Compatible Pair of Nontrivial Actions for Some Cyclic Groups of 2-Power Order

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Abstract. Compatible action played a very important verification before the nonabelian tensor product can be computed. This paper gives the some exact number of compatible pairs of actions for some cyclic groups of 2-power order. Some necessary and sufficient number theoretical conditions for a pair of cyclic groups of 2-power order with nontrivial actions act compatibly on each other are used to investigate some properties to find the exact number of compatible pairs of actions. Algorithms in Groups, Algorithms and Programming (GAP) software are constructed to create more examples on selected case. New results on compatible pair of nontrivial actions of order two and four are given.

Keywords: Compatible Actions, Cyclic Groups, Nonabelian Tensor Product

INTRODUCTION

Let *G* and *H* be groups which act on each other and each of which acts on itself by conjugation, then the actions are compatible if ${s_h}g' = {s(h(s^{-1}g'))}$ and ${hg}h' = {h(s(h^{-1}h'))}$. This compatible action played a very important verification before the nonabelian tensor product can be computed. Brown and Loday [1] were the first to introduce the concept of the nonabelian tensor product of groups with the compatible actions. Compatible conditions are the prerequisite before nonabelian tensor product can be calculated.

This construction of the nonabelian tensor product has its origins in the algebraic K-theory and in homotopy theory. Several authors, starting with Brown, Johnson and Robertson [2], have studied group theoretical aspects of nonabelian tensor products extensively. A paper by Kappe [3] gives an overview of known results and literature.

Many authors are working with nonabelian tensor square (where H = G), and the compatibly conditions can be checked easily. It is interesting to check the compatible conditions when calculating tensor product since the groups are not the same or when the actions are nontrivial. Ellis and McDermott [4] had done this but specifically for quaternion group of order 32. Visscher [5] in 1998 had provided necessary and sufficient conditions for a pair of cyclic groups to act compatibly. He gave complete compatible conditions when one of the actions is trivial or both actions are trivial in cyclic groups of *p*-power order. In this research, all compatible actions involving at least one of the actions having order greater than two will be determined as a continuation from Visscher's.

Groups, Algorithms and Programming (GAP) software [6] is used to determine the conditions in which the actions satisfy the compatibility conditions in some finite cyclic groups.

SOME PREPARATORY RESULTS ON COMPATIBILITY CONDITIONS

The nonabelian tensor product $G \otimes H$ is defined only if g and h act on each other in such way that the actions satisfy certain compatibility condition. Start with the definition of an action a group g on a group h.

Definition 2.1 [5]

Let *G* and *H* be groups. An action of *G* on *H* is a mapping, $\Phi: G \to Aut(H)$ such that

 $\Phi(gg')(h) = \Phi(g)(\Phi(g')(h))$

for all $g, g' \in G$ and $h \in H$.

For a group g to act on a group h, by using the fact that the identity in g acts as the identity mapping on h, thus this implies that all elements in g act as automorphism of h on h. The definition of a compatible action is given as follows.

Definition 2.1 [5]

Let G and H be groups which act on each other. These mutual actions are said to be compatible with each other and with the actions of G and H on themselves by conjugation if

$$\binom{{}^{g}h}{g}g' = {}^{g}\binom{h}{g^{-1}}g'$$
 and $\binom{h}{g}h' = {}^{h}\binom{g}{h^{-1}}h'$

for all $g, g' \in G$ and $h \in H$.

Since we are focusing on finite cyclic groups, the compatibility conditions which hold for abelian groups will be reduce to ${}^{(s_h)}g' = {}^hg'$ and ${}^{(hg)}h' = {}^gh'$, for all $g, g' \in G$ and $h \in H$. For compatibility of actions of G and H on each other, it suffices for the compability conditions to hold on the generators of G and H. Visscher [7] has done this and he gave some necessary and sufficient number theoretical conditions for a pair of finite cyclic groups to act compatibly on each other as stated in Proposition 2.3.

Proposition 2.3 [5]

Let $G = \langle x \rangle = C_m$ and $H = \langle y \rangle = C_n$ be finite cyclic groups. Then there exist mutual actions of *G* and *H* on each other such that $y = x^k$ and $x = y^l$ for $k, l \in \mathbb{Z}$ if and only if conditions (i) and (ii) below are satisfied. These action are compatible if and only condition (iii) is satisfied as well.

- (i) (k,m) = (l,n) = 1
- (ii) $k^n \equiv 1 \mod m$ and $l^m \equiv 1 \mod n$
- (iii) $k^{l-1} \equiv 1 \mod m$ and $l^{k-1} \equiv 1 \mod n$.

In the next section, all compatible actions involving at least one of the actions having order greater than two will be determined.

COMPATIBLE CONDITIONS FOR NONTRIVIAL ACTION

In the previous section, we can see that Visscher had done the classification of compatibility condition focusing on cyclic groups and provide necessary and sufficient conditions for the general case without any specific condition.

Now, our aim is to determine the compatible conditions if the order of the action is more than two. All computations are doing by GAP. New GAP coding has been built to calculate the compatible actions.

Let $G = \langle x \rangle = C_m$ and $H = \langle y \rangle = C_m$ be finite cyclic groups. Let the actions of *G* and *H* on each other be such that ${}^y x = x^k$ and ${}^x y = y^l$ for $k, l \in \mathbb{Z}$. Now consider that both actions have order four. The compatible conditions that are satisfied by *k* and *l* are given in **Table 1** when G = H and in **Table 2** when $G \neq H$.

Groups	k	l		Groups	k	l
$G = H = C_{2^4}$	5	5		$G = H = C_{2^7}$	33	33
Z	13	13		2	97	97
$G = H = C_{2^5}$	9	9		$G = H = C_{2^8}$	65	65
2	25	25		2	193	193
$G = H = C_{2^6}$	17	17		$G = H = C_{2^9}$	129	129
2*	49	49]	2	385	385

TABLE 1. Values of k and l in which the action of order four satisfying the compatible conditions when G = H.

Groups	k	l	Groups	k	l		Groups	k	l
$G = C_{a^4}$	5	9	$G = C_{25}$	9	5]	$G = C_{26}$	17	5
2- and	5	25	2-	9	13		2°	17	13
	13	9	and	25	5		and	49	5
$H = C_{2^5}$	13	25	$H = C_{2^4}$	25	13		$H = C_{2^4}$	49	13
$G = C_{2^4}$	5	17	$G = C_{2^5}$	9	17		$G = C_{2^6}$	17	9
and	5	49	and	9	49		and	17	25
	13	17		25	17			49	9
$H = C_{2^6}$	13	49	$H = C_{2^6}$	25	49		$H = C_{2^5}$	49	25
$G = C_{2^4}$	5	33	$G = C_{2^5}$	9	33		$G = C_{2^6}$	17	33
and	5	97	and	9	97		and	17	97
	13	33		25	33			49	33
$H = C_{2^7}$	13	97	$H = C_{2^7}$	25	97		$H = C_{2^7}$	49	97
$G = C_{2^4}$	5	65	$G = C_{2^5}$	9	65		$G = C_{2^6}$	17	65
and	5	193	and	9	193		and	17	193
	13	65		25	65			49	65
$H = C_{2^8}$	13	193	$H = C_{2^8}$	25	193		$H = C_{2^8}$	49	193
$G = C_{2^4}$	5	129	$G = C_{2^5}$	9	129		$G = C_{2^6}$	17	129
and	5	385	and	9	385		and	17	385
	13	129		25	129	-		49	129
$H = C_{2^9}$	13	385	$H = C_{2^9}$	25	385		$H = C_{2^9}$	49	385
Groups	l.	1	Groups	k	1	-	Groups	l.	1
Groups	k 33	<i>l</i>	Groups	k	<i>l</i>	-	Groups	k	<i>l</i>
$\frac{\text{Groups}}{G = C_{2^7}}$	k 33 33	l 5 13	$\frac{\text{Groups}}{G = C_{2^8}}$	k 65 65	<i>l</i> 5	-	$\frac{\text{Groups}}{G = C_{2^9}}$	k 129 129	<i>l</i> 5
$Groups$ $G = C_{2^7}$ and	k 33 33 97	<i>l</i> 5 13 5	$\frac{\text{Groups}}{G = C_{2^8}}$ and	k 65 65 193	<i>l</i> 5 13 5	-	$\frac{\text{Groups}}{G = C_{2^9}}$ and	k 129 129 385	<i>l</i> 5 13 5
$Groups$ $G = C_{2^7}$ and $H = C_{2^4}$	k 33 33 97 97	<i>l</i> 5 13 5 13	$Groups$ $G = C_{2^8}$ and $H = C_{2^4}$	k 65 65 193 193	<i>l</i> 5 13 5 13	-	$Groups$ $G = C_{2^9}$ and $H = C_{2^4}$	k 129 129 385 385	<i>l</i> 5 13 5 13
$Groups$ $G = C_{2^{7}}$ and $H = C_{2^{4}}$ $G = C_{2^{4}}$	k 33 33 97 97 33	$ \begin{array}{r}l\\5\\13\\5\\13\\9\end{array} $	$Groups$ $G = C_{2^8}$ and $H = C_{2^4}$ $G = C_{2^4}$	k 65 65 193 193 65	<i>l</i> 5 13 5 13 9	-	Groups $G = C_{2^9}$ and $H = C_{2^4}$ $G = C_{2^4}$	k 129 129 385 385 129	<i>l</i> 5 13 5 13 9
$Groups$ $G = C_{2^{7}}$ and $H = C_{2^{4}}$ $G = C_{2^{7}}$	k 33 33 97 97 97 33 33	<i>l</i> 5 13 5 13 9 25	$Groups$ $G = C_{2^8}$ and $H = C_{2^4}$ $G = C_{2^8}$	k 65 193 193 65 65	<i>l</i> 5 13 5 13 9 25	-	$Groups$ $G = C_{2^9}$ and $H = C_{2^4}$ $G = C_{2^9}$	k 129 129 385 385 129 129	<i>l</i> 5 13 5 13 9 25
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$Groups$ $G = C_{2^{7}}$ and $H = C_{2^{4}}$ $G = C_{2^{7}}$ and $H = C_{2^{5}}$	k 33 37 97 97 33 37 97 33 33 97 97 97 97 97 97 97 97 97	<i>l</i> 5 13 5 13 9 25 9 25	$Groups$ $G = C_{2^8}$ and $H = C_{2^4}$ $G = C_{2^8}$ and $H = C_{2^5}$	k 65 193 193 65 65 193 193	<i>l</i> 5 13 5 13 9 25 9 25	-	$Groups$ $G = C_{2^9}$ and $H = C_{2^4}$ $G = C_{2^9}$ and $H = C_{2^5}$	k 129 129 385 385 129 129 385 385 385	<i>l</i> 5 13 5 13 9 25 9 25
Groups $G = C_{2^7}$ and $H = C_{2^4}$ $G = C_{2^7}$ and $H = C_{2^5}$ $G = C_{2^7}$	k 33 97 97 97 33 33 97 97 33	<i>l</i> 5 13 5 13 9 25 9 25 17	Groups $G = C_{2^8}$ and $H = C_{2^4}$ $G = C_{2^8}$ and $H = C_{2^5}$ $G = C_{2^8}$	k 65 193 193 65 65 193 65 65 193 65 65 193 65	<i>l</i> 5 13 5 13 9 25 9 25 17		$Groups$ $G = C_{2^9}$ and $H = C_{2^4}$ $G = C_{2^9}$ and $H = C_{2^5}$ $G = C_{2^9}$	k 129 129 385 385 129 129 385 385 385 129	<i>l</i> 5 13 5 13 9 25 9 25 17
$Groups$ $G = C_{2^{7}}$ and $H = C_{2^{4}}$ $G = C_{2^{7}}$ and $H = C_{2^{5}}$ $G = C_{2^{7}}$ ord	k 33 97 97 97 33 33 97 97 97 33 33 33	<i>l</i> 5 13 5 13 9 25 9 25 17 49	$Groups$ $G = C_{2^8}$ and $H = C_{2^4}$ $G = C_{2^8}$ and $H = C_{2^5}$ $G = C_{2^8}$ ord	$ \begin{array}{r} k \\ 65 \\ 193 \\ 193 \\ 65 \\ 193 \\ 193 \\ 193 \\ 65 \\ 65 \\ 65 \\ 65 \\ \end{array} $	<i>l</i> 5 13 5 13 9 25 9 25 9 25 17 49		$Groups$ $G = C_{2^9}$ and $H = C_{2^4}$ $G = C_{2^9}$ and $H = C_{2^5}$ $G = C_{2^9}$ and	k 129 129 385 385 129 129 385 385 385 129 129	<i>l</i> 5 13 5 13 9 25 9 25 9 25 17 49
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Groups $G = C_{2^7}$ and $H = C_{2^4}$ $G = C_{2^7}$ and $H = C_{2^5}$ $G = C_{2^7}$ and $H = C_{2^6}$ $G = C_{2^7}$ and $H = C_{2^6}$ $G = C_{2^7}$ and $H = C_{2^8}$ $G = C_{2^7}$ and $H = C_{2^8}$ $G = C_{2^7}$ and	k 33 97 97 33 97 97 33 97 97 33 33 97 97 33 33 97 97 33 33 97 97 33 33 97 97 33 97 97 33 97 97 93 97 97	<i>l</i> 5 13 9 25 9 25 17 49 17 49 65 193 65 193 65 129 385 129	$\begin{tabular}{ c c c c c } \hline G &= C_{2^8} \\ and \\ H &= C_{2^4} \\ \hline G &= C_{2^8} \\ and \\ H &= C_{2^5} \\ \hline G &= C_{2^8} \\ and \\ H &= C_{2^6} \\ \hline G &= C_{2^8} \\ and \\ H &= C_{2^7} \\ \hline G &= C_{2^8} \\ and \\ H &= C_{2^7} \\ \hline G &= C_{2^8} \\ and \\ H &= C_{2^7} \\ \hline G &= C_{2^8} \\ and \\ H &= C_{2^8} \\ and \\ H &= C_{2^8} \\ \hline G &= C_{2^8} \\ and \\ H &= C_{2^8} \\ \hline G &= $	$\begin{array}{c} k \\ 65 \\ 65 \\ 193 \\ 193 \\ 65 \\ 65 \\ 193 \\ 193 \\ 65 \\ 65 \\ 193 \\ 193 \\ 65 \\ 65 \\ 193 \\ 193 \\ 65 \\ 65 \\ 193 \\ 193 \\ 65 \\ 65 \\ 193 \\ 193 \\ 65 \\ 65 \\ 193 \\ 193 \\ 65 \\ 65 \\ 193 \\ 193 \\ 65 \\ 65 \\ 193 \\ 193 \\ 65 \\ 65 \\ 193 \\ 193 \\ 65 \\ 65 \\ 193 \\ 193 \\ 65 \\ 65 \\ 193 \\ 193 \\ 193 \\ 193 \\ 65 \\ 65 \\ 193$	<i>l</i> 5 13 9 25 9 25 17 49 17 49 33 97 33 97 129 385 129		$Groups$ $G = C_{2^9}$ and $H = C_{2^4}$ $G = C_{2^9}$ and $H = C_{2^5}$ $G = C_{2^9}$ and $H = C_{2^6}$ $G = C_{2^9}$ and $H = C_{2^7}$ $G = C_{2^9}$ and $H = C_{2^7}$	k 129 129 385 385 129 385 385 385 385 385 129 385	$ \begin{array}{r} l\\ 5\\ 13\\ 5\\ 13\\ 9\\ 25\\ 9\\ 25\\ 17\\ 49\\ 17\\ 49\\ 33\\ 97\\ 33\\ 97\\ 65\\ 193\\ 65\\ \end{array} $

TABLE 2. Values of k and l in which the action of order four satisfying the compatible conditions when $G \neq H$.

This lead the following results:

Theorem 3.1

Let $C_{2^{\alpha}} = \langle x \rangle$ and $C_{2^{\beta}} = \langle y \rangle$ be cyclic groups of 2-power order. Furthermore, let $C_{2^{\alpha}}$ and $C_{2^{\beta}}$ act on each other so that $y = x^k$ and $x = y^l$ for $k, l \in \mathbb{N}$ with (k, 2) = (l, 2) = 1. Then $C_{2^{\alpha}}$ and $C_{2^{\beta}}$ act compatibly on each other when $\alpha = \beta$ with $y^4 x = x$ and $x^4 y = y$, if and only if k and l are congruent to one of the following:

(i)
$$k = l \equiv 1 + 2^{\alpha - 2} \mod 2^{\alpha}$$
;

 $k = l \equiv 1 - 2^{\alpha - 2} \mod 2^{\alpha}$ (ii)

Theorem 3.2

Let $C_{2^{\alpha}} = \langle x \rangle$ and $C_{2^{\beta}} = \langle y \rangle$ be cyclic groups of 2-power order. Furthermore, let $C_{2^{\alpha}}$ and $C_{2^{\beta}}$ act on each other so that $y = x^k$ and $x = y^l$ for $k, l \in \mathbb{N}$ with (k, 2) = (l, 2) = 1. Then $C_{2^{\alpha}}$ and $C_{2^{\beta}}$ act compatibly on each other when $\alpha \neq \beta$ with $y^4 x = x$ and $x^4 y = y$, if and only if k and l are congruent to one of the following:

- $k \equiv 1 + 2^{\alpha 2} \mod 2^{\alpha}$ and $l \equiv 1 + 2^{\beta 2} \mod 2^{\beta}$; (i)
- $k \equiv 1 + 2^{\alpha 2} \mod 2^{\alpha}$ and $l \equiv 1 2^{\beta 2} \mod 2^{\beta}$; (ii)
- $k \equiv 1 2^{\alpha 2} \mod 2^{\alpha}$ and $l \equiv 1 + 2^{\beta 2} \mod 2^{\beta}$; (iii)
- $k \equiv 1 2^{\alpha 2} \mod 2^{\alpha}$ and $l \equiv 1 2^{\beta 2} \mod 2^{\beta}$. (iv)

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Next, let the actions of G and H be on each other such that $y = x^k$ and $x = y^l$ for $k, l \in \mathbb{Z}$. Now consider that one of the actions has order four and the other one has order two. The compatible conditions that are satisfied by kand *l* are given in **Table 3** when G = H and in **Table 4** when $G \neq H$.

Groups Groups Groups k k 1 k 9 9 3 3 3 9 9 9 5 9 $G = H = C_{2^4}$ $G = H = C_{2^4}$ $G = H = C_{2^4}$ 11 11 9 11 9 9 13 13 9 9 13 9 7 17 7 17 7 17 9 $G = H = C_{2^5}$ 9 17 17 9 17 $G = H = C_{2^5}$ $G = H = C_{2^5}$

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TABLE 3. Values of k for which the action has order four and l for which the action had order two satisfying the compatible conditions when the groups G and H are the same.

Groups	k	l		Groups	k	l		Groups	k	l
$G = C_{2^4}$	3	17		$G = C_{2^5}$	7	9		$G = C_{2^6}$	15	9
and	5	17		and	9	9		and	17	9
and	11	17		and	23	9		and	47	9
$H = C_{2^5}$	13	17		$H = C_{2^4}$	25	9		$H = C_{2^4}$	49	9
$G = C_{2^4}$	3	33		$G = C_{2^5}$	7	33		$G = C_{2^6}$	15	17
and	5	33		and	9	33		and	17	17
	11	33			23	33			47	17
$H = C_{2^6}$	13	33		$H = C_{2^6}$	25	33		$H = C_{2^5}$	49	17
$G = C_{2^4}$	3	65		$G = C_{2^5}$	7	65		$G = C_{2^6}$	15	65
and	5	65		and	9	65		and	17	65
	11	65			23	65			47	65
$H = C_{2^7}$	13	65		$H = C_{2^7}$	25	65	-	$H = C_{2^7}$	49	65
$G = C_{2^4}$	3	129		$G = C_{2^5}$	7	129		$G = C_{2^6}$	15	129
and	5	129		and	9	129	-	and	17	129
	11	129			23	129			47	129
$H = C_{2^8}$	13	129		$H = C_{2^8}$	25	129		$H = C_{2^8}$	49	129
$G = C_{2^4}$	3	257		$G = C_{2^5}$	7	257		$G = C_{2^6}$	15	257
and	5	257	-	and	9	257		and	17	257
	11	257			23	257			47	257
$H = C_{2^9}$	13	257		$H = C_{2^9}$	25	257		$H = C_{2^9}$	49	257
Crowns	1.	1	1	Crowns	I.	1	1	Crowns	I.	1
Groups	<u>k</u>	l		Groups	k	l		Groups	k	
$Groups$ $G = C_{2^7}$	k 31 33	<i>l</i> 9		$\frac{\text{Groups}}{G = C_{2^8}}$	k 63	l 9 9		$Groups$ $G = C_{2^9}$	k 127 129	<i>l</i> 9
$Groups$ $G = C_{2^7}$ and	k 31 33 95	<i>l</i> 9 9		$\frac{\text{Groups}}{G = C_{2^8}}$ and	k 63 65 191	<i>l</i> 9 9		$\frac{\text{Groups}}{G = C_{2^9}}$ and	k 127 129 383	<i>l</i> 9 9
$Groups$ $G = C_{2^7}$ and $H = C_{2^4}$	k 31 33 95 97	<i>l</i> 9 9 9 9		$Groups$ $G = C_{2^8}$ and $H = C_{2^4}$	k 63 65 191 193	<i>l</i> 9 9 9 9		$Groups$ $G = C_{2^9}$ and $H = C_{2^4}$	k 127 129 383 385	<i>l</i> 9 9 9
Groups $G = C_{2^7}$ and $H = C_{2^4}$ $G = C_{2^7}$	k 31 33 95 97 31	<i>l</i> 9 9 9 9 17		$Groups$ $G = C_{2^8}$ and $H = C_{2^4}$ $G = C_{3^8}$	k 63 65 191 193 63	<i>l</i> 9 9 9 9 17		Groups $G = C_{2^9}$ and $H = C_{2^4}$ $G = C_{2^9}$	k 127 129 383 385 127	<i>l</i> 9 9 9 9 17
$Groups$ $G = C_{2^{7}}$ and $H = C_{2^{4}}$ $G = C_{2^{7}}$	k 31 33 95 97 31 33	<i>l</i> 9 9 9 9 9 17 17		Groups $G = C_{2^8}$ and $H = C_{2^4}$ $G = C_{2^8}$	k 63 65 191 193 63 65	<i>l</i> 9 9 9 9 17 17		Groups $G = C_{2^9}$ and $H = C_{2^4}$ $G = C_{2^9}$	k 127 129 383 385 127 129	<i>l</i> 9 9 9 9 9 17 17
$Groups$ $G = C_{2^{7}}$ and $H = C_{2^{4}}$ $G = C_{2^{7}}$ and	k 31 33 95 97 31 33 95	<i>l</i> 9 9 9 17 17 17		$Groups$ $G = C_{2^8}$ and $H = C_{2^4}$ $G = C_{2^8}$ and	k 63 65 191 193 63 65 191	<i>l</i> 9 9 9 17 17 17		$Groups$ $G = C_{2^9}$ and $H = C_{2^4}$ $G = C_{2^9}$ and	k 127 129 383 385 127 129 383	<i>l</i> 9 9 9 17 17 17
$Groups$ $G = C_{2^{7}}$ and $H = C_{2^{4}}$ $G = C_{2^{7}}$ and $H = C_{2^{5}}$	k 31 33 95 97 31 33 95 97	<i>l</i> 9 9 9 17 17 17 17		$Groups$ $G = C_{2^8}$ and $H = C_{2^4}$ $G = C_{2^8}$ and $H = C_{2^5}$	k 63 65 191 193 63 65 191 193	<i>l</i> 9 9 9 17 17 17 17		$Groups$ $G = C_{2^9}$ and $H = C_{2^4}$ $G = C_{2^9}$ and $H = C_{2^5}$	k 127 129 383 385 127 129 383 385	<i>l</i> 9 9 9 17 17 17 17
Groups $G = C_{2^7}$ and $H = C_{2^4}$ $G = C_{2^7}$ and $H = C_{2^5}$ $G = C_{2^7}$	k 31 33 95 97 31 33 95 97 31	<i>l</i> 9 9 9 17 17 17 17 17 33		$Groups$ $G = C_{2^8}$ and $H = C_{2^4}$ $G = C_{2^8}$ and $H = C_{2^5}$ $G = C_{2^8}$	k 63 65 191 193 63 65 191 193 63	<i>l</i> 9 9 9 17 17 17 17 17 33		$Groups$ $G = C_{2^9}$ and $H = C_{2^4}$ $G = C_{2^9}$ and $H = C_{2^5}$ $G = C_{2^9}$	k 127 129 383 385 127 129 383 385 385 127	<i>l</i> 9 9 9 17 17 17 17 33
$Groups$ $G = C_{2^{7}}$ and $H = C_{2^{4}}$ $G = C_{2^{7}}$ and $H = C_{2^{5}}$ $G = C_{2^{7}}$ ord	k 31 33 95 97 31 33 95 97 31 33	<i>l</i> 9 9 9 17 17 17 17 17 33 33		$Groups$ $G = C_{2^8}$ and $H = C_{2^4}$ $G = C_{2^8}$ and $H = C_{2^5}$ $G = C_{2^8}$ and	k 63 65 191 193 63 65 191 193 63 63 63 63 63 63 63 65	<i>l</i> 9 9 9 17 17 17 17 17 33 33		$Groups$ $G = C_{2^9}$ and $H = C_{2^4}$ $G = C_{2^9}$ and $H = C_{2^5}$ $G = C_{2^9}$ and	k 127 129 383 385 127 129 383 385 127 129	<i>l</i> 9 9 9 17 17 17 17 17 33 33
$\begin{tabular}{ c c c c }\hline Groups & \\ \hline G = C_{2^7} & \\ and & \\ \hline H = C_{2^4} & \\ \hline G = C_{2^7} & \\ and & \\ \hline H = C_{2^5} & \\ \hline G = C_{2^7} & \\ and & \\ \hline \end{array}$	k 31 33 95 97 31 33 95 97 31 33 95	<i>l</i> 9 9 9 17 17 17 17 17 33 33 33		$Groups$ $G = C_{2^8}$ and $H = C_{2^4}$ $G = C_{2^8}$ and $H = C_{2^5}$ $G = C_{2^8}$ and	k 63 65 191 193 63 65 191 193 63 65 191 193 63 65 191	l 9 9 9 17 17 17 17 17 33 33 33 33		Groups $G = C_{2^9}$ and $H = C_{2^4}$ $G = C_{2^9}$ and $H = C_{2^5}$ $G = C_{2^9}$ and	k 127 129 383 385 127 129 383 385 127 129 383	<i>l</i> 9 9 9 17 17 17 17 17 33 33 33
Groups $G = C_{2^7}$ and $H = C_{2^4}$ $G = C_{2^7}$ and $H = C_{2^5}$ $G = C_{2^7}$ and $H = C_{2^5}$ $G = C_{2^7}$ and $H = C_{2^6}$	k 31 33 95 97 31 33 95 97 31 33 95 97 31 33 95 97 31 33 95 97 31 33 95 97	<i>l</i> 9 9 9 17 17 17 17 17 33 33 33 33		$Groups$ $G = C_{2^8}$ and $H = C_{2^4}$ $G = C_{2^8}$ and $H = C_{2^5}$ $G = C_{2^8}$ and $H = C_{2^6}$	k 63 65 191 193 63 65 191 193 63 65 191 193 63 65 191 193 63 65 191 193	<i>l</i> 9 9 9 17 17 17 17 17 33 33 33 33 33		$Groups$ $G = C_{2^9}$ and $H = C_{2^4}$ $G = C_{2^9}$ and $H = C_{2^5}$ $G = C_{2^9}$ and $H = C_{2^6}$	k 127 129 383 385 127 129 383 385 127 129 383 385 385	$ \begin{array}{c} l \\ 9 \\ 9 \\ 9 \\ 9 \\ 17 \\ 17 \\ 17 \\ 17 \\ 33 \\ 33 \\ 33 \\ 33 \\ 33 \end{array} $
$\begin{tabular}{ c c c c }\hline Groups & \\ \hline G = C_{2^7} & \\ and & \\ \hline H = C_{2^4} & \\ \hline G = C_{2^7} & \\ and & \\ \hline H = C_{2^5} & \\ \hline G = C_{2^7} & \\ and & \\ \hline H = C_{2^6} & \\ \hline G = C_{2^7} & \\ \hline \end{array}$	k 31 33 95 97 31 33 95 97 31 33 95 97 31	<i>l</i> 9 9 9 17 17 17 17 17 33 33 33 33 129		$Groups$ $G = C_{2^8}$ and $H = C_{2^4}$ $G = C_{2^8}$ and $H = C_{2^5}$ $G = C_{2^8}$ and $H = C_{2^6}$ $G = C_{2^8}$	k 63 65 191 193 63 65 191 193 63 65 191 193 63	$ \begin{array}{c} l \\ 9 \\ 9 \\ 9 \\ 17 \\ 17 \\ 17 \\ 17 \\ 33 \\ 33 \\ 33 \\ 33 \\ 65 \\ \end{array} $		$Groups$ $G = C_{2^9}$ and $H = C_{2^4}$ $G = C_{2^9}$ and $H = C_{2^5}$ $G = C_{2^9}$ and $H = C_{2^6}$ $G = C_{2^9}$	k 127 129 383 385 127 129 383 385 127 129 383 385 127 129 383 385 127	$ \begin{array}{c} l \\ 9 \\ 9 \\ 9 \\ 9 \\ 17 \\ 17 \\ 17 \\ 17 \\ 33 \\ 33 \\ 33 \\ 33 \\ 65 \\ \end{array} $
Groups $G = C_{2^7}$ and $H = C_{2^4}$ $G = C_{2^7}$ and $H = C_{2^5}$ $G = C_{2^7}$ and $H = C_{2^6}$ $G = C_{2^7}$ and $H = C_{2^6}$ $G = C_{2^7}$ and	k 31 33 95 97 31 33 95 97 31 33 95 97 31 33 95 97 31 33 95 97 31 33	<i>l</i> 9 9 9 17 17 17 17 17 17 33 33 33 33 33 129 129		$Groups$ $G = C_{2^8}$ and $H = C_{2^4}$ $G = C_{2^8}$ and $H = C_{2^5}$ $G = C_{2^8}$ and $H = C_{2^6}$ $G = C_{2^8}$ and $H = C_{2^6}$ $G = C_{2^8}$ and	k 63 65 191 193 63 65 191 193 63 65 191 193 63 65 191 193 63 63 63 63 63 63 63 65	$ \begin{array}{c} l \\ 9 \\ 9 \\ 9 \\ 9 \\ 17 \\ 17 \\ 17 \\ 17 \\ 33 \\ 33 \\ 33 \\ 33 \\ 65 \\ 65 \\ 65 \\ \end{array} $		$Groups$ $G = C_{2^9}$ and $H = C_{2^4}$ $G = C_{2^9}$ and $H = C_{2^5}$ $G = C_{2^9}$ and $H = C_{2^6}$ $G = C_{2^9}$ and $G = C_{2^9}$ and $G = C_{2^9}$ and $G = C_{2^9}$ and	k 127 129 383 385 127 129 383 385 127 129 383 385 127 129 383 385 127 129 383 385 127 129 383 385 127 129	<i>l</i> 9 9 9 17 17 17 17 17 33 33 33 33 65 65
Groups $G = C_{2^7}$ and $H = C_{2^4}$ $G = C_{2^7}$ and $H = C_{2^5}$ $G = C_{2^7}$ and $H = C_{2^6}$ $G = C_{2^7}$ and $H = C_{2^6}$ $G = C_{2^7}$ and $H = C_{2^6}$	k 31 33 95 97 31 33 95 97 31 33 95 97 31 33 95 97 31 33 95 97 31 33 95 97 31 33 95	<i>l</i> 9 9 9 17 17 17 17 17 33 33 33 33 129 129 129		$Groups$ $G = C_{2^8}$ and $H = C_{2^4}$ $G = C_{2^8}$ and $H = C_{2^5}$ $G = C_{2^8}$ and $H = C_{2^6}$ $G = C_{2^8}$ and $H = C_{2^6}$	k 63 65 191 193 63 65 191 193 63 65 191 193 63 65 191 193 63 65 191 193 63 65 191	<i>l</i> 9 9 9 17 17 17 17 17 33 33 33 33 65 65 65 65		$Groups$ $G = C_{2^9}$ and $H = C_{2^4}$ $G = C_{2^9}$ and $H = C_{2^5}$ $G = C_{2^9}$ and $H = C_{2^6}$ $G = C_{2^9}$ and $H = C_{2^6}$	k 127 129 383 385 127 129 383 385 127 129 383 385 127 129 383 385 127 129 383 385 127 129 383 385 127 129 383	$ \begin{array}{c} l \\ 9 \\ 9 \\ 9 \\ 9 \\ 17 \\ 17 \\ 17 \\ 17 \\ 33 \\ 33 \\ 33 \\ 33 \\ 65 \\ 65 \\ 65 \\ 65 \\ 65 \\ \end{array} $
Groups $G = C_{2^7}$ and $H = C_{2^4}$ $G = C_{2^7}$ and $H = C_{2^5}$ $G = C_{2^7}$ and $H = C_{2^6}$ $G = C_{2^7}$ and $H = C_{2^6}$ $G = C_{2^7}$ and $H = C_{2^8}$	k 31 33 95 97 31 33 95 97 31 33 95 97 31 33 95 97 31 33 95 97 31 33 95 97 31 33 95 97	<i>l</i> 9 9 9 17 17 17 17 17 33 33 33 33 33 129 129 129		Groups $G = C_{2^8}$ and $H = C_{2^4}$ $G = C_{2^8}$ and $H = C_{2^5}$ $G = C_{2^8}$ and $H = C_{2^6}$ $G = C_{2^8}$ and $H = C_{2^6}$ $G = C_{2^8}$ and $H = C_{2^7}$	k 63 65 191 193 63 65 191 193 63 65 191 193 63 65 191 193 63 65 191 193 63 65 191 193	$ \begin{array}{c} l \\ 9 \\ 9 \\ 9 \\ 9 \\ 17 \\ 17 \\ 17 \\ 17 \\ 33 \\ 33 \\ 33 \\ 33 \\ 65 \\ 65 \\ 65 \\ 65 \\ 65 \\ 65 \\ 65 \\ 65$		Groups $G = C_{2^9}$ and $H = C_{2^4}$ $G = C_{2^9}$ and $H = C_{2^5}$ $G = C_{2^9}$ and $H = C_{2^6}$ $G = C_{2^9}$ and $H = C_{2^6}$ $G = C_{2^9}$ and $H = C_{2^6}$ $G = C_{2^9}$ and $H = C_{2^7}$	$\begin{array}{r} k \\ 127 \\ 129 \\ 383 \\ 385 \\ 127 \\ 129 \\ 383 \\ 385 \\ 127 \\ 129 \\ 383 \\ 385 \\ 127 \\ 129 \\ 383 \\ 385 \\ 127 \\ 129 \\ 383 \\ 385 \\ 385 \\ 385 \\ \end{array}$	$ \begin{array}{c} l \\ 9 \\ 9 \\ 9 \\ 9 \\ 17 \\ 17 \\ 17 \\ 17 \\ 33 \\ 33 \\ 33 \\ 33 \\ 65 \\ 65 \\ 65 \\ 65 \\ 65 \\ 65 \\ 65 \\ 65$
$\begin{tabular}{ c c c c c } \hline Groups & \\ \hline G = C_{2^7} & \\ and & \\ \hline H = C_{2^4} & \\ \hline G = C_{2^7} & \\ and & \\ \hline H = C_{2^5} & \\ \hline G = C_{2^7} & \\ and & \\ \hline H = C_{2^6} & \\ \hline G = C_{2^7} & \\ and & \\ \hline H = C_{2^8} & \\ \hline G = C_{2^7} & \\ \hline G = C_{2^7} & \\ \hline \end{array}$	k 31 33 95 97 31 33 95 97 31 33 95 97 31 33 95 97 31 33 95 97 31 33 95 97 31 33 95 97 31 33 95 97 31	<i>l</i> 9 9 9 17 17 17 17 17 17 17 17 17 17 17 17 12 9 129 129 129 129 129		Groups $G = C_{2^8}$ and $H = C_{2^4}$ $G = C_{2^8}$ and $H = C_{2^5}$ $G = C_{2^8}$ and $H = C_{2^6}$ $G = C_{2^8}$ and $H = C_{2^6}$ $G = C_{2^8}$ and $H = C_{2^7}$ $G = C_{2^8}$ and $H = C_{2^7}$ $G = C_{2^8}$	k 63 65 191 193 63 65 191 193 63 65 191 193 63 65 191 193 63 65 191 193 63 65 191 193 63 65 191 193 63	$ \begin{array}{c} l \\ 9 \\ 9 \\ 9 \\ 9 \\ 17 \\ 17 \\ 17 \\ 17 \\ 33 \\ 33 \\ 33 \\ 33 \\ 65 \\ 65 \\ 65 \\ 65 \\ 65 \\ 65 \\ 257 \\ \end{array} $		Groups $G = C_{2^9}$ and $H = C_{2^4}$ $G = C_{2^9}$ and $H = C_{2^5}$ $G = C_{2^9}$ and $H = C_{2^6}$ $G = C_{2^9}$ and $H = C_{2^6}$ $G = C_{2^9}$ and $H = C_{2^7}$ $G = C_{2^9}$ and $H = C_{2^7}$ $G = C_{2^9}$	$\begin{array}{r} k \\ 127 \\ 129 \\ 383 \\ 385 \\ 127 \\ 129 \\ 383 \\ 385 \\ 127 \\ 129 \\ 383 \\ 385 \\ 127 \\ 129 \\ 383 \\ 385 \\ 127 \\ 129 \\ 383 \\ 385 \\ 127 \\ 129 \\ 383 \\ 385 \\ 127 \\ 1$	$ \begin{array}{c} l \\ 9 \\ 9 \\ 9 \\ 9 \\ 17 \\ 17 \\ 17 \\ 17 \\ 33 \\ 33 \\ 33 \\ 33 \\ 65 \\ 65 \\ 65 \\ 65 \\ 65 \\ 129 \\ \end{array} $
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TABLE 4. Values of k for which the action has order four and l for which the action has order two satisfying the compatible conditions when the groups G and H are different.

The result is given in the following theorem.

Theorem 3.3

Let $G = \langle x \rangle \cong C_{2^{\alpha}}$ and $H = \langle y \rangle \cong C_{2^{\beta}}$ be cyclic groups of 2-power order. Furthermore, let *G* and *H* act on each other so that ${}^{y}x = x^{k}$ and ${}^{x}y = y^{l}$ for $k, l \in \mathbb{N}$ with (k, 2) = (l, 2) = 1. Then *G* and *H* act compatibly on each other with the action of *H* on *G* is order 4 and the action of *H* on *G* of order 2, if and only if *k* and *l* are congruent to one of the following:

- (i) $k \equiv 1 + 2^{\alpha 2} \mod 2^{\alpha}$ and $l \equiv 1 2^{\beta 1} \mod 2^{\beta}$;
- (ii) $k \equiv 1 2^{\alpha 2} \mod 2^{\alpha}$ and $l \equiv 1 2^{\beta 1} \mod 2^{\beta}$;
- (iii) $k \equiv -1 + 2^{\alpha 2} \mod 2^{\alpha}$ and $l \equiv 1 2^{\beta 1} \mod 2^{\beta}$;
- (iv) $k \equiv -1 2^{\alpha 2} \mod 2^{\alpha}$ and $l \equiv 1 2^{\beta 1} \mod 2^{\beta}$.

CONCLUSION

In this paper, three theorems have been stated accordingly to the order of the actions that satisfy compatible actions, which covered when the both actions have order 4, and when one of the actions has order 4 while the other has order 2

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