Compatible Pair of Nontrivial Actions for Some Cyclic Groups of 2-Power Order

Sahimel Azwal Sulaiman[a]**,** Mohd Sham Mohamad[b] and Yuhani Yusof[c]

[a,b,c]*Futures and Trends Research Groups, Faculty of Industrial Sciences & Technology, Universiti Malaysia Pahang, Lebuhraya Tun Razak, 26300 Gambang, Kuantan, Pahang Darul Makmur*

[a]*titus1704@yahoo.com*, [b]*mohdsham@ump.edu.my*, [c]*yuhani@ump.edu.my*

Abstract. Compatible action played a very important verification before the nonabelian tensor product can be computed. This paper gives the some exact number of compatible pairs of actions for some cyclic groups of 2-power order. Some necessary and sufficient number theoretical conditions for a pair of cyclic groups of 2-power order with nontrivial actions act compatibly on each other are used to investigate some properties to find the exact number of compatible pairs of actions. Algorithms in Groups, Algorithms and Programming (GAP) software are constructed to create more examples on selected case. New results on compatible pair of nontrivial actions of order two and four are given.

Keywords: Compatible Actions, Cyclic Groups, Nonabelian Tensor Product

INTRODUCTION

Let G and H be groups which act on each other and each of which acts on itself by conjugation, then the actions are compatible if ${}^{({}^{g}h)}g' = {}^{g}({}^{h}({}^{g^{-1}}g'))$ and ${}^{({}^{h}g)}h' = {}^{h}({}^{g}({}^{h^{-1}}h'))$. This compatible action played a very important verification before the nonabelian tensor product can be computed. Brown and Loday [1] were the first to introduce the concept of the nonabelian tensor product of groups with the compatible actions. Compatible conditions are the prerequisite before nonabelian tensor product can be calculated.

This construction of the nonabelian tensor product has its origins in the algebraic K-theory and in homotopy theory. Several authors, starting with Brown, Johnson and Robertson [2], have studied group theoretical aspects of nonabelian tensor products extensively. A paper by Kappe [3] gives an overview of known results and literature.

Many authors are working with nonabelian tensor square (where $H = G$), and the compatibly conditions can be checked easily. It is interesting to check the compatible conditions when calculating tensor product since the groups are not the same or when the actions are nontrivial. Ellis and McDermott [4] had done this but specifically for quaternion group of order 32. Visscher [5] in 1998 had provided necessary and sufficient conditions for a pair of cyclic groups to act compatibly. He gave complete compatible conditions when one of the actions is trivial or both actions are trivial in cyclic groups of p-power order. In this research, all compatible actions involving at least one of the actions having order greater than two will be determined as a continuation from Visscher's.

Groups, Algorithms and Programming (GAP) software [6] is used to determine the conditions in which the actions satisfy the compatibility conditions in some finite cyclic groups.

SOME PREPARATORY RESULTS ON COMPATIBILITY CONDITIONS

The nonabelian tensor product $G \otimes H$ is defined only if g and h act on each other in such way that the actions satisfy certain compatibility condition. Start with the definition of an action a group g on a group h.

Definition 2.1 [5]

Let G and H be groups. An action of G on H is a mapping, $\Phi : G \to \mathrm{Aut}(H)$ such that

$$\Phi(gg')(h) = \Phi(g)(\Phi(g')(h))$$

for all $g, g' \in G$ and $h \in H$.

For a group *g* to act on a group *h*, by using the fact that the identity in *g* acts as the identity mapping on *h*, thus this implies that all elements in *g* act as automorphism of *h* on *h*. The definition of a compatible action is given as follows.

Definition 2.1 [5]
Let *G* and *H* be groups which act on each other. These mutual actions are said to be compatible with each other and with the actions of *G* and *H* on themselves by conjugation if

$$^{({}^{g}h)}g' = {}^{g}({}^{h}({}^{g^{-1}}g')) \text{ and } {}^{({}^{h}g)}h' = {}^{h}({}^{g}({}^{h^{-1}}h'))$$

for all $g, g' \in G$ and $h \in H$.

Since we are focusing on finite cyclic groups, the compatibility conditions which hold for abelian groups will be reduce to $^{({}^{g}h)}g' = {}^{h}g'$ and $^{({}^{h}g)}h' = {}^{g}h'$, for all $g, g' \in G$ and $h \in H$. For compatibility of actions of *G* and *H* on each other, it suffices for the compability conditions to hold on the generators of *G* and *H*. Visscher [7] has done this and he gave some necessary and sufficient number theoretical conditions for a pair of finite cyclic groups to act compatibly on each other as stated in Proposition 2.3.

Proposition 2.3 [5]
Let $G = \langle x \rangle = C_m$ and $H = \langle y \rangle = C_n$ be finite cyclic groups. Then there exist mutual actions of *G* and *H* on each other such that ${}^{y}x = x^k$ and ${}^{x}y = y^l$ for $k, l \in \mathbb{Z}$ if and only if conditions (i) and (ii) below are satisfied. These action are compatible if and only condition (iii) is satisfied as well.

(i) $(k, m) = (l, n) = 1$

(ii) $k^n \equiv 1 \bmod m$ and $l^m \equiv 1 \bmod n$

(iii) $k^{l-1} \equiv 1 \bmod m$ and $l^{k-1} \equiv 1 \bmod n$.

In the next section, all compatible actions involving at least one of the actions having order greater than two will be determined**.**

COMPATIBLE CONDITIONS FOR NONTRIVIAL ACTION

In the previous section, we can see that Visscher had done the classification of compatibility condition focusing on cyclic groups and provide necessary and sufficient conditions for the general case without any specific condition.

Now, our aim is to determine the compatible conditions if the order of the action is more than two. All computations are doing by GAP. New GAP coding has been built to calculate the compatible actions.

Let $G = \langle x \rangle = C_m$ and $H = \langle y \rangle = C_m$ be finite cyclic groups. Let the actions of *G* and *H* on each other be such that ${}^{y}x = x^k$ and ${}^{x}y = y^l$ for $k, l \in \mathbb{Z}$. Now consider that both actions have order four. The compatible conditions that are satisfied by *k* and *l* are given in **Table 1** when $G = H$ and in **Table 2** when $G \neq H$.

TABLE 1. Values of *k* and *l* in which the action of order four satisfying the compatible conditions when $G = H$.

Groups	k	l		Groups	k	l
$G = H = C_{2^4}$	5	5		$G = H = C_{2^7}$	33	33
	13	13			97	97
$G = H = C_{2^5}$	9	9		$G = H = C_{2^8}$	65	65
	25	25			193	193
$G = H = C_{2^6}$	17	17		$G = H = C_{2^9}$	129	129
	49	49			385	385

TABLE 2. Values of *k* and *l* in which the action of order four satisfying the compatible conditions when $G \neq H$.

Groups	k	l	Groups	k	l	Groups	k	l
$G = C_{2^4}$ and $H = C_{2^5}$	5	9	$G = C_{2^5}$ and $H = C_{2^4}$	9	5	$G = C_{2^6}$ and $H = C_{2^4}$	17	5
	5	25		9	13		17	13
	13	9		25	5		49	5
	13	25		25	13		49	13
$G = C_{2^4}$ and $H = C_{2^6}$	5	17	$G = C_{2^5}$ and $H = C_{2^6}$	9	17	$G = C_{2^6}$ and $H = C_{2^5}$	17	9
	5	49		9	49		17	25
	13	17		25	17		49	9
	13	49		25	49		49	25
$G = C_{2^4}$ and $H = C_{2^7}$	5	33	$G = C_{2^5}$ and $H = C_{2^7}$	9	33	$G = C_{2^6}$ and $H = C_{2^7}$	17	33
	5	97		9	97		17	97
	13	33		25	33		49	33
	13	97		25	97		49	97
$G = C_{2^4}$ and $H = C_{2^8}$	5	65	$G = C_{2^5}$ and $H = C_{2^8}$	9	65	$G = C_{2^6}$ and $H = C_{2^8}$	17	65
	5	193		9	193		17	193
	13	65		25	65		49	65
	13	193		25	193		49	193
$G = C_{2^4}$ and $H = C_{2^9}$	5	129	$G = C_{2^5}$ and $H = C_{2^9}$	9	129	$G = C_{2^6}$ and $H = C_{2^9}$	17	129
	5	385		9	385		17	385
	13	129		25	129		49	129
	13	385		25	385		49	385

Groups	k	l	Groups	k	l	Groups	k	l
$G = C_{2^7}$ and $H = C_{2^4}$	33	5	$G = C_{2^8}$ and $H = C_{2^4}$	65	5	$G = C_{2^9}$ and $H = C_{2^4}$	129	5
	33	13		65	13		129	13
	97	5		193	5		385	5
	97	13		193	13		385	13
$G = C_{2^7}$ and $H = C_{2^5}$	33	9	$G = C_{2^8}$ and $H = C_{2^5}$	65	9	$G = C_{2^9}$ and $H = C_{2^5}$	129	9
	33	25		65	25		129	25
	97	9		193	9		385	9
	97	25		193	25		385	25
$G = C_{2^7}$ and $H = C_{2^6}$	33	17	$G = C_{2^8}$ and $H = C_{2^6}$	65	17	$G = C_{2^9}$ and $H = C_{2^6}$	129	17
	33	49		65	49		129	49
	97	17		193	17		385	17
	97	49		193	49		385	49
$G = C_{2^7}$ and $H = C_{2^8}$	33	65	$G = C_{2^8}$ and $H = C_{2^7}$	65	33	$G = C_{2^9}$ and $H = C_{2^7}$	129	33
	33	193		65	97		129	97
	97	65		193	33		385	33
	97	193		193	97		385	97
$G = C_{2^7}$ and $H = C_{2^9}$	33	129	$G = C_{2^8}$ and $H = C_{2^9}$	65	129	$G = C_{2^9}$ and $H = C_{2^8}$	129	65
	33	385		65	385		129	193
	97	129		193	129		385	65
	97	385		193	385		385	193

This lead the following results:

Theorem 3.1
Let $C_{2^\alpha} = \langle x \rangle$ and $C_{2^\beta} = \langle y \rangle$ be cyclic groups of 2-power order. Furthermore, let C_{2^α} and C_{2^β} act on each other so that ${}^y x = x^k$ and ${}^x y = y^l$ for $k, l \in \mathbb{N}$ with $(k,2) = (l,2) = 1$. Then C_{2^α} and C_{2^β} act compatibly on each other when $\alpha = \beta$ with ${}^{y^4} x = x$ and ${}^{x^4} y = y$, if and only if *k* and *l* are congruent to one of the following:

(i) $k = l \equiv 1 + 2^{\alpha-2} \bmod 2^\alpha$;
(ii) $k = l \equiv 1 - 2^{\alpha-2} \bmod 2^\alpha$.

Theorem 3.2
Let $C_{2^\alpha} = \langle x \rangle$ and $C_{2^\beta} = \langle y \rangle$ be cyclic groups of 2-power order. Furthermore, let C_{2^α} and C_{2^β} act on each other so that ${}^y x = x^k$ and ${}^x y = y^l$ for $k, l \in \mathbb{N}$ with $(k,2) = (l,2) = 1$. Then C_{2^α} and C_{2^β} act compatibly on each other when $\alpha \neq \beta$ with ${}^{y^4} x = x$ and ${}^{x^4} y = y$, if and only if *k* and *l* are congruent to one of the following:

(i) $k \equiv 1 + 2^{\alpha-2} \bmod 2^\alpha$ and $l \equiv 1 + 2^{\beta-2} \bmod 2^\beta$;
(ii) $k \equiv 1 + 2^{\alpha-2} \bmod 2^\alpha$ and $l \equiv 1 - 2^{\beta-2} \bmod 2^\beta$;
(iii) $k \equiv 1 - 2^{\alpha-2} \bmod 2^\alpha$ and $l \equiv 1 + 2^{\beta-2} \bmod 2^\beta$;
(iv) $k \equiv 1 - 2^{\alpha-2} \bmod 2^\alpha$ and $l \equiv 1 - 2^{\beta-2} \bmod 2^\beta$.

Next, let the actions of *G* and *H* be on each other such that ${}^y x = x^k$ and ${}^x y = y^l$ for $k, l \in \mathbb{Z}$. Now consider that one of the actions has order four and the other one has order two. The compatible conditions that are satisfied by *k* and *l* are given in **Table 3** when $G = H$ and in **Table 4** when $G \neq H$.

TABLE 3. Values of *k* for which the action has order four and *l* for which the action had order two satisfying the compatible conditions when the groups *G* and *H* are the same.

Groups	k	l		Groups	k	l		Groups	k	l
$G = H = C_{2^4}$	3	9		$G = H = C_{2^4}$	3	9		$G = H = C_{2^4}$	3	9
	5	9			5	9			5	9
	11	9			11	9			11	9
	13	9			13	9			13	9
$G = H = C_{2^5}$	7	17		$G = H = C_{2^5}$	7	17		$G = H = C_{2^5}$	7	17
	9	17			9	17			9	17
	23	17			23	17			23	17
	25	17			25	17			25	17

TABLE 4. Values of k for which the action has order four and l for which the action has order two satisfying the compatible conditions when the groups G and H are different.

Groups	k	l	Groups	k	l	Groups	k	l
$G=C_{2^4}$ and $H=C_{2^5}$	3	17	$G=C_{2^5}$ and $H=C_{2^4}$	7	9	$G=C_{2^6}$ and $H=C_{2^4}$	15	9
	5	17		9	9		17	9
	11	17		23	9		47	9
	13	17		25	9		49	9
$G=C_{2^4}$ and $H=C_{2^6}$	3	33	$G=C_{2^5}$ and $H=C_{2^6}$	7	33	$G=C_{2^6}$ and $H=C_{2^5}$	15	17
	5	33		9	33		17	17
	11	33		23	33		47	17
	13	33		25	33		49	17
$G=C_{2^4}$ and $H=C_{2^7}$	3	65	$G=C_{2^5}$ and $H=C_{2^7}$	7	65	$G=C_{2^6}$ and $H=C_{2^7}$	15	65
	5	65		9	65		17	65
	11	65		23	65		47	65
	13	65		25	65		49	65
$G=C_{2^4}$ and $H=C_{2^8}$	3	129	$G=C_{2^5}$ and $H=C_{2^8}$	7	129	$G=C_{2^6}$ and $H=C_{2^8}$	15	129
	5	129		9	129		17	129
	11	129		23	129		47	129
	13	129		25	129		49	129
$G=C_{2^4}$ and $H=C_{2^9}$	3	257	$G=C_{2^5}$ and $H=C_{2^9}$	7	257	$G=C_{2^6}$ and $H=C_{2^9}$	15	257
	5	257		9	257		17	257
	11	257		23	257		47	257
	13	257		25	257		49	257

Groups	k	l	Groups	k	l	Groups	k	l
$G=C_{2^7}$ and $H=C_{2^4}$	31	9	$G=C_{2^8}$ and $H=C_{2^4}$	63	9	$G=C_{2^9}$ and $H=C_{2^4}$	127	9
	33	9		65	9		129	9
	95	9		191	9		383	9
	97	9		193	9		385	9
$G=C_{2^7}$ and $H=C_{2^5}$	31	17	$G=C_{2^8}$ and $H=C_{2^5}$	63	17	$G=C_{2^9}$ and $H=C_{2^5}$	127	17
	33	17		65	17		129	17
	95	17		191	17		383	17
	97	17		193	17		385	17
$G=C_{2^7}$ and $H=C_{2^6}$	31	33	$G=C_{2^8}$ and $H=C_{2^6}$	63	33	$G=C_{2^9}$ and $H=C_{2^6}$	127	33
	33	33		65	33		129	33
	95	33		191	33		383	33
	97	33		193	33		385	33
$G=C_{2^7}$ and $H=C_{2^8}$	31	129	$G=C_{2^8}$ and $H=C_{2^7}$	63	65	$G=C_{2^9}$ and $H=C_{2^7}$	127	65
	33	129		65	65		129	65
	95	129		191	65		383	65
	97	129		193	65		385	65
$G=C_{2^7}$ and $H=C_{2^9}$	31	257	$G=C_{2^8}$ and $H=C_{2^9}$	63	257	$G=C_{2^9}$ and $H=C_{2^8}$	127	129
	33	257		65	257		129	129
	95	257		191	257		383	129
	97	257		193	257		385	129

The result is given in the following theorem.

Theorem 3.3
Let $G = \langle x \rangle \cong C_{2^\alpha}$ and $H = \langle y \rangle \cong C_{2^\beta}$ be cyclic groups of 2-power order. Furthermore, let *G* and *H* act on each other so that ${}^{y}x = x^k$ and ${}^{x}y = y^l$ for $k, l \in \mathbb{N}$ with $(k,2) = (l,2) = 1$. Then *G* and *H* act compatibly on each other with the action of *H* on *G* is order 4 and the action of *H* on *G* of order 2, if and only if *k* and *l* are congruent to one of the following:

(i) $k \equiv 1 + 2^{\alpha-2} \bmod 2^\alpha$ and $l \equiv 1 - 2^{\beta-1} \bmod 2^\beta$;

(ii) $k \equiv 1 - 2^{\alpha-2} \bmod 2^\alpha$ and $l \equiv 1 - 2^{\beta-1} \bmod 2^\beta$;

(iii) $k \equiv -1 + 2^{\alpha-2} \bmod 2^\alpha$ and $l \equiv 1 - 2^{\beta-1} \bmod 2^\beta$;

(iv) $k \equiv -1 - 2^{\alpha-2} \bmod 2^\alpha$ and $l \equiv 1 - 2^{\beta-1} \bmod 2^\beta$.

CONCLUSION

In this paper, three theorems have been stated accordingly to the order of the actions that satisfy compatible actions, which covered when the both actions have order 4, and when one of the actions has order 4 while the other has order 2

ACKNOWLEDGMENTS

The authors would like to thank the Research & Innovation Department, Universiti Malaysia Pahang for the financial funding through RACE Grant, RDU141307.

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