Composition Operators and Classical Function Theory

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Declaration

I declare that this written submission represents my project work, and where ideas or words of others have been included, I have adequately referenced the original sources. I own the mistake, if any, crept into this report and do not hold anybody or any reference responsible for such mistakes.

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Abstract

(1) The section **Linear Fractional Prologue** is a context, to be consulted as needed , on the basic properties and classification of linear fractional transformations. Linear fractional maps play a vital role in my work, both as agents for changing coordinates and transforming settings.

(2) In the section **Fourier series**, I discuss how to construct a inner product from given Fourier series , the Dirichlet Kernel and its properties. Then I give the proof of Plancharal theorem and Parseval's theorem which play a good role through out my project.

(3) This **Littlewood's Theorem** section is most important part of my work. After developing some of the basic properties of H^2, here we shows that every composition operator acts boundedly on the Hilbert space. As pointed out above, this is essentially Littlewood's Subordination Principle. I present Littlewood's original proof - a beautiful argument that is perfectly transparent in its beauty, but utterly baffling in its lack of geometric insight. Much of conclusion can be regarded as an effort to understand the geometric underpinning of this theorem.

(4) Having established that every composition operator is bounded on H^2, we turn to the most natural follow-up question: "Which composition operators are compact?" The Chapter **Compactness:Introduction** sets out the motivation for this problem. The property of "boundedness" for composition operators means that each one takes bounded subsets of H^2 to bounded subsets. The question above asks us to specify precisely how much the inducing map ϕ has to compress the unit disc into itself in order to insure that the operator C_ϕ compresses bounded subsets of H^2 into relatively compact ones.

(5) In Chapter **Compactness and Univalence** we discover that the geometric soul of Littlewood's Theorem is bound up in the Schwarz Lemma. Armed with this insight, we are able to characterize the univalently induced compact composition operators, obtaining a compactness criterion that leads directly to the Julia-Caratheodory Theorem on the angular derivative.

(6) In Chapter **The Angular Derivative**, I give the idea of the proof of Julia-Caratheodory Theorem in a way that emphasizes its geometric content, especially its connection with the Schwarz Lemma.

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Chapter 1

Introduction

The study of the composition operators links some of the most basic we can ask about linear operators with beautiful classical results from analytic-function theory. The process invests old theorems with new meanings and bestows upon functional analysis in intriguing class of concrete linear operators.

The setting is the simplest one consistent with serious "function-theoretic operator theory": The unit disc U of the complex plane, and the Hilbert space H^2 of functions holomorphic on U with square summable power series coefficients. To each holomorphic function φ that takes U into itself we associate the composition operator C_φ defined by

$$C_\varphi f = f \circ \varphi \qquad (f \ is \ holomorphic \ function \ on \ the \ disc)$$

and set for ourselves the goal of discovering the connection between the function theoretic properties of φ and behaviour of C_φ on H^2 . Here is a summary of major notational and linguistic conventions used throughout my project.

1.1 Notation and Terminology

Here is a summary of major notational and linguistic conventions used throughout my project.
1. Disc and Half Plane : The symbol U denotes the unit disc of the complex plane (or simply the the disc). $\bar{U}$ denotes the closed unit disc and ∂U is the unit circle. π is the open right half plane $(w : Re(w) > 0)$ and $\bar{\pi}$ is closure of π .
2.The space of the holomorphic function: We denote $H(U) = (f : U \longrightarrow \mathbb{C})$, the space of holomorphic function on U and this space is always understood to be endowed with the topology κ of uniform convergence on compact subset of U and the notation $f_n \longrightarrow^{\kappa} f$ means that the sequence f_n of functions converges to uniformly on every compact subset of U.
3. Hilbert space: "Hilbert space" always means separable Hilbert space. The norm in any Hilbert space is denoted by "$\|.\|$," and the inner product by " $< .,. >$ ". We have to work with these Hilbert space: The usual Lebesgue space L^2, square integrable with respect to arc-length measure.: and l^2, the space of square summable (one sided) complex sequences.
4. Operators: The term <u>operators</u> always means bounded linear transformation and <u>finite-rank operators</u> are bounded linear transformation with finite dimensional range. That means finite-rank operator is a bounded linear operator between Banach spaces whose range are finite dimensional. The symbol $\|.\|$ originally used for the Hilbert space norm, will also be used for the norm of the bounded linear operator, or for any other norm that we want to discuss.
5.Special automorphism: We will frequently employ the linear fractional transformation $\alpha : U \longrightarrow U$ such that

$$\alpha_p(z) = \frac{p - z}{1 - \bar{p}z} \qquad for \quad p \in U\cdot$$

This is the special automorphism (a bi-holomorphic mapping of the set onto itself) of U that interchanges the origin and the point p.
6. Iterates: If φ is a holomorphic self mapping of U and n is a positive integer, then the n-th iterates of φ is the n-fold composition of φ with itself. We always denote this map by φ_n :

$$\varphi_n = \varphi \circ \varphi \circ \varphi \circ \ \ldots . \varphi \quad (n\ times).$$

7.Estimates: We frequently write estimate for non-negative functions that look like

$$A(x) \leq const.B(x),$$

for some range of x. In such inequalities the constant is always positive and finite, and is allowed to one occurrence to another, but it never depends on x. when we write

$$A(x) \approx B(x) \quad (for\ some\ range\ of\ x)$$

We mean

$$const.A(x) \leq B(x) \leq const.A(x)$$

for relevant values of x. Frequently occurring instances of this are the simple equivalences:

$$1 - x^2 \approx 1 - x \qquad (0 \le x \le 1)$$

and

$$1 - x \approx log\frac{1}{x} \qquad (as\ x \longrightarrow 1-)\cdot$$

1.2 Some Essential Definition

We have to discuss some definitions which will be used throughout my project.

Definition 1.2.1. (Normed Linear Space) Let, V be a vector space over the field $F(\mathbb{R} or \mathbb{C})$. A norm on V is a mapping(or function) $\|.\|$ from V to $\mathbb{R}^+$,

$$\|.\| : V \longrightarrow \mathbb{R}^+$$

satisfying the following three axioms:
(N1) $\|u\| = 0 \Longrightarrow u = 0,\ for\ u \in V$ $[Positivity]$
(N2) $\|\lambda u\| = |\lambda|\|u\|,\ \forall u \in V\ and\ \ \forall \lambda \in F$ $[Homogeneity]$
(N3)$\|u + v\| \le \|u\| + \|v\|,\ \forall u, v \in V$ $[Triangleineqality]$.
we call the pair $(V, \|.\|)$ a normed linear space.

Definition 1.2.2. (Banach Space)
A complete normed linear space is called Banach space ·

Definition 1.2.3. (Inner Product Space) Let V be a vector space over a field $F(\mathbb{R}$ or $\mathbb{C})$. By an inner product on V, we mean a mapping

$$f := < . \ , \ . >: V \times V \longrightarrow F, \quad (u, v) \longmapsto < u, v >= f(u, v)$$

that assigns for each $(u, v) \in V \times V$ a value in F , denoted by ¡u,v¿, called the inner product of u and v, such that for each $u, v, w \in V\ and\ \lambda \in F$ we have
$(I1) < u, u > \ge 0\ and\ < u, u >= 0 \Longrightarrow u = 0.$ $[Positivity]$
$(I2) < u, v >= \overline{< v, u >}$ $[Conjugate\ symmetric]$
$(I3) < \lambda u, v >= \lambda < u, v >$ $[Homogeneous]$
$(I4) < u, v + w >=< u, v > + < u, w >$ $[Additivity]$

we call the pair $(V, <, >)$ a inner product space.

Definition 1.2.4. (Hilbert Space)
An inner product space V is called a Hilbert space if it is complete with respect to the induced norm $\|.\|_V$. That means, a vector space V over the field F is a Hilbert space iff the following two conditions hold:
(a) there is an inner product on V.
(b) every Cauchy sequence with respect to the induced norm is convergent.

Note 1.2.1. The function $\|.\|$ defined by $\|u\| = \sqrt{< u, u >}$ makes V into a normed linear space.

Definition 1.2.5. (Linear Operator) Let X and Y be two linear spaces over the same field F. Then the mapping $T : X \longrightarrow Y$ is called linear operator (or mapping) if-
(1)$T(x_1 + x_2) = T(x_1) + T(x_2)$
(2)$T(\alpha x_1) = \alpha T(x_1) \quad \forall x_1, x_2 \in X, \quad \alpha \in F.$

Definition 1.2.6. (Bounded Linear Operator) Let X and Y be two normed linear spaces over the same field $F(\mathbb{R} \text{ or } \mathbb{C})$ and $T : D(T) \longrightarrow Y$ a linear operator where $D(T) \subset X$. The operator T is said to be bounded if $\exists$ a real no c such that,

$$\forall x \in D(T), \qquad \|T(x)\|_Y \leq c\|x\|_X.$$

Chapter 2

Linear Fractional Prologue

2.1 Properties

A linear fractional transformation is a mapping $T_A : \mathbb{C} \longrightarrow \mathbb{C}$ of the form

$$T_A(z) = \frac{az+b}{cz+d} \qquad (1)$$

subject to the condition ad-bc $\neq 0$, which is necessary and sufficient condition for T to be a non constant. We denote the set of all such maps by LFT($\hat{\mathbb{C}}$), where the notation is intended to call attention to the fact that, with the obvious convention about the point at ∞, each linear fractional transformation can be regarded as a one-to-one holomorphic mapping of the Riemann Sphere $\hat{\mathbb{C}}$ onto itself.

Group properties:
$LFT(\hat{\mathbb{C}})$ is a group under composition. Each of its members maps every circle on the sphere (i.e. every circle or line in the plane) to another circle, and given any pair of circles, some members of $LFT(\hat{\mathbb{C}})$ maps one onto the other. The same is true for the set of triples of distinct points of the sphere. In the language of group theory :
$LFT(\hat{\mathbb{C}})$ acts transitively on the set of circles of $\hat{\mathbb{C}}$, and triply transitively on the points.

Matrix representation:
Each non-singular 2 $\times$ 2 complex matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

gives rise to a linear fractional transformation T_A by means of definition (1) above. So, easily we can say that $T_A = T_{\lambda A}$ for any $\lambda \in \mathbb{C}$.

Definition 2.1.1. If ad-bc=1 in the definition (1), then T is in standard form.
Actually there are two standard forms, since the determinant is not changed when all coefficients are replaced by their negatives.
The utility of matrices in dealing with linear fractional transformations comes from the fact that

$T_A \circ T_B = T_{AB}$. Borrowing again from the group theory, we say that $S, T \in LFT(\hat{\mathbb{C}})$ are conjugate if there exist $V \in LFT(\hat{\mathbb{C}})$ such that $S = V \circ T \circ V^{-1}$. Thus :
Conjugate linear fractional transformations corresponds to similar matrices.

2.2 Fixed Points

The linear fractional transformation $\frac{az+b}{cz+d}$ fixes the point ∞ iff c=0, in which case ∞ is the only fixed point iff a=d and b≠0 . Otherwise the fixed point equation is a quadratic with solutions

$$\alpha, \beta = \frac{(a-d) \pm [(a-d)^2 + 4bc]^{\frac{1}{2}}}{2c}$$

From this equation we can say that :
(a) If c≠0 and $(a-d)^2 + 4bc \neq 0$ then we will get two distinct finite fixed points.
(b) If $c \neq 0$ and $(a-d)^2 + 4bc = 0$ then we will get only one finite fixed point.
(c) If c=0 and a-d≠0 then we will get two fixed points, one of these will be ∞ and other one will be a finite fixed point.
(d) If c=0 and a-d=0 then we will get only one fixed point which will be ∞.

The Trace: If T(z)=$\frac{az+b}{cz+d}$ is in standard form (i.e. ad-bc=1), then the define of the trace of T to be $\chi(T)$=$\pm(a+d)$.

Example 2.2.1. T has ∞ as its only fixed point on the sphere iff T(z)=z+b, in which case $|\chi(T)|$=2.

If T has only finite fixed points, then th equation written above for these fixed point can be at least partially expressed in terms of the trace:

$$\alpha, \beta = \frac{(a-d) \pm [\chi(T)^2 + 4]^{\frac{1}{2}}}{2c} \tag{2}$$

This equation and the given remark about maps with unique fixed point at ∞ shows that
T∈ $LFT(\hat{\mathbb{C}})$ has a unique fixed point in $\mathbb{C}$ iff $|\chi(T)|$ =2.

2.3 Derivatives at the Fixed Points

: If T∈ $LFT(\hat{\mathbb{C}})$ is in standard form, then

$$T'(z) = \frac{ad-bc}{(cz+d)^2} = \frac{1}{(cz+d)^2}$$

and now using (2) the above equation shows that the derivative of T at its fixed points can be represented in terms of the trace :

$$T'(\alpha), T'(\beta) = \frac{1}{(\frac{(a-d) \pm \sqrt{\chi(T)^2 - 4}}{2c} + d)^2}$$

$$= \frac{4}{(\chi(T) \pm \sqrt{\chi(T)^2 - 4})^2} \tag{3}$$

where the ambiguity in the sign of the trace is absorbed by the fact that the right hand side is a perfect square. From this it follows that

$$T'(\alpha) = \frac{1}{T'(\beta)} \qquad \textit{and} \qquad T'(\alpha) + T'(\beta) = \chi(T)^2 - 2 \tag{4}$$

In the case T has fixed point at ∞ and another finite one, it must have the form T(z)=az+b , in which case we define $T'(\infty) = (1/T)'(0)^{-1} = \frac{1}{a}$. Hence , the relation (4) holds. In summary we can write:

Theorem(Fixed points and derivatives):

Suppose $T \in LFT(\hat{\mathbb{C}})$ then these are equivalent:

- $|\chi(T)| = 2$
- $T' = 1$ at a fixed point of T.
- T has just one fixed point on $\hat{\mathbb{C}}$.

If T has two distinct fixed points, then its derivative at these points are reciprocals and their sum is $\chi(T)^2 - 2$.

2.4 Classification

A map T $\in LFT(\hat{\mathbb{C}})$ is called *parabolic* if it has a single fixed point in $\hat{\mathbb{C}}$. Suppose T is parabolic and has its fixed point at $\alpha \in \mathbb{C}$.

If T is not parabolic, there are two fixed points $\alpha, \beta \in \hat{\mathbb{C}}$.

Multipliers furnish the following classification of non-parabolic maps.

Definition 2.4.1. Suppose $T \in LFT(\hat{\mathbb{C}})$ is neither parabolic nor identity. Let $\lambda \neq 1$, be the multiplier of T. Then T is called

- Elliptic if $|\lambda| = 1$.
- Hyperbolic if $\lambda > 0$.
- Loxodromic if T is neither elliptic nor parabolic.

Theorem:(Classification by the trace)

Suppose T is linear fractional map that is not the identity. Then T is loxodromic iff its trace $\chi(T)$ is not real. If $\chi(T)$ is real, T is

- Hyperbolic $\iff$ $|\chi(T)| > 2$
- *Parabolic* $\iff$ $|\chi(T)| = 2$
- *Elleptic* $\iff$ $|\chi(T)| < 2$

Important results on Fixed points:
(i) Every bilinear transformation with two finite fixed points $\alpha,\ \beta$ can be put in the form

$$\frac{w-\alpha}{w-\beta} = \lambda\frac{z-\alpha}{z-\beta}$$

(a) If $|\lambda| = 1$, then the transformation is elliptic.
(b) If $\lambda > 0$ ($\neq 1$), then the transformation is hyperbolic.
(ii) Every bilinear transformation which has only one fixed point α can be put in the form

$$\frac{1}{w-\alpha} = \frac{1}{z-\alpha} + \lambda$$

In this case transformation is parabolic.

Now I am clearing these transformation by some examples:

Example 2.4.1. **w=**$\frac{z}{2-z}$
Solution: The fixed points are given by

$$z = \frac{z}{2-z} \quad or, \quad 2z - z^2 = z$$

$$\text{or, } \ z(z-1) = 0 \qquad \Longrightarrow \ z = 0, 1$$

Hence 0 and 1 are the two fixed points in this case.To obtain the normal form we can write,

$$w = \frac{z}{2-z} \quad and \quad w - 1 = \frac{2z-2}{2-z}$$

$$\therefore \frac{w}{w-1} = \frac{z}{2z-2} \qquad = \frac{1}{2}\ \frac{z}{z-1}$$

which is the required normal form.
So here, $\quad \lambda = \frac{1}{2} > 0.$
Hence, this transformation is hyperbolic.

Example 2.4.2. **w=**$\frac{3z-4}{z-1}$
Solution: The fixed points are given by,

$$z = \frac{3z-4}{z-1}$$

$$or, z^2 - z = 3z - 4 \quad or, (z-2)^2 = 0$$

$$\Longrightarrow \ z = 2, 2$$

i.e. z=2 is the only fixed point in this case. To obtain the normal form, we have

$$w = \frac{3z-4}{z-1} \qquad \Longrightarrow \ wz - w = 3z - 4$$

$$\Longrightarrow \ (w-2)(z-2) + 2z + 2w - w - 3z = 0$$

$$\Longrightarrow \ (w-2)(z-2) - (z-2) + (w-2) = 0$$

$$\implies 1 - \frac{1}{w-2} + \frac{1}{z-2} \quad or, \frac{1}{w-2} = \frac{1}{z-2} + 1,$$

which is required normal form.
Here, the transformation is parabolic.

Example 2.4.3. w=$\frac{z-1}{z+1}$
Solution: The fixed points are given by

$$z = \frac{z-1}{z+1} \quad \implies z^2 + 1 = 0 \quad \implies z = \pm i$$

Hence i and $-i$ are two fixed points in this case. to obtain the normal form we have,

$$w = \frac{z-1}{z+1}$$

$$\implies w - i = \frac{z-1}{z+1} - i \; and \; w + i = \frac{z-1}{z+1} + i$$

$$\implies \frac{w-i}{w+i} = \frac{z-1-iz-i}{z-1+iz+i} = \frac{(1-i)(z-i)}{1+i)(z+i)}$$

$$\implies \frac{w-i}{w+i} = \frac{(1-i)^2(z-i)}{(1+i)(1-i)(z+i)} = -i\frac{z-i}{z+i}$$

which is required normal form.
Here $\lambda = -i \implies |\lambda| = 1$
Hence the transformation is elliptic.

Example 2.4.4. w=$\frac{(2+i)z-2}{z+i}$
Solution: The fixed points are given by

$$z = \frac{(2+i)z-2}{z+i} \quad \implies z^2 - 2z + 2 = 0$$

$$\therefore \qquad z = \frac{2 \pm \sqrt{(4-8)}}{2} = 1 \pm i$$

Hence 1+i and 1-i are the two fixed points in this case.
To obtain the normal form, we have

$$w = \frac{(2+i)z-2}{z+i}$$

$$\therefore \qquad w - (1+i) = \tfrac{(2+i)z-2}{z+i} - (1+i)$$

$$= \frac{2z + iz - 2 - z - iz - i + 1}{z+i} = \frac{z-(1+i)}{z+i}$$

and

$$w - (1-i) = \tfrac{(2+i)z-2}{z+i} - (1-i)$$

$$= \frac{2z + iz - 2 - z - iz - i - 1}{z+i}$$

$$= \frac{z-(1-i)+2i(z+i-1)}{z+i}$$

$$= \frac{(1+2i)[z-(1-i)]}{z+i}$$

$$\implies \qquad \tfrac{w-(1+i)}{w-(1-i)} = \tfrac{z-(1+i)}{(1+2i)[z-(1-i)]}$$

$$= \frac{1-2i}{5}\frac{z-(1+i)}{z-(1-i)}$$

which is the required normal form.
Here, $\lambda = \frac{1-2i}{5}$. So , neither λ is real nor $|\lambda|$=1 .
Hence , the transformation is loxodromic.

Chapter 3

Fourier Series

We have to know some relation from the Fourier Series which will be used through out my project. For this purpose I am giving a summary of Fourier Series.

3.1 Basics of Fourier Series

Notation:
C[a,b]- The space of all continuous function on [a,b].
$C_{2\pi}[-\pi,\pi] = \{f \in C[-\pi,\pi] :\ f(-\pi) = f(\pi)\}$
Now defining the inner product of f and g s.t.

$$< f, g >:= \int_a^b f(t)\overline{g(t)}dt \quad \forall\ f, g \in C[a, b]$$

Definition 3.1.1. A function is of the form

$$P(x) = a_0 + \sum_{k=1}^{n}(a_k \cos kx + b_k \sin kx)$$

is called Trigonometric polynomial.
clearly, P(x)$\in$ $C_{2\pi}[-\pi,\pi]$

Note 3.1.1. Let,

$$P(x) = a_0 + \sum_{k=1}^{n}(a_n \cos kx + b_n \sin kx)$$

We know that

$$\cos kx = \frac{e^{ikx} + e^{-ikx}}{2}\ and\ \sin kx = \frac{e^{ikx} - e^{-ikx}}{2i}$$

Then,

$$P(x) = a_0 + \sum_{k=1}^{n}[a_k \frac{e^{ikx} + e^{-ikx}}{2} + b_k \frac{e^{ikx} - e^{-ikx}}{2i}]$$

$$= a_0 + \sum_{k=1}^{n}[(\frac{a_k - ib_k}{2})e^{ikx} + (\frac{a_k + ib_k}{2})e^{-ikx}]$$

Now assuming,

$$c_0 = a_0, \quad c_k = \frac{a_k - ib_k}{2} \quad and \quad c_{-k} = \frac{a_k + ib_k}{2}$$

then

$$\begin{aligned} P(x) &= c_0 + \sum_{k=1}^{n} (c_k e^{ikx} + c_{-k} e^{-ikx}) \\ &= c_0 + \sum_{k=1}^{n} c_k e^{ikx} + \sum_{k=1}^{n} c_{-k} e^{-ikx} \\ &= c_0 + \sum_{k=1}^{n} c_k e^{ikx} + \sum_{k=-n}^{-1} c_k e^{ikx} \\ &= \sum_{k=-n}^{n} c_k e^{ikx} \end{aligned}$$

Recall: Now $(C_{2\pi}[-\pi, \pi], <,>)$ is an inner product space where

$$< f, g >:= \int_{-\pi}^{\pi} f(t)\overline{g(t)}dt \quad \forall \ f, g \in C_{2\pi}[-\pi, \pi]$$

$$||f||_2 := (\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(t)|^2 dt)^{\frac{1}{2}}$$

Note 3.1.2. We know that , $e^{imt} \in C_{2\pi}[-\pi, \pi] \ \ \forall \ m \in \mathbb{Z}$ then,

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{imt} e^{-ilt} dt = \begin{cases} 1 & \text{if} \ \ m = l \\ 0 & \text{if} \ \ m \neq l \end{cases}$$

- Now let

$$P(x) = \sum_{k=-N}^{N} c_k e^{ikx}$$

then

$$P(x)e^{-imx} = \sum_{k=-N}^{N} c_k e^{ikx} e^{-imx}$$

$$\implies \qquad \frac{1}{2\pi} \int_{-\pi}^{\pi} P(x)e^{-imx} dx = \sum_{k=-N}^{N} \frac{1}{2\pi} \int_{-\pi}^{\pi} c_k e^{ikx} e^{-imx} dx$$

$$\implies \qquad \frac{1}{2\pi} \int_{-\pi}^{\pi} P(x)e^{-imx} dx = c_m$$

Definition 3.1.2.
Fourier Series and Coefficient: Let $f \in C_{2\pi}[-\pi, \pi]$
We call $\hat{f}(m)$ as a m-th <u>Fourier coefficient</u> of f.
And the sum $\sum_{m=-\infty}^{\infty} \hat{f}(m)e^{imx}$ is called <u>Fourier Series</u> of f.

Definition 3.1.3. A series of the form $\sum_{n=-\infty}^{\infty} a_n e^{inx}$ is called trigonometric series.

The above series can be written as

$$a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx).$$

Definition 3.1.4. Let $f(x)$ be a single valued function defined in the interval $[-\pi, \pi]$.
If f(x) is bounded, it be integrable in$[-\pi, \pi]$.
If f(x) is unbounded, the improper integral $\int_{-\pi}^{\pi} f(x)dx$ be absolutely convergent. Then the trigonometric series,

$$a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

is called the Fourier Series corresponding to the function $f(x)$, where

$$a_0 = \frac{2}{\pi}\int_{-\pi}^{\pi} f(x)dx,$$

$$a_n = \frac{2}{\pi}\int_{-\pi}^{\pi} f(x)\cos nxdx, \quad b_n = \frac{2}{\pi}\int_{-\pi}^{\pi} f(x)\sin nxdx.$$

Note 3.1.3. Without deciding whether the series converges to f(x) or not, We use the notation '$\sim$' and write $f(x) \sim \quad a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$ where '$\sim$' means is not actually equal to but generates or series corresponding to the function f(x).
However if the Fourier corresponding to f(x) converges to f(x) then we can write,

$$f(x) = \quad a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx).$$

Partial Sum of the Fourier Series Let, $f \in C_{2\pi}[-\pi, \pi]$ and if $\sum_{n=-\infty}^{\infty} \hat{f}(n)e^{inx}$ be the Fourier series of f then

$$S_N(f)(x) := \sum_{n=-N}^{N} \hat{f}(n)e^{inx}$$

is called the N-th partial sum of the Fourier series of f.
Here, for each $n \in \mathbb{N}$
• $S_N(f)$ is continuous.
• $S_N(f)$ is 2π periodic function, i.e. $S_N(f)(x + 2\pi) = S_N(f)(x)$.

An important consequence of the uniform convergent theorem is " if the Fourier Series of f converges uniformly to f, then f must be continuous on $[-\pi, \pi]$ with $f(\pi) = f(-\pi)$ ". Then the following basic question arise:
Is there an important relationship between f(x) and its Fourier series?
Does the partial sum $S_N(f)(x)$ approximate f(x) for large values of N in some sense ?
Does the Fourier series $\sum_{n=-N}^{N} \hat{f}(n)e^{inx}$ converges to f(x)?
We are not aiming to discuss these questions, but wish to show there exist a continuous 2π periodic function with a divergent Fourier series. We remark that the problem of deciding whether or not the Fourier series $\sum_{n=-N}^{N} \hat{f}(n)e^{inx}$ converges at a specific point(or, everywhere) is difficult as it usually

requires some degree of smoothness (differentiability and some uniform convergence condition).

3.2 Dirichlet Kernel and its properties

Let $\sum_{n=-\infty}^{\infty} \hat{f}(n)e^{inx}$ be the Fourier series of f $\in C_{2\pi}[-\pi,\pi]$ and

$$S_N(f)(x) = \sum_{n=-N}^{N} \hat{f}(n)e^{inx}, \quad x \in [-\pi,\pi]$$

be the N th partial sum of Fourier series f(x).
Then

$$\begin{aligned} S_N(f)(x) &= \sum_{n=-N}^{N} (\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-int}dt)e^{inx} \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)(\sum_{n=-N}^{N} e^{-int}e^{inx})dt \\ &= \int_{-\pi}^{\pi} f(t)(\frac{1}{2\pi}\sum_{n=-N}^{N} e^{-in(x-t)})dt \\ &= \int_{-\pi}^{\pi} f(t)D_N(x-t)dt \end{aligned}$$

where $D_N(x) = \frac{1}{2\pi}\sum_{n=-N}^{N} e^{inx}$.
Observe that,

$$\begin{aligned} D_N(x) &= \frac{1}{2\pi}\sum_{n=-N}^{N} e^{inx} \\ &= \frac{1}{2\pi}e^{-iNx}\sum_{n=0}^{2N} e^{inx} \\ &= \frac{1}{2\pi}e^{-iNx}\sum_{n=0}^{2N} e^{(ix)^n} \\ &= \frac{1}{2\pi}e^{-iNx}\frac{1-(e^{ix})^{2N+1}}{1-e^{ix}} \\ &= \frac{1}{2\pi}\frac{e^{-iNx}-e^{i(N+1)x}}{1-e^{ix}} \\ &= \frac{1}{2\pi}\frac{e^{-i(N+\frac{1}{2})x}-e^{i(N+\frac{1}{2})x}}{e^{-\frac{ix}{2}}-e^{\frac{ix}{2}}} \end{aligned}$$

$\implies D_N(x) = \frac{1}{2\pi}\frac{\sin(N+\frac{1}{2})x}{\sin\frac{x}{2}}$ $\forall x \in [-\pi,\pi]$ is called the Dirichlet's Kernel. So,

$$S_N f(x) = \int_{-\pi}^{\pi} f(t)D_N(x-t)dt$$

Now the properties of Dirichlet's kernel are:
(i) D_N is even function.
(ii) $\int_{-\pi}^{\pi} D_N(x)dx = 1 \quad \forall\ N \in \mathbb{N}$
(iii) $\lim_{n\longrightarrow\infty} \int_{-\pi}^{\pi} |D_N(x)|dx = \infty$.

3.3 Some Important Theorem Related to Composition Operators

Theorem 3.3.1. *Let $f \in C_{2\pi}[-\pi,\pi]$ and $S_N(f)$ be the N-th partial sum of $\sum_{n=-\infty}^{\infty} \hat{f}(n)e^{inx}$ and $t_N(x) = \sum_{n=-N}^{\mathbb{N}} c_n e^{inx}$. Then,*

$$||f - S_N(f)||_2 \leq ||f - t_N||_2.$$

Moreover the equality holds $\iff$ $\hat{f} = c_n \ \ \forall\ -N \leq n \leq N$.

Proof:

$$\begin{aligned} ||f - t_N||_2^2 &=< f - t_N, f - t_N > \\ &=< f,f > - < f,t_N > - < t_N, f > + < t_N, t_N > \\ &= ||f||_2^2 - 2Re < f, t_N > +||t_N||_2^2 \qquad (1) \end{aligned}$$

Now,

$$\begin{aligned} ||t_N||_2^2 &=< t_N, t_N > \\ &= \langle \sum_{n=-\mathbb{N}}^{\mathbb{N}} c_n e^{inx}, \sum_{m=-N}^{\mathbb{N}} c_n e^{imx} \rangle \\ &= \sum_{n=-N}^{N} c_n \sum_{m=-N}^{N} \overline{c_m} < e^{inx}, e^{imx} > \\ &= \sum_{n=-N}^{N} |c_n|^2 \qquad \because < e^{inx}, e^{imx} >= \begin{cases} 1 & if \ \ m = n \\ 0 & if \ \ m \neq n \end{cases} \end{aligned}$$

Again,

$$\begin{aligned} < f, t_N > &=< f, \sum_{n=-\mathbb{N}}^{\mathbb{N}} c_n e^{inx} > \\ &= \sum_{n=-N}^{N} \bar{c}_n < f, e^{inx} > \\ &= \sum_{n=-N}^{N} \bar{c}_n \frac{1}{2\pi} \int_{-\pi}^{-\pi} f(x)e^{-inx}dx \\ &= \sum_{n=-N}^{N} \bar{c}_n \hat{f}(n) \end{aligned}$$

$$And \qquad ||f-S_N(f)||_2^2 =< f-S_N(f), f-S_N(f) >$$

$$=< f,f > - < f, S_N(f) > - < S_N(f), f > + < S_N(f), S_N(f) >$$

$$= ||f||_2^2 - 2Re < f, S_N(f) > + \sum_{n=-N}^{N} |\hat{f}(n)|^2$$

$$= ||f||_2^2 - 2\sum_{-N}^{N} |\hat{f}(n)|^2 + \sum_{n=-N}^{N} |\hat{f}(n)|^2$$

$$= ||f||_2^2 - \sum_{n=-N}^{N} |\hat{f}(n)|^2.$$

Now from equation (1) we can write ,

$$||f-t_N||_2^2 = \ ||f||_2^2 - 2Re \sum_{n=-N}^{N} \bar{c}_n \hat{f}(n) + \sum_{n=-N}^{N} |c_n|^2$$

$$= ||f||_2^2 - \sum_{n=-N}^{N} |\hat{f}(n)|^2 + \sum_{n=-N}^{N} |\hat{f}(n)|^2 - \sum_{n=-N}^{N} (\bar{c}_n \hat{f}(n) + c_n \overline{\hat{f}(n)}) + \sum_{n=-N}^{N} |c_n|^2$$

$$= ||f-S_N(f)||_2^2 + \sum_{n=-N}^{N} (|\hat{f}(n)|^2 - \bar{c}_n \hat{f}(n) - c_n \overline{\hat{f}(n)} + |c_n|^2)$$

$$= ||f-S_N(f)||_2^2 + \sum_{n=-N}^{N} |c_n - \hat{f}(n)|^2$$

$$\implies \qquad ||f-S_N(f)||_2^2 \le ||f-t_(N)||_2^2$$

$$\implies \qquad ||f-S_N(f)||_2 \le ||f-t_(N)||_2$$

Equality holds $\iff \sum_{n=-N}^{N} |c_n - \hat{f}(n)|^2 = 0$

$$\implies \qquad c_n = \hat{f}(n) \quad \forall -N \le n \le N.$$

Corollary 3.3.2. $f \in C_{2\pi}[-\pi,\pi]$ then $\sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2 \le ||f||_2^2$.

Proof: $\because \qquad ||f-S_N(f)||_2 \ge 0$

$$\therefore \qquad ||f-S_N(f)||_2^2 =< f-S_N(f), f-S_N(f) >$$

$$=< f,f > - < f, S_N(f) > - < S_N(f), f > + < S_N(f), S_N(f) >$$

$$= ||f||_2^2 - 2Re < f, S_N(f) > + \sum_{n=-N}^{N} |\hat{f}(n)|^2$$

$$= ||f||_2^2 - 2\sum_{-N}^{N} |\hat{f}(n)|^2 + \sum_{n=-N}^{N} |\hat{f}(n)|^2$$

$$= ||f||_2^2 - \sum_{n=-N}^{N} |\hat{f}(n)|^2 \qquad \geq \qquad 0$$

$\implies$ $\qquad \sum_{n=-N}^{N} |\hat{f}(n)|^2 \leq ||f||_2^2 \quad \forall\ n \in \mathbb{N}$

$\implies$ $\qquad \sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2 \leq ||f||_2^2$

this inequality is called $\underline{Bessel's\ \ inequality}$.

Note 3.3.3. (1) Let $f \in C_{2\pi}[-\pi, \pi]$ and $P(x) = \sum_{n=-N}^{N} c_n e^{inx}$ then

$$||f - S_N(f)||_2 \leq ||f - P||_2.$$

(2) Given $f \in C_{2\pi}[-\pi, \pi]$, $\exists$ a sequence of trigonometric polynomial say P_n , such that $P_n \longrightarrow f$ in $(C_{2\pi}[-\pi, \pi], <, >)$ as $n \longrightarrow \infty$.

Theorem 3.3.4. *Let $f \in C_{2\pi}[-\pi, \pi]$. Then we have the followings :*

(i) $S_N(f) \longrightarrow f$ as $N \longrightarrow \infty$ in $(C_{2\pi}[-\pi, \pi], <, >)$.

(ii)$||f||_2^2 = \sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2$ *[Plancharal Theorem]*

(iii)If $g \in C_{2\pi}[-\pi, \pi]$, then$< f, g >= \sum_{-\infty}^{\infty} \hat{f}(n)\overline{\hat{g}(n)}$. *[Parseval's Theorem]*
i.e.

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\overline{g(x)}dx = \sum_{-\infty}^{\infty} \hat{f}(n)\overline{\hat{g}(n)}$$

Proof:
(i) Given $f \in C_{2\pi}[-\pi, \pi]$, let $\epsilon > 0$ the $\exists$ a trigonometric polynomial $P(x)$=$\sum_{n=-N}^{N} c_n e^{inx}$ such that

$$||f - P||_2 < \epsilon$$

Also we know that

$$||f - S_N(f)||_2 \leq ||f - P||_2 \quad \forall\ N \in \mathbb{N}$$

$\implies$ $\qquad ||S_N(f) - f||_2 \leq \epsilon \quad \forall N \geq M$

Hence ,

$$||S_N(f) - f||_2 \longrightarrow 0 \ as \ \ N \longrightarrow \infty$$

$\implies$ $\qquad S_N(f) \longrightarrow f \ \ as \ \ N \longrightarrow \infty \quad in \ \ (C_{2\pi}[-\pi, \pi], <, >).$

(ii) Now from (i) we can write ,

$$S_N(f) \longrightarrow f \ as \ N \longrightarrow \infty \quad in \ \ (C_{2\pi}[-\pi, \pi], <, >)$$

$\Longrightarrow \qquad ||S_N(f)||_2 \longrightarrow ||f||_2 \quad as \quad N \longrightarrow \infty$

$\Longrightarrow \qquad ||S_N(f)||_2^2 \longrightarrow ||f||_2^2 \quad as \quad N \longrightarrow \infty$

$\Longrightarrow \qquad \sum_{n=-N}^{N} |\hat{f}(n)|^2 \longrightarrow ||f||^2 \quad as \quad N \longrightarrow \infty$

$\Longrightarrow \qquad ||f||^2 = \sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2.$

(iii) Let $f, g \in C_{2\pi}[-\pi, \pi]$ and

$$f(x) \sim \sum_{n=-\infty}^{\infty} \hat{f}(n)e^{inx} \ and \quad g(x) \sim \sum_{n=-\infty}^{\infty} \hat{g}(n)e^{inx}$$

Since $f \in C_{2\pi}[-\pi, \pi]$, so from (i) we have ,

$$S_N(f) \longrightarrow f \ as \ N \longrightarrow \infty \quad in \ \ (C_{2\pi}[-\pi, \pi], <, >)$$

Now,

$$\begin{aligned} ||S_N(f)\bar{g} - f\bar{g}||_2^2 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} |S_N(f)(x)\overline{g(x)} - f(x)\overline{g(x)}|^2 dx \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} |(S_N(f)(x) - f(x))\overline{g(x)}|^2 dx \\ &\leq ||S_N(f) - f||_2^2 \ \ ||g||_\infty^2 \longrightarrow 0 \ as \ N \longrightarrow \infty \end{aligned}$$

$\Longrightarrow \qquad S_N(f)\bar{g} \longrightarrow f\bar{g} \ as \ N \longrightarrow \infty \ in \ (C_{2\pi}[-\pi, \pi])$

Then,

$$\begin{aligned} &|\frac{1}{2\pi}\int_{-\pi}^{\pi} S_N(f)(x)\overline{g(x)} - f(x)\overline{g(x)}dx| \\ &\leq \frac{1}{2\pi}\int_{-\pi}^{\pi} |S_N(f)(x) - f(x)||g(x)|dx \\ &\leq \frac{1}{2\pi}(\int_{-\pi}^{\pi} |S_N(f)(x) - f(x)|^2)^{\frac{1}{2}}(\int_{-\pi}^{\pi} |g(x)|^2 dx)^{\frac{1}{2}} \\ &\longrightarrow 0 \quad as \quad N \longrightarrow \infty \end{aligned}$$

$$\Longrightarrow \qquad \frac{1}{2\pi}\int_{-\pi}^{\pi} S_N f(x)\overline{g(x)}dx \longrightarrow \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\overline{g(x}dx \qquad as \ N \longrightarrow \infty \quad (\star)$$

Now observe that,

$$\begin{aligned} &\frac{1}{2\pi}\int_{-\pi}^{\pi} S_N f(x)\overline{g(x)}dx \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \sum_{n=-N}^{N} \hat{f}(n) \ e^{inx} \ \overline{g(x)}dx \end{aligned}$$

$$= \sum_{n=-N}^{N} \hat{f}(n) \; \frac{1}{2\pi} \int_{-\pi}^{\pi} \; \overline{g(x) e^{-inx}} dx$$

$$= \sum_{n=-N}^{N} \hat{f}(n) \; \overline{\hat{g}(n)} \qquad (\star\star)$$

Now using $(\star)$ *and* $(\star\star)$ *we have ,*

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \overline{g(x} dx = \sum_{n=-N}^{N} \hat{f}(n) \; \overline{\hat{g}(n)} \quad as \quad N \longrightarrow \infty$$

$$\Longrightarrow \qquad < f, g >= \sum_{n=-\infty}^{\infty} \hat{f}(n) \; \overline{\hat{g}(n)}$$

Moreover,

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx = \sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2.$$

Chapter 4

Littlewood's Theorem

In this chapter I introduce the Hilbert space H^2 of analytic function, discuss its norm, and give the original proof that every composition operator takes H^2 boundedly into itself.

4.1 The Hardy Space H^2

Definition 4.1.1. A function

$$f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n \in H(U) \tag{1}$$

belongs to the Hardy Space H^2 if its sequence of the power series coefficients are square summable, i.e.

$$H^2 = \{f \in H(U) : \sum_{n=0}^{\infty} |\hat{f}(n)|^2 < \infty\}$$

where U=Open unit disc =$\{z \in \mathbb{C} : |z| < 1\}$
and $H(U)$= The space of the holomorphic function

$$= \{f : U \longrightarrow \mathbb{C} \; holomorphic\}.$$

Example 4.1.1. $f(z) = sinz, \quad f(z) = cosz, \quad f(z) = e^z$
these all are holomorphic in open disc and also square summable.

Note 4.1.2. Every square-summable sequence $\{\hat{f}(n)\} = \{\hat{f}(0), \hat{f}(1), \hat{f}(2), ...\}$ of the complex numbers is the coefficient sequence of an H^2-function; if $\{a_n\}_{n=0}^{\infty}$ is square summable, then it is bounded, so the corresponding power series $\sum_{n=0}^{\infty} a_n z^n$ converges on U to an analytic function that belongs to H^2.

By the uniqueness theorem for power series , the map that associate the function f with the sequence $\{\hat{f}(n)\}$ is therefore a vector isomorphism of H^2 onto l^2.
where l^2=The Hilbert space of square summable complex sequences.

Now we can turn the, H^2 into a Hilbert space by declaring the map to to be an isometry :

$$||f|| = (\sum_{n=0}^{\infty} |\hat{f}(n)|^2)^{\frac{1}{2}} \qquad (f \in H^2)$$

So some properties of H^2 follow readily from this definition. Now using this the next result shows, the H^2 function cannot grow too rapidly.

Growth Estimate: For $f \in \ H^2$

$$|f(z)| \leq \frac{||f||}{\sqrt{1-|z|^2}} \qquad (2)$$

for each $z \ \in \ U$.

Proof: The equation (1) gives the power series representation of f as

$$f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n$$

Now applying the Cauchy-Schwarz inequality to this power series representation of f, we obtain for each $z \in U$,

$$\begin{aligned}
|f(z)| &\leq \sum_{n=0}^{\infty} |\hat{f}(n)||z|^n \\
&\leq (\sum_{n=0}^{\infty} |\hat{f}(n)|^2)^{\frac{1}{2}} \ (\sum_{n=0}^{\infty} |z|^{2n})^{\frac{1}{2}} \\
&= ||f|| \ [1 + |z|^2 + |z|^4 + ...]^{\frac{1}{2}} \qquad [\because ||f|| = (\sum_{n=0}^{\infty} |\hat{f}(n)|^2)^{\frac{1}{2}}] \\
&= ||f||/(\frac{1}{1-|z|^2})^{\frac{1}{2}} \qquad [\because |z| \leq 1] \\
&= \frac{||f||}{\sqrt{1-|z|^2}}.
\end{aligned}$$

Corollary 4.1.3. Every norm convergent sequence in H^2 converges (to the same limit) uniformly on compact subset of U.

Proof: Suppose $\{f_n\}$ is a sequence in H^2 norm-convergent to a function $f \ \in H^2$
i.e. $||f_n - f|| \longrightarrow 0$.
Now for $0 < R < 1$, from the Growth estimate
for each fixed n,

$$\begin{aligned}
\sup_{|z| \leq R} |f_n(z) - f(z)| &\leq \sup_{|z| \leq R} \frac{||f_n - f||}{\sqrt{1-|z|^2}} \\
&\leq \frac{||f_n - f||}{\sqrt{1-R^2}} \qquad [\because \ 0 < |z| \leq R < 1]
\end{aligned}$$

So $f_n \ \longrightarrow f$ uniformly on the closed disc $|z| \leq R$.

Since R is arbitrary, $f_n \longrightarrow f$ uniformly on every compact subset of U.

Note 4.1.4. It is also easy to see from the definition that H^2 contains some of unbounded functions. For example,

$$\log \frac{1}{1-z} = \sum_{n=1}^{\infty} \frac{z^n}{n} \in H^2.$$

It is unbounded because,
$as\ z \longrightarrow 1^-, \quad \log 1 - z \longrightarrow -\infty$
$\Longrightarrow \quad asz \longrightarrow 1^-, \qquad \log \frac{1}{1-z} \longrightarrow \infty$

However the definition of H^2 in terms of the coefficients more often obscures than reveals. Here two important facts are arise :

- $H^\infty \subset H^2$. More generally, if $b \in H^\infty$ and $f \in H^2$ then the pointwise product $bf \in H^2$. (where the symbol H^∞ denotes the space of bounded analytic functions on U)

- If φ is a holomorphic self-map of U, then $f \circ \varphi \in H^2$. (This statement is known as Littlewood's Theorem.)

Both statements say something about linear operators. The first one asserts that the operator of "multiplication by b",

$$M_b f = bf \qquad (f \in H^2)$$

takes H^2 into itself, while the result we are calling Littlewood's Theorem says the same thing about the *composition operator* C_φ:

$$C_\varphi f = f \circ \varphi \qquad (f \in H^2).$$

4.2 H^2 via Integrals Means

Suppose $f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n$ is a function of holomorphic on U.

Writing $z = re^{i\theta}$, and using the orthogonality of the functions $\{e^{in\theta}\}_0^\infty$ in L^2, for $0 \leq r < 1$ we have,

$$f(re^{i\theta}) = f_r(\theta) = \sum_{n=0}^{\infty} \hat{f}(n) r^n e^{in\theta}$$

$$M_2^2(f,r) =^{def} \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(re^{i\theta})|^2 d\theta$$

$$= \sum_{n=0}^{\infty} |\hat{f}(n)|^2 r^{2n} \qquad [By\ Parseval's\ theorem]$$

$$\leq \sum_{n=0}^{\infty} |\hat{f}(n)|^2 \qquad [\because 0 \leq r < 1]$$

If $f \in H^2$ then

$$M_2^2(f,r) \leq \sum_{n=0}^{\infty} |\hat{f}(n)|^2 = ||f||^2$$

$$\Longrightarrow \qquad M_2(f,r) \leq ||f||$$

So $M_2(f,r)$ is bounded by H^2 norm of f.

Conversely, if

$$\lim_{r \longrightarrow 1^-} M_2(f,r) = M < \infty$$

then for each non-negative integer N, the N-th partial sum of the power series of f

$$\sum_{n=0}^{N} |\hat{f}(n)|^2 r^{2n} \leq \sum_{n=0}^{\infty} |\hat{f}(n)|^2 r^{2n} \leq M^2$$

So sending r to 1 we see that each partial sum of the series for $||f||^2$ is bounded by M^2.
Hence this ia true for the whole series .
Thus $f \in H^2$ and $||f|| \leq M$
This complete the derivation of the alternative expression for the H^2 norm. i.e. $||f|| = \infty$ whenever $f \notin H^2$.

Proposition: Suppose f is a holomorphic on U. Then as $r \longrightarrow 1^-$ the mean $M_2(f,r)$ increases to $||f||$:

$$||f||^2 = \lim_{r \longrightarrow 1^-} \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(re^{i\theta}))|^2 d\theta, \tag{4}$$

thus $f \in H^2$ if and only if $M_2(f,r)$ is bounded for $0 \leq r < 1$.

To test the utility of these results, let us return to the two important facts left hanging at the end of the last section.
For each $b \in H^\infty$ let

$$||b||_\infty = \sup_{z \in U} |b(z)|.$$

Since the integral of the large function is larger , So we can write $M_2(b,r) \leq ||b||_\infty, \ \ \forall\ 0 < r < 1$.
Since $H^\infty \subset H^2$
$\therefore\ b \in H^2 \ \ with \ \ ||b|| \leq ||b||_\infty$
Now

$$\begin{aligned}
||bf||^2 &= \lim_{r \longrightarrow 1^-} \frac{1}{2\pi} \int_{-\pi}^{\pi} |(bf)(re^{i\theta})|^2 d\theta \\
&= \lim_{r \longrightarrow 1^-} \frac{1}{2\pi} \int_{-\pi}^{\pi} |b(re^{i\theta})|^2 f(re^{i\theta})|^2 d\theta \\
&\leq \lim_{r \longrightarrow 1^-} (\sup_{b \in U} |b(z)|)^2 \ \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(re^{i\theta}))|^2 d\theta
\end{aligned}$$

$\Longrightarrow$ $$||bf||^2 \leq \lim_{r \longrightarrow 1^-} ||b||_\infty^2 \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(re^{i\theta}))|^2 d\theta = ||b||_\infty^2 ||f||^2$$

$\Longrightarrow$ $$||bf|| \leq ||b||_\infty ||f|| \qquad (b \in H^\infty,\ f \in H^2) \qquad (5)$$

$\Longrightarrow$ for each $f \in H^2$, the pointwise product $bf \in H^2$.

4.3 Littlewood's Subordination Principle

Fortunately, none of this discouraged Littlewood from using the tools at hand to construct the beautiful proof we are going to present. Everything revolves around the special case $\varphi(0) = 0$, after which the result follows by means of the manageable conformal change of variable. The case $\varphi(0) = 0$, which is only one Littlewood actually considered, furnishes two surprises. First, the proof requires only the fact that, since φ maps U into itself, so from (5) the multiplication operator M_φ acts contractively on H^2,

$$||M_\varphi f|| \leq ||f|| \qquad (f \in H^2) \qquad (6)$$

Second, the contractive property of M_φ get passed on to C_φ.

Recall: A linear operator T on a Hilbert space H is said to be bounded if it takes the unit ball B into a bounded set., and that the norm of such an operator is defined to be

$$||T|| = sup\{||Tf|| : f \in B\}.$$

So the equation (5) gives , if $b \in H^\infty$ then the operator of multiplication by b is bounded on H^2, and has norm $\leq ||b||_\infty$.

Note 4.3.1. The bounded linear operators on a Hilbert space H are precisely the continuous ones. If $||Tf|| \leq ||f||$ for each $f \in H$(i.e. $||T|| \leq 1$) then T is called a contraction on that Hilbert space H. Thus the multiplication operator M_φ is a contraction on H^2.

Littlewood's Subordination Principle(1925):
Statement:suppose φ is a holomorphic self-map of U, with $\varphi(0) = 0$. Then for each $f \in H^2$,

$$C_\varphi f \in H^2 \ \ and \ \ ||C_\varphi f|| \leq ||f||$$

Proof:The backward shift operator B , defined on H^2 by

$$Bf(z) = \sum_{n=0}^{\infty} \hat{f}(n+1) z^n \qquad \left(f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n \in H^2\right)$$

From the define of $Bf(z)$ for any $f \in H(U)$ we can write following two identities:

$$f(z) = f(0) + zBf(z) \qquad (z \in U) \tag{7}$$

$$B^n f(0) = \hat{f}(n) \qquad (n = 0, 1, 2, ...) \tag{8}$$

To begin the proof assuming f is a holomorphic polynomial. Then,

$$\begin{aligned} |f \circ \varphi(z)| &= |f(\varphi(z))| \\ &= |a_n \varphi(z)^n + a_{n-1}\varphi(z)^{n-1} + ... + a_1\varphi(z) + a_0| \\ &\leq |a_n||\varphi(z)|^n + |a_{n-1}||\varphi(z)|^{n-1} + ... + |a_1||\varphi(z)| + |a_0| \\ &< |a_n| + |a_{n-1}| + ... + |a_0| \qquad (\because |\varphi(z)| < 1) \end{aligned}$$

$\implies \quad f \circ \varphi$ is bounded on U.
So from the previous theorem we can say that

$$f \circ \varphi \in H^2 \qquad i.e. \qquad C_\varphi f \in H^2.$$

Now it is remains to proof that , $\quad ||C_\varphi|| \leq ||f||$
We begin this norm estimate by substituting $\varphi(z)$ for z in (7), to obtain

$$f(\varphi(z)) = f(0) + \varphi(z)(Bf)(\varphi(z))$$

Rewriting in the language of composition and multiplication operators as

$$C_\varphi f = f(0) + M_\varphi C_\varphi Bf \tag{9}$$

Now

$$\begin{aligned} |||C_\varphi f||^2 &= < C_\varphi f, C_\varphi f >=< f \circ \varphi, f \circ \varphi > \\ &=< f(0) + M_\varphi C_\varphi Bf, f(0) + M_\varphi C_\varphi Bf > \\ &= |f(0)|^2 + |M_\varphi C_\varphi Bf|^2 + 2Re < M_\varphi C_\varphi Bf, f(0) > \end{aligned}$$

Since $\varphi(0) = 0$ then we can say $< M_\varphi C_\varphi Bf, f(0) >= 0$. So,

$$\begin{aligned} ||C_\varphi f||^2 &= |f(0)|^2 + ||M_\varphi C_\varphi Bf||^2 \\ &\leq |f(0)|^2 + ||C_\varphi Bf||^2 \end{aligned} \tag{10}$$

where the last inequality follows from (6), the contractivity property of M_φ.

Now successively substitute $Bf, B^2f, B^3f, ...$ for f in (10) we get,

$$
\begin{array}{ccc}
||C_\varphi Bf||^2 & \leq & |Bf(0)|^2 + ||C_\varphi B^2 f||^2 \\
||C_\varphi B^2 f||^2 & \leq & |B^2 f(0)|^2 + ||C_\varphi B^3 f||^2 \\
: & : & : \\
: & : & : \\
||C_\varphi B^n f||^2 & \leq & |B^n f(0)|^2 + ||C_\varphi B^{n+1} f||^2.
\end{array}
$$

Putting all these inequalities together , we get

$$||C_\varphi f||^2 \leq \sum_{k=0}^{n} |B^k f(0)|^2 + ||C_\varphi B^{n+1} f||^2$$

for each non-negative integer n. Since f is a polynomial and choosing n be the degree of f, then $B^{n+1}f = 0$. and this reduces the last inequality to

$$
\begin{aligned}
||C_\varphi f||^2 &\leq \sum_{k=0}^{n} |B^k f(0)|^2 \\
&= \sum_{k=0}^{n} |\hat{f}(k)|^2 \qquad (using \;\; 8) \\
&= ||f||^2
\end{aligned}
$$

which shows that C_φ is an H^2-norm contraction , at least on the vector space of holomorphic polynomials.

For finishing the proof, suppose $f \in H^2$ is not a polynomial. Let f_n be the n-th partial sum of its Taylor series $f(z) = \sum_{n=o}^{\infty} \hat{f}(n) z^n$.
Then $f_n \longrightarrow f$ in the norm of H^2 i.e. $||f_n||_2. \longrightarrow ||f||_2$.
Since, every norm convergent sequence in H^2 converges uniformly to the same limit on compact subset of U.
So, $f_n \longrightarrow f$ in the compact subset κ of U.
Hence , $f_n \circ \varphi \longrightarrow f \circ \varphi$ in the compact subset κ.
Clearly $||f_n|| \leq ||f||$.
And we have just shown that, $||f_n \circ \varphi|| \leq ||f_n||$. ($f_n$ is the n th partial sum so it is n-th degree polynomial.)

Thus recalling the abbreviation $M_2(f, r)$ for the L^2 norm of f over the circle of radius r and for fixed $0 < r < 1$ we have ,

$$M_2(f \circ \varphi, r) = \lim_{n \longrightarrow \infty} M_2(f_n \circ \varphi, r) \qquad (\because f_n \circ \varphi \longrightarrow f \circ \varphi \; in \; \kappa)$$

$$\leq \lim_{n \longrightarrow \infty} \sup ||f_n \circ \varphi||$$

$$\leq \lim_{n \longrightarrow \infty} \sup ||f_n|| \qquad (\because ||f_n \circ f|| \leq ||f_n||)$$

$$\leq ||f|| \qquad (\because ||f_n|| \leq ||f||).$$

Now,

$$||f \circ \varphi||^2 = \lim_{r \longrightarrow 1^-} M_2^2(f \circ \varphi, r) \leq ||f||^2$$

$$\implies \qquad ||f \circ \varphi|| \leq ||f||$$

$$i.e. \qquad ||C_\varphi f|| \leq ||f|| \qquad (for\ each\ f \in H^2)$$

Hence complete the proof.

4.3.1 Proof of Littlewood's Theorem

Littlewood's Theorem: Suppose φ is a holomorphic self-map of U. Then C_φ is a bounded operator on H^2, and

$$||C_\varphi|| \leq \sqrt{\frac{1 + |\varphi(0)|}{1 - |\varphi(0)|}}$$

Automorphism-induced composition operators: To prove the C_φ is bounded even when φ does not fix the origin, we utilize the conformal automorphisms to move points of U from where they are to where we want them. For each $p \in U$, the special automorphism $\alpha_p : U \longrightarrow U$ s.t.

$$\alpha_p(z) = \frac{p - z}{1 - \bar{p}z}, \tag{11}$$

. This is the special automorphism of U which interchanges p with origin, and is its own inverse. Writing $p = \varphi(0)$. Then the holomorphic function $\psi = \alpha_p \circ \varphi$ takes U into itself and fixes the origin i.e. $\psi(0) = 0$. By the self-inverse property of α_p we have

$$\varphi = \alpha_p \circ \psi,$$

and this translate the operator equation $C_\varphi = C_\psi C_{\alpha_p}$.

We have just seen that, C_ψ is bounded (in fact a contraction, from the Littlewood's Subordination principle), and we know that the product of two bounded operator is always bounded. So to proof C_φ is a bounded operator, we have to show C_{α_p}. Thus, the boundedness of C_ψ on H^2 will follow from the following lemma.

Lemma 4.3.2. *For each $p \in U$ the operator C_{α_p} is bounded on H^2. Moreover*

$$||C_{\alpha_p}|| \leq \sqrt{\frac{1+|p|}{1-|p|}}.$$

Proof: *Suppose that f is a holomorphic in a neighbourhood of the closed unit disc, say in RU for some $R > 1$.Then the limit in formula (4) can be passed inside the integral sign, with the result that,*

$$||f||^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(e^{i\theta})|^2 d\theta$$

Then

$$\begin{aligned}
||f \circ \alpha_p||^2 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(\alpha_p(e^{i\theta}))|^2 d\theta \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(e^{it})|^2 |\alpha_p{}'(e^{it})| dt \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(e^{it})|^2 | \frac{1-|p|^2}{|1-\bar{p}e^{it}|^2} dt \qquad [\because \quad \alpha_p(e^{it}) = \frac{p-e^{it}}{1-\bar{p}e^{it}}] \\
&\leq \frac{1-|p|^2|}{(1-|p|)^2} \quad (\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(e^{it})|^2 |dt) \\
&= \frac{1+|p|}{1-|p|} ||f||^2
\end{aligned}$$

$\Longrightarrow$ *the desired inequality holds for all functions holomorphic in RU; in particular it holds for polynomial. It remains only to transfer the results to the rest of H^2. And to show this we have to repeat the argument which used to finish the proof of Littlewood's Subordination principle.*
(Hints: we have take the N-th partial sum of the Taylor series of f and that N-th partial sum will treat as polynomial.)

Note 4.3.3. At this point we have assembled everything we need to show that composition operators act boundedly on H^2.

Littlewood's Theorem: Suppose φ is a holomorphic self-map of U. Then C_φ is a bounded operator on H^2, and

$$||C_\varphi|| \leq \sqrt{\frac{1+|\varphi(0)|}{1-|\varphi(0)|}}.$$

Proof: From Automorphism-induced composition operators, we have $C_\varphi = C_\psi C_{\alpha_p}$, where $\varphi(0) = p$ and $\psi(0) = 0$.
Now from the last lemma and Littlewood's Subordination Principle show that, both operators of the right-hand side are bounded on H^2.
Hence, C_φ is the product of bounded operator on H^2, and therefore C_φ is itself bounded.

$\because ||C_{\alpha_p}|| \leq \sqrt{\frac{1+|p|}{1-|p|}}$ and C_ψ is contraction. So we can write,

$$||C_\varphi|| \leq ||C_\psi|| ||C_{\alpha_p}|| \leq \sqrt{\frac{1+|\varphi(0)|}{1-|\varphi(0)|}}$$

Chapter 5

Compactness: Introduction

Having established that every composition operator is bounded on H^2, now we turn to the most natural follow-up question:

Which composition operators are compact ?

The property "boundedness" for composition operators means that each one takes bounded subset of H^2 to bounded subsets.
The question above asks us to specify precisely how much the inducing map ϕ has to compress the unit disc into itself in order to ensure that the operator C_ϕ compresses bounded subsets of H^2 into relatively compact ones.

5.1 Compact Operators

Definition 5.1.1. Relatively compact: A subset of a topological space is said to be relatively compact if its closure in the space is compact.

A linear operator T on a Hilbert space H is said to be compact if it maps every bounded set into relatively compact one. It is not necessary to check every bounded set here; since translation and multiplication by a scalar are homeomorphism of H, it is enough to test only on the unit ball.

By Heine-Borel theorem, every linear transformation on a finite dimensional Hilbert space is compact. Similarly, on an infinite dimensional Hilbert space, every bounded operator with finite dimensional range is compact.

My first observation is that th compact operators are precisely those that can be approximated by such finite rank operators.

Actually compact operators on Hilbert space is an extension of the concept of a matrix acting on a finite-dimensional Vector Space; i.e. Compact operator are precisely the closure of finite rank operators in the topology induced by the operator norm.

Finite Rank Approximation Theorem: Suppose T is a bounded linear operator on a Hilbert space H. Then T is compact iff there is a sequence F_n of finite rank bounded operators such that $||T - F_n|| \longrightarrow 0$.

Proof: Suppose first that , T is compact on H. Let e_n ba an orthonormal basis for H and consider the projection operators

$$P_n f = \sum_{k=0}^{n} \langle f, e_k \rangle e_k \qquad (f \in H)$$

where " $<,>$ " denotes inner product in H.
So clearly, $||P_n f|| \leq ||f||$ for each n
$\implies$ P_n is a contraction on H and $||P_n f - f|| \longrightarrow 0$ for each $f \in H$.
Let B denote the unit ball of H (open or closed).
We are assuming that $T(B)$ is relatively compact in H. sine a absolutely fundamental fact about sequences of transformations on a metric space:

Pointwise Convergence + Equicontinuity $\implies$ Uniform convergence on compact subsets.

We just observed that, $P_n \longrightarrow I$ (the identity map) pointwise on H.
Since, the operators P_n are all contractions, the whole family is equicontinuous on bounded ses, so applying the equicontinuity principle state above to the closure of $T(B)$, which are assuming is compact, so we can say that $P_n \longrightarrow I$ uniformly on $T(B)$.
i.e. $P_n T \longrightarrow T$ on B, which means that $||P_n T - T|| \longrightarrow 0$. Since $P_n T$ is bounded finite rank operator for each n, this establishes the desired approximation.

Conversely, suppose some sequence F_n of bounded finite rank operators converges in operator norm to T. We need to show that $T(B)$ is relatively compact in H.
Another result metric space theory makes short work of this proof:
A subset K of a metric space X is relatively compact iff for every $\epsilon \geq 0$ there is a finite set of points $N_\epsilon \subset X$ such that each point of K lies at most ϵ distance for N_ϵ.

The set N_ϵ is often called "totally bounded." So a set is totally bounded iff it is relatively compact. If K is relatively compact, we get N_ϵ by covering the closure of K by open ϵ- balls, extracting a finite subcovering, and choosing as N_ϵ the centers.
In other way , if we have N_ϵ, then for any open covering of the closure of K, a finite covering subordinate to the original one, from which follows the compactness of that closure.
Returning to Hilbert space , let $\epsilon \geq 0$ be given, and fix a value of n so that $||F_n - T|| < \epsilon/2$. Let N be an $\epsilon/2$-net for $F_n(B)$. Then it is easy to check that N be an ϵ-net for $T(B)$.
Now from the above characterization it follows that, $T(B)$ is relatively compact.

5.2 First Class of Example

The most drastic way ϕ can compress the unit disc is to take it to a point, in which case the resulting composition operator has one dimensional range(the space of constant functions), and is therefore compact. The next result shows that this compactness persists if we merely assume that $\phi(U)$ is relatively compact in U .

First Compactness Theorem:If $||\phi||_\infty < 1$ then C_ϕ is a compact operator on H^2.

Proof:For each positive integer n define the operator

$$T_n f = \sum_{k=0}^{n} \hat{f}(k)\phi^k \qquad (f \in H^2).$$

Thus T_n maps H^2 onto the linear span of the first n powers of ϕ.
By our comparison of H^2 and H^∞ norms, T_n is therefore a bounded, finite rank operator on H^2 and we can obtain $||T_n|| \leq \sqrt{n+1}$.
Our claim is $||C_\phi - T_n|| \longrightarrow 0$.

Now,
$$\begin{aligned}
||(C_\phi - T_n)f|| &= || \textstyle\sum_{k=n+1}^{\infty} \hat{f}(k)\phi^k||_2 \\
&\leq \sum_{k=n+1}^{\infty} |\hat{f}(k)| \; ||\phi^k||_2 \\
&\leq \sum_{k=n+1}^{\infty} |\hat{f}(k)| \; ||\phi||_\infty^k \\
&\leq (\sum_{k=n+1}^{\infty} |\hat{f}(k)|^2)^{1/2} (\sum_{k=n+1}^{\infty} ||\phi||_\infty^{2k})^{1/2} \\
&\leq \frac{||\phi||_\infty^{n+1}}{\sqrt{1 - ||\phi||_\infty^2}} ||f||.
\end{aligned}$$

The given condition is $||\phi||_\infty \leq 1$. Thus

$$||C_\phi - T_n|| \leq \frac{||\phi||_\infty^{n+1}}{\sqrt{1 - ||\phi||_\infty^2}} \longrightarrow 0 \qquad as \quad n \longrightarrow \infty.$$

This exhibits that C_ϕ is an operator norm limit of finite rank operators, so it is compact on H^2.

This results shows that H^2 supports a lot of compact composition operators. In order to state the improved result without distracting complications, we need a boundary version of the integral representation of the H^2.

5.3 A Better Compactness Theorem

The H^2 norm revisited: Since the polynomials are dense in H^2 it seems reasonable that some form of this boundary representation of the norm should carry over to all of H^2. If $f \in H^2$, then because the coefficients are square summable , the Fourier series $\sum_{n=0}^{\infty} \hat{f}(n)e^{in\theta}$ converges in L^2 to some $f^* \in L^2$, so clearly the equation

$$||f||^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(e^{i\theta})|^2 d\theta \qquad (1)$$

holds with $f(e^{i\theta})$ replaced on the right by $f^*(e^{i\theta})$. What makes the formula really useful in the study of H^2 is something much deeper.

The Radial Limit Theorem: Suppose $f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n$ is a function in H^2, and f^* is the function in L^2 with the Fourier Series $\sum_{n=0}^{\infty} \hat{f}(n) e^{in\theta}$. Then

$$\lim_{r \longrightarrow 1^-} f(re^{i\theta}) = f^*(e^{i\theta})$$

for almost every $e^{i\theta} \in \partial U$, and the H^2 norm of f is theL^2 norm of f^*.
The deep part of the theorem is the existence and identification of the radial limit function f^*. From now on we will use this boundary form of the H^2 norm whenever it is convenient, always writing $f(e^{i\theta})$ instead of $f^*(e^{i\theta})$ for the radial limit.

The better compactness theorem: In the proof of "First Compactness Theorem" we used the fact that the supremum norm dominates the H^2 norm. The calculation would have looked like as follows:

$$\begin{aligned} ||(C_\phi - T_n)f|| &\leq \sum_{k=n+1}^{\infty} |\hat{f}(k)| ||\phi^k|| \\ &\leq (\sum_{k=n+1}^{\infty} |\hat{f}(k)|^2)^{1/2} (\sum_{k=n+1}^{\infty} ||\phi^k||^2)^{1/2} \\ &\leq (\sum_{k=n+1}^{\infty} ||\phi^k||^2)^{1/2} ||f|| \\ \Longrightarrow \quad ||C_\phi - T_n|| &\leq (\sum_{k=n+1}^{\infty} ||\phi^k||^2)^{1/2}. \end{aligned}$$

and as before, this implies the compactness of C_ϕ provided that

$$\sum_{n=0}^{\infty} ||\phi^n||^2 < \infty. \qquad (2)$$

Condition (2) can in turn be rewritten as follows, where we use the boundary form of the H^2 norm discussed in the last section.

$$\begin{aligned} \infty &> \sum_{n=0}^{\infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} |\phi(e^{i\theta})|^{2n} d\theta \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{n=0}^{\infty} |\phi(e^{i\theta})|^{2n} d\theta \quad [by \;\; Fubini] \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{1 - |\phi(e^{i\theta})|^2} d\theta \end{aligned}$$

where the interchange of integration and summation is justified by positivity, and the summation of the geometric series is already justified by its convergence.

The Hilbert-Schmidt Theorem for composition operators:
If

$$\int_{-\pi}^{\pi} \frac{1}{1-|\phi(e^{i\theta})|^2} d\theta < \infty \tag{3}$$

then C_ϕ is compact on H^2.

Remark:The heart of our proof showed the integral condition (3) to be equivalent to (2), which can be rewritten $\sum ||C_\phi(z^n)||^2 < \infty$.

A operator T on a Hilbert space H is called a Hilbert-Schmidt operator if, for some orthonormal basis $\{e_n\}$ of H ,if

$$\sum_{n=0}^{\infty} ||Te_n||^2 < \infty.$$

The argument that deduced the compactness of C_ϕ from (2) works in general and shows:
Every Hilbert-Schmidt operator is compact.
The title of the Theorem above comes from the fact that its proof shows C_ϕ to be a Hilbert-Schmidt operator whenever ϕ satisfies (3).The Hilbert-Schmidt condition (2) does not depend on the particular choice of orthonormal basis, shows that our proof actually characterizes the Hilbert-Schmidt composition operators as the ones for which ϕ satisfies condition (3).
In the last section we showed that C_ϕ is compact whenever $||\phi||_\infty < 1$. Our Hilbert Schmidt Theorem allows for a significant improvement.

The Polygonal Compactness Theorem: If ϕ maps the unit disc into a polygon inscribed in the unit circle, then C_ϕ is compact on H^2.

The proof will show that, C_ϕ is actually a Hilbert Schmidt operator. The major step involves proving the result for an important class of example.
Lens Maps: For $0 < \alpha < 1$ define ϕ_α to be holomorphic self-map of U i.e. $\phi_\alpha : U \longrightarrow U$ that we will get by using the Mobius transformation

$$\sigma(z) = \frac{1+z}{1-z} \tag{4}$$

to take U onto the right half-plane $\Pi = \{z \in \mathbb{C} : Re\ z > 0\}$, then employing the α-th power to squeeze the half plane onto the sector $\{|argw| < \alpha\pi/2\}$, and completing the task by mapping back to U via σ^{-1}. The result is:

$$\phi_\alpha(z) = \frac{\sigma(z)^\alpha - 1}{\sigma(z)^\alpha + 1} \tag{5}$$

Because ϕ_α takes the unit disc onto the lens-shaped region L_α, we call it "lens map". Our first asserts that each lens map induces a Hilbert-Schmidt operator on H^2.

Lemma: Each lens map satisfies the Hilbert-Schmidt condition (3).
Proof: For convenience we write ϕ instead of ϕ_α. Since ϕ fixes the points ± 1 and sends every other point of ∂U into U. it is enough to examine that the integrability 0f $(1 - |\phi(e^{i\theta})|^2)^{-1}$ over small arcs centered at ± 1, and by symmetry it is enough to consider just one of these points, say $+1$.

To study the behavior of ϕ near this point, observe that

$$1 - \phi(z) = \frac{2}{\sigma(z)^{\alpha} + 1}$$

$$\Longrightarrow \qquad \sigma(e^{i\theta}) = icot(\theta/2)$$

so for $|\theta| < \pi/2$,

$$|\sigma(e^{i\theta})| = |cot(\theta/2)| \leq \frac{2}{|\theta|},$$

whereupon

$$|1 - \phi(e^{i\theta})| \geq \frac{2}{|\sigma(e^{i\theta})|^{\alpha} + 1} \geq constant\ |\theta|^{\alpha}.$$

Since $0 < \alpha < 1$ this estimate shows that the function $[1 - \phi(e^{i\theta})]^{-1}$ is integrable over the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

Now each point $\phi(e^{i\theta})$ lies between the real axis and a line through the point $+1$ that makes an angle $\alpha\pi/2$ with that axis. Now using the law of cosines we can write

$$1 - |\phi(e^{i\theta})|^2 \geq \ constant.|1 - \phi(e^{i\theta})|$$

for all θ near 0 i.e. the distance from $\phi(e^{i\theta})$ to the unit circle is about the same as its distance to the point $+1$. Thus $(1 - |\phi(e^{i\theta})|^2)^{-1}$ is integrable in an interval centered about $\theta = 0$, and this complete the proof of Hilbert Schmidt Theorem.

Proof of the Polygonal Compactness Theorem:
From the last proof of Hilbert-Schmidt theorem for composition operator showed that, for any holomorphic self-map ϕ of U,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1}{1 - |\phi(e^{i\theta})|^2} d\theta = \sum_{n=0}^{\infty} ||C_{\phi}(z^n)||^2 \tag{6}$$

where we allow the possibility that one side of the equation (and therefore both sides) might be infinite.

To begin the proof, suppose that, ϕ maps the unit disc into one side of the lenses L_{α} i.e. $\phi : U \longrightarrow L_{\alpha}$ is defined. Then $\psi = {\phi_{\alpha}}^{-1} \circ \phi$ is a holomorphic self-map of U, and $\phi = \phi_{\alpha} \circ \psi$. Thus $C_{\phi} = C_{\psi} C_{\phi_{\alpha}}$, so

$$||C_{\phi}(z^n)|| \leq ||C_{\psi}||\ ||C_{\phi_{\alpha}}(z^n)|| \qquad \forall\, n \in \mathbb{Z}^{+}.$$

Our lemma about lens maps along with (6) above shows that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1}{1 - |\phi(e^{i\theta})|^2} d\theta \leq ||C_{\psi}||^2 \sum_{n=0}^{\infty} ||C_{\phi_{\alpha}}(z^n)||^2 < \infty \tag{7}$$

This shows that anything that maps the unit disc into a lens induced a Hilbert-Schmidt operator.

Now for the general case, the factorization argument above shows that it is enough to consider maps ϕ that take the unit disc conformally onto polygons inscribed in the unit circle. Each such ϕ extends to a homeomorphism from the closed disc onto the closure of the polygon.

Consider a vertex of the polygon, which, without loss of generality, we may assume to be the point $+1$. We may also assume this point is fixed by ϕ. Thus the map $\chi = (1+\phi)/2$ fixes $+1$ and takes the disc into a lens L_α for some a sufficiently close to 1, so by the work of the last paragraph the function $(1-|\chi(e^{i\theta})|^2)^{-1}$ is integrable over the unit circle. Now as $\theta \longrightarrow 0$, both $\phi(e^{i\theta})$ and $\chi(e^{i\theta})$ approach $+1$, while staying inside L_α (i.e. they approach $+1$ "non-tangentially"), so we have for all sufficiently small θ,

$$1-|\chi(e^{i\theta})|^2 \approx |1-\chi(e^{i\theta})| = |\frac{1-\phi(e^{i\theta})}{2}| \approx \frac{1-|\phi(e^{i\theta})|^2}{2}$$

Thus, the reciprocal of the function on the right is integrable over an interval symmetric about $\theta = 0$.

The function $(1-|\phi(e^{i\theta})|^2)^{-1}$ is therefore integrable over an interval centered about the pre-image of each vertex of the polygon, so it is therefore integrable over the whole unit circle. Our Hilbert-Schmidt Theorem now shows that C_ϕ is compact on H^2.

5.4 Compactness and Weak Convergence

When studying compactness in metric spaces it often helps to express everything in terms of sequential convergence. The same holds for the study of compact operators. The definition of compactness for Hilbert space operators can be rephrased to read something like "compact operators are the ones that take weakly convergent sequences into norm convergent ones."

Weak Convergence Theorem: For ϕ a holomorphic self-map of U, the following statements are equivalent:
(a) C_ϕ is a compact operator on H^2.
(b) If $\{f_n\}$ is a sequence that is bounded in H^2 and converges to zero uniformly on compact subsets of U, then $||C_\phi f_n|| \longrightarrow 0$.
Proof: The key to this proof is the fundamental growth condition (2), which asserts that H^2 convergence implies pointwise convergence on U, and that bounded subsets of H^2 are, as classes of functions, uniformly bounded on compact subsets of U. Let B denote the closed unit ball in H^2.
$(a) \implies (b)$: Assuming that C_ϕ is a compact operator, i.e., that $C_\phi(B)$ is a relatively compact subset of H^2.
We are giving a sequence $\{f_n\}$ that lies in MB (the ball of radius M), and converges to zero uniformly on compact subsets of U i.e. $f_n \longrightarrow 0 \; in \; k$.
Claim is $||C_\phi f_n|| \longrightarrow 0$
i.e. it suffices to show that the zero-function is the unique limit point of the sequence $\{C_\phi f_n\}$ for the norm topology.

since $f_n \longrightarrow 0$ in k which implies $f_n \circ \phi \longrightarrow 0$ in k

$$\implies \qquad C_\phi f_n \longrightarrow 0.$$

Since H^2 convergence $\implies$ pointwise convergence in U.
So zero is the only possible limit point.
Since C_ϕ is compact and $\{f_n\}$ lies in MB.
Now by the compactness of C_ϕ the set $\{C_\phi f_n\}$ is relatively compact, so there must be a limit point.
Hence the theorem is proved.
$(b) \implies (a)$: Suppose $\{f_n\}$ is a sequence of functions in B. We have to show that the image sequence $\{C_\phi f_n\}$ has a convergent subsequence.
Since the functions in B are bounded uniformly on compact subsets of U.
By Montel's Theorem we can say $\exists$ a subsequence $\{g_k = f_{n_k}\}$ that converges uniformly on compact subsets of U to a holomorphic function g.
<u>Claim</u> $g \in H^2$.
Indeed, for each $0 < r < 1$,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |g(re^i\theta)|^2 d\theta = \lim_{k \longrightarrow \infty} \frac{1}{2\pi}\int_{-\pi}^{\pi} |g_k(re^{i\theta)}|^2 d\theta \leq \sup_k ||g_k||^2 \leq 1$$

$$\implies ||g|| \leq 1 \qquad \implies g \in H^2.$$

Then the sequence $\{g_k - g\}$ is bounded on H^2 and $g_k - g \longrightarrow 0$ in k .
By the hypothesis (b) we can say that,
$||C_\phi(g_k - g)|| \longrightarrow 0$
$\implies$ The image sequence $\{C_\phi f_n\}$ has a convergent subsequence.
i.e. From finite rank approximation theorem we can say that, C_ϕ is compact operator.

5.5 Non-Compact Composition Operators

We use the "sequential" characterization of compactness to show that C_ϕ can fail to be compact if $\phi(e^{i\theta})$ approaches the boundary of U either too quickly or too often. Our first example shows that C_ϕ can fail to be compact even if $|\phi(e^{i\theta})| = 1$ at a single point $e^{i\theta} \in \partial U$.

Example: (The values of ϕ approaches the boundary too quickly).
For $0 < \lambda < 1$, Let, $\phi(z) = \lambda z + (1 - \lambda)$. Then C_ϕ is not compact on H^2.
Proof: For each fixed $0 < r < 1$, define

$$f_r(z) = \frac{\sqrt{1 - r^2}}{1 - rz}, \qquad z \in U$$

Since,

$$||f|| = \left(\sum_{n=0}^{\infty} |\hat{f}(n)|^2\right)^{1/2}$$

So, $||f_r|| = 1$ i.e. each of these functions has norm 1.
And as $r \longrightarrow 1^-$, $f_r \longrightarrow 0$ uniformly on the compact subset of U.
Now

$$C_\phi f_r = f_r \circ \phi = f_r(\lambda z + 1 - \lambda) = \frac{\sqrt{1-r^2}}{1 - r + r\lambda - r\lambda z}$$

$$= \frac{\sqrt{1-r^2}}{1-r+r\lambda}\left[1 + \frac{r\lambda}{1-r+r\lambda} z + \left(\frac{r\lambda}{1-r+r\lambda}\right)^2 z^2 + ...\right]$$

Now,

$$||C_\phi f_r||^2 = ||f_r \circ \phi||^2 = \frac{1-r^2}{(1-r+r\lambda)^2} \frac{1}{1 - \frac{r^2\lambda^2}{(1-r+r\lambda)^2}}$$

$$= \frac{1-r^2}{(1-r+2r\lambda)(1-r)} = \frac{1+r}{1+r(2\lambda-1)} \qquad (o < r < 1)$$

So as, $r \longrightarrow 1-$

$$||f_r \circ \phi||^2 \longrightarrow \frac{2}{2\lambda} = \frac{1}{\lambda}$$

$$\Longrightarrow ||C_\phi f_r|| \longrightarrow \lambda^{-1/2} \neq 0$$

So by the previous theorem, C_ϕ is not compact.

If a map induces a non-compact operator, then any map whose values approach the unit circle "faster" should also induce a non compact operator. The theorem below formalizes this idea, and allows us to turn results for specific classes of maps like the ones above into general compactness theorems.

Comparison Principle for Compactness:
Suppose, ϕ and ψ are holomorphic self-maps of U with ϕ univalent and $\psi(U) \subset \phi(U)$. If C_ϕ is compact on H^2, then so is C_ψ.
Proof: We use an argument similar to the one that occured in the proof of Polygonal Compactness Theorem.
Since ϕ is univalent
that means, ϕ^{-1} is also analytic in U and $\psi(U) \subset \phi(U)$.
So, $\chi = \phi^{-1} \circ \psi$ which takes $U \longrightarrow U$ holomorphically.
Thus $\psi = \phi \circ \chi$
$\Longrightarrow \quad C_\psi = C_\chi C_\phi$.

If S and T are operators on a Hilbert space H with S bounded and T compact, then both ST and TS are compact.
The bounded operators preserve both boundedness and relative compactness of subsets of H.
Since C_ϕ is compact and C_χ is bounded , so from the above complement we can say that $C_\chi C_\phi = C_\psi$ is compact. This completes the proof.

We can use our new Comparison Principle to generalize the class of examples that led off this section.

Corollary: Suppose ϕ is a univalent self-map of U, and that $\phi(U)$ contains a disc in U that is tangent to the unit circle. Then C_ϕ is not compact.
Proof: We may suppose, without loss of generality, that the disc (saying Δ) is tangent to the unit circle at $+1$.
Therefore if λ be the radius of disc Δ,then we have $0 < \lambda < 1$ and $\Delta = \lambda U + (1-\lambda) \subset \phi(U)$.
Thus Δ is the image of U under the map $\psi(z) = \lambda z + (1-\lambda)$ and $\psi(U) \subset \phi(U)$.
Now from first result of this section C_ψ is not compact. By the Comparison Principle, C_ϕ is not compact.

$\star$ If $\phi(z)$ approaches to the unit circle 'to closely', is not compact as a univalent map contains a disc which tangent to a unit circle in U.

Remarks: (a) In the Comparison Principle we cannot do without the univalence of ϕ. Indeed, there exists a map that takes U onto itself in no more than two-to-one fashion, but nonetheless induces a compact composition operator.
(b) Later, we will show that non-compactness persists if $\psi(U)$ contains a domain whose boundary approaches the unit circle "as smoothly as the curve $y = x^\alpha$ approaches the real axis $(1 < \alpha \leq 2)$."
The corollary above deals with the case $\alpha = 2$, while the Polygonal Compactness Theorem shows that the result fails for $\alpha = 1$.
(c) Subordination. In the proofs of both the Polygonal Compactness Theorem and the above Comparison Principle we used the fact that:
If ϕ and ψ are holomorphic self-maps of U with ϕ is univalent and $\psi(U) \subset \phi(U)$, then $\psi = \phi \circ \chi$ where χ is a holomorphic self-map of U .
More generally, if f and g are any two holomorphic functions, with $f = go \circ \chi$ where χ is a holomorphic self-map of U, we say that f is subordinate to g (in the usual definition of subordination it is also required that $\chi(0) = 0$, but here we ignore this .)

The above results assert that a composition operator cannot be compact if the values of its inducing map approach the unit circle too quickly, even if this only happens at a single point. Here is an apparently different way to defeat compactness.

Proposition:(The values of ϕ approach the boundary too often)
Suppose ϕ is a holomorphic self-map of U for which the set

$$E(\phi) = \{\theta \in [-\pi, \pi] : |\phi(e^{i\theta})| = 1\}$$

has positive Lebesgue measure. Then C_ϕ is not compact on H^2.
Proof: Set $E = E(\phi)$.
Clearly each monomial $z^n (n \geq 0)$ belongs to the unit ball of H^2,and the whole sequence $\{z^n\} \longrightarrow 0$ uniformly on compact subsets of U. On the other hand,

$$||C_\phi(z^n)||^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |\phi(e^{i\theta})|^{2n} d\theta \geq \frac{1}{2\pi} \int_E |\phi(e^{i\theta})|^{2n} d\theta \geq \frac{1}{2\pi} |E| > 0$$

$$\Longrightarrow \qquad ||C_\phi(z^n)|| \neq 0$$

So, C_ϕ is not compact.

Summary: We have seen that ϕ is compact if $\phi(z)$ stays inside an inscribed polygon, but that it is not compact whenever $\phi(z)$ approaches to the unit circle "too often," in the sense that $\phi(e^{i\theta} = 1$ for θ in a set of positive measure, or "too closely," as is the case for a univalent inducing map whose image contains a disc that is tangent to the unit circle. These results all suggest that C_ϕ is compact if and only if $\phi(z)$ does not get too close to the unit circle too often.

Chapter 6

Compactness and Univalence

We are now ready to classify the univalent self-maps of U that induce compact composition operators on H^2. A fragment of operator theoretic folk-wisdom will help us guess the answer:
If a "big-oh" condition describes a class of bounded operators, then the corresponding "little-oh" condition picks out the subclass of compact operators.

6.1 The H^2-Norm via Area Integrals

We have employed each of the following formulas for the norm of a function $f \in H^2$:

$$||f||^2 = \sum_{n=0}^{\infty} |\hat{f}(n)|^2 = \lim_{r \longrightarrow 1^-} \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(re^{i\theta})|^2 d\theta$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(e^{i\theta})|^2 d\theta$$

This section will contribute one more item to the list: a representation of the norm by an integral over the unit disc itself. In what follows we write dA for two dimensional Lebesgue measure, restricted to the unit disc, and normalized to have mass one ($dA = \frac{1}{\pi} dxdy$).
Proposition:(Area Integral estimate for the H^2 norm)
For $f \in H(U)$;

$$\frac{1}{2}||f - f(0)||^2 \leq \int_U |f'(z)|^2 (1 - |z|^2) dA(z) \leq ||f - f(0)||^2 \tag{1}$$

Proof:

$$\int_U |f'(z)|^2 .(1 - |z|^2) dA(z) = 2 \int_0^1 \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |f'(re^{i\theta})|^2 d\theta \right) (1 - r^2) r dr$$

$$= 2 \int_0^1 \left(\sum_{n=1}^{\infty} n^2 .|\hat{f}(n)|^2 r^{2n-2} \right) (1 - r^2) r dr$$

$$= 2 \sum_{n=1}^{\infty} n^2 |\hat{f}(n)|^2 \int_{n=0}^{1} (r^{2n-2} - r^{2n}) r dr$$

$$= 2\sum_{n=1}^{\infty} n^2|\hat{f}(n)|^2 \left[\frac{r^{2n}}{2n} - \frac{r^{2n+2}}{2n+2}\right]_0^1$$

$$= 2\sum_{n=1}^{\infty} n^2|\hat{f}(n)|^2 \frac{1}{2n(n+1)}$$

$$= \sum_{n=1}^{\infty} \frac{n}{n+1}|\hat{f}(n)|^2$$

Now

$$||f - f(0)||^2 = \sum_{n=0}^{\infty} |\hat{f}(n) - \hat{f}(0)|^2$$

$$= \sum_{n=1}^{\infty} |\hat{f}(n)|^2$$

So, easily we can say that,
the quantity $\sum_{n=1}^{\infty} \frac{n}{n+1}|\hat{f}(n)|^2$ lies between $||f - f(0)||^2$ and $\frac{1}{2}||f - f(0)||^2$. Hence the theorem is proved.

6.2 The Theorem

Littlewood's Theorem revisited:
Armed with the area integral representation of the H^2 norm, we can now give the "right" proof of Littlewood's theorem, at least for univalent inducing map $\phi(0) = 0$.
For $f \in H^2$, we substitute $f \circ \phi$ for f in the equation (1),

$$\frac{1}{2}||f \circ \phi - f(0)||^2 \leq \int_U |(f \circ \phi)'(z)|^2(1 - |z|^2)dA(z)$$

$$= \int_U |f'(\phi(z))|^2(1 - |z|^2)|\phi'(z)|^2 dA(z)$$

$$\leq \int_U |f'(\phi(z))|^2(1 - |\phi(z)|^2)|\phi'(z)|^2 dA(z) \qquad \text{(By Schwarz lemma)}$$

$$= \int_{\phi(U)} |f'(w)|^2(1 - |w|^2)|dA(w) \qquad \text{(Putting } w = \phi(z))$$

$$\leq ||f - f(0)||^2 \qquad \leq ||f||^2$$

$$\implies \qquad ||C_\phi - f(0)||^2 \leq 2||f||^2$$

Now,

$$||C_\phi f|| \leq ||C_\phi f - f(0)|| + ||f(0)||$$

$$\leq 2||f|| + ||f|| = 3||f||$$

Hence C_ϕ is bounded on H^2.
At first glance this method of proving boundedness for composition operators seems to lose a lot, since we actually know that C_ϕ is a contraction whenever $\phi(0) = 0$, even if ϕ is not univalent.

But the method provides what we need most: the "big-oh" condition that stands behind Littlewood's Theorem; it is nothing but the Schwarz Lemma, disguised in the form
$1-|z|^2 = O(1-|\phi(z)|^2) as |z| \longleftrightarrow 1-$, where the "big-oh" constant is 1.
According to the Folk Wisdom dispensed at the beginning of this chapter, the corresponding "little-oh" condition should tell a lot about compactness.
This intuition is confirmed by the next result, which is the main result of this chapter.

6.3 The Univalent Compactness Theorem:

Statement: Suppose ϕ is univalent self map of U. then C_ϕ is compact on H^2 iff

$$\lim_{|z|\longrightarrow 1^-} \frac{1-|\phi(z)|}{1-|z|} = \infty \tag{2}$$

6.3.1 Proof of sufficiency:

We are assuming ϕ is univalent and satisfies the condition (2)
i.e. ϕ satisfies $\lim_{|z|\longrightarrow 1^-} \frac{1-|\phi(z)|}{1-|z|} = \infty$.
To show C_phi is compact in H^2, we employ the the sequential charaterization of compactness.
It is sufficient to show that, if $\{f_n\}$ is sequence that is bounded in H^2 and converges to 0 uniformly on the compact subset of U then $||c_\phi f_n|| \longrightarrow 0$.
So without loss of generality let , $||f_n| \leq 1 \forall n$.

Let $\epsilon > 0$ be given then the condition (2) guarantees a number $0 < r < 1$ s.t.

$$1-|z|^2 \leq \epsilon(1-|\phi(z)|^2) \text{ for } r < |z| < 1 \tag{3}$$

We fix this this for the remainder of the proof. According to the area integral estimate (1) of the H^2 norm:

$$\frac{1}{2}||C_\phi f_n - f_n(\phi(0))||^2 \leq \int_{rU} + \int_{U-rU} |(f_n \circ \phi)'(z)|^2(1-|z|^2)dA(z)$$

Given $f_n \longrightarrow 0$ uniformly on compact subset of U.
i.e. $f_n \circ \phi \longrightarrow 0$ uniformly on the compact subset of U.
i.e. $C_\phi f_n \longrightarrow 0$ uniformly on every compact subset of U.
$\Longrightarrow \quad (f_n \circ \phi)' \longrightarrow 0$ uniformly on every compact subset of U. So the first integral of above converges to 0.
Now for the second integral,

$$\begin{aligned}
||C_\phi f_n - f_n(\phi(0))||^2 &\leq o(1) + \epsilon \int_{U-rU} |f_n(\phi(z))\phi'(z)|^2(1-|\phi(z)|^2)dA(z) \\
&\leq o(1) + \epsilon \int_U |f_n'(\phi(z))|^2(1-|\phi(z)|^2)|\phi'(z)|^2 dA(z) \\
&\leq o(1) + \epsilon \int_U |f_n'(w)|^2(1-|w|^2)|dA(w) \qquad \text{(putting } \phi(z) = w) \\
&\leq o(1) + 2\epsilon||f_n - f_n(0)||^2 \qquad \text{(by (1) again)}
\end{aligned}$$

$$\leq o(1) + 2\epsilon,$$

where in the last line we used the fact that $||f_n - f_n(0)|| \leq ||f_n|| \leq 1$ for each n.
Since $f_n(\phi(0)) \longrightarrow 0$, so the above estimate shows that
$limsup_n||C_\phi f_n|| \leq 2\epsilon$.
$\Longrightarrow \quad ||C_\phi f_n|| \longrightarrow 0 \quad$ because ϵ was an arbitrary positive number
$\Longrightarrow$ C_ϕ is compact.

The more subtle part of the theorem is the proof that condition (2) is necessary for compactness. There are several paths to this result, each of which requires a new idea. We choose one based on elementary operator theory.

6.3.2 The Adjoint Operator

The scene now shifts to an abstract (separable) Hilbert space H. Recall that the norm of each element $y \in H$ can be expressed in terms of the inner product by

$$||y|| = \sup_{x\in B} | < x, y > | \qquad (4)$$

where B is the unit ball in H, and the supremum is attained at the unit vector $x = y/||y||$ (assuming $y \neq 0$). In particular, the linear functional induced on H by y:

$$x \mapsto < x, y > \qquad (x \in H)$$

is bounded linear functional on H of norm $||y||$.
Now the Reisz representation Theorem asserts that each bounded linear functional on H is induced in this way by some (necessarily unique) vector $y \in H$.
If T is a bounded linear operator on H, and $y \in H$, then the linear functional

$$x \mapsto < Tx, y > \qquad (x \in H)$$

is bounded, so there is a unique vector in H, which we denote by T^*y, that represents this functional in equation (4). The operator T^* so defined on H is called the adjoint of T; its definition can be summarized like this:

$$< x, T^*y > = < Tx, y > \qquad (x, y \in H).$$

Clearly T^* is a linear transformation on H, and (4) implies that $||T*|| = ||T||$. It is also routine to check that

$$(T1 + T2)^* = T^* + T^* \text{ and } (cT)^* = cT^*.$$

where the symbol T, with or without subscripts, denotes a bounded linear operator on H and c is a complex number. In short, we have proved the following result, where $\mathcal{L}(\mathcal{H})$ denotes the space of bounded linear operators on H, taken in the operator norm.

Lemma: The map $T \longrightarrow T^*$ is a conjugate-linear isometry on $\mathcal{L}(\mathcal{H})$.

The adjoint of a finite rank operator: Suppose T is a bounded operator of rank one on $\mathcal{H}$. This means that for some $x, y \in \mathcal{H}$,

$$Tz =< z, y > x(z \in \mathcal{H})$$

Using equation (5) we easily compute for each $z, w \in \mathcal{H}$,

$$< z, T^*w >=< Tz, w >=<< z, y > x, w >$$

$$=< z, y >< x, w >=< z, \overline{< x, w >}y >$$

$$=< z, < w, x > y >,$$

from which it follows that $T^*w =< w, x > y$.
It is customary to write the one dimensional operator T as $\bigotimes$., so the result just proved can be succiently rephrased as follows.

Lemma: The adjoint of a rank one operator has rank one; in fact if $x, y \in \mathcal{H}$, then $(x \bigotimes y)* = y \bigotimes x$.

Since every finite rank operator is a sum of rank one operators, the Lemma and the linear nature of the adjoint operation yield this:
Corollary: The adjoint of a finite rank bounded operator again has finite rank.
All the results developed in this section now combine to show that the adjoint operation preserves compactness.

Proposition: The adjoint of a compact operator is compact.

Proof: Suppose T is a compact operator on $\mathcal{H}$.
By the approximation theorem we that there exists a sequence F_n of bounded finite rank operators such that $||T - F_n|| \longrightarrow O$. Since the adjoint operation is additive and isometric in the operator norm,

$$\lim_n ||T^* - {F_n}^*|| = \lim_n ||(T - F_n)^*|| = 0$$

Since each of the operators ${F_n}^*$ is of finite rank and bounded.
So by the approximation theorem shows that T^* is compact.

Adjoint composition operators and reproducing kernels: Our second computation involves the adjoint of a composition operator. Although there is no good description of the adjoint that works for all composition operators on all H^2 functions, we can always compute its action on an important special family of functions in H^2: the reproducing kernels.
For each point $p \in U$, let

$$k_p(z) =^{def} \frac{1}{1 - \bar{p}z} = \sum_{n=0}^{\infty} \bar{p}^n z^n.$$

So clearly $k_p \in H^2$. It is called reproducing kernel for the point p, and it gets the name from the fact that for each $f \in H^2$,

$$< f, k_p >= \sum_{n=0}^{\infty} \hat{f}(n)p^n = f(p). \tag{6}$$

Lemma: $C_\phi^* k_p = k_{\phi(p)}$ for each $p \in U$.
Proof: For each $f \in H^2$ we have

$$< f, C_\phi^* k_p >=< C_\phi f, k_p >= C_\phi f(p) = f(\phi(p)) =< f, k_{\phi(p)} >$$

$$\implies C_\phi^* k_p = k_{\phi(p)}.$$

6.3.3 Proof of Necessity

TheoremNecessary condition for compactness Suppose ϕ is a holomorphic self-map of U and that C_ϕ is compact on H^2. Then

$$\lim_{|z| \longrightarrow 1^-} \frac{1 - |\phi(z)|}{1 - |z|} = \infty$$

Proof: For each $p \in U$, let

$$f_p(z) = \frac{k_p}{||k_p||} = \frac{\sqrt{1 - |p|^2}}{1 - \bar{p}z},$$

the normalized reproducing kernel for p.
We are going to show that,

$$||C_\phi^* f_p|| \longrightarrow 0 \ as \ |p| \longrightarrow 1^-. \tag{7}$$

This will finish the proof, since from the last lemma we can say that

$$||C^* \phi f_p||^2 = (1 - |p|^2)||k_{\phi(p)}||^2 = \frac{1 - |p|^2}{1 - |\phi(p)^2|}$$

. To prove (7), recall that the adjoint operator C_ϕ^* inherits the compactness of C_ϕ.
Thus the collection of C_ϕ^* images of normalized reproducing kernels is a relatively compact subset of H^2, so every sequence of these images has a convergent subsequence.
We need only show that, the zero function is the only possible limit of such a subsequence. Suppose $|p_n| \longrightarrow 1^-$
$mboxand\ C *_p hi f_{p_n} \longrightarrow g$ in the H^2 norm.
We'll be finished if we can show that $g = 0$.
To see this, let h be any polynomial.
Then the continuity of the inner product gives,

$$< g, h >= \lim_n < C_\phi^* f_{p_n}, h >$$

$$= \lim_n \sqrt{1 - |p_n|^2} < C^* k_{p_n}, h >$$

$$= \lim_n \sqrt{1-|p_n|^2} < k_{\phi(p_n)}, h >$$
$$= \lim_n \sqrt{1-|p_n|^2}\ \overline{h(\phi(p_n))}$$
$$= 0$$

where the third line follows from the Lemma at the end of the last section, and the last one from the fact that h is bounded on U.
Thus g is orthogonal to every polynomial.
Since the polynomials form a dense subset of H^2, it follows that g is the zero function.
With this result, the proof of the Univalent Compactness Theorem is complete.

6.4 Compactness and Contact

The results of Chapter 5 suggested a strong connection, at least for univalently induced composition operators, between compactness and the "degree of contact" that the image of the inducing map has with the unit circle.
It was shown, for example, that the operator is compact if this image is confined to an inscribed polygon, and non-compact if the image contains a disc tangent to the circle. The Univalent Compactness Theorem allows us to considerably refine these results. In this section we show that a univalently induced composition operator will fail to be compact whenever, for some $\alpha > 1$, its image approaches the unit circle "faster than $y = x^\alpha$ approaches the real axis."
In the other direction, we give an example that shows that the corners in Polygonal Compactness Theorem can be rounded off "just a little" without loss of compactness.

Contact with the boundary: The first order of business is to decide how to measure the order of contact made by a region in U with the unit circle.
For simplicity we consider only contact at the point +1; all the arguments work with obvious modifications for any other point of ∂U.
Let $\gamma : [0, \pi] \longrightarrow [0, 1)$ be a continuous function with $\gamma(0) = 0$ but $\gamma(\theta) > 0$ otherwise.
We use γ to define a Jordan curve Γ in U by means of the polar equation

$$1 - r = \gamma(|\theta|) \qquad\qquad (|\theta| \leq \pi).$$

Thus Γ is symmetric about the real axis, and lies in U except for a single point of intersection with the unit circle at + 1.

For a positive number α, let us agree to call Γ an α- curve at +1 if $\theta^{-\alpha}\gamma(\theta)$ has a (finite) non-zero limit as $\theta \longrightarrow 0$.

For example, a triangle that is symmetric about the real axis and lies in U except for a vertex at +1 is a 1-curve, while a circle properly contained in $\bar{U}$, and tangent to ∂U at + 1 is a 2-curve.

Finally, we say Γ approaches the unit circle smoothly at +1 if $\theta^{-1}\gamma(\theta) \longrightarrow 0\ as\ \theta \longrightarrow 0^+$. Thus, every α-curve for $\alpha > 1$ approaches ∂U smoothly.

We say a region $\omega \subset U$ has contact α with the unit circle at + 1 if it contains an α-curve at + 1 (we could be more precise and say ω has contact at least a at + 1). If ω merely contains a curve that approaches the boundary smoothly at + 1 then we say the region has smooth contact with ∂

at that point.

We will find it useful to express these definitions in terms of distances, both in the unit disc and the right half-plane.

Lemma 6.4.1. *Suppose $\alpha \geq 1$ Then Γ is an $\alpha-$curve if and only if*

$$\lim \frac{1-|z|}{|1-z|^{\alpha}} \qquad\qquad (z \longrightarrow 1,\ z \in \Gamma)$$

exits (finitely) and is non-zero.
Thus Γ is an α-curve if and only if for each of its points, the distance to the boundary is comparable to the α-th power of the distance to +1.
Proof: *For $z = re^{i\theta} \in \Gamma$, we calculate*

$$1-|z|^2 = (1-re^{i\theta})\overline{(1-re^{i\theta})}$$

$$= 1 + r^2 - 2rcos\theta$$

$$= (1-r)^2 + r(2sin\theta/2)^2$$

Hence for $|\theta| \longrightarrow 0$,

$$\left(\frac{|1-z|^{\alpha}}{1-|z|}\right)^{2/\alpha} = \frac{\gamma(|\theta|)^2 + (1+o(1))\theta^2}{\gamma(|\theta|)^{2/\alpha}}$$

$$= \gamma(|\theta|)^{2(1-1/\alpha)} + (1+o(1))\left(\frac{|\theta|}{\gamma(|\theta|)^{1/\alpha}}\right)^2$$

If $\alpha = 1$ then 1st summand of the last line is $\equiv 1$, while if $\alpha > 1$ then it converges to 0 as $|\theta| \longrightarrow 0$. So the 2nd summand is also finite (non-zero) by the definition of α-curve.
This establishes our assertion.

We will be constructing univalent self-maps of the unit disc by working instead in the right half-plane Π, and then returning to the disc through the change of variable $w = \tau(z) = \frac{1+z}{1-z}$. Thus we need to know how the concept of "α-curve" fares under this change of scene. To make sense out of what is going to happen, it helps to keep in mind that τ transforms line segments through +1 into other line segments, but it also transforms circles tangent to ∂U at +1 into (vertical) lines.

Lemma 6.4.2. *Suppose γ and Γ are as above. Let $\tilde{\Gamma}$ be the image of γ under the map τ. Then $\Gamma$is an α-curve iff*

$$\lim \frac{Rew}{|w|^{2-\alpha}} \qquad\qquad (w \longrightarrow \infty, w \in \tilde{\Gamma})$$

exists and is non-zero.
Proof: *The change of variable can be written $z = \frac{w-1}{w+1}$, from which follows two important distance formulas:*

$$1 - z = \frac{2}{w+1} \qquad \text{and} \qquad 1-|z|^2 = |\frac{2}{w+1}|^2 Re\ w. \tag{8}$$

These show that as $z \longrightarrow 1$, (equivalently: as $w \longrightarrow \infty$),

$$\frac{1-z}{|1-z|^{\alpha}} = \frac{1}{1+|z|}\frac{1-|z|^2}{|1-z|^{\alpha}}$$

$$= \left(\frac{1}{2}+o(1)\right)|\frac{2}{w+1}|^{2-\alpha} Re\ w$$

$$= 2^{1-\alpha}(1+o(1))\frac{Re\ w}{|w|^{2-\alpha}},$$

So, by previous lemma we can say that $\frac{Re\ w}{|w|^{2-\alpha}}$ which yields the desired result.

A class of examples: We can now write down for each $1 < \alpha < 2$ examples of univalent maps ψ for which $\psi(U)$ is a Jordan domain whose boundary is, near the point $+1$, an α-curve, and for which C_ϕ is not compact on H^2. we introduce two additional parameters $a, b > 0$ for later use (inviting the reader to set them both equal to 1 in the proof below), and define

$$\Psi(w) = \Psi_{\alpha,a,b}(w) = a + w + bw^{2-\alpha}$$

where the principal branch of the argument is used to define the fractional power on the right. Clearly Ψ maps the right half-plane into itself. Let $\psi = \psi_{\alpha,a,b}$ be the corresponding holomorphic self-map of U .

Proposition: For each $1 < \alpha < 2$ and $a, b > 0$ the map ψ has these properties:
(a) ψ is univalent on $\overline{U}$, and $\psi(\overline{U}) \subset U \cup \{1\}$.
(b)$\psi(\partial U)$ is an α-curve at +l.
(c) C_ϕ is not compact on H^2.
Proof:
(a) We work in the right half-plane.
Univalence follows from the fact that the derivative of Ψ has positive real part. More precisely note that

$$\Psi'(w) = 1 + (2-\alpha)bw^{1-\alpha} = 1 + \frac{(2-\alpha)b}{w^{\alpha-1}}$$

has positive real part in $\overline{\Pi} - \{0\}$.
Hence if w_1 and w_2 are distinct points of that set, and L is the line segment joining the points,then

$$\Psi(w_2) - \Psi(w_1) = \int_L \Psi'(\zeta)d\zeta = (w_2 - w_1)\int_0^1 \Psi'(tw_2 + (1-t)w_1)dt.$$

Since the integrand in the last integral is strictly positive, so is the integral, hence $\Psi(w_1) \neq \Psi(w_2)$. The argument works as well for $w_1 = 0$ since the singularity in the derivative is integrable, and clearly Ψ extends continuously to $\hat{C}$, with $\Psi(\infty) = \infty$.

(b)By the definition of Ψ, we have for each real y,

$$Re\Psi(iy) = a + bc(\alpha)|y|^{2-\alpha}$$

where $c(\alpha) = cos(\frac{\pi}{2}(2-\alpha))$, which, because $0 < 2-\alpha < 1$, is a positive number. Now the definition of Ψ shows that $|\Psi(iy)| = (1+0(1))|y|$ as $|y| \longrightarrow \infty$, hence

$$\lim_{|y|\longrightarrow\infty} \frac{\text{Re } \Psi(iy)}{|\Psi(iy)|^{2-\alpha}} = bc(\alpha) \qquad (9)$$

This result, along with Lemma 2 above shows that the boundary of $\phi(U)$ is an α-curve at $+1$.

(c) To prove that C_ϕ is not compact on H^2 it suffices, by the Univalent Compactness Theorem of the last chapter, to show that as $x \longrightarrow 1$ (in the unit interval) we have

$$\liminf_{x\longrightarrow 1-} \frac{1-\psi(x)}{1-x} < \infty,$$

This is easy: for $0 < x < 1$ set $u = (1+x)/(1-x)$, the corresponding point of the positive real axis, use the first of formulas (8) to calculate:

$$\frac{1-\psi(x)}{1-x} = \frac{2}{1+\Psi(u)} \cdot \frac{1+u}{2}$$

$$= \frac{1+u}{a+1+u+bu^{2-\alpha}}$$

Since $1 < \alpha < 2$ the last expression converges to 1 as $u \longrightarrow \infty$, that is, as $x \longrightarrow 1$.

Improved Non-compactness Theorem: If ϕ is univalent and $\phi(U)$ has contact $\alpha > 1$ with the unit circle at some point, then C_ϕ is not compact on H^2 .

Proof: Without loss of generality we may assume that the point of contact is $+1$. Thus we are assuming that the image of ϕ contains an α-curve at $+1$.

By the Comparison Principle, it suffices to prove that the Riemann map of the unit disc onto the region bounded by this α-curve induces a non-compact composition operator. So without loss of generality we may assume that ϕ is this Riemann map.

By Lemmas 1 and 2, and (9) above, we can choose the parameter b small enough so that the part of the boundary of $\psi_{\alpha,0,b}(U)$ that lies in some neighborhood V of $+1$ is contained in $\phi(U)$.

We claim that a sufficiently large choice of the "translation" parameter a forces $\psi = \psi_{\alpha,a,b}$ to map U into V, and therefore completely into $\phi(U)$, at which point the non-compactness of C_ϕ will guarantee, via the Comparison Principle, that C_ϕ is not compact.

The argument is best visualized in the right half-plane, where the boundary of $\phi(U)$ becomes a simple curve in Π that is symmetric about the real axis and heads out to ∞, and the neighborhood V becomes a neighborhood (which we still denote by V) of ∞, i.e. the exterior of some half-disc with center at the origin. Let $\Omega = \psi_{\alpha,0,b}(U)$ and write Φ for the counterpart of ϕ acting on Π.

We have previously chosen the dilation parameter b so that $\Omega \cap V \subset \Phi(\Pi)$. Now it is easy to check that Ω is taken into itself by horizontal translation. Indeed, Ω is symmetric about the real axis and its upper boundary has the form $y = A + Bx^\gamma$, where $A, B > 0$ and $\gamma > 1$. The translation

property follows from this symmetry and the fact that the upper boundary curve is the graph of a monotone increasing function.
Now choose $a > 0$ so that $a + \Omega \subset V$. Then by the work of the last paragraph,

$$\Psi_{\alpha,a,b}(\Pi) = a + \Omega \subset \Phi(\Pi),$$

and the proof is complete.

We close this chapter by using the Univalent Compactness Theorem to construct an example of a compact composition operator whose inducing map takes the disc onto a domain that touches the boundary smoothly.

Example of "smooth compactness": There exist univalent self-map ϕ of U such that
(a) $\phi(\overline{U}) \subset U \cup \{+1\}$,
(b) $\phi(U)$ contacts ∂U smoothly at $+1$,
(c) C_ϕ is compact on H^2.
Proof: Instead of the unit disc, we work in the half-disc

$$\Delta = \{w \in \Pi : |w| < \frac{1}{2e}\}$$

with the holomorphic function

$$f(w) = -cw\ logw \qquad (w \in \Delta)$$

where $c > 0$, and and the principal branch of the logarithm is employed on the right.
One easily checks that f maps Δ into a bounded subset of the right half-plane, so the constant c can be chosen so that $f(\Delta) \subset \Delta$.
Moreover, f' has positive real part on $\overline{\Delta}\backslash\{0\}$, so as in the proof of the Proposition, f is univalent on $\overline{\Delta}$.

Now let τ be a univalent map taking Δ onto U, with $\tau(0) = +1$. This map extends to a homeomorphism of the corresponding closed regions-as can be seen by either writing it down as a composition of elementary mappings, or quoting Caratheodory's Extension Theorem .

Thus, the Reflection Principle insures that τ extends analytically to a mapping that takes the entire disc $|w| < 1/2e$ univalently onto a simply connected domain containing both U and the point + 1.

Let ϕ be the univalent self-map of U that corresponds, via τ, to f on Δ.
Since τ is analytic with non-vanishing derivative in a full neighborhood of the origin, distance estimates in transfer over to corresponding distance estimates in U.
In particular, if $|y| < 1/2e$ and $e^{i\theta} = \tau(iy)$, then

$$\begin{aligned}\frac{1 - |\phi(e^{i\theta})|}{|1 - \phi(e^{i\theta})|} &= \frac{\text{dist.}(\phi(e^{i\theta}), \partial U)}{\text{dist.}(\phi(e^{i\theta}), +1} \\ &\leq \text{const.}\frac{\text{dist.}(f(iy), \partial\Pi)}{\text{dist.}(f(iy), 0)}\end{aligned}$$

$$
\begin{aligned}
&= \text{const.}\frac{\text{Re}f(iy)}{|f(iy)|} \\
&= \text{const.}\frac{\pi|y|/2}{\{(\pi|y|/2)^2 + (|y|\ \log|y|)^2\}^{1/2}} \\
&\leq \frac{\text{const.}}{-\log|y|} \\
&\longrightarrow 0
\end{aligned}
$$

as $|y| \longrightarrow 0$, i.e. as $|\theta| \longrightarrow 0$.
Thus the image of the unit disc under ϕ approaches the boundary smoothly at $+1$.

The compactness of C_ϕ on H^2, will follow from the Univalent Compactness Theorem once we show that

$$\frac{1 - |\phi(z)|}{1 - |z|} \longrightarrow \infty \tag{10}$$

as z tends to any point of ∂U. We need only check this limit at the point $+1$, since the closure of $\phi(U)$ approaches the unit circle nowhere else.
For $z \in U$ let $z = \tau(w)$, and estimate as above,

$$
\begin{aligned}
\frac{1 - |\phi(z)|}{1 - |z|} &\geq \text{const.}\frac{\text{dist.}(f(w), \partial\Pi)}{\text{dist.}(w, \partial\Pi)} \\
&= \text{const.}\frac{\text{Re}f(w)}{\text{Re}w} \\
&= \text{const.}\frac{(\text{Re } w)(-\log(|w|)) + (\text{Im } w)\text{arg } w}{\text{Re } w} \\
&\geq \text{const. } \log(1/|w|), \\
&\longrightarrow \infty
\end{aligned}
$$

as $w \longrightarrow 0$, i.e. as $z \longrightarrow +1$.
where the last inequality follows from our use of the principal branch of the argument, which insures that both argument and imaginary part always have the same sign.
So C_ϕ is compact on H^2.

Note: The idea behind this proof can be modified to produce Hilbert-Schmidt composition operators induced by univalent mappings that take the unit disc onto sub domains that contact the boundary smoothly. Used in conjunction with the Comparison Principle, these examples produce "rounded corners" versions of the Polygonal Compactness Theorem.

Chapter 7

The Angular Derivative

The condition
$$\lim_{|z|\longrightarrow 1^-} \frac{1-|\phi(z)|}{1-|z|} = \infty \qquad (1)$$
which characterizes compactness for univalently induced composition operators. Because this condition involves the limit of a difference quotient, one might suspect that its real meaning is wrapped up in the boundary behavior of the derivative of ϕ. This is exactly what happens: we will see shortly that condition (1) is the hypothesis of the classical Julia-Caratheodory Theorem, which characterizes the existence of the "angular derivative" of ϕ at points of ∂U, and provides a compelling geometric interpretation of (1) in terms of "conformality at the boundary."
After discussing its connection with the compactness problem, we present a proof of the Julia-Caratheodory Theorem that emphasizes the role of hyperbolic geometry. The following terminology describes the limiting behavior involved in this circle of ideas.
Definition:
(a) A sector (in U) at a point $w \in \partial U$ is the region between two straight lines in U that meet at w and are symmetric about the radius to w.
(b) If f is a function defined on U and $w \in \partial U$, then
$$\angle \lim_{z\longrightarrow w} f(z) = L$$
means that $f(z) \longrightarrow L$ as $z \longrightarrow w$ through any sector at w. When this happens, we say L is non -tangential (or angular) limit of f at w.

7.1 The Definition

We say a holomorphic self-map ϕ of U has an angular derivative at $w \in \partial U$, if for some $\eta \in \partial U$
$$\angle \lim_{z\longrightarrow w} \frac{\eta - \phi(z)}{w - z}$$
exists finitely. We call the limit the angular derivative of ϕ at w, and denote it by $\phi'(w)$.
Warning: The existence of the angular derivative $\Longrightarrow$ that ϕ has angular limit at w. We are

requiring that η be a point of the unit circle, so regardless of how smooth ϕ may be at the boundary, our definition demands:
"ϕ cannot have an angular derivative at any boundary point at which it fails to have an angular limit of modulus one."
The work so far shows that for the compactness problem, the important phenomena occur as $\phi(z)$ approaches the boundary. For example, according to our definition, the function $\phi(z) = z/2$ (which induces a compact composition operator) has an angular derivative nowhere on ∂U. While the map $\phi(z) = (1+z)/2$, which has an angular derivative at the point $+1$ (and, according to our definition, nowhere else) induces a non-compact operator.
These examples raise the possibility that the results of the last chapter might be restated in terms of the angular derivative. This is exactly what is going to happen. The "necessary" part of the program follows immediately from the definitions, and as before, does not require univalence.
Proposition: If ϕ has an angular derivative at a point $w \in \partial U$ then C_ϕ is not compact on H^2.
Proof: Letting η denote the angular limit of ϕ at w we have,

$$\liminf_{|z|\longrightarrow 1^-} \frac{1-|\phi(z)|}{1-|z|} \leq \liminf_{|z|\longrightarrow 1^-} \frac{1-|\phi(rw)|}{1-r} \leq \liminf_{|z|\longrightarrow 1^-} \frac{\eta-|\phi(rw)|}{w-rw}| = |\phi'(w)|$$

$\Longrightarrow$ C_ϕ is not compact by the necessary condition of the compactness.

7.2 The Julia-Carathéodory Theorem

In addition to raising the issue of compactness, the definition of the angular derivative suggests that ϕ has some kind of conformality at the boundary points where it exists, and it further raises the possibility that the derivative of ϕ might also have a non-tangential limit at w.

Statement of The Julia-Carathéodory Theorem: Suppose ϕ is a holomorphic selfmap of U, and $w \in \partial U$. Then the following three statements are equivalent:

(JC 1) $\liminf_{z\longrightarrow w} \frac{1-|\phi(z)|}{1-|z|} = \delta < \infty$.
(JC 2) $\angle\lim_{z\longrightarrow w} \frac{\eta-\phi(z)}{w-z}$ exists for some $\eta \in \partial U$.
(JC 3) $\angle\lim_{z\longrightarrow w} \phi'(z)$ exists and $\angle\lim_{z\longrightarrow w} \phi(z) = \eta \in \partial U$.
<u>Moreover</u>
• $\delta > 0$ in (JC 1),
• the boundary points η in (JC 2) and (JC 3) are the same, and
• the limit of the difference quotient in (JC 2) coincides with that of the derivative in (JC 3), with both equal to $w\overline{\eta}\delta$.
Since condition (JC 1) is just the compactness criterion of the last chapter, this allows the results of that chapter to be restated in terms of the angular derivative.

Angular derivative Criteria for Compactness:Suppose ϕ is a holomorphic self-map of U,
(a) If C_ϕ is compact on H^2 then ϕ has an angular derivative at no point of ∂U.
(b) If ϕ is univalent and has no angular derivative at any point of ∂U, then C_ϕ is compact on H^2.

To appreciate the purely function-theoretic power of the Julia-Caratheodory Theorem, observe how, almost as an afterthought, it asserts that if, on a sequence z_n of points in U that converges to a boundary point w, the images $\phi(z_n)$ tend to the boundary rapidly enough, then regardless of how sparse or tangential z_n may be, the function ϕ must have a radial limit at w. So even ignoring what it says about derivatives, the Julia-Caratheodory Theorem already yields a non-trivial result about boundary behavior.

Remark: Condition (JC 2) implies that ϕ is "non-tangentially conformal" at w. To understand this conformality, we may without loss of generality take $w = \eta$. Then (JC 1) and the fact that $\delta > 0$, make it possible to recycle the proof from elementary complex analysis that "holomorphic plus non vanishing derivative implies conformal." The argument yields this:
"If a smooth curve in U ends at a point $w \in \partial U$, at which it makes an angle $\alpha < \pi/2$ with the radius to that point, then the same is true of the image curve at the boundary point η."
In particular, the image of the radius itself meets the unit circle perpendicularly, and if two non-tangential curves intersect at w at some angle, then their images intersect at the same angle.

First applications: As an example of what the Julia-Caratheodory Theorem contributes to the compactness problem, note how it clarifies the intuition behind the Polygonal Compactness Theorem: if ϕ takes U into a polygon inscribed in the unit circle, then at no vertex preimage does ϕ have the conformality demanded by the Angular Derivative Criterion. Because of our requirement that the angular derivative can only exist at points whose (radial) pre-image is on the unit circle, it exists at none of the other points either, so C_ϕ is compact.
For another example, recall the assertion of the "First Compactness Theorem" :
"C_ϕ is compact whenever $||\phi||_\infty < 1$."

Corollary: If ϕ is univalent and has no radial limit of modulus 1, then C_ϕ is compact.

Proof of easy parts of the JC theorem

- Before beginning to prove the Julia-Carathoodory Theorem, we need to isolate the main issue. The "upward implications" of the theorem are routine. For example, if both function and derivative converge as in (JC 3), then we integrate the derivative to get (JC 2), obtaining in the process the same limit we had for the derivative.
- The implication (JC 2) $\implies$ (JC 1) is immediate if we let z tend to w along the radius to w, take absolute values, and use the "reverse triangle inequality." This also shows that the quantity δ on the right-hand side of (JC 1) is dominated by the magnitude of the limit in (JC 2).
- The implication (JC 2) $\implies$ (JC 3) requires a little more care, but it does not pose a real problem.
- The heart of the Julia-Caratheodory Theorem is the implication (JC 1) $\implies$ (JC 2).

7.3 The Invariant Schwarz Lemma

The subject of this section, also called the Schwarz-Pick Lemma, is what results when you subject the Schwarz Lemma to a conformal change of variable. In order to state the result efficiently, we need a conformally invariant way of measuring distance in the unit disc.

<u>Schwarz Pick Lemma:</u> A variant of Schwarz Lemma can be stated that is invariant under analytic automorphism on U. The variant is known Schwarz-Pick theorem.

Now , recall the special conformal automorphisms α_p ,

$$\alpha_p(z) = \frac{p-z}{1-\bar{p}z}.$$

Definition. The pseudo-hyperbolic distance between points p and q of U is:

$$d(p,q) = |\alpha_p(q)| = \left|\frac{p-q}{1-\bar{p}q}\right|. \tag{2}$$

The pseudo-hyperbolic distance is actually a metric on U that induces the usual Euclidean topology; however our work requires only the following easily verified observations:
(a) For each pair of points $p, q \in U$ we have $d(p,q) = d(q,p)$, and $d(p,q) < 1$.
(b) $d(p,q) = 0 \implies p = q$.
(c) d is continuous when viewed as a real-valued function on $U \times U$.
(d) For each compact subset K of U,

$$\lim_{|q| \longrightarrow 1^-} \inf_{p \in K} d(p,q) = 1.$$

Property (d) asserts that the pseudo-hyperbolic distance from a point to a fixed compact set tends to 1 as the point tends to the boundary. This is obvious if K is a single point, and not difficult to prove in general.
For example, it is rendered perfectly transparent by the formula

$$1 - d(p,q)^2 = \frac{(1-|p|^2)(1-|q|^2)}{|1-\bar{p}q|^2}, \tag{3}$$

which is itself the of a straightforward calculation. This formula will prove very useful in our further study of the pseudo-hyperbolic distance.

With the pseudo-hyperbolic distance in our corner, we can state a simple but far-reaching generalization of the Schwarz Lemma.

The Invariant Schwarz Lemma. If ϕ is a holomorphic self-map of U, then for every pair of points $p, q \in U$ we have

$$d(\phi(p), \phi(q)) \leq d(p,q).$$

Moreover there is equality here for some pair of points if and only if there is equality for all pairs, and this happens if and only if ϕ is a conformal automorphism of$U$.

Proof: If $p = \phi(p) = 0$ then we are talking about the original statement of the Schwarz Lemma. Otherwise, let $b = \phi(p)$ and consider the map $\alpha_b \circ \phi \circ \alpha_p$, which takes the disc into itself, and fixes the origin. Upon applying the Schwarz Lemma to this map, evaluating at the point $z = \alpha_p(q)$, and noting that the automorphism α_p is its own inverse, we obtain the inequality

$$|\alpha_b \circ \phi(q)| \leq |\alpha_p(q)|,$$

which is precisely what we want. The case of equality follows from the corresponding part of the original Schwarz Lemma.

This form of the Schwarz Lemma asserts that holomorphic self-maps of U that are not automorphisms strictly decrease all pseudo-hyperbolic distances. To make a geometric statement out of this we have to examine the balls associated with the pseudo-hyperbolic distance.
For $p \in U$ and $0 < r < 1$ the r-ball centered at p is

$$\Delta(p, r) = \left\{ z : \left| \frac{z - p}{1 - \bar{p}z} \right| < r \right\} = \alpha_p(rU),$$

We call $\Delta(p, r)$ the pseudo-hyperbolic disc of (pseudo-) center p and (pseudo-) radius r. Since $\Delta(p, r)$ is the image of the disc rU under a conformal automorphism of U, it is an ordinary open disc.
In terms of these discs, the statement of the Invariant Schwarz Lemma becomes

$$\phi(\Delta(p, r)) \subset \Delta(\phi(p), r).$$

If $p = 0$ the assertion is that $\phi_(rU) \subset rU$, which is of course just the geometric interpretation of the original Schwarz Lemma.

7.4 A Boundary Schwarz Lemma

In this section we push the Invariant Schwarz Lemma "out to the boundary." The idea is to examine pseudo-hyperbolic discs whose centers tend to a point w of the unit circle, and whose radii tend to one, and find the condition on centers and radii that guarantees the convergence of such a family of discs to a disc tangent to the unit circle at w.

Equation (3) for the pseudo-hyperbolic distance, allows the definition of pseudo-hyperbolic disc to be rewritten as:

$$\Delta(p, r) = \left\{ z : |1 - \bar{z}p|^2 < \frac{1 - |p|^2}{1 - r^2}(1 - |z|^2) \right\}. \tag{4}$$

This equation wants to tell us something about the limiting behavior of discs.
For if $p \longrightarrow w \in \partial U$ and $r \longrightarrow 1$ in such a way that

$$\frac{1 - |p|}{1 - r} \longrightarrow \lambda \in (0, \infty),$$

then the expression on the right-hand side of the inequality in (4) converges to $\lambda(1 - |z|^2)$, while the one on the left goes to $|1 - z\bar{w}|^2$. Therefore $\Delta(p, r)$ must be converging (somehow) to the set $H(w, \lambda)$ defined by

$$H(w, \lambda) =: \left\{ z : |1 - z\bar{w}|^2 < \lambda(1 - |z|^2) \right\}. \tag{5}$$

Upon completing the square in the inequality on the right, we find that $H(w, \lambda)$ is the Euclidean disc centered at $w/(1+\lambda)$, of radius $\lambda/(l+\lambda)$. In particular, $H(w, \lambda)$ is tangent to ∂U at the point w, it expands as λ increases, and as $\lambda \longrightarrow \infty$ it fills up the whole unit disc. We call $H(w, \lambda)$ a horodisc at w.

The Disc Convergence Lemma: Suppose $w \in \partial U$, and p_n is a sequence of points in U that converges to w. Suppose $0 < r_n \longrightarrow 1$ in such a way that

$$\lambda = \lim_n \frac{1 - |p_n|}{1 - r_n}.$$

Then

$$H(w, \lambda) \subset \liminf_n \Delta(p_n, r_n) \subset \limsup_n \Delta(p_n, r_n) \subset \overline{H(w, \lambda)}.$$

Here the lim sup of a sequence of sets is the collection of points that belong to infinitely many of the sets, and the corresponding lim inf is the collection of points that belong to all the sets from some index onward.

The Lemma says that if we treat the converging discs as if they behaved like points, our error will be confined to boundary points.

Now we are in position to use the Disc Convergence Lemma to get a boundary version of the Invariant Schwarz Lemma.

Julia's Theorem: Suppose ϕ is a non-constant holomorphic self-map of U, and that and w are points of ∂U . Suppose further that $\{p_n\}$ is a sequence of points in U that converges to w in such a way that both $\phi(p_n) \longrightarrow \eta$ and

$$\frac{1 - |\phi(p_n)|}{1 - |p_n} \longrightarrow \delta < \infty \qquad (6)$$

Then:

(a)$\delta > 0$,

(b)$\phi(H(w, \lambda)) \subset H(\eta, \lambda\delta)$ for every $\lambda > 0$, and

(c)$\angle lim_{z \longrightarrow w} \phi(z) = \eta$.

Proof: (a) We first show that $\delta > 0$. Note that if ϕ were to fix the origin then the Schwarz Lemma would tell us straightaway that $\delta \geq 1$. The same idea works when we apply the Invariant Schwarz Lemma with $q = 0$, and it yields

$$d(\phi(p), \phi(0)) \leq d(p, 0) = |p|$$

for every $p \in U$. Upon rewriting this inequality using the identity (3), and doing a little algebra, we obtain

$$\frac{|1 - \overline{\phi(p)}\phi(0)|^2}{1 - |\phi(0)|^2} \leq \frac{1 - |\phi(p)|^2}{1 - |p|^2}.$$

Now the triangle inequality shows that

$$\frac{1 - |\phi(0)|}{1 + |\phi(0)|} \leq \frac{|1 - \overline{\phi(p)}\phi(0)|^2}{1 - |\phi(0)|^2}$$

and upon putting the last two inequality together, setting $p = p_n$ and letting $n \longrightarrow \infty$, we obtain

$$\frac{1 - |\phi(0)|}{1 + |\phi(0)|} \leq \frac{1 - |\phi(p_n)|^2}{1 - |p_n|^2} \longrightarrow \delta$$

hence $\delta > 0$ as desired.

Corollary: If $4\{p_n\}$ is a sequence in U that converges to $w \in \partial U$ au, and on which the quotients $\frac{(1-|\phi(p_n)|)}{(1-|p_n|)}$ are bounded, then $\{\phi(p_n)\}$ converges to some point $\eta \in \partial U$, and ϕ has angular limit η at w.

Note: Now using this last corollary and Julia theorem we can prove that the remaining two part of the Julia- Caratheodory Theorem.

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