)} = a^{(i+1)} \upharpoonright i$ implies

$$\begin{aligned} \mathsf{E}\, a^{(i)} &= \mathsf{E}(a^{(i+1)} \upharpoonright i) \\ &= \mathsf{E}\, a^{(i+1)} \wedge i \\ &\leq i. \end{aligned}$$

(4) By Proposition 2.3(11), we have immediately $a^{(i+1)} = a^{(i)}$. The thesis follows by induction.
(5) By (3), $\mathsf{E}\, a^{(i)} \leq i$, for all $i \in \omega$. On the other hand, if s is E-unbounded, the case $\mathsf{E}\, a^{(i)} < i$ never occurs, otherwise (4) would lead to a contradiction. Therefore the thesis follows.
(6) ($\Leftarrow$) Let a be a projective limit of s, hence $a \upharpoonright i = a^{(i)}$. Define $j(i) = i$ in Equation (9). We have, by point (1), $a^{(i+k)} \upharpoonright i = a^{(i)}$, for any k, hence we have $a \upharpoonright i = a^{(i+k)} \upharpoonright i$: this proves that a is a limit for s.
($\Rightarrow$) Suppose a is a limit for s. Then for any i there exists $j(i)$ such that for any k, $a^{(j(i)+k)} \upharpoonright i = a \upharpoonright i$. On the other hand $a^{(j(i)+k)} \upharpoonright i = a^{(i)}$, by (1), since $i \leq j(i) + k$, hence we conclude $a \upharpoonright i = a^{(i)}$. □

We will prove that the three notions of completeness are equivalent: a tiered object $\mathbf{A}$ is complete if and only if it is Cauchy complete, if and only if it is projectively complete.
We start with a lemma.

Lemma 4.2. In a tiered object $\mathbf{A}$, for any Cauchy sequence $s = \langle a^{(i)} \rangle_{i \in \omega}$ there is a unique projective sequence $\pi(s) = \langle c^{(i)} \rangle_{i \in \omega}$ such that $s \simeq \pi(s)$.

Proof:
Given a Cauchy sequence $s = \langle a^{(i)} \rangle_{i \in \omega}$ we define $\pi(s) = \langle c^{(i)} \rangle_{i \in \omega}$ by:

$$c^{(i)} =_{\text{def}} a^{(\nu(i))} \upharpoonright i \text{ for the least } \nu(i) \text{ such that } i \leq \nu(i) \text{ and } \forall k \in \omega \left(a^{(\nu(i))} \upharpoonright i = a^{(\nu(i)+k)} \upharpoonright i \right) \quad (12)$$

Observe that

$$(\dagger) \;\; n \leq m \implies \nu(n) \leq \nu(m).$$

In fact:
(a) $a^{(\nu(m))} \upharpoonright n = a^{(\nu(m)+k)} \upharpoonright n$ for all $k \in \omega$, (use restriction to n and m as from Definition 2.2(2));
(b) $n \leq \nu(m)$ (since by definition $m \leq \nu(m)$);
then conclude using the minimality of $\nu(n)$ with respect to (a) and (b).
$\pi(s)$ is projective, since

$$\begin{aligned} c^{(i+1)} \upharpoonright i &= a^{(\nu(i+1))} \upharpoonright i \\ &= a^{(\nu(i))} \upharpoonright i && \text{by definition of } \nu(i) \\ &= c^{(i)}. \end{aligned}$$

We have $s \simeq \pi(s)$. In fact, given $i \in \omega$, for any $k \in \omega$ we have $a^{(\nu(i)+k)} \restriction i = c^{(i)}$, by definition of $\nu(i)$. On the other hand, for any $p \in \omega$,

$$\begin{aligned} c^{(\nu(i)+p)} \restriction i &= (a^{(\nu(\nu(i)+p))} \restriction (\nu(i)+p)) \restriction i \\ &= a^{(\nu(\nu(i)+p))} \restriction i && \text{since } i \leq \nu(i)+p \\ &= a^{(\nu(i))} \restriction i && \text{since } i \leq \nu(i)+p, \text{ hence } \nu(i) \leq \nu(\nu(i)+p) \text{ by } (\dagger) \\ &= c^{(i)}. \end{aligned}$$

So we have $s \simeq \pi(s)$, since for any k, p, we have proved $a^{(\nu(i)+k)} \restriction i = c^{(\nu(i)+p)} \restriction i$.
Let now $s' = \langle d^{(i)} \rangle_{i \in \omega}$ be another projective sequence such that $s \simeq s'$. Then it follows $\pi(s) \simeq s'$, being $\simeq$ an equivalence relation. So, for any $i \in \omega$, let $j(i)$ such that for any $p, k \in \omega$, $c^{(j(i)+k)} \restriction i = d^{(j(i)+p)} \restriction i$. By Proposition 4.1(1) we obtain $c^{(i)} = d^{(i)}$, since $j(i) + k \geq i$, hence we have the thesis. □

It is known from the works of Walters [34, 35] that, in general, the sheaf condition on $\mathbf{A}$, namely the existence of least upper bounds of compatible subsets (that is chains) of A, amounts to Cauchy completeness (see also [33, 32]). We can now prove this in an elementary way, as a consequence of the fact that the three notion of completeness for a tiered object coincide.

Proposition 4.2. For a pointed tiered object $\mathbf{A}$ the following are equivalent:

1. $\mathbf{A}$ is projectively complete;
2. $\mathbf{A}$ is Cauchy complete;
3. $\mathbf{A}$ is complete.

Proof:
We will prove (1) ⇒ (2) ⇒ (3) ⇒ (1).
(1) ⇒ (2) Let $\mathbf{A}$ be projectively complete, and let s be a Cauchy sequence. Consider the projective sequence $\pi(s)$ defined in Lemma 4.2. Let a be its projective limit. By Proposition 4.1(6) a is also the limit of $\pi(s)$. Since $s \simeq \pi(s)$, we have that a is also the limit of s, proving that $\mathbf{A}$ is Cauchy complete.
(2) ⇒ (3) Let $\mathbf{A}$ be Cauchy complete. Let B a chain in $\mathbf{A}$. We have to prove that B has a least upper bound. If B is empty or finite the thesis is immediate: $\perp$ or the greatest element of B respectively is the least upper bound. Otherwise, define a sequence $s = \langle a^{(n)} \rangle_{n \in \omega}$. Let $b_0 \in B$ arbitrary. Then

$$\begin{aligned} a^{(0)} &= b_0 \\ a^{(n+1)} &= \text{any } b \in B \text{ such that } \mathsf{E}\, b > \mathsf{E}\, a^{(n)}. \end{aligned}$$

Note that $a^{(n+1)}$ is defined, by Proposition 2.3(10), and that for any $b \in B$ there exists $n(b) \in \omega$ such that $b \sqsubseteq a^{(n(b))}$ (take $n(b)$ large enough so that $\mathsf{E}\, b \leq \mathsf{E}\, a^{(n(b))}$ and use the fact that B is a chain). Moreover $a^{(n)} \sqsubseteq a^{(n+1)}$, since B is a chain and $\mathsf{E}\, a^{(n)} < \mathsf{E}\, a^{(n+1)}$. By induction we have $a^{(n)} \sqsubseteq a^{(n+k)}$ for any $n, k \in \omega$. The sequence s is Cauchy. In fact, by the last statement above and Proposition 2.3(5),

$a^{(i)} \restriction i \sqsubseteq a^{(i+k)} \restriction i$, for any $i \in \omega$. Now observe that by definition, for any $i \in \omega$, we have $\mathsf{E}\, a^{(i)} \geq i$. Therefore it follows

$$\begin{aligned} \mathsf{E}(a^{(i)} \restriction i) &= \mathsf{E}(a^{(i)}) \wedge i \\ &= i && \text{by the remark above} \\ &= \mathsf{E}(a^{(i+k)}) \wedge i && \text{as above} \\ &= \mathsf{E}(a^{(i+k)} \restriction i). \end{aligned}$$

This implies s is Cauchy (by choosing $j(i) = i$ in Equation (9)). Let $a^{(\infty)}$ be the limit of s. We prove that $a^{(\infty)}$ is the least upper bound of B. Given $b \in B$, consider the element $a^{(n(b))}$ defined above. Let $m = \mathsf{E}\, a^{(n(b))}$. Then by definition of limit, there exists $j(m) \in \omega$ such that for all $k \in \omega$, $a^{(\infty)} \restriction m = a^{(j(m)+k)} \restriction m$. On the other hand $a^{(j(m)+k)} \restriction m = a^{(n(b))}$, since $a^{(n(b))} \sqsubseteq a^{(j(m)+k)}$, hence we have $a^{(\infty)} \restriction m = a^{(n(b))}$, which proves $a^{(n(b))} \sqsubseteq a^{(\infty)}$. Now we conclude $b \sqsubseteq a^{(\infty)}$ by transitivity. Therefore we have proved that $a^{(\infty)}$ is an upper bound for B. The thesis follows from Proposition 2.4(1).
(3) $\Rightarrow$ (1) Let $\mathbf{A}$ be complete. Let $s = \langle a^{(i)} \rangle_{i \in \omega}$ a projective sequence. Then s is an ascending chain, by Proposition 4.1(2). Let $a = \bigsqcup s$. Then a is also the projective limit of s, since for any $n \in \omega$

$$\begin{aligned} a \restriction n &= (\bigsqcup s) \restriction n \\ &= \bigsqcup_{i \in \omega} (a^{(i)} \restriction n) && \text{by Proposition 2.1(3)} \\ &= a^{(n)} && \text{by Proposition 4.1(1).} \end{aligned}$$

□

5. Projective completion of tiered objects

Two standard completions are possible for tiered object: ideal completion or metric completion, according to whether we choose the partial order or the metric as the tool for the completion process. For a comparison between the two approaches see [27]. In general, the two completions do not coincide. Anyway tiered objects afford a simpler kind of completion, based on projective sequences, which we call *projective completion*. The projective completion of a tiered object is a new tiered object isometric to the standard metric completion: we just state this result without proving it, since it will not be used in the remainder of the paper. Instead we will prove that the projective completion and the ideal completion of a *finitary* tiered object are isomorphic: this result is in order, since it connects tiered objects and their projective completions with the results of Adámek [1]. In Section 7 we will see an application of this fact. Projective completion is based on the property of a tiered object shown in Lemma 4.2: for any Cauchy sequence there exists a unique projective sequence that "behaves in the same way": this sequence can be chosen as the canonical representative of all the $\simeq$-related Cauchy sequences. As a consequence, the completion process is carried out by simply taking the points of finite extent together with the projective sequences, without any need of quotienting the sequences (see [22], Chapter 6). As a further consequence, the embedding of a finitary tiered object in its projective completion is set-theoretic inclusion.

We start with the definition of finitary tiered object.

Definition 5.1. 1. Given a tiered object $\mathbf{A}$, let $A^* =_{\text{def}} \{a \in A \mid \mathsf{E}\, a \in \omega\}$. $\mathbf{A}^*$ is defined as the tiered object whose underlying set is A^*, with restriction and extent induced by $\mathbf{A}$.

2. A tiered object $\mathbf{A}$ is *finitary* if $\mathbf{A} = \mathbf{A}^*$.

Remark 5.1. Observe that any projective sequence is formed of elements in A^*.

We now define the projective completion. The standard Cauchy completion (see for instance [22], Chapter 6), is based on taking $\simeq$-equivalence classes of Cauchy sequences. Tiered objects allow to chose canonical representatives of such classes, namely the projective sequences $\pi(s)$ of Lemma 4.2. This simplifies the construction of the completion: elements of finite extent are put directly into the projective completion; then we add all the projective sequences. In particular, equivalence classes of elements $a \in A$ of infinite extents are represented by the projective sequences $\langle a \upharpoonright n\rangle_{n\in\omega}$.

Definition 5.2. Let $\mathbf{A} = \langle A, \upharpoonright, \mathsf{E}\rangle$ be a tiered object. Let $\mathcal{C}^\infty(\mathbf{A})$ be the set of E-unbounded projective sequences. Define the *projective completion* of $\mathbf{A}$, $\mathbf{A}^\infty = \langle A^\infty, \upharpoonright^\infty, \mathsf{E}^\infty\rangle$ by:
- $A^\infty = A^* \cup \mathcal{C}^\infty(\mathbf{A})$
- $x \upharpoonright^\infty p = \begin{cases} x \upharpoonright p & \text{if } x \in A^* \\ a^{(p)} & \text{if } x = \langle a^{(n)}\rangle_{n\in\omega} \in \mathcal{C}^\infty(\mathbf{A}) \text{ and } p \in \omega \\ x & \text{if } x \in \mathcal{C}^\infty(\mathbf{A}) \text{ and } p = \infty \end{cases}$
- $\mathsf{E}^\infty\, x = \begin{cases} \mathsf{E}\, x & \text{if } x \in A^* \\ \infty & \text{if } x \in \mathcal{C}^\infty(\mathbf{A}) \end{cases}$

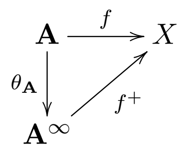
The following lemma is easily proven.

Lemma 5.1. If $\mathbf{A}$ is pointed, then $\mathbf{A}^\infty$ is a complete tiered object.

Proof:
We just prove that $\mathbf{A}^\infty$ is complete. By Proposition 4.2 it is enough to prove that $\mathbf{A}^\infty$ is projectively complete. In this proof we use the notation $[x^{(i)}]_{i\in\omega}$ for projective sequences in $\mathbf{A}^\infty$, and the standard notation $\langle a^{(i)}\rangle_{i\in\omega}$ for projective sequences in $\mathbf{A}$. So, let $s = [x^{(i)}]_{i\in\omega}$ be a projective in $\mathbf{A}^\infty$. By Remark 5.1, the $x^{(i)}$'s have finite extent in $\mathbf{A}^\infty$, hence they are in A^*. Therefore $s' = \langle x^{(i)}\rangle_{i\in\omega}$ is projective in $\mathbf{A}$, hence it is an element of $\mathbf{A}^\infty$. Finally we have $\langle x^{(i)}\rangle_{i\in\omega} \upharpoonright^\infty p = x^{(p)}$, which proves that s' is the projective limit for s. □

The next lemma shows that for finitary tiered objects, projective completions are isomorphic to ideal completions, since they satisfy the universal property of ideal completions. This result is simple but important, since it allows to connect the results of the coming sections to Adámek [1]. Moreover, note that the embedding of a finitary $\mathbf{A}$ in its projective completion is given simply by the set-theoretic inclusion.

Lemma 5.2. Let $\mathbf{A}$ be a finitary pointed tiered object. Let $\theta_{\mathbf{A}} : \mathbf{A} \to \mathbf{A}^\infty$ be the inclusion. Then $\theta_{\mathbf{A}}$ is monotone and for any cpo X and monotone function $f : \mathbf{A} \to X$, there exists a unique continuous extension $f^+ : \mathbf{A}^\infty \to X$ such that the following diagram commutes:



Proof:
First of all note that it makes sense to speak of continuous functions from $\mathbf{A}^\infty$, since by Lemma 5.1 and Corollary 2.1 $\mathbf{A}^\infty$ is a cpo. The commutativity of the diagram forces the definition of f^+ on $\mathbf{A}$: $f^+(\theta_{\mathbf{A}}(a)) = f(a)$
Now observe that for any sequence $s = \langle a^{(n)} \rangle_{n\in\omega} \in \mathcal{C}^\infty(\mathbf{A})$, $s = \bigsqcup_{n\in\omega} a^{(n)}$ (the sup is of course taken in $\mathbf{A}^\infty$). Then we define f^+ by continuous extension

$$f^+(s) = \bigsqcup_{n\in\omega} f(a^{(n)}).$$

$\square$

6. Limits, colimits and tiered objects

In this section we investigate a particular relation between limits and colimits which arises when we consider sequences of the shape $\xi = \langle X_n, j_n : X_{n+1} \to X_n \rangle_{n\in\omega}$ and $\eta = \langle X_n, i_n : X_n \to X_{n+1} \rangle_{n\in\omega}$. In general no meaningful relation between $\varprojlim \xi$ and $\varinjlim \eta$ holds. But if $\langle \eta, \xi \rangle$ is *tierable pair* (see below Definition 6.1), then $\varinjlim \eta$ is a (pointed) tiered object whose projective completion is $\varprojlim \xi$. Conversely, any pointed tiered object $\mathbf{A}$ induces a tierable pair which allows to recover $\mathbf{A}$ (see Theorem 6.1). An interesting application of the results in the present section occurs when considering sequences generated by certain inclusion preserving functors (such as polynomial functors or the finite powerset): in such a case final coalgebras arise as projective completions of initial algebras. We will study this case in Section 7.
Given $f, g : A \to B$, let $\mathcal{E}(f, g) = \{a \in A \mid f(a) = g(a)\}$, that is $\mathcal{E}(f, g)$ is the (domain of the) equalizer of f and g.

Definition 6.1. Consider two sequences $\eta = \langle X_n, i_n : X_n \to X_{n+1} \rangle$ and $\xi = \langle X_n, j_n : X_{n+1} \to X_n \rangle_{n\in\omega}$. We say that the pair $\langle \eta, \xi \rangle$ is *tierable* if X_0 is the one-point set and the following properties are satisfied for all $n \in \omega$:

1. $j_n \circ i_n = \mathrm{Id}_{X_n}$.
2. $\mathcal{E}(j_{n+1}, i_n \circ j_n \circ j_{n+1}) \subseteq (i_{n+1} \circ i_n)(X_n)$.

In the following, given $n \leq p \in \omega$, $i_{n,p} = i_{p-1} \circ \ldots \circ i_n$, $j_{n,p} = j_n \circ \ldots \circ j_{p-1}$.

Lemma 6.1. Let $\langle \eta, \xi \rangle$ be tierable. Let $n, q, p, r \in \omega$, with $q < p \leq r$. Then

1. i_n is mono, j_n is epi.
2. $j_{p,r} \circ i_{p,r} = \mathrm{Id}_{X_p}$.
3. $\forall x \in X_{n+2}. j_{n+1}(x) \in i_n(X_n) \implies i_{n+1}(j_{n+1}(x)) = x$.
4. $\forall x \in X_r$, if $i_{q,p}(j_{q,r}(x)) = j_{p,r}(x)$, then $i_{q,r}(j_{q,r}(x)) = x$.
5. Let $\sigma = \langle x^{(k)} \rangle_{k\in\omega} \in \varprojlim \xi$. If $i_p(x^{(p)}) = x^{(p+1)}$, then $i_{p,r}(x^{(p)}) = x^{(r)}$.

Proof:
(1) and (2) are immediate.
(3) Let $j_{n+1}(x) = i_n(y)$, for some $y \in X_n$. Then $i_n(j_n(j_{n+1}(x))) = i_n(j_n(i_n(y))) = i_n(y)$, because of Definition 6.1(1), so $x \in \mathcal{E}(j_{n+1}, i_n \circ j_n \circ j_{n+1})$. By Definition 6.1(2) we have $x \in i_{n+1}(i_n(X_n))$. Therefore there is $y' \in X_n$ such that $x = i_{n+1}(i_n(y'))$. We have

$$
\begin{aligned}
i_{n+1}(j_{n+1}(x)) &= i_{n+1}(j_{n+1}(i_{n+1}(i_n(y')))) \\
&= i_{n+1}(i_n(y')) && \text{by Definition 6.1(1)} \\
&= x
\end{aligned}
$$

(by the way, as a consequence of previous point (1), $y = y'$).
(4) We reason by induction on $r - p$. If $r = p$ we have the thesis since $j_{p,p} = \mathrm{Id}_{X_p}$. Consider the case $r > p$, and let $i_{q,p}(j_{q,r}(x)) = j_{p,r}(x)$. Let $y = j_{p+1,r}(x)$. We have

$$
\begin{aligned}
j_p(y) &= j_p(j_{p+1,r}(x)) \\
&= j_{p,r}(x) \\
&= i_{q,p}(j_{q,r}(x)) && \text{by the hypothesis} \\
&= i_{p-1}(i_{q,p-1}(j_{q,r}(x))).
\end{aligned}
$$

Therefore $j_p(y) \in i_{p-1}(X_{p-1})$, hence by point (3) we have $i_p(j_p(y)) = y$. Recalling the definition of y, we have $i_p(j_p(j_{p+1,r}(x))) = j_{p+1,r}(x)$. Therefore

$$
\begin{aligned}
j_{p+1,r}(x) &= i_p(j_p(j_{p+1,r}(x))) && \text{by above} \\
&= i_p(j_{p,r}(x)) \\
&= i_p(i_{q,p}(j_{q,r}(x))) && \text{by hypothesis} \\
&= i_{q,p+1}(j_{q,r}(x)).
\end{aligned}
$$

Now we can apply induction hypothesis, since $r - (p+1) < r - p$, and conclude $x = i_{q,r}(j_{q,r}(x))$.
(5) We will prove the equivalent statement that for any $r \geq p$, $i_r(x^{(r)}) = x^{(r+1)}$. We reason by induction on $r - p$. If $p = r$ the thesis follows from the hypothesis. Let $r > p$. Then $i_{r-1}(x^{(r-1)}) = x^{(r)}$ by induction, and moreover $x^{(r)} = j_r(x^{(r+1)})$. So $j_r(x^{(r+1)}) \in i_{r-1}(X_{r-1})$ and by (3) it follows $i_r(j_r(x^{(r+1)})) = x^{(r+1)}$. We conclude $x^{(r+1)} = i_r(x^{(r)})$, since by definition $x^{(r)} = j_r(x^{(r+1)})$. □

As expected, a pointed tiered object induces a tierable pair.

Definition 6.2. Let $\mathbf{A} = \langle A, \upharpoonright, \mathsf{E}\rangle$ be a tiered object. We define the sequence $\xi_{\mathbf{A}} = \langle A_n, j_n : A_{n+1} \to A_n\rangle_{n\in\omega}$ as follows:
- $A_n = \{a \in A \mid \mathsf{E}\, a \leq n\}$;
- $\forall a \in A_{n+1}.\ j_n(a) = a \upharpoonright n$.
Finally we define the sequence $\eta_{\mathbf{A}} =_{\text{def}} \langle A_n, \subseteq\rangle_{n\in\omega}$, and $\alpha_{\mathbf{A}}$ as the pair $\langle \xi_{\mathbf{A}}, \eta_{\mathbf{A}}\rangle$.

From Definition 2.2 the following simple lemma follows.

Lemma 6.2. $\alpha_{\mathbf{A}}$ is tierable.

Consider a sequence $\eta = \langle X_n, i_n : X_n \to X_{n+1}\rangle_{n\in\omega}$. We fix the notation for the colimit of η. Let $X_\eta = (\bigcup_{n\in\omega} X_n \times \{n\})/\sim$, where $\sim$ is the equivalence relation generated by the pairs $\{(\langle x, n\rangle, \langle y, n+1\rangle) \mid y = i_n(x)\}$. Namely

$$\langle x, n\rangle \sim \langle y, m\rangle \quad \Leftrightarrow \quad \begin{array}{l} n \leq m \;\&\; y = i_{n,m}(x) \text{ or} \\ m < n \;\&\; x = i_{m,n}(y). \end{array}$$

The colimit of η is given by $\langle X_\eta, \langle \rho_n : X_n \to X_\eta\rangle_{n\in\omega}\rangle$, where $\rho_n(x) = [\langle x, n\rangle]_\sim$, for any $x \in X_n$.
We define $\rho'_n : X_\eta \to X_n$ by $\rho'_n(\bar{u}) = x$, if $\bar{u} = [\langle x, n\rangle]_\sim$. Observe that ρ'_n is well-defined, since, given n, x is unique: this is a consequence of the injectivity of i_n. Actually it is easy to prove that $\rho'_n \circ \rho_n = \text{Id}_{X_n}$.
In the following we will omit the subscript $\sim$ in the notation for the equivalence classes.

Definition 6.3. Let $\alpha = \langle \eta, \xi\rangle$ be tierable. We define a structure $\mathbf{X}_\alpha = \langle X_\alpha, \upharpoonright_\alpha, \mathsf{E}_\alpha\rangle$ as follows:
- $X_\alpha = X_\eta$;
- $[\langle x, k\rangle] \upharpoonright_\alpha n = \begin{cases} [\langle j_{n,r}(i_{k,r}(x)), n\rangle] & \text{if } n \in \omega, \text{ where } r \text{ is any natural number such that } r \geq n, k \\ [\langle x, k\rangle] & \text{if } n = \infty \end{cases}$
- $\mathsf{E}_\alpha[\langle x, k\rangle] = \min\{p \mid i_{p,k}(j_{p,k}(x)) = x\}$.

Before proving that $\mathbf{X}_\alpha$ is a finitary pointed tiered object, we need a technical lemma.

Lemma 6.3. Let α be tierable, $k \in \omega$. Then

1. The restriction $\upharpoonright_\alpha$ and the extent E_α are well-defined.
2. If $n \geq k$, $x \in X_k$, then $[\langle x, k\rangle] \upharpoonright_\alpha n = [\langle x, k\rangle]$.
3. If $n \leq k$, $x \in X_k$, then $[\langle x, k\rangle] \upharpoonright_\alpha n = [\langle j_{n,k}(x), n\rangle]$.

Proof:
(1) Let $[\langle x, k\rangle] \in X_\eta$. We prove that, when $n \in \omega$, $[\langle x, k\rangle] \upharpoonright_\alpha n = [\langle j_{n,r}(i_{k,r}(x)), n\rangle]$ depends neither on the choice of $r \geq n, k$, nor on the representative of the equivalence class. In fact, let $r' \geq n, k$. Suppose $r \leq r'$. Then

$$\begin{aligned} j_{n,r'}(i_{k,r'}(x)) &= j_{n,r}(j_{r,r'}(i_{r,r'}(i_{k,r}(x)))) \\ &= j_{n,r}(i_{k,r}(x)) && \text{by Lemma 6.1(2).} \end{aligned}$$

The case $r' < r$ is treated by swapping r and r' above. So $[\langle x, k\rangle] \upharpoonright_\alpha n$ does not depend on the choice of r. Now, let $\langle y, h\rangle \sim \langle x, k\rangle$. Suppose $k \leq h$, hence $y = i_{k,h}(x)$. Then, choosing $r \geq h, k, n$, we have

$$\begin{aligned} j_{n,r}(i_{h,r}(y)) &= j_{n,r}(i_{h,r}(i_{k,h}(x))) \\ &= j_{n,r}(i_{k,r}(x)). \end{aligned}$$

The case $h < k$ is similar. Therefore $[\langle x, k\rangle] \upharpoonright_\alpha n$ does not depend on the representative of $[\langle x, k\rangle]$.
We now prove that also E_α is well-defined. Let $\langle x, k\rangle \sim \langle y, h\rangle$, with $k \leq h$. Then $\mathsf{E}_\alpha[\langle x, k\rangle] = \mathsf{E}_\alpha[\langle y, h\rangle]$ since the two sets $\{p \mid i_{p,k}(j_{p,k}(x)) = x\} = \{p \mid i_{p,h}(j_{p,h}(y))\}$ coincide (this last fact is an easy consequence of $y = i_{k,h}(x)$).
The case $k > h$ is similar.
(2) The thesis is trivial if $n = \infty$. Otherwise

$$\begin{aligned} [\langle x, k\rangle] \upharpoonright_\alpha n &= [\langle i_{k,n}(x), n\rangle] && \text{choosing } r = n \text{ in the definition of } \upharpoonright_\alpha \\ &= [\langle x, k\rangle] && \text{by definition of } \sim . \end{aligned}$$

(3) Choosing $r = k$ in the definition of $\upharpoonright_\alpha$ we have immediately $[\langle x, k\rangle] \upharpoonright_\alpha n = [\langle j_{n,k}(x), n\rangle]$. □

Lemma 6.4. Given a tierable $\alpha = \langle \eta, \xi\rangle$, $\mathbf{X}_\alpha$ is a finitary pointed tiered object. Moreover, $\langle X_\eta, \langle \rho_n\rangle_{n\in\omega}\rangle$, is the colimit of η.

Proof:
The second statement is immediate, by definition of X_η. The unique point in X_0 is the bottom element.
We prove the three conditions of Definition 2.2.
Let $\mathsf{E}_\alpha[\langle x, k\rangle] = q$. Then, by definition of E_α, $q \leq k$ and (†) $i_{q,k}(j_{q,k}(x)) = x$. By Lemma 6.3(3) we have $[\langle x, k\rangle] \upharpoonright_\alpha q = [\langle j_{q,k}(x), q\rangle]$. By (†) above, $\langle j_{q,k}(x), q\rangle \sim \langle x, k\rangle$ hence we obtain condition (1) of Definition 2.2.
Consider now condition (2) of Definition 2.2. In what follows we will omit the subscript of the restriction $\upharpoonright_\alpha$. Let $x \in X_k$, and $q \leq p \in \omega$. We can assume, without loss of generality, that $k \geq q, p$, by Lemma 6.3(1). Let $\bar{x}$ be an abbreviation for $[\langle x, k\rangle]$. We have

$$\begin{aligned} (\bar{x} \upharpoonright q) \upharpoonright p &= [\langle j_{q,k}(x), q\rangle] \upharpoonright p && \text{by Lemma 6.3(3)} \\ &= [\langle j_{q,k}(x), q\rangle] && \text{by 6.3(2)} \\ &= \bar{x} \upharpoonright q \\ &= \bar{x} \upharpoonright q \wedge p. \end{aligned}$$

On the other hand

$$\begin{aligned} (\bar{x} \upharpoonright p) \upharpoonright q &= [\langle j_{p,k}(x), p\rangle] \upharpoonright q && \text{by Lemma 6.3(3)} \\ &= [\langle j_{q,p}(j_{p,k}(x)), q\rangle] && \text{again by Lemma 6.3(3)} \\ &= [\langle j_{q,k}(x), q\rangle] \\ &= \bar{x} \upharpoonright q \wedge p && \text{as above.} \end{aligned}$$

We now consider condition (3) of Definition 2.2. The case $p = \infty$ is trivial, so let $p \in \omega$. Consider the following two sets:

$$B_1 = \{q \mid \bar{x} \upharpoonright q = \bar{x}\};$$
$$B_2 = \{q \mid (\bar{x} \upharpoonright p) \upharpoonright q = \bar{x} \upharpoonright p\}.$$

Let $q_1 = \min B_1 (= \mathsf{E}_\alpha\, \bar{x})$, and $q_2 = \min B_2 = (\mathsf{E}_\alpha(\bar{x} \upharpoonright p))$. We have to prove that $q_1 \wedge p = q_2$. Let $q \in B_1$. We will prove that $q \in B_2$. In fact, there are two cases. If $p \leq q$, then $(\bar{x} \upharpoonright p) \upharpoonright q = \bar{x} \upharpoonright (p \wedge q) = \bar{x} \upharpoonright p$, so $q \in B_2$. If $p > q$, then

$$\begin{aligned}(\bar{x} \upharpoonright p) \upharpoonright q &= \bar{x} \upharpoonright (p \wedge q) \\ &= (\bar{x} \upharpoonright q) \upharpoonright p \\ &= \bar{x} \upharpoonright p && \text{since } q \in B_1.\end{aligned}$$

Again, we conclude $q \in B_2$. So we proved $B_1 \subseteq B_2$. Moreover $p \in B_2$. These two facts imply $q_1 \wedge p \leq q_2$.

We prove the opposite inequality. Let $q \in B_2$, and suppose $q < p$. By definition of B_2, $(\bar{x} \upharpoonright p) \upharpoonright q = \bar{x} \upharpoonright p$. Using Lemma 6.3(3), this implies $[\langle j_{q,k}(x), q\rangle] = [\langle j_{p,k}(x), p\rangle]$, that is $i_{q,p}(j_{q,k}(x)) = j_{p,k}(x)$. Applying Lemma 6.1(4), we have $i_{q,k}(j_{q,k}(x)) = x$. So we have

$$\begin{aligned}\bar{x} \upharpoonright q &= [\langle j_{q,k}(x), q\rangle && \text{by Lemma 6.3(3)} \\ &= [\langle x, k\rangle] && \text{by above} \\ &= \bar{x}.\end{aligned}$$

Therefore $q \in B_1$. This proves that $q_1 \wedge p \leq q_2$.

□

As an immediate consequence of Lemma 5.1 we have

Corollary 6.1. Given a tierable sequence α, $\mathbf{X}_\alpha^\infty =_{\mathrm{def}} (\mathbf{X}_\alpha)^\infty$ is a complete tiered object.

$\mathbf{X}_\alpha^\infty$ can be viewed as a cone over ξ, by considering the projections $\pi_n : X_\alpha^\infty \to X_n$ induced by the restriction:

$$\forall x \in X_\alpha^\infty.\ \pi_n(x) = \rho'_n(x \upharpoonright_\alpha^\infty n).$$

It follows immediately from the definition of $\upharpoonright_\alpha^\infty$ that $\mathbf{X}_\alpha^\infty$ is a cone for α.

Next theorem shows that, as expected, the tierable sequence associated to a complete tiered object $\mathbf{A}$, allows to recover $\mathbf{A}$.

Theorem 6.1. Let $\mathbf{A}$ be a complete tiered object. Then $\mathbf{X}_{\alpha_\mathbf{A}}^\infty =_{\mathrm{def}} (\mathbf{X}_{\alpha_\mathbf{A}})^\infty \cong \mathbf{A}$.

Proof:

Observe that $X_{\alpha_\mathbf{A}} = \{x \in A \mid \mathsf{E}\, x \in \omega\}$. We consider the map $\phi : A \to X_{\alpha_\mathbf{A}}^\infty$ defined by:

$$\phi(a) = \begin{cases} a & \text{if } \mathsf{E}\, a \in \omega \\ \langle a \upharpoonright n\rangle_{n\in\omega} & \text{if } \mathsf{E}\, a = \infty \end{cases}$$

By completeness of $\mathbf{A}$, ϕ is a bijection, and by routine checks it is easily seen that it respects restriction and extent. □

Theorem 6.2. Let $\eta = \langle X_n, i_n \rangle_{n\in\omega}$, $\xi = \langle X_n, j_n \rangle_{n\in\omega}$, $\alpha = \langle \eta, \xi \rangle$ be tierable. Then

$$\mathbf{X}^{\infty}_{\alpha} \cong \varprojlim \xi.$$

Proof:
Let $\langle C, (\beta_n : C \to X_n)_{n\in\omega} \rangle$ be another cone for ξ, so for any $n \in \omega$, $\beta_n = j_n \circ \beta_{n+1}$ or, more generally,

$$\forall p \geq q \in \omega.\ \beta_q = j_{q,p} \circ \beta_p \quad (\dagger).$$

We will prove that for any $c \in C$ there exists a unique $\bar{x}_c \in \mathbf{X}^{\infty}_{\alpha}$ such that

$$\beta_n(c) = \pi_n(\bar{x}_c) \quad (*).$$

For any $c \in C$, let $s_c = \langle \bar{x}_n \rangle_{n\in\omega}$, where for any $n \in \omega$, $\bar{x}_n = \rho_n(\beta_n(c))$. Observe that s_c is projective in $\mathbf{X}_{\alpha}$, since

$$\begin{aligned}
\bar{x}_{n+1} \restriction_{\alpha} n &= [\langle \beta_{n+1}(c), n+1 \rangle] \restriction n \\
&= [\langle j_n(\beta_{n+1}(c), n \rangle] && \text{by Lemma 6.3(3)} \\
&= [\langle \beta_n(c), n \rangle] && \text{by } (\dagger) \\
&= \bar{x}_n.
\end{aligned}$$

If s_c is $\mathbf{E}$-unbounded, then the only possibility, in order to satisfy $(*)$, is to define $\bar{x}_c = s_c$. In the other case, there exists a least $q \in \omega$ such that $\bar{x}_q = \bar{x}_{q+1}$. Let $\bar{x}_c = \bar{x}_q$. Then, for any $n \leq q$,

$$\begin{aligned}
\pi_n(\bar{x}_c) &= \pi_n(\rho_q(\beta_q(c))) \\
&= \rho'_n(\rho_q(\beta_q(c)) \restriction^{\infty}_{\alpha} n) \\
&= \rho'_n([\langle j_{n,q}(\beta_q(c)), q \rangle]) && \text{by Lemma 6.3(3)} \\
&= j_{n,q}(\beta_q(c)) \\
&= \beta_n(c).
\end{aligned}$$

If $n > q$, first recall that by Lemma 6.1(5), $\bar{x}_q = \bar{x}_{q+1}$ implies $i_{q,n}(\beta_q(c)) = \beta_n(c)$ $(\flat)$. Then

$$\begin{aligned}
\pi_n(\bar{x}_c) &= \pi_n(\rho_q(\beta_q(c))) \\
&= \rho'_n(\rho_q(\beta_q(c)) \restriction^{\infty}_{\alpha} n) \\
&= \rho'_n([\langle \beta_q(c), q \rangle] \restriction^{\infty}_{\alpha} n) \\
&= \rho'_n([\langle \beta_q(c), q \rangle]) && \text{by Lemma 6.3(2)} \\
&= \rho'_n([\langle i_{q,n}(\beta_q(c)), n \rangle]) && \text{since } i_{q,n}(\beta_q(c)) \sim \beta_q(c) \\
&= \rho'_n([\langle \beta_n(c), n \rangle]) && \text{by } (\flat) \text{ above} \\
&= j_{n,q}(\beta_q(c)) && \text{as above} \\
&= \beta_n(c).
\end{aligned}$$

Therefore $\bar{x}_c$ satisfies $(*)$. We prove the uniqueness of $\bar{x}_c$. If $\bar{y} \in \mathbf{X}_\alpha^\infty$ is another element which satisfies $(*)$, it follows $\mathsf{E}_\alpha^\infty\, \bar{y} \le q$ from Proposition 2.3(7), since

$$\begin{aligned}\bar{y} \upharpoonright_\alpha^\infty q &= \rho_q(\pi_q(\bar{y}))\\ &= \rho_q(\beta_q(c)) \text{ by } (*)\\ &= \rho_{q+1}(\beta_{q+1}(c)) \text{ by definition of } q \text{ (and } \bar{x}_n)\\ &= \rho_{q+1}(\pi_{q+1}(\bar{y})) \text{ by } (*)\\ &= \bar{y} \upharpoonright_\alpha^\infty q+1.\end{aligned}$$

On the other hand we have by definition of $\bar{x}_c$, $\mathsf{E}_\alpha^\infty\, \bar{x}_c = q$. Therefore

$$\begin{aligned}\bar{x}_c &= \bar{x}_c \upharpoonright_\alpha^\infty q\\ &= [\langle \beta_q(c), q\rangle]\\ &= \rho_q(\beta_q(c))\\ &= y \upharpoonright_\alpha^\infty q\\ &= y && \text{by Proposition 2.3(12), since } \mathsf{E}_\alpha^\infty y \le q.\end{aligned}$$

Summing up, we have a unique way for defining $\phi : C \to \mathbf{X}_\alpha^\infty$ in order to satisfy $\beta_n(c) = \pi_n(\phi(c))$ for any $n \in \omega$ and $c \in C$:

$$\phi(c) = \begin{cases} \langle \beta_n(c)\rangle_{n\in\omega} & \text{if } \langle \rho_n(\beta_n(c))\rangle_{n\in\omega} \text{ is } \mathsf{E}\text{-unbounded}\\ \rho_q(\beta_q(c)) & \text{otherwise, where } q = \min\{p \mid \beta_p(c) = \beta_{p+1}(c)\}\end{cases}$$

□

7. Applications

7.1. Tiered (co)algebras

Consider the functor $F(X) = \{\epsilon\} + \{0,1\} \times X$. Its final coalgebra is the set of finite and infinite binary sequences, which is isomorphic to the ideal completion (under prefix ordering) of its initial algebra, consisting of the the set of finite binary sequences. A general account of this phenomenon is given by Adámek [1]: under certain hypotheses on functors $F : \mathbf{Set} \to \mathbf{Set}$, final F-coalgebras are isomorphic to *ideal* completions of initial F-algebras. In this section we apply the results of Section 6, and construe final coalgebras $\mathcal{Y}$ as *projective* completions of initial algebras $\mathcal{X}$, looking at the latter as finitary tiered objects,

$$\mathcal{Y} \cong \mathcal{X}^\infty. \tag{13}$$

This result is a specialization of the theory of Adámek because, despite our use of projective instead of ideal completion, Lemma 5.2 shows that the two kinds of completion yield isomorphic cpos in the case of finitary tiered objects. Therefore, equation (13) (which summarizes Theorem 7.1 and Corollary 7.1) may not be considered a theoretic improvement of [1]. Nevertheless we believe to be of some interest to expose the details, as the use of projective completion substantiates in a direct way the naive intuition that in many cases final coalgebras can be defined in a set-theoretic way out of initial algebras, by taking all the points in the initial algebras (the "finite points") and adding the projective sequences (the "infinite points").

Definition 7.1. Let $F : \mathbf{Set} \to \mathbf{Set}$ a functor. We say that F is *tiering* if the following conditions hold:

1. F preserves inclusions;
2. $F(\emptyset) \neq \emptyset$;
3. F respects equalizers in the following sense: for every $f, g : Y \to Y'$, $\mathcal{E}(F(f), F(g)) \subseteq F(\mathcal{E}(f, g))$;
4. for any $f : Y \to Y'$, $F(f)^{-1}(F(\emptyset)) \subseteq F(\emptyset)$.

As examples of tiering functors, we mention those polynomial functors F which have the shape $F(X) = A + \ldots$, where A is a fixed non-empty set; a further example is given by the finite powerset functor. Consider for instance the polynomial functor $F(X) = A + (B \times X)$, where A and B are fixed sets. It is easy to prove that (1), (2) and (3) are satisfied. Moreover, (4) holds, since given a morphism $f : X \to Y$, $F(\emptyset) = A$ and we have that $(F(f))^{-1}(A) = A$. These argumens apply to any polynomial functor. Also the finite powerset functor $\mathcal{P}_f$ satisfies the conditions above.
A remark is in order. In Definition 7.1 we could require the weaker condition that functors preserve mono instead of inclusion (and modify accordingly the other points in the definition): also in that case the main results of this section, namely Theorem 7.1 and Corollary 7.1 still hold: in fact they are a direct consequence of the results in Section 6, where tierable pairs $\langle \eta, \xi \rangle$ are considered. We choose to work with inclusion preserving functors, because on the one hand they capture the relevant cases of polynomial functors and finite powerset, on the other hand they make the construction particularly smooth.

Let F be a ω-continuous and ω-cocontinuous tiering functor on **Set**. Let d_0 any element of $F(\emptyset)$: we define $X_0 = \{d_0\}$. X_0 is a terminal object in **Set** and enjoys the property $X_0 \subseteq F(X_0)$, since $\emptyset \subseteq X_0$ implies $F(\emptyset) \subseteq F(X_0)$ and moreover $X_0 \subseteq F(\emptyset)$ by definition. Therefore we can build the sequence $\sigma_0 = \langle F^n(X_0), \subseteq \rangle_{n \in \omega}$. Observe that

Remark 7.1. Zipping together the two sequences $\sigma_\emptyset$ and σ_0 we have

$$\emptyset \subseteq X_0 \subseteq F(\emptyset) \subseteq F(X_0) \subseteq F^2(\emptyset) \subseteq F^2(X_0) \ldots$$

which shows that the two sequences have the same colimit. As known, the definition of the colimit could be given without quotienting, by taking simply $X_{\sigma_0} = \bigcup_{n \in \omega} F^n(X_0)$.

On the other hand we can define the inverse sequence $\tau_0 = \langle F^n(X_0), F^n(j_0) : F^{n+1}(X_0) \to F^n(X_0) \rangle_{n \in \omega}$, where j_0 is unique to the terminal object. We will prove that the pair $\alpha_0 = \langle \sigma_0, \tau_0 \rangle$ is tierable, and we will apply the results of the previous section. In particular, it will follow that $|\mathbf{X}_{\alpha_0}| = X_{\sigma_0}$ is the initial algebra by Lemma 6.4 and the remark above. On the other hand, its projective completion, namely $\mathbf{X}^\infty_{\alpha_0}$, is $\varprojlim \tau_0$ by Theorem 6.2, hence it is the final coalgebra, since X_0 is a terminal object.

Lemma 7.1. Let F be a tiering functor over $\mathbf{Set}$. Let $X_0 \subseteq F^n(\emptyset)$ be any one-point set. With the notation above, α_0 is tierable.

Proof:
We prove that condition (1) of Definition 6.1 holds. Exploiting the fact that i_n are inclusions, this amounts

to show that $\forall n \in \omega . x \in X_n \implies j_n(x) = x$. We reason by induction on n. We have $j_0(d_0) = d_0$, hence the thesis holds for $n = 0$. Observe that the following diagram is commutative:

$$\begin{array}{ccc} X_0 & \xrightarrow{\subseteq} & F(X_0) \\ & \searrow^{id} & \downarrow j_0 \\ & & X_0 \end{array}$$

As a consequence, for any $n \in \omega$, the following diagram is commutative too:

$$\begin{array}{ccc} X_n = F^n(X_0) & \xrightarrow{\subseteq} & F^{n+1}(X_0) = X_{n+1} \\ & \searrow^{id} & \downarrow F^n(j_0)=j_n \\ & & F^n(X_0) = X_n \end{array}$$

Now we prove by induction that condition (2) of Definition 6.1 holds, that is for any $n \in \omega$, $\mathcal{E}(j_{n+1}, j_n \circ j_{n+1}) \subseteq X_n$. First we prove that $X_0 = \mathcal{E}(j_1, j_0 \circ j_1)$. We have $d_0 \in \mathcal{E}(j_1, j_0 \circ j_1)$. In fact $j_0(j_1(d_0)) = d_0$. On the other hand observe that the commutativity of any diagram

$$\begin{array}{ccc} \emptyset & \xrightarrow{\subseteq} & X \\ & \searrow^{\subseteq} & \downarrow f \\ & & Y \end{array}$$

implies the commutativity of

$$\begin{array}{ccc} F(\emptyset) & \xrightarrow{\subseteq} & F(X) \\ & \searrow^{\subseteq} & \downarrow F(f) \\ & & F(Y) \end{array}$$

This last diagram implies that $F(f)_{|F(\emptyset)}$ behaves as the identity. Therefore $j_1(d_0) = F(j_0)(d_0) = d_0$, since $d_0 \in F(\emptyset)$. This proves $d_0 \in \mathcal{E}(j_1, j_0 \circ j_1)$.
Now suppose that for some x, $j_1(x) = j_0(j_1(x))$. Then $j_1(x) = d_0$, hence $x \in F(j_0)^{-1}(F(\emptyset))$. Then $x \in F(\emptyset)$, by Definition 7.1(4). By the same argument used for d_0, we have that $F(j_0)(x) = x$, that is $j_1(x) = x$, hence $x = d_0$. We have so proved $\mathcal{E}(j_1, j_0 \circ j_1) = X_0$. Now suppose the thesis holds for n. We have

$$\begin{aligned} \mathcal{E}(j_{n+2}, j_{n+1} \circ j_{n+2}) &= \mathcal{E}(F(j_{n+1}, F(j_n) \circ F(j_{n+1}) \\ &\subseteq F(\mathcal{E}(j_{n+1}, j_n \circ j_{n+1})) && \text{by Definition 7.1(3)} \\ &\subseteq F(X_n) \\ &= X_{n+1}. \end{aligned}$$

□

Theorem 7.1. Let $F : \mathbf{Set} \to \mathbf{Set}$ be a ω-continuous tiering functor. Then the final coalgebra is the complete tiered object $\mathbf{X}^{\infty}_{\alpha_0}$.

Proof:
Since X_0 is chosen to be a terminal object, the final coalgebra is $\varprojlim \tau_0$. By Lemma 7.1 α_0 is tierable, hence $\mathbf{X}_{\alpha_0}^{\infty}$ is a complete tiered object, and by Theorem 6.2 it is the limit of τ_0. □

Corollary 7.1. Let $F : \mathbf{Set} \to \mathbf{Set}$ be a ω-continuous and ω-cocontinuous tiering functor. Then it is possible to define the final coalgebra $\mathcal{Y}$ as the projective completion of the initial algebra $\mathcal{X}$, so that $\mathcal{X} \subseteq \mathcal{Y}$.

Proof:
By Lemma 6.4 $\mathbf{X}_{\alpha_0}$ is the colimit of $\langle F^n(X_0), \subseteq\rangle_{n\in\omega}$, which coincides with the initial algebra by Remark 7.1, and by the previous theorem $\mathbf{X}_{\alpha_0}^{\infty}$ is the final coalgebra. So the thesis follows from the very definition of $\mathbf{X}_{\alpha_0}^{\infty}$ as projective completion of $\mathbf{X}_{\alpha_0}$. □

7.2. A proof principle for tiered objects

We show now how the properties of (complete) tiered objects can be exploited in justifying a general proof principle for their elements. We shall see that the approximation structure given by tiering allows to resolve the apparent circularity implicit in some natural approaches to proving properties of infinite objects.

Properties of elements of a tiered object $\mathbf{A}$ shall be identified with subsets of $\mathbf{A}$ enjoying some simple closure properties that have been suggested by the treatment of the interpretation of recursive types (which has been the initial motivation for the present work) [3, Chapter 10].

Definition 7.2. A subset $P \neq \emptyset$ of a complete tiered objects $\mathbf{A}$ is *complete and uniform* if it satisfies the following two properties:

Uniformity if $a \in P$ then $a \upharpoonright p \in P$ for all $p \in \Omega$,

Completeness if $\{a^{(i)}\}_{i\in\omega} \subseteq P$ is an $\sqsubseteq$-increasing chain, then $\bigsqcup_{i\in\omega} a^{(i)} \in P$.

If $P \subseteq \mathbf{A}$ is complete and uniform, define

$$[\![a \in P]\!] =_{\text{def}} \bigvee\{p \in \Omega \mid a \upharpoonright p \in P\}.$$

By Proposition 2.4(3), we have

$$p \leq [\![a \in P]\!] \Leftrightarrow a \upharpoonright p \in P. \tag{14}$$

Complete and uniform relations over complete tiered objects can be defined as complete and uniform sets of tuples (that however do not belong to products as characterized, e.g., in [14, §4.8(iv)]):

Definition 7.3. Given complete tiered objects $\mathbf{A}, \mathbf{B}$, $\mathbf{A}\cdot\mathbf{B}$ is defined as the product $A \times B$ with $\mathsf{E}\langle a, b\rangle = \mathsf{E}\, a \sqcup \mathsf{E}\, b$ and $\langle a, b\rangle \upharpoonright p = \langle a \upharpoonright p, b \upharpoonright p\rangle$.

It is straightforward to show that $\mathbf{A} \cdot \mathbf{B}$ is a complete tiered object. In particular, the diagonal $\Delta_{\mathbf{A}} =_{\text{def}} \{\langle a, a'\rangle \mid a, a' \in A, a = a'\}$ is a complete and uniform subset of $\mathbf{A} \cdot \mathbf{A}$, so we can define an Ω-valued *equivalence*

$$\begin{aligned}[\![a \equiv b]\!] &=_{\text{def}} [\![\langle a, b\rangle \in \Delta_{\mathbf{A}}]\!] \\ &= \bigvee\{p \in \Omega \mid a \upharpoonright p = b \upharpoonright p\}.\end{aligned} \tag{15}$$

Then we have the following equivalent version of the separation property:

$$p \leq [\![a \equiv b]\!] \Leftrightarrow a \restriction p = b \restriction p. \tag{16}$$

The following notion of *operation* and its characterization in terms of restriction are borrowed from [14, 5.1,5.4]:

Definition 7.4. Given tiered objects $\mathbf{A}, \mathbf{B}$, an *operation* is any mapping $f : A \to B$ such that

$$[\![a \equiv b]\!] \leq [\![f(a) \equiv f(b)]\!].$$

Proposition 7.1. For complete tiered objects $\mathbf{A}, \mathbf{B}$, $f : A \to B$ is an operation if and only if

$$f(a) \restriction p = f(a \restriction p) \restriction p$$

for all $p \in \Omega$ and $a \in A$.

An operation $f : A \to B$ is *guarded* if

$$f(a) \restriction (p+1) = f(a \restriction p) \restriction (p+1)$$

for all $p \in \Omega$ and $a \in A$. Assume that $f : A \to A$ is guarded and $\mathbf{A}$ is complete. Then build inductively the sequence:

$$\begin{cases} x^{(0)} & = \bot \\ x^{(n+1)} & = f\left(x^{(n)}\right) \restriction (n+1) \end{cases}$$

For all $i \in \omega$, $x^{(i)} = x^{(i+1)} \restriction i$, so $x^{(\infty)}$ exists and satisfies

$$x^{(\infty)} \restriction n = x^{(n)}.$$

For all $n \in \omega$

$$f(x^{(\infty)}) \restriction n = x^{(\infty)} \restriction n = x^{(n)}$$

therefore $\lim_{i \to \infty} x_{(i)}$ is the (unique) fixed point of f, by the separation property. Summarizing, we have the following version of the Banach fixed point theorem for complete tiered objects, which has been proved several times in the literature in slightly different forms (see for example [12, 29]):

Proposition 7.2. If $\mathbf{A}$ is a complete tiered object, every guarded operation $f : A \to A$ has a unique fixed point.

For guarded operations we also have the following proof principle, that applies to complete and uniform properties:

Proposition 7.3. Assume that $\mathbf{A}$ is a complete tiered object and $f : \mathbf{A} \to \mathbf{A}$ is a guarded operation. Then, for any complete and uniform $P \subseteq A$ such that $f(P) \subseteq P$ and any $a \in A$:

$$([\![f(a) \in P]\!] \leq [\![a \in P]\!]) \Rightarrow a \in P. \tag{17}$$

Proof:
Observe that the premise is equivalent to

$$\forall n \in \omega \, (n \leq [\![f(a) \in P]\!] \Rightarrow n \leq [\![a \in P]\!]) \, .$$

which in turn is equivalent, by (14), to

$$\forall n \in \omega \, (f(a) \restriction n \in P \Rightarrow a \restriction n \in P) \, ,$$

then use induction on $n \in \omega$: $a \restriction 0 = \bot \in P$, for every $a \in A$. Assume now that $a \restriction n \in P$: then $f(a \restriction n) \in P$ because $f(P) \subseteq P$, so also $f(a) \restriction (n+1) = f(a \restriction n) \restriction (n+1) \in P$ by uniformity of P and because f is guarded. Finally, by the premise, $a \restriction (n+1) \in P$, therefore $a = \bigsqcup_{n \in \omega} (a \restriction n) \in P$ by completeness of P. □

The interesting aspect of this result is that the stratification implicit in the premise of (17) is essential to prove that $a \in P$ (and therefore also that $f(a) \in P$, as $F(P) \subseteq P$). A related use of stratification is exploited in the coinductive proof system for subtyping recursive types described in [7].

Remark 7.2. The proof principle of Proposition 7.3 can also be read as a special form of fixpoint induction [36]: in fact, under the assumptions of (17), the unique fixed point of f, which exists by Proposition 7.2, is in P. Note, in passing, that in this form the principle and its justification look very similar to the metric coinduction principle of [23, 24].

We show now an application of Proposition 7.3 to the proof of the following well-known equality in the theory of lazy lists [11]:

$$\mathtt{nat} = \mathtt{from}(0), \tag{18}$$

where

$$\begin{aligned} \mathtt{nat} &= 0 : \mathtt{inc}(\mathtt{nat}) \\ \mathtt{inc}(\mathtt{nil}) &= \mathtt{nil} \\ \mathtt{inc}(x : xs) &= x + 1 : \mathtt{inc}(xs) \\ \mathtt{from}(n) &= n : \mathtt{from}(n+1). \end{aligned}$$

Lazy lists are the completion (defined as in Section 5) of the (finitary) tiered object of finite lists, where restriction and extent are given by the usual `take` and `length` functions [16]:

$$\begin{aligned} \ell \restriction 0 &= \mathtt{nil} \\ \mathtt{nil} \restriction n+1 &= \mathtt{nil} \\ (x : \ell) \restriction n+1 &= x : (\ell \restriction n) \end{aligned}$$

$$\begin{aligned} \mathbf{E}(\mathtt{nil}) &= 0 \\ \mathbf{E}(x : \ell) &= 1 + \mathbf{E}(\ell) \end{aligned}$$

It is easy to check that these clauses define a tiering on finite lists, that can easily be lifted to a tiering on lazy lists. Now, for any lazy list xs, take $f(xs) = 0 : \mathtt{inc}(xs)$, and observe that this is a guarded operation. In order to apply Proposition 7.3, we have to show that, for all $n \in \omega$:

$$[0 : \mathtt{inc}(\mathtt{nat}) \upharpoonright n+1 = 0 : \mathtt{inc}(\mathtt{from}(0)) \upharpoonright n+1] \Longrightarrow [\mathtt{nat} \upharpoonright n+1 = \mathtt{from}(0) \upharpoonright n+1]$$

and the premise is equivalent to

$$\mathtt{nat} \upharpoonright n+1 = 0 : \mathtt{inc}(\mathtt{from}(0)) \upharpoonright n+1 \tag{19}$$

We first prove:

Lemma 7.2. For all $n, p \in \omega$,

$$\mathtt{inc}(\mathtt{from}(n)) \upharpoonright p = \mathtt{from}(n+1) \upharpoonright p.$$

Proof:
By induction on p. The Lemma is obviously true when $p = 0$; assume now that $p > 0$:

$$\begin{aligned}
\mathtt{inc}(n : \mathtt{from}(n+1)) \upharpoonright p = & \\
&= (n+1 : \mathtt{inc}(\mathtt{from}(n+1))) \upharpoonright p \\
&= n+1 : (\mathtt{inc}(\mathtt{from}(n+1)) \upharpoonright p-1) \\
&= n+1 : (\mathtt{from}(n+2) \upharpoonright p-1) && \text{by induction hypothesis} \\
&= (n+1 : \mathtt{from}(n+2)) \upharpoonright p \\
&= \mathtt{from}(n+1) \upharpoonright p.
\end{aligned}$$

□

Then we calculate:

$$\begin{aligned}
\mathtt{nat} \upharpoonright n+1 = & \\
&= 0 : \mathtt{inc}(\mathtt{from}(0)) \upharpoonright n+1 && \text{by (19)} \\
&= 0 : (\mathtt{inc}(\mathtt{from}(0)) \upharpoonright n) \\
&= 0 : (\mathtt{from}(1)) \upharpoonright n) && \text{by Lemma 7.2} \\
&= (0 : \mathtt{from}(1)) \upharpoonright n+1 \\
&= \mathtt{from}(0) \upharpoonright n+1.
\end{aligned}$$

By Proposition 7.3 we conclude that $\mathtt{nat} = \mathtt{from}(0)$.

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