Exact Solutions of Accelerated Flows for a Generalized Burgers' Fluid, I: The Case $\gamma < \dfrac{\lambda^2}{4}$

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Abstract: An analysis is presented to develop the exact solutions for the accelerated flows of a generalized Burgers' fluid when the relaxation time satisfying the condition $\gamma < \lambda^2/4$. The corresponding expressions for the velocity field and associated tangential stress are obtained by using Laplace transform for the problems of flow induced by constantly accelerated plate. The obtained solutions are presented through simple or multiple integrals in terms of Bessel functions. The corresponding solutions for Burgers' fluid are recovered as special case of the solutions obtained here.

Keywords: Generalized Burgers's Fluid, Laplace Transform, Exact Solutions.

1. Introduction

The flow induced in a fluid due to sudden motion or because of continues oscillation of a flat plate is usually known in the literature as Stokes' first and second problems, respectively [1,2]. The exact solutions of an unsteady flow for unsteady Navier-Stokes equation are always handy and of fundamental importance. These solutions not only provide an explicit solution to a problem that has physical relevance but can also be used to check the stability analysis of the solutions. Further, such solutions can be used for testing the efficiency of algorithms/complex numerical schemes for flows in complicated flow domains. In the literature there is fairly large number of flows of Newtonian fluids for which the exact analytical solutions are possible. However, for non-Newtonian fluids such exact solutions are rare. This is because of the reason that the governing equations of non-Newtonian fluids are much more complicated and of higher order as compare to Navier-Stokes equations.

The purpose of this paper is to determine the exact solutions corresponding to generalized Burgers' fluid when the motion in the fluid is induced because of the constant acceleration of the plate. More exactly, we intend to extend the analysis in [3] to a larger class of fluids. The present analysis is not only an attempt towards enhancement of the theory of generalized Burgers' fluid but the present fluid model has its significant importance especially in describing the behavior of asphalt and asphalt concrete [4]. Further the model under discussion has also been used to model other geological structures, such as Olivine rocks and the propagation of seismic waves in the interior of the earth [5,6]. Such models have also been successfully used to describe the motion of the earth's mantle [7].

Several analytical methods are available for finding the exact solutions of non-Newtonian fluids. Here, Laplace transform method has been employed to find the exact solutions of the proposed problems. Despite of the fact that the Laplace transform method does not work always, for example, the problems involving second grade fluid. This is due to incompatibility between the prescribed data (For further details see Bandelli [8,9]). On the other hand, the Laplace transform technique has also been successfully used by several authors to determine the exact solutions of rate type viscoelastic fluids [10-23]. Generally, for these fluids a new initial condition is necessary apart from the condition that the fluid is initially at rest. In this article, we try to establish the exact analytic solutions for the accelerated flows of a generalized Burgers' fluid when the relaxation times satisfying the condition $\gamma < \lambda^2/4$. The analytical expressions for the velocity fields and associated tangential stresses are determined by means of Laplace transform. The corresponding solutions for Burgers' fluid are recovered as a special case of the solutions obtained here.

2. Governing Equations

For an incompressible flow, the governing equations are

$$\operatorname{div}\mathbf{V} = \mathbf{0}, \tag{1}$$

$$\rho\frac{d\mathbf{V}}{dt} = \operatorname{div}\mathbf{T}, \tag{2}$$

where Eq. (1) is the continuity equation and Eq. (2) is the momentum equation in the absence of body forces. In these equations $\mathbf{V}$ is the velocity field, ρ is the density, $\mathbf{T}$ is the Cauchy stress tensor and for a generalized Burgers' fluid, it is given by

$$\mathbf{T} = -p\mathbf{I} + \mathbf{S}, \tag{3}$$

$$\mathbf{S} + \lambda\frac{\delta\mathbf{S}}{\delta t} + \gamma\frac{\delta^2\mathbf{S}}{\delta t^2} = \mu\left(\mathbf{A} + \lambda_r\frac{\delta\mathbf{A}}{\delta t} + \eta\frac{\delta^2\mathbf{A}}{\delta t^2}\right),$$

where p is the pressure, $\mathbf{I}$ is the identity tensor, $\mathbf{S}$ is the extra stress tensor, μ is the dynamic viscosity, λ and $\lambda_r < \lambda$ are respectively the relaxation and retardation times, and γ and η are the material constants of generalized

Burgers' fluid. The first Rivlin-Ericksen tensor $\mathbf{A}$ and the upper convected time derivative $\delta/\delta t$ are defined as

$$\mathbf{A} = \mathbf{L} + \mathbf{L}^T, \mathbf{L} = \nabla\mathbf{V}, \tag{4}$$

$$\frac{\delta^2\mathbf{S}}{\delta t^2} = \frac{\delta}{\delta t}\left(\frac{\delta\mathbf{S}}{\delta t}\right) = \frac{\delta}{\delta t}\left(\frac{d\mathbf{S}}{dt} - \mathbf{LS} - \mathbf{SL}^T\right), \tag{5}$$

where d/dt denotes the usual material time derivative and $\mathbf{L}$ is the velocity gradient. For the problem under discussion, we assume that the velocity field and the shear stress are of the following forms:

$$\mathbf{V} = v(y,t)\mathbf{i},\ \mathbf{S} = \mathbf{S}(y,t), \tag{6}$$

where $\mathbf{i}$ is the unit vector along the $x-$coordinate direction and v is the $x-$component of velocity field $\mathbf{V}$. Invoking Eq. (6) the continuity equation (1) is identically satisfied and in view of initial conditions $\mathbf{S}(y,0) = \partial_t\mathbf{S}(y,0) = \mathbf{0}$, Eq. (3) gives

$$\left(1 + \lambda\partial_t + \gamma\partial_t^2\right)T(y,t) = \mu\left(1 + \lambda_r\partial_t + \eta\partial_t^2\right)\partial_y v(y,t), \tag{7}$$

and $S_{xz} = S_{yy} = S_{zz} = 0$, where $S_{xy} = T(y,t)$ is the tangential stress.

In the absence of external pressure gradient, Eq. (2) reduces to the relevant equation

$$\partial_y T(y,t) = \rho\partial_t v(y,t). \tag{8}$$

3. Flow due to Constantly Accelerated Plate

Consider an incompressible generalized Burgers' fluid occupying the upper half space of (x,y) plane. The fluid is bounded by a rigid plate at $y=0$ such that the positive $y-$axis is taken normal to the plate and $x-$axis is taken parallel to the plate. Initially, both the fluid and the plate are at rest. At time $t=0^+$, the plate is brought to a constant acceleration $f(t) = At$ and motion in the fluid is induced in the direction parallel to $x-$axis. Under these assumptions, the flow is governed by Eqs. (7) and (8) along with initial and boundary conditions:

$$v(y,0) = \partial_t v(y,0) = 0, T(y,0) = \partial_t T(y,0) = 0,\ y > 0, \tag{9}$$

$$v(0,t) = f(t), \tag{10}$$

$$v(y,t) \text{ and } T(y,t) \to 0 \text{ as } y \to \infty. \tag{11}$$

Introducing the following dimensionless variables

$$\tau = \frac{t}{\lambda},\ \xi = \frac{y}{c\lambda},\ U = \frac{v}{\lambda A} \text{ and } S = \frac{T}{\rho AcV}, \tag{12}$$

into Eqs. (7)-(11), we get

$$\left(1 + \partial_\tau + \beta\partial_\tau^2\right)S(\xi,\tau) = \left(1 + \alpha\partial_\tau + \beta_0\partial_\tau^2\right)\partial_\xi U(\xi,\tau); \xi,\tau > 0, \tag{13}$$

$$\partial_\xi S(\xi,\tau) = \partial_\tau U(\xi,\tau);\ \xi,\tau > 0, \tag{14}$$

$$U(\xi,0) = \partial_\tau U(\xi,0) = 0,\ S(\xi,0) = \partial_\tau S(\xi,0) = 0;\ \xi > 0, \tag{15}$$

$$U(0,\tau) = \tau, \tag{16}$$

$$U(\xi,\tau),\ S(\xi,\tau) \to 0 \text{ as } \xi \to \infty,\ \tau > 0. \tag{17}$$

where

$$c = \sqrt{\frac{\mu}{\rho\lambda}},\ \alpha = \frac{\lambda_r}{\lambda},\ \beta = \frac{\gamma}{\lambda^2} \text{ and } \beta_0 = \frac{\eta}{\lambda^2}. \tag{18}$$

Applying Laplace transform to Eqs. (13) and (14) along with boundary conditions given by Eqs. (16) and (17) in view of initial conditions (15) we obtain

$$\left(1 + q + \beta q^2\right)\bar{S}(\xi,q) = \left(1 + \alpha q + \beta_0 q^2\right)\frac{d\bar{U}(\xi,q)}{d\xi}, \tag{19}$$

$$\frac{d\bar{S}(\xi,q)}{d\xi} = q\bar{U}(\xi,q), \tag{20}$$

$$\bar{U}(0,q) = \frac{1}{q^2}, \tag{21}$$

$$\bar{U}(\xi,q),\ \bar{S}(\xi,q) \to 0 \text{ as } \xi \to \infty. \tag{22}$$

Eliminating $\bar{S}(\xi,q)$ between Eqs. (19) and (20), we get the following differential equation

$$\frac{d^2\bar{U}(\xi,q)}{d\xi^2} - \frac{q\left(\beta q^2 + q + 1\right)}{\left(\beta_0 q^2 + \alpha q + 1\right)}\bar{U}(\xi,q) = 0. \tag{23}$$

Now solving Eq. (23) and by using Eqs. (21) and (22) we get

$$\bar{U}(\xi,q) = \frac{1}{q^2}\exp\left(-\xi\sqrt{\frac{q\left(\beta q^2 + q + 1\right)}{(\beta_0 q^2 + \alpha q + 1)}}\right). \tag{24}$$

Incorporating Eq. (24) into Eq. (19), we obtain the following expression for $\bar{S}(\xi,q)$

$$\begin{aligned}\bar{S}(\xi,q) = &-\frac{1}{q^2}\sqrt{\frac{q\left(\beta q^2 + q + 1\right)}{(\beta_0 q^2 + \alpha q + 1)}} \\ &\times\exp\left(-\xi\sqrt{\frac{q\left(\beta q^2 + q + 1\right)}{(\beta_0 q^2 + \alpha q + 1)}}\right).\end{aligned} \tag{25}$$

3.1 Calculation of the Dimensionless Velocity

In order to determine $U(\xi,\tau) = L^{-1}\{\bar{U}(\xi,q)\}$, we decompose $\bar{U}(\xi,q)$ as follows:

$$\bar{U}(\xi,q) = \bar{U}_1(q)\bar{U}_2(\xi,q), \tag{26}$$

where

$$\begin{aligned}&\bar{U}_1(q) = \frac{w(q)}{q^2}, \bar{U}_2(\xi,q) = \frac{1}{w(q)}\exp\left(-\frac{\xi}{\sqrt{w(q)}}\right), \\ &w(q) = \frac{\beta_0 q^2 + \alpha q + 1}{q\left(\beta q^2 + q + 1\right)}.\end{aligned} \tag{27}$$

If $U_1(\tau)=L^{-1}\{\overline{U}_1(q)\}$ and $U_2(\xi,\tau)=L^{-1}\{\overline{U}_2(\xi,q)\}$, then by convolution theorem [24], we have

$$U(\xi,\tau)=\left(U_1*U_2\right)(\tau)=\int_0^{\tau}U_1(\tau-s)U_2(\xi,s)ds$$
$$=\int_0^{\tau}U_1(s)U_2(\xi,\tau-s)ds. \tag{28}$$

To find $U_1(\tau)$, there are three possible different cases for γ, i.e. $\gamma<\lambda^2/4, \gamma=\lambda^2/4$ and $\gamma>\lambda^2/4$. Here, we will only consider the case $\gamma<\lambda^2/4$ for which the quadratic equation $\beta q^2+q+1=0$, has real and distinct roots and the function $\overline{U}_1(q)$ takes the following form

$$\begin{aligned}\overline{U}_1(q)=&\frac{\beta_0 q_1^2 q_2^2+q_2^2+q_1q_2+q_1^2-\alpha q_1 q_2^2-\alpha q_1^2 q_2}{\beta\left(q_1q_2\right)^3}\\&\times\frac{1}{q}+\frac{\alpha q_1q_2-q_1-q_2}{\beta\left(q_1q_2\right)^2}\frac{1}{q^2}+\frac{1}{\beta q_1q_2}\frac{1}{q^3}\\&+\frac{\left(-\alpha q_1+\beta_0q_1^2+1\right)}{\beta q_1^3\left(q_1-q_2\right)}\frac{1}{q+q_1}\\&+\frac{\left(-\alpha q_2+\beta_0q_2^2+1\right)}{\beta q_2^3\left(q_2-q_1\right)}\frac{1}{q+q_2},\end{aligned} \tag{29}$$

where $q_{1,2}=\left(1\pm\sqrt{1-4\beta}\right)/2\beta$. Consequently, the inverse Laplace transform of Eq. (29) gives the following expression for $U_1(\tau)$

$$\begin{aligned}U_1(\tau)=\Bigg\{&\frac{\beta_0 q_1^2 q_2^2+q_2^2+q_1q_2+q_1^2-\alpha q_1 q_2^2-\alpha q_1^2 q_2}{\beta\left(q_1q_2\right)^3}\\&+\frac{\alpha q_1q_2-q_1-q_2}{\beta\left(q_1q_2\right)^2}\tau+\frac{1}{\beta q_1q_2}\frac{\tau^2}{2}+\frac{\left(-\alpha q_1+\beta_0q_1^2+1\right)}{\beta q_1^3\left(q_1-q_2\right)}e^{-q_1\tau}\\&+\frac{\left(-\alpha q_2+\beta_0q_2^2+1\right)}{\beta q_2^3\left(q_2-q_1\right)}e^{-q_2\tau}\Bigg\},\end{aligned} \tag{30}$$

or simply

$$U_1(\tau)=\left\{\begin{aligned}&\left(1+\beta_0-\alpha-\beta\right)-\left(1-\alpha\right)\tau\\&+\frac{\tau^2}{2}-\frac{\alpha_1}{q_1}e^{-q_1\tau}-\frac{\alpha_2}{q_2}e^{-q_2\tau}\end{aligned}\right\}. \tag{31}$$

where $\alpha_k=\frac{\alpha q_k-\beta_0q_k^2-1}{q_k\left(q_k-2\right)}$ for $k=1,2.$

Using the inversion formula for compound functions [24], we write $U_2(\xi,\tau)$ as follows:

$$U_2(\xi,\tau)=\int_0^{\infty}f\left(\xi,u\right)g\left(u,\tau\right)du$$
$$=\frac{1}{2\sqrt{\pi\xi}}\int_0^{\infty}\int_0^{\infty}\frac{z\sqrt{z}}{u\sqrt{u}}\exp\left(-\frac{z^2}{4u}\right)J_1\left(2\sqrt{\xi z}\right)g\left(u,\tau\right)dz\,du, \tag{32}$$

where $J_1(\cdot)$ is the Bessel function of first kind of order one.

To find $g(u,\tau)$, we decompose $w(q)$ in the following form

$$w(q)=\frac{1}{q}+\frac{\beta_1}{q+q_1}-\frac{\beta_2}{q+q_2}, \tag{33}$$

where β_1 and β_2 are given as

$$\beta_1=-\frac{1}{2}-\frac{\varepsilon\left(-1+2\alpha+\varepsilon\right)}{\left(-1+2\alpha+\varepsilon+2\beta_0\right)}+\frac{3\left(-1+2\alpha\right)\varepsilon}{\left(-2+4\alpha+4\beta_0\right)}, \tag{34}$$

and

$$\beta_2=\frac{1}{2}-\frac{\varepsilon\left(1-2\alpha+\varepsilon\right)}{\left(1-2\alpha+\varepsilon-2\beta_0\right)}+\frac{3\left(-1+2\alpha\right)\varepsilon}{\left(-2+4\alpha+4\beta_0\right)}, \tag{35}$$

with $\varepsilon=\frac{1-2\alpha-2\beta_0}{\sqrt{1-4\beta}}$. For the case $\gamma<\lambda^2/4$, we will discuss the following four possibilities.

Case 1: $\lambda_r=\frac{\lambda}{2}$ or $\lambda_r\neq\frac{\lambda}{2}$ and $\gamma<\lambda_r\left(\lambda-\lambda_r\right)$

In this case $\beta_1<0$ and $\beta_2>0$ and taking into consideration Eq. (33), $g(u,\tau)$ can be written as

$$g(u,\tau)=\delta(\tau)+g_1(u,\tau), \tag{36}$$

where

$$\begin{aligned}g_1(u,\tau)=&-\sqrt{\frac{u}{\tau}}J_1\left(2\sqrt{u\tau}\right)+\sqrt{\frac{-\beta_1u}{\tau}}e^{-q_1\tau}I_1\left(2\sqrt{-\beta_1u\tau}\right)\\&+\sqrt{\frac{\beta_2u}{\tau}}e^{-q_2\tau}I_1\left(2\sqrt{\beta_2u\tau}\right)-\int_0^{\tau}\frac{u\sqrt{-\beta_1}}{\sqrt{s(\tau-s)}}e^{-q_1(\tau-s)}\\&\times J_1\left(2\sqrt{us}\right)I_1\left(2\sqrt{-\beta_1u(\tau-s)}\right)ds\\&-\int_0^{\tau}\frac{u\sqrt{\beta_2}}{\sqrt{s(\tau-s)}}e^{-q_2(\tau-s)}J_1\left(2\sqrt{us}\right)I_1\left(2\sqrt{\beta_2u(\tau-s)}\right)ds\\&+\int_0^{\tau}\frac{u\sqrt{-\beta_1\beta_2}}{\sqrt{s(\tau-s)}}e^{-q_1s-q_2(\tau-s)}I_1\left(2\sqrt{-\beta_1us}\right)\\&\times I_1\left(2\sqrt{\beta_2u(\tau-s)}\right)ds\\&-\int_0^{\tau}\int_0^{s}\frac{u\sqrt{-\beta_1\beta_2u}}{\sqrt{\sigma(s-\sigma)(\tau-s)}}e^{-q_1(s-\sigma)-q_2(\tau-s)}J_1\left(2\sqrt{u\sigma}\right)\\&\times I_1\left(2\sqrt{-\beta_1u(s-\sigma)}\right)I_1\left(2\sqrt{\beta_2u(\tau-s)}\right)d\sigma\,ds,\end{aligned} \tag{37}$$

where $\delta(\cdot)$ is the Dirac delta function and $I_1(\cdot)$ is the modified Bessel function of the first kind of order one.

Invoking Eq. (36) into Eq. (32), having in mind Eq. (28), the expression for the dimensionless velocity is

$$
\begin{aligned}
U(\xi,\tau) &= \frac{U_1(\tau)}{2\sqrt{\pi\xi}} \int_0^\infty\int_0^\infty \frac{z\sqrt{z}}{u\sqrt{u}} \exp\left(-\frac{z^2}{4u}\right) \\
&\times J_1\left(2\sqrt{\xi z}\right) dz du + \frac{1}{2\sqrt{\pi\xi}} \int_0^\tau\int_0^\infty\int_0^\infty \frac{z\sqrt{z}}{u\sqrt{u}} \\
&\times \exp\left(-\frac{z^2}{4u}\right) J_1\left(2\sqrt{\xi z}\right) \\
&\times U_1(\tau - s) g_1(u,s) dz\, du\, ds,
\end{aligned} \tag{38}
$$

where $U_1(\tau)$ and $g_1(u,\tau)$ are given by Eqs. (31) and (37).

Case 2: $\lambda_r \in \left[0,\frac{\lambda}{2}\right)$ and $\gamma > \lambda_r(\lambda - \lambda_r)$

In this case both constants β_1 and β_2 are positive. Adopting similar procedure as in the first case, as a result we found an expression for $U(\xi,\tau)$ same as given by Eq. (38) with

$$
\begin{aligned}
g_2(u,\tau) &= -\sqrt{\frac{u}{\tau}} J_1\left(2\sqrt{u\tau}\right) + \sqrt{\frac{\beta_2 u}{\tau}} e^{-q_2\tau} I_1\left(2\sqrt{\beta_2 u\tau}\right) \\
&-\sqrt{\frac{\beta_1 u}{\tau}} e^{-q_1\tau} J_1\left(2\sqrt{\beta_1 u\tau}\right) \\
&-\int_0^\tau \frac{u\sqrt{\beta_2}}{\sqrt{s(\tau-s)}} e^{-q_2(\tau-s)} J_1\left(2\sqrt{us}\right) I_1\left(2\sqrt{\beta_2 u(\tau-s)}\right) ds \\
&+\int_0^\tau \frac{u\sqrt{\beta_1}}{\sqrt{s(\tau-s)}} e^{-q_1(\tau-s)} J_1\left(2\sqrt{us}\right) J_1\left(2\sqrt{\beta_1 u(\tau-s)}\right) ds \\
&-\int_0^\tau \frac{u\sqrt{\beta_1\beta_2}}{\sqrt{s(\tau-s)}} e^{-q_2 s - q_1(\tau-s)} I_1\left(2\sqrt{\beta_2 us}\right) J_1\left(2\sqrt{\beta_1 u(\tau-s)}\right) ds \\
&+\int_0^\tau\int_0^s \frac{u\sqrt{\beta_1\beta_2 u}}{\sqrt{\sigma(s-\sigma)(\tau-s)}} e^{-q_1(\tau-s)-q_2(s-\sigma)} J_1\left(2\sqrt{u\sigma}\right) \\
&\times J_1\left(2\sqrt{\beta_1 u(\tau-s)}\right) I_1\left(2\sqrt{\beta_2 u(s-\sigma)}\right) d\sigma\, ds
\end{aligned} \tag{39}
$$

instead of $g_1(u,\tau)$.

Case 3: $\lambda_r \in \left(\frac{\lambda}{2},\lambda\right)$ and $\gamma > \lambda_r(\lambda - \lambda_r)$

Here both β_1 and β_2 are negative and the corresponding velocity $U(\xi,\tau)$ have the same form as given by Eq. (38) with

$$
\begin{aligned}
g_3(u,\tau) &= -\sqrt{\frac{u}{\tau}} J_1\left(2\sqrt{u\tau}\right) + \sqrt{\frac{-\beta_1 u}{\tau}} e^{-q_1\tau} I_1\left(2\sqrt{-\beta_1 u\tau}\right) \\
&-\sqrt{\frac{-\beta_2 u}{\tau}} e^{-q_2\tau} J_1\left(2\sqrt{-\beta_2 u\tau}\right) \\
&-\int_0^\tau \frac{u\sqrt{-\beta_1}}{\sqrt{s(\tau-s)}} e^{-q_1(\tau-s)} J_1\left(2\sqrt{us}\right) I_1\left(2\sqrt{-\beta_1 u(\tau-s)}\right) ds \\
&+\int_0^\tau \frac{u\sqrt{-\beta_2}}{\sqrt{s(\tau-s)}} e^{-q_2(\tau-s)} J_1\left(2\sqrt{us}\right) J_1\left(2\sqrt{-\beta_2 u(\tau-s)}\right) ds \\
&-\int_0^\tau \frac{u\sqrt{\beta_1\beta_2}}{\sqrt{s(\tau-s)}} e^{-q_1 s - q_2(\tau-s)} I_1\left(2\sqrt{-\beta_1 us}\right) J_1\left(2\sqrt{-\beta_2 u(\tau-s)}\right) ds \\
&+\int_0^\tau\int_0^s \frac{u\sqrt{\beta_1\beta_2 u}}{\sqrt{\sigma(s-\sigma)(\tau-s)}} e^{-q_1(s-\sigma)-q_2(\tau-s)} J_1\left(2\sqrt{u\sigma}\right) \\
&\times I_1\left(2\sqrt{-\beta_1 u(s-\sigma)}\right) J_1\left(2\sqrt{-\beta_2 u(\tau-s)}\right) d\sigma\, ds
\end{aligned} \tag{40}
$$

instead of $g_1(u,\tau)$.

Case 4: $\lambda_r \neq \frac{\lambda}{2}$ and $\gamma = \lambda_r(\lambda - \lambda_r)$

In this case for $\varepsilon = \pm 1$, we have respectively

$$
\begin{aligned}
&q_3 = \frac{1}{\alpha+\beta_0}, q_4 = \frac{1}{1-\alpha-\beta_0}, \\
&\beta_3 = \frac{\beta_0(1+\alpha+\beta_0)}{(\alpha+\beta_0)(2\alpha+2\beta_0-1)}, \beta_4 = \frac{1+2\alpha^2+(3\alpha+\beta_0)(\beta_0-1)}{(\alpha+\beta_0-1)(2\alpha+2\eta-1)}, \\
&\alpha_3 = \frac{\beta_0(1+\alpha+\beta_0)}{2\alpha+2\beta_0-1}, \alpha_4 = \frac{1+2\alpha^2+(3\alpha+\beta_0)(\beta_0-1)}{1-2\alpha-2\beta_0}
\end{aligned} \tag{41}
$$

and

$$
\begin{aligned}
q_3 &= \frac{1}{1-\alpha-\beta_0}, q_4 = \frac{1}{1+\beta_0}, \\
\beta_3 &= -\frac{1+2\alpha^2+3\alpha(\beta_0-1)-\beta_0+\beta_0^2}{(\alpha+\beta_0-1)(2\alpha+2\beta_0-1)}, \\
\beta_4 &= \frac{\beta_0(1+\alpha+\beta_0)}{(\alpha+\beta_0)(2\alpha+2\eta-1)}, \\
\alpha_3 &= \frac{1+2\alpha^2+(3\alpha+\beta_0)(\beta_0-1)}{1-2\alpha-2\beta_0}, \\
\alpha_4 &= \frac{\beta_0(1+\alpha+\beta_0)}{2\alpha+2\beta_0-1}.
\end{aligned} \tag{42}
$$

In both cases $\varepsilon = \pm 1$, $w(q)$ has the same expression as given by Eq. (33). Thus, in view of above restrictions the Laplace inverse of $\overline{U}_1(q)$ results in

$$
U_1(\tau) = \left\{ \begin{aligned} &(1+\beta_0-\alpha-\beta)-(1-\alpha)\tau+\frac{\tau^2}{2} \\ &-\frac{\alpha_3}{q_3}e^{-q_3\tau}-\frac{\alpha_4}{q_4}e^{-q_4\tau} \end{aligned} \right\}. \tag{43}
$$

Thereby, the expression for the dimensionless velocity $U(\xi,\tau)$ in view of Eq. (28) is given by Eq. (38) with

$$g_4(u,\tau)=-\sqrt{\frac{u}{\tau}}J_1\left(2\sqrt{u\tau}\right)+\sqrt{\frac{-\beta_3 u}{\tau}}e^{-q_3\tau}I_1\left(2\sqrt{-\beta_3 u\tau}\right)$$
$$+\sqrt{\frac{\beta_4 u}{\tau}}e^{-q_4\tau}I_1\left(2\sqrt{\beta_4 u\tau}\right)$$
$$-\int_0^\tau \frac{u\sqrt{-\beta_3}}{\sqrt{s(\tau-s)}}e^{-q_3(\tau-s)}J_1\left(2\sqrt{us}\right)I_1\left(2\sqrt{-\beta_3 u(\tau-s)}\right)ds$$
$$-\int_0^\tau \frac{u\sqrt{\beta_4}}{\sqrt{s(\tau-s)}}e^{-q_4(\tau-s)}J_1\left(2\sqrt{us}\right)I_1\left(2\sqrt{\beta_4 u(\tau-s)}\right)ds$$
$$+\int_0^\tau \frac{u\sqrt{-\beta_3\beta_4}}{\sqrt{s(\tau-s)}}e^{-q_3 s-q_4(\tau-s)}I_1\left(2\sqrt{-\beta_3 us}\right)$$
$$\times I_1\left(2\sqrt{\beta_4 u(\tau-s)}\right)ds$$
$$-\int_0^\tau\int_0^s \frac{u\sqrt{-\beta_3\beta_4 u}}{\sqrt{\sigma(s-\sigma)(\tau-s)}}e^{-q_3(s-\sigma)-q_4(\tau-s)}J_1\left(2\sqrt{u\sigma}\right)$$
$$\times I_1\left(2\sqrt{-\beta_3 u(s-\sigma)}\right)I_1\left(2\sqrt{\beta_4 u(\tau-s)}\right)d\sigma\, ds, \tag{44}$$

instead of $g_1(u,\tau)$ and $U_1(\tau)$ is given by Eq. (43).

3.2 Calculation of Dimensionless Shear Stress

For shear stress we write Eq. (25) in the following form

$$\bar{S}(\xi,q)=-\bar{S}_1(q)\bar{S}_2(\xi,q), \tag{45}$$

where

$$\bar{S}_1(q)=\frac{\beta_0 q^2+\alpha q+1}{q^2\left(\beta q^2+q+1\right)},$$
$$\bar{S}_2(\xi,q)=\frac{1}{\sqrt{w(q)}}\exp\left(-\frac{\xi}{\sqrt{w(q)}}\right), \tag{46}$$

and $w(q)$ is given by Eq. (33).

If $S_1(\tau)=L^{-1}\{\bar{S}_1(q)\}$ and $S_2(\xi,\tau)=L^{-1}\{\bar{S}_2(\xi,q)\}$, then by convolution product we can write

$$S(\xi,\tau)=-(S_1*S_2)(\tau)=-\int_0^\tau S_1(\tau-s)S_2(\xi,s)ds$$
$$=-\int_0^\tau S_1(s)S_2(\xi,\tau-s)ds. \tag{47}$$

To find $S_1(\tau)$, we arrange $\bar{S}_1(q)$ in the following form

$$\bar{S}_1(q)=\frac{\alpha q_1 q_2-q_1-q_2}{\beta(q_1q_2)^2}\frac{1}{q}+\frac{1}{\beta q_1 q_2}\frac{1}{q^2}+\frac{-\beta_0 q_1^2+\alpha q_1-1}{q_1(q_1-2)}$$
$$\times\frac{1}{q+q_1}+\frac{-\beta_0 q_2^2+\alpha q_2-1}{q_2(q_2-2)}\frac{1}{q+q_2}. \tag{48}$$

The Laplace inverse of Eq. (48) gives

$$S_1(\tau)=\alpha_1 e^{-q_1\tau}+\alpha_2 e^{-q_2\tau}+\tau+\alpha-1, \tag{49}$$

where q_1, q_2, α_1 and α_2 having similar expressions as given before. Adopting similar procedure as for $U(\xi,\tau)$, one obtains the following expression for the shear stress

$$S(\xi,\tau)=-\frac{S_1(\tau)}{2\sqrt{\pi}}\int_0^\infty\int_0^\infty \frac{z}{u\sqrt{u}}\exp\left(-\frac{z^2}{4u}\right)J_0\left(2\sqrt{\xi z}\right)dz\, du$$
$$-\frac{1}{2\sqrt{\pi}}\int_0^\tau\int_0^\infty\int_0^\infty \frac{z}{u\sqrt{u}}\exp\left(-\frac{z^2}{4u}\right)J_0\left(2\sqrt{\xi z}\right) \tag{50}$$
$$\times S_1(\tau-s)g_k(u,s)dz\, du\, ds,$$

where $g_k(u,\tau)$ with $k=1,2,3,$ are given by Eqs. (37), (39) or (40) and $S_1(\tau)$ is given by Eq. (49). In the last case

$$S_1(\tau)=\alpha_3 e^{-q_3\tau}+\alpha_4 e^{-q_4\tau}+\tau+\alpha-1, \tag{51}$$

where α_3,α_4,q_3 and q_4 are same as given in Eqs. (41) and (42). Consequently, the expression for $S(\xi,\tau)$ is given by

$$S(\xi,\tau)=-\left[\frac{\alpha_3 e^{-q_3\tau}+\alpha_4 e^{-q_4\tau}+\tau+\alpha-1}{2\sqrt{\pi}}\right]$$
$$\times\int_0^\infty\int_0^\infty \frac{z}{u\sqrt{u}}\exp\left(-\frac{z^2}{4u}\right)J_0\left(2\sqrt{\xi z}\right)dzdu$$
$$-\frac{1}{2\sqrt{\pi}}\int_0^\tau\int_0^\infty\int_0^\infty \frac{z}{u\sqrt{u}}\exp\left(-\frac{z^2}{4u}\right)J_1\left(2\sqrt{\xi z}\right)g_1(u,s) \tag{52}$$
$$\times\left[\alpha_3 e^{-q_3(\tau-s)}+\alpha_4 e^{-q_4(\tau-s)}+(\tau-s)+\alpha-1\right]dz\, du\, ds,$$

where $g_1(u,\tau)$ is same as given by Eq. (37).

4. Special Case

Taking the limit of Eqs. (38) and (50), as $\eta\to 0$ and hence $\beta_0\to 0,$ we get the following expressions for the velocity and shear stress

$$U(\xi,\tau)=\frac{U(\tau)}{2\sqrt{\pi\xi}}\int_0^\infty\int_0^\infty \frac{z\sqrt{z}}{u\sqrt{u}}\exp\left(-\frac{z^2}{4u}\right)J_1\left(2\sqrt{\xi z}\right)dzdu$$
$$+\frac{1}{2\sqrt{\pi\xi}}\int_0^\tau\int_0^\infty\int_0^\infty \frac{z\sqrt{z}}{u\sqrt{u}}\exp\left(-\frac{z^2}{4u}\right)J_1\left(2\sqrt{\xi z}\right) \tag{53}$$
$$\times U(\tau-s)g(u,s)dz\, du\, ds,$$

$$S(\xi,\tau)=-\frac{S(\tau)}{2\sqrt{\pi}}\int_0^\infty\int_0^\infty \frac{z}{u\sqrt{u}}\exp\left(-\frac{z^2}{4u}\right)\times J_0\left(2\sqrt{\xi z}\right)dz\, du$$
$$-\frac{1}{2\sqrt{\pi}}\int_0^\tau\int_0^\infty\int_0^\infty \frac{z}{u\sqrt{u}}\times\exp\left(-\frac{z^2}{4u}\right)J_0\left(2\sqrt{\xi z}\right) \tag{54}$$
$$\times S(\tau-s)g(u,s)dz\, du\, ds,$$

with

$$U(\tau)=\left\{\begin{array}{l}(1-\alpha-\beta)-(1-\alpha)\tau+\dfrac{\tau^2}{2}\\ -\dfrac{e^{-q_1\tau}}{q_1}-\dfrac{e^{-q_2\tau}}{q_2}\end{array}\right\}, \tag{55}$$

$$S_1(\tau)=\alpha_1 e^{-q_1\tau}+\alpha_2 e^{-q_2\tau}+\tau+\alpha-1, \tag{56}$$

where $\alpha_k=\frac{\alpha q_k-1}{q_k(q_k-2)}$.

The expression for $g(u,\tau)$ is same as for $g_1(u,\tau)$ given by Eq. (37) with the following values of β_1,β_2 and ε

$$\beta_1=\frac{\varepsilon-1}{2},\beta_2=\frac{\varepsilon+1}{2},\ \varepsilon=\frac{1-2\alpha}{\sqrt{1-4\beta}}. \tag{57}$$

It is important to note that the limiting solutions (53) and (54) of present study are similar to Khan *et al*. [3, Eqs. (41) and (53)].

5. Concluding Remarks

This article reports the exact solution for the accelerated flows of a generalized Burgers' fluid when the relaxation time satisfy the condition $\gamma<\lambda^2/4$. The flow is induced by uniform accelerated motion of the plate. The Laplace transform method was used for finding the expressions for velocity field and adequate tangential stress. The solutions are obtained for four different cases. The solutions obtained in [3] are recovered as a special case.

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