CHAPTER 3 : A NEW EFFICIENT APPROXIMATION OF CONCENTRATION PARAMETER

3.1 Introduction

The aim of this chapter is to propose an efficient approximation for mean resultant length $A(\kappa)$ and concentration parameter $\hat{\kappa}$. For that reason, a brief discussion on the general construction of mean resultant length $A(\kappa)$ is given in Section 3.2, and the existence approximation solution of the concentration parameter is listed for large and small κ in Section 3.3. A new formula of $A(\kappa)$ is constructed using the reconstruction of summation series of $I_0(\kappa)$ using two approaches namely, piecewise approximation and maximum likelihood estimator. The new approximation method is given in Section 3.4.

Furthermore, new approximation solutions of $\hat{\kappa}$ is also proposed using two approaches in Section 3.5. First, we consider the power series expansion of the mean resultant length and the estimate of the concentration parameter may be obtained by the roots of a polynomial function. Detailed description is given in Section 3.6. Secondly, we consider the power series expansion of the reciprocal of a Bessel function in the log-likelihood function of the concentration parameter and the estimate of concentration parameter may be obtained by minimizing the negative value of the log-likelihood function. Detailed description is given in Section 3.7.

The concentration parameter from both approaches may be estimated, for example, using the polyroot function and minimum sum function in the `SPlus` package. The efficiency of new proposed method is then tested using a simulation study with random data, and again with applications data. Detailed descriptions are given in Section 3.8 to Section 3.10. Result, discussion and conclusion of new proposed method are provided at the end of this chapter.

3.2 The General Construction of Mean Resultant Length

A brief discussion on the general construction of mean resultant length of von Mises distribution can be found in Mardia and Jupp (2000), and Jammalamadaka and SenGupta (2001). Recall that, the von Mises distribution is symmetrical about mean direction μ and has pdf

$$f(\theta;\mu,\kappa)=\frac{1}{2\pi I_0(\kappa)}e^{\kappa\cos(\theta-\mu)},\quad 0\le\theta<2\pi, \tag{2.12}$$

where $0\le\mu<2\pi$ and $\kappa\ge 0$ are parameters. Thus, the distribution is invariant under the transformation

$$\theta\mapsto\mu-\theta, \tag{3.1}$$

and its density has the following property,

$$f(\theta-\mu)=f(\mu-\theta). \tag{3.2}$$

Generally, the value of mean direction $\bar{\theta}$ and mean resultant length $\bar{R}$ are obtained from the first trigonometric moment about zero direction. The p^{th} trigonometric moment about zero direction is given by

$$\phi_p=\alpha_p+i\beta_p=E\left(e^{ip\theta}\right)=\int_0^{2\pi}e^{ip\theta}dF(\theta),\quad p=0,\pm1,\pm2,\ldots \tag{3.3}$$

where

$$\phi_0 = 1, \qquad \left|\phi_p\right| \le 1,$$

complex conjugate of ϕ_p, $\overline{\phi}_p = \phi_{-p}$,

$$\alpha_p = E\left(\cos p\theta\right) = \int_0^{2\pi} \cos p\theta dF\left(\theta\right),$$

$$\beta_p = E\left(\sin p\theta\right) = \int_0^{2\pi} \sin p\theta dF\left(\theta\right),$$

$F\left(\theta\right) = \int_0^{2\pi} \frac{1}{2\pi I_0\left(\kappa\right)} e^{\kappa\cos\left(\theta-\mu\right)} d\theta$ is cumulative density function,

and $\alpha_{-p} = \alpha_p, \quad \beta_{-p} = \beta_p, \quad \left|\alpha_p\right| \le 1, \quad \left|\beta_p\right| \le 1$.

Hence, for $p \ge 0$,

$$\phi_p = \rho_p e^{i\mu_p}, \tag{3.4}$$

where $\rho_p = \sqrt{\alpha_p^2 + \beta_p^2}$, $\mu_p = \tan^{-1}\left(\frac{\beta_p}{\alpha_p}\right)$ and $\rho_p \ge 0$. When $p = 1$,

$$\phi_1 = \rho_1 e^{i\mu_1} = \rho e^{i\mu}. \tag{3.5}$$

Also, the p^{th} trigonometric moment about mean direction is defined by

$$\overline{\phi}_p = E\left(e^{ip\left(\theta-\mu\right)}\right) = \int_0^{2\pi} e^{ip\left(\theta-\mu\right)} dF\left(\theta-\mu\right) = \overline{\alpha}_p + i\overline{\beta}_p, \tag{3.6}$$

where

$\overline{\alpha}_p = E\left(\cos p\left(\theta-\mu\right)\right) = \int_0^{2\pi} \cos p\left(\theta-\mu\right) dF\left(\theta-\mu\right)$, and

$$\overline{\beta}_p = E\left(\sin p\left(\theta-\mu\right)\right) = \int_0^{2\pi} \sin p\left(\theta-\mu\right) dF\left(\theta-\mu\right).$$