Multi-parametric programming

novel theory & algorithmic developments

by

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Abstract

Multi-parametric programming is a mathematical theory to address optimisation problems involving varying parameters. Based on sensitivity and singularity theories, multi-parametric programming derives the optimum solution of the optimisation variables as analytical continuous functions of the varying parameters. In this thesis, novel theory and algorithmic developments are presented for the solution of various classes of multi-parametric programs with relevance to real-life applications. In Part 1, Advances in global optimisation, particular focus is given to the following classes of global optimisation problems in the context of multi-parametric programming: (i) bilevel programming, (ii) multi-level hierarchical and decentralised programming and (iii) multi-parametric mixed integer linear programming. In Part 2, Advances in robust optimisation & control, the foundations towards a comprehensive general theory for robust optimal control are described in detail. These involve two steps, the effective solution of (iv) constrained dynamic programming, and the (v) robust re-formulation of the original model-based predictive control problem.

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And finally, to the two people I can imagine dedicating this book to, my best friends, companions and parents, Nice and Jorge.

London, June 6, 2008

Dedicated to my best friends,

Nice and Jorge

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1. Introduction

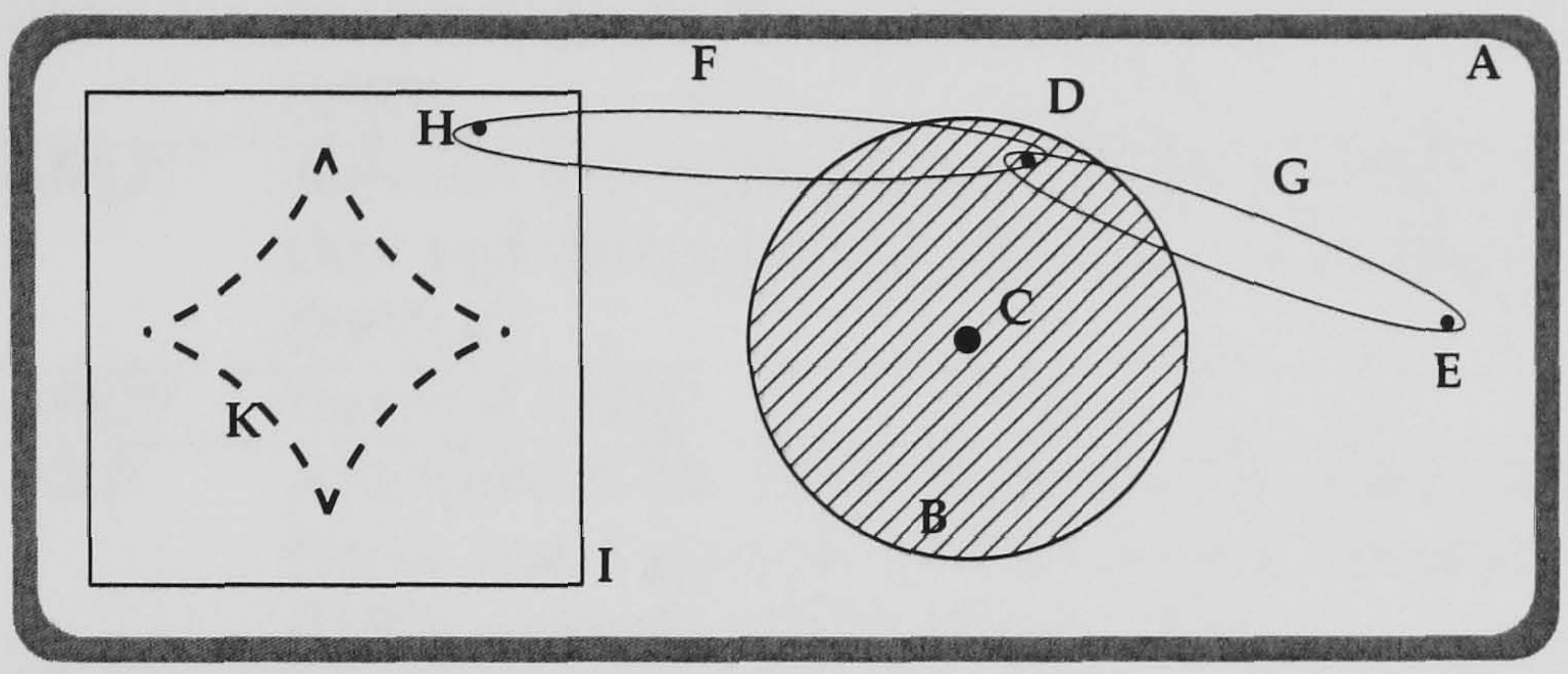
"Dans le monde réellement renversé,
le vrai est un moment du faux."
Guy Debord

Multi-parametric programming refers to a class of optimisation problems which involve some type of bounded uncertainty and/or variability within the mathematical model. A typical and general multi-parametric program is of the following form:

$$
\begin{aligned}
z(\theta) = \min_{x,y} \ & f(x,y,\theta), \\
s.t. \ & h(x,y,\theta) = 0, \\
& g(x,y,\theta) \leq 0, \\
& x \in X \subseteq \mathbb{R}^n, \quad y \in \{0,1\}^m, \\
& \theta \in \Theta,
\end{aligned}
\tag{1.1}
$$

where, x is the vector of continuous optimisation variables, y is the vector of binary optimisation variables and θ is the vector of bounded parameters ($\theta^L \leq \theta \leq \theta^U$); f is a scalar function and h, g are general vectorial functions. The objective of multi-parametric programming is to understand how does the optimal solution of problem (1.1) vary with θ (Dantzig *et al.*, 1967).

Varying constraints and/or objective may result in quantitative and qualitative changes in the solution. In a quantitative change, the solution moves smoothly to a neighbouring point, with the same set of active constraints. Whereas, a leap into a region with a different set of active constraints, corresponds to a quantitative and qualitative change of the solution. For instance, regard the Zeeman's catastrophe machine, Figure 1.1.



Legend: **A** - board; **B** - rotating disk secured at its centre by a pin **C**; **D** - a pin stuck into the disk; **E** - a pin stuck into the board; **F,G** - rubber bands; **H** - pencil; **I** - piece of paper.
As the pencil moves within the paper, the rotating disk settles itself in a certain position. Then, certain small moves of the pencil correspond to jumps in the position of the disk, that if marked draw curve **K**; which corresponds to the boundary of a qualitative change in the system behaviour (Arnold, 1984). In the multi-parametric programming literature the coordinates of the pencil are called state parameters and curve **K** delimits a *Critical Region*.

Figure 1.1.: Zeeman's catastrophe machine.

Multi-parametric programming has recently received considerable attention in the open literature (see Pistikopoulos *et al.*, 2007*a*), especially due to its important application to model predictive control (MPC) - see Pistikopoulos *et al.* (2007*b*). Various classes of (1.1) have been studied, see Table 1.1, with corresponding important developments and applications in control, see Table 1.2, and other areas, Table 1.3.

Despite the above major advances, many important classes of Problem (1.1) have not yet been fully addressed. In this thesis we aim at different classes of multi-parametric non-convex programs.

Table 1.1.: Classes of multi-parametric programming algorithms.

mp-LP	Saaty and Gass (1954); Gass and Saaty (1954); Gal and Nedoma (1972); Gal (1975); Dua (2000); Borrelli *et al.* (2003); Bemporad and Filippi (2003); Filippi (2004).
mp-QP	Dua *et al.* (2002); Bemporad *et al.* (2002); Tøndel *et al.* (2003).
mp-MILP	Acevedo and Pistikopoulos (1997); Kosmidis (1999); Dua and Pistikopoulos (2000); Li and Ierapetritou (2007*a*,*b*).
mp-MIQP	Dua *et al.* (2002).
mp-NLP	Armacost (1974, 1976); Kyparisis and Fiacco (1987); Fiacco and Kyparisis (1988); Dua and Pistikopoulos (1999); Acevedo and Salgueiro (2003).
mp-MINLP	Pertsinidis (1992); Pertsinidis *et al.* (1998); Dua and Pistikopoulos (1999).
mp-GO	Dua *et al.* (2004).
mp-DO	Sakizlis *et al.* (2002, 2003, 2004*a*,*b*).

Table 1.2.: Applications of multi-parametric programming to MPC.

Hierarchical decentralised control	Faísca *et al.* (2007*a*).
Hybrid control	Sakizlis *et al.* (2002); Borrelli *et al.* (2005); Morari and Barić (2006); Baotić *et al.* (2006).
Linear discrete systems	Pistikopoulos *et al.* (2000); Dua *et al.* (2002); Bemporad *et al.* (2002); Tøndel *et al.* (2003); Sakizlis *et al.* (2005).
Non-linear control	Sakizlis *et al.* (2007).
Robust control	Sakizlis *et al.* (2004*b*,*c*); Bemporad *et al.* (2003); Kouramas *et al.* (2008*a*).

Table 1.3.: Other applications of multi-parametric programming.

Drug delivery systems	Dua and Pistikopoulos (2005); Dua *et al.* (2006).
Dynamic programming	Baotić *et al.* (2006); Faísca *et al.* (2008).
Game theory	Faísca *et al.* (2007*b*,*a*).
Pro-active scheduling	Ryu *et al.* (2004); Ryu and Pistikopoulos (2007).
Supply chain	Pistikopoulos *et al.* (2007).

1.1. Goals and thesis organisation

The main goals of this thesis are the development of novel theory and algorithms for classes of multi-parametric programming which involve the solution of global optimisation problems. Examples illustrating the applicability of the new developments are (i) the plant selection problem, (ii) optimal control of multilevel systems, (iii) process design with uncertainty and (iv) robust control of linear discrete systems.

◇ **Plant selection problem**

Companies commonly take decisions at two different levels: (i) headquarters and (ii) manufacturing plants. The manufacturing plants aim to minimise the operating cost, whereas the headquarters aim to maximise the overall profit; which, originates a loop of operational and strategic decisions, Figure 1.2.

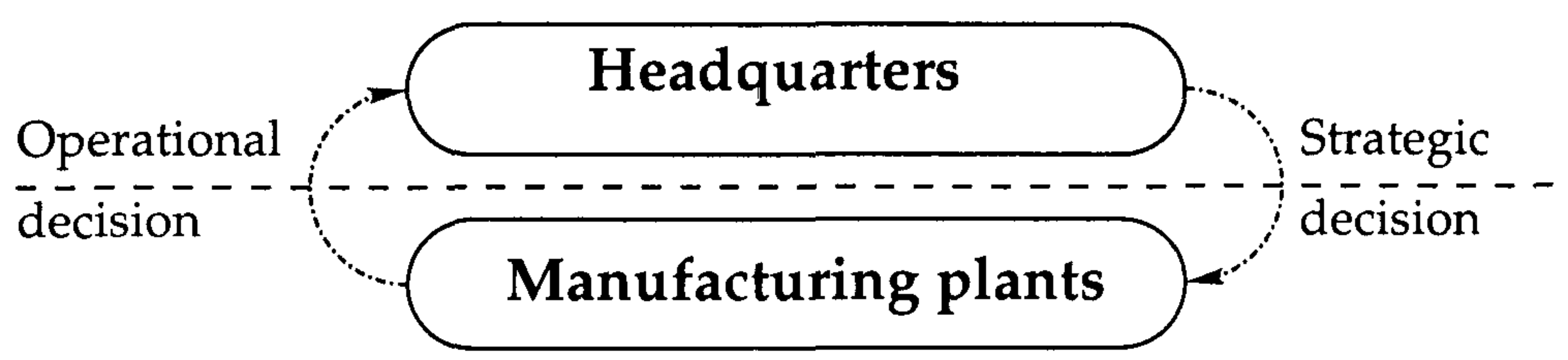


Figure 1.2.: Plant selection problem.

This hierarchical decision problem is formulated as a bilevel programming problem (Floudas, 2000). Whilst, the inner problem corresponds to the minimisation of the operating cost, the outer problem corresponds to the global maximisation of the profit. Although the headquarters have control over the full set of optimisation variables, they are constrained by the action of the manufacturing plants. The challenges are:

- the existence of an hierarchy of decision makers. The mathematical modelling of the interaction between the two levels of decision is complex and it is widely known to result in a global optimisation problem;
- the presence of logical variables. The headquarters have to take strategic decisions, e.g. which plant to open and which products to assign. Obviously, the mixed integer nature of the problem further increases the problem complexity.

◇ **Optimal control of multi-level systems**

In many hierarchical control strategies a dynamic system is optimised within a complex structure with different objectives at different levels, Figure 1.3. As

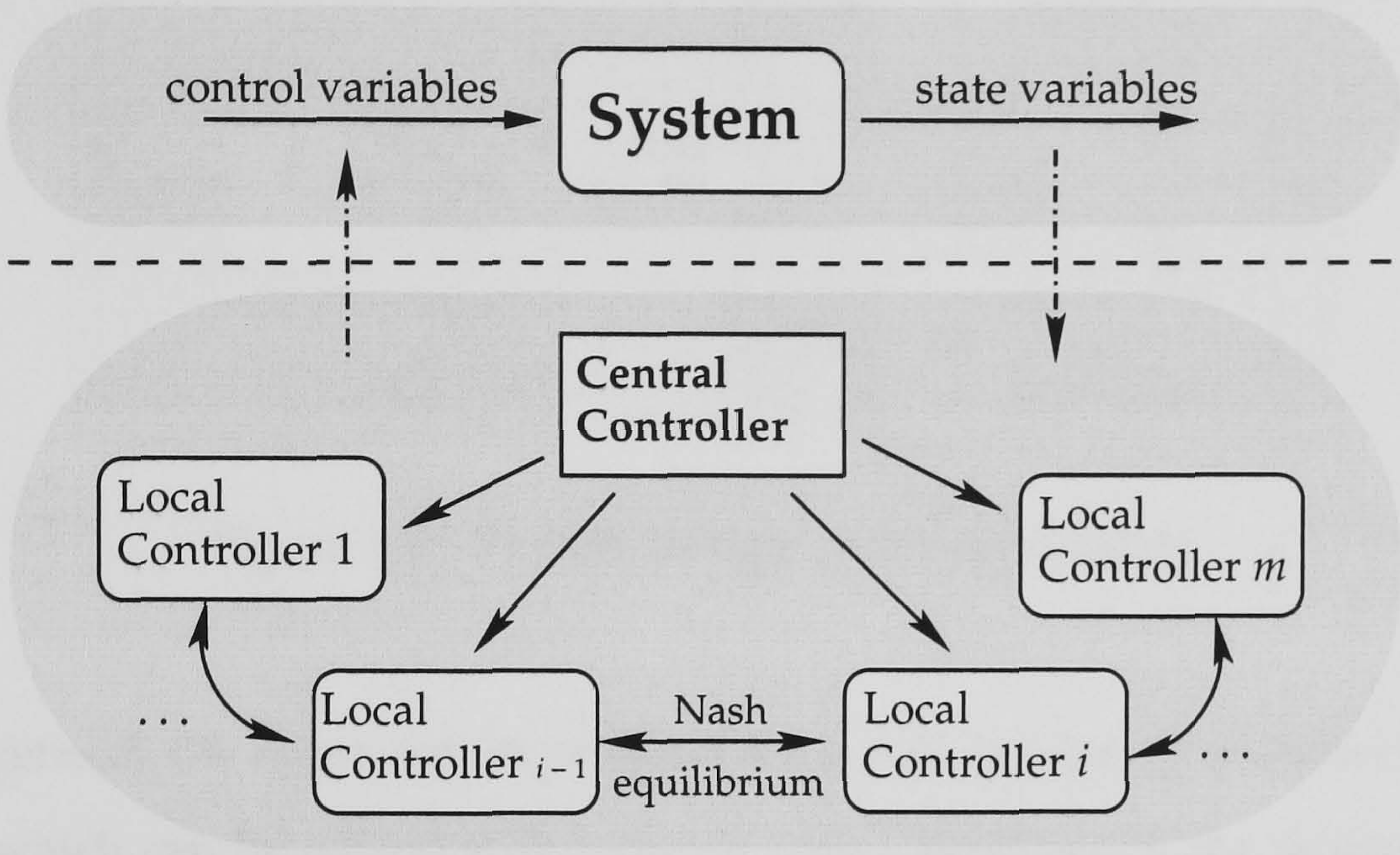


Figure 1.3.: Optimal control of multi-level systems.

we will see, this class of problems is posed as a multi-level optimisation problem, which establishes a complex network of information (Başar and Olsder, 1982). In this thesis, we assume that different agents at the same level reach a *Nash* equilibrium, whereas agents between different levels reach a type of *Stackelberg* equilibrium. The main challenges are:

- the coordination of multiple agents within a complex structure. Again, the resulting problem is formulated as a global optimisation problem;
- the computation of *Nash* and *Stackelberg* equilibria in the presence of constraints. Originally, the theory of games does not consider the existence of constraints.

◇ **Process design with uncertainty**

Process design problems are a very well-studied topic, both in academic and industry communities (Biegler *et al.*, 1997). The most common solution strategy is to formulate a superstructure and solve the resulting optimisation problem, Figure 1.4.

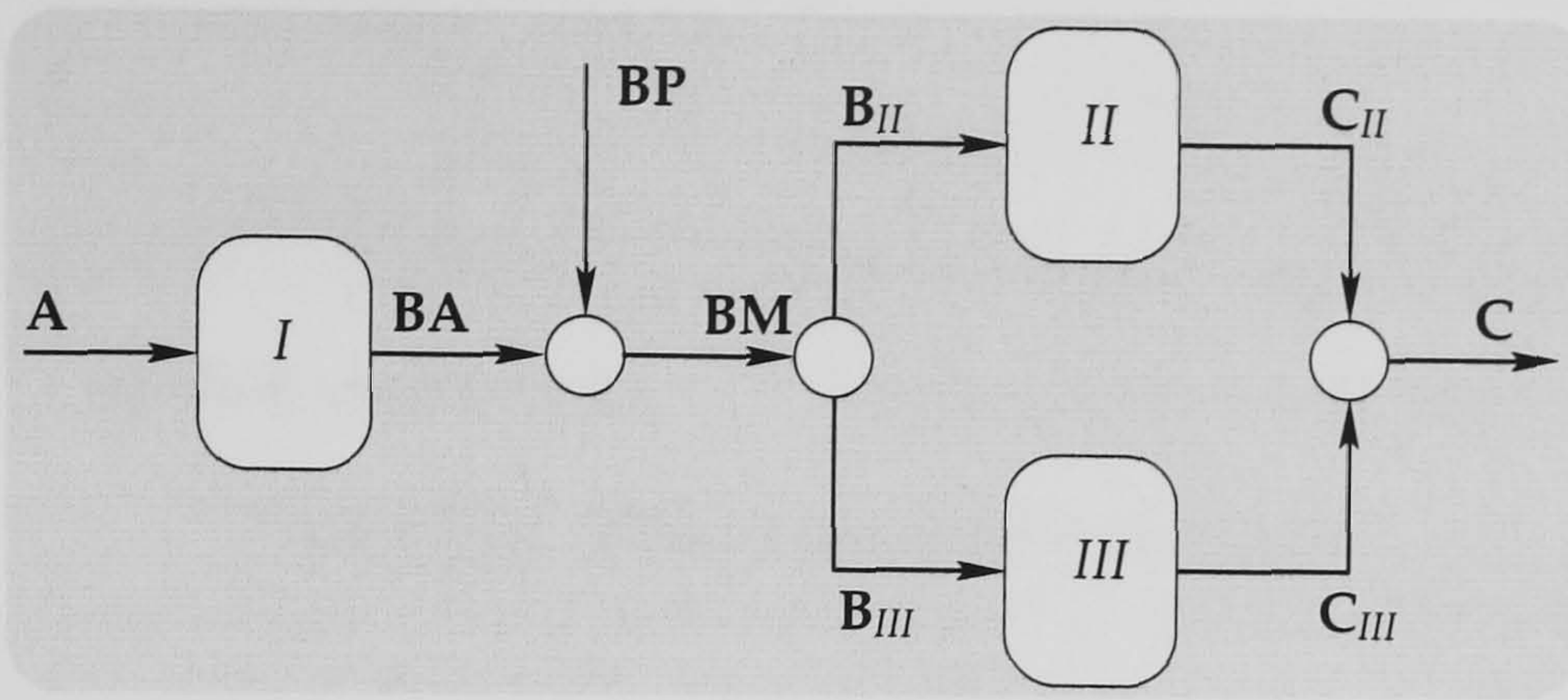


Figure 1.4.: Process design under uncertainty.

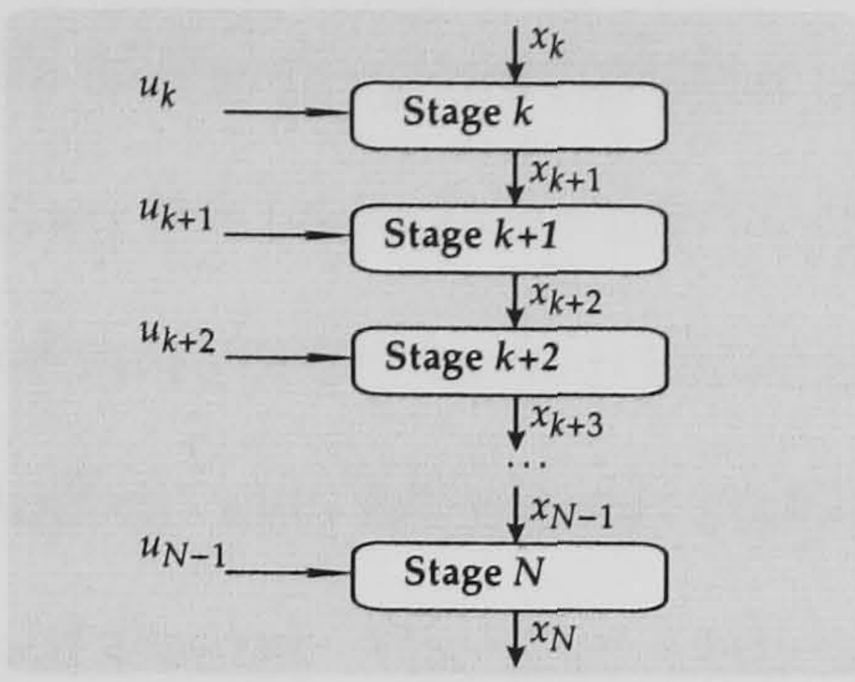
In general, the optimisation problem is a mixed integer optimisation problem; which can be solved with existing tools. However, if the designer assumes that the cost function and/or constraints may vary during operation, varying parameters have to be considered in the mathematical model. Therefore, the challenge is:

- the coexistence of varying parameters in the objective function and in the constraints. It gives rise to bilinearities and consequently to a global optimisation problem.

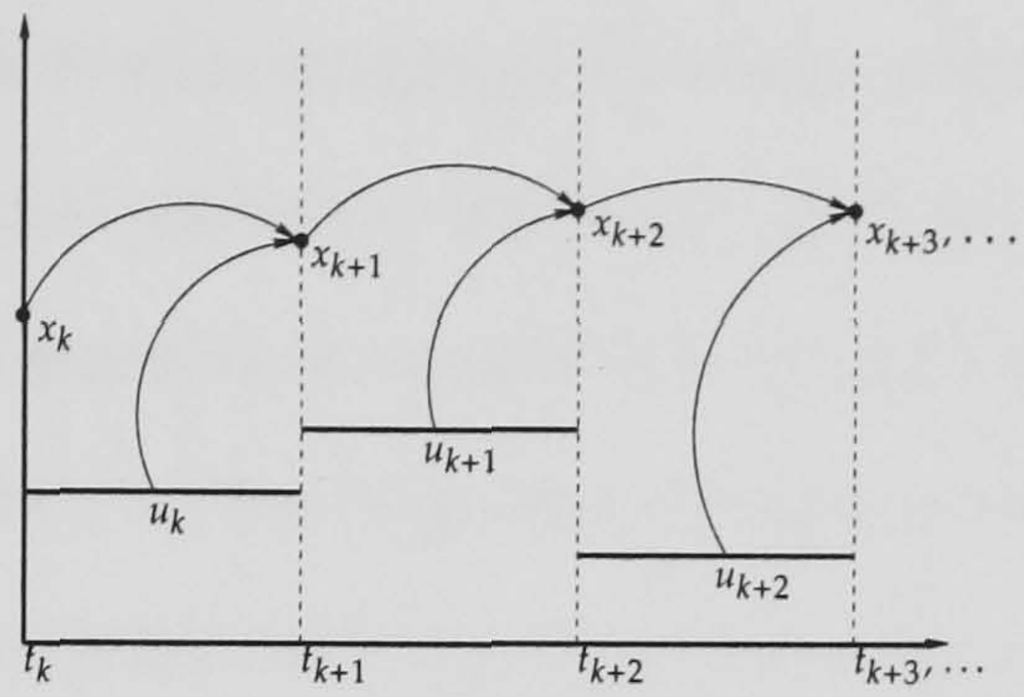
◇ **Robust optimal control of discrete linear systems**

Dynamic Programming is well-documented as being a powerful tool to solve multi-stage decision problems (Bellman, 2003). Based on the optimality principle, the original problem disassembles into a set of problems of lower dimensionality, thereby significantly reducing the complexity of obtaining the solution, Figure 1.5.

Dynamic programming is very popular in the design of optimal control policies for discrete linear systems. Nevertheless, the standard strategy neither addresses the presence of hard constraints nor considers the existence of uncertainty in the model. Thus, the challenges are:



(a) Multi-stage process.



(b) Time horizon of actions.

Figure 1.5.: Multi-stage decision process.

- the development of a novel approach for the solution of dynamic programming problems in the presence of hard constraints. Since the optimal decision laws are non-linear, the resulting optimisation problems are non-convex;
- the development of a novel approach whose solution is immune against uncertainty in the discrete linear model. The direct robust re-formulation of the original constrained optimal control problem is well known for originating a global optimisation problem.

The thesis is organised in the following way. In Chapter 2 (Part I) we propose a global optimisation approach for the solution of various classes of bilevel programming problems based on recently developed parametric programming algorithms. We first describe how we can recast and solve the inner (follower's) problem of the bilevel formulation as a multi-parametric programming problem, with parameters being the (unknown) variables of the outer (leader's) problem. By inserting the obtained rational reaction sets in the upper level problem the overall problem is transformed into a set of independent quadratic, linear or mixed integer linear programming problems, which can be solved to global optimality. In particular, we solve bilevel quadratic and bilevel mixed integer linear problems, with or without right-hand-side uncertainty. A number of examples are presented to illustrate the steps and details of the proposed global optimisation strategy. Subsequently, Chapter 3 outlines the foundations of a general global optimisation strategy for the solution of multilevel hierarchical and general decentralised multilevel problems, based on our recent developments on multi-parametric programming and control theory. The core idea is to recast each optimisation subproblem, present in the hierarchy, as a multi-parametric programming problem, with parameters being the optimisation variables belonging to the remaining subproblems. This then transforms the multilevel problem into single-level linear/convex optimisation problems. For decentralised systems, where more than one optimisation problem is present at each level of the hierarchy, Nash equilibrium is considered. A three person dynamic optimisation problem is presented to illustrate the mathematical developments.

In Chapter 4, we present a novel global optimisation approach for the general solution of multi-parametric mixed integer linear programs (mp-MILPs). We describe an optimisation procedure which iterates between a (master) mixed integer nonlinear program and a (slave) multi-parametric program. Moreover, we explain how to overcome the presence of bilinearities, responsible for the non-convexity of the multi-parametric program, in two classes of mp-MILPs, with (i) varying parameters in the objective function and (ii) simultaneous presence of varying parameters in the objective function and the right-hand side of the constraints. Examples are provided to illustrate the solution steps.

In Part II, we focus on the new developments in robust optimisation and control. Chapter 5 presents a new algorithm for solving complex multi-stage optimisation problems involving hard constraints and uncertainties, based on dynamic and multi-parametric programming techniques. Each echelon of the dynamic programming procedure, typically employed in the context of multi-stage optimisation models, is interpreted as a multi-parametric optimisation problem, with the present states and future decision variables being the parameters, while the present decisions the corresponding optimisation variables. This reformulation significantly reduces the dimension of the original problem, essentially to a set of lower dimensional multi-parametric programs, which are sequentially solved. Furthermore, the use of sensitivity analysis circumvents non-convexities that naturally arise in constrained dynamic programming problems.

In Chapter 6, we describe the foundations of a novel optimisation framework for the solution of the linear quadratic regulation problem of parametric uncertain systems. Based on dynamic and multi-parametric programming techniques, the procedure recast the original problem into a robust formulation considering the worst-case variation in the system's dynamic model. Moreover, we describe how the robust formulation, which preserves the original linear-quadratic program, is solved using the multi-parametric dynamic programming algorithm for linear time-invariant systems, developed in Chapter 5. The solution steps are illustrated with the double integrator example considering path and input constraints, plus parametric uncertainty in the linear state transition model.

Part I.

Advances in global optimisation

2. Parametric global optimisation for bilevel programming

In this chapter, we propose a global optimisation approach for the solution of various classes of bilevel programming problems based on recently developed parametric programming algorithms. We first describe how we can recast and solve the inner (follower's) problem of the bilevel formulation as a multi-parametric programming problem, with parameters being the (unknown) variables of the outer (leader's) problem. By inserting the obtained rational reaction sets in the upper level problem the overall problem is transformed into a set of independent quadratic, linear or mixed integer linear programming problems, which can be solved to global optimality. In particular, we solve bilevel quadratic and bilevel mixed integer linear problems, with or without right-hand-side uncertainty. A number of examples are presented to illustrate the steps and details of the proposed global optimisation strategy.

2.1. Introduction

Multilevel optimisation problems have attracted considerable attention from the scientific and economic community in recent years. Due to its many applications, multilevel and in particular bilevel programming have evolved significantly. Bilevel programming problems (BLPP) involve a hierarchy of two optimisation problems, of the following form (Vicente, 1992; Migdalas *et* *al.*, 1997; Floudas, 2000; Gümüş and Floudas, 2001; Dempe, 2003):

$$\begin{aligned} &\min_{x,y} F(x,y), \\ s.t.\ \ &G(x,y) \leq 0, \\ &x \in X, \\ &y \in \text{argmin}\{f(x,y) : g(x,y) \leq 0, y \in Y\}, \end{aligned} \tag{2.1}$$

where, $X \subseteq \mathbb{R}^{nx}$ and $Y \subseteq \mathbb{R}^{ny}$ are both compact convex sets; F and f are real functions: $\mathbb{R}^{(nx+ny)} \rightarrow \mathbb{R}$; G and g are vectorial real functions, $G : \mathbb{R}^{(nx+ny)} \rightarrow \mathbb{R}^{nu}$ and $g : \mathbb{R}^{(nx+ny)} \rightarrow \mathbb{R}^{nl}$; $nx,\ ny \in \mathbb{N}$ and $nu,\ nl \in \mathbb{N} \cup \{0\}$. The following definitions are associated to Problem (2.1):

Relaxed feasible set (or constrained region),

$$\Omega = \{x \ \in \ X, y \ \in \ Y : G(x,y) \leq 0, g(x,y) \leq 0\}; \tag{2.2}$$

Lower level feasible set,

$$\mathrm{C}(x) = \{y \ \in \ Y : g(x,y) \leq 0\}; \tag{2.3}$$

Follower's rational reaction set,

$$\mathrm{M}(x) = \{y \ \in \ Y : y \in argmin\{f(x,y) : y \in \mathrm{C}(x)\}\}; \tag{2.4}$$

Inducible region,

$$IR = \{x \ \in \ X, y \ \in \ Y : (x,y) \in \Omega, y \in \mathrm{M}(x)\}. \tag{2.5}$$

Applications of bilevel and multilevel programming include design optimisation problems in process systems engineering (Clark and Westerberg, 1990; Clark, 1990); design of transportation networks (LeBlanc and Boyce, 1985); agricultural planning (Fortuny-Amat and McCarl, 1981); management of multi-divisional firms (Ryu *et al.*, 2004) and hierarchical decision-making structures (Fortuny-Amat and McCarl, 1981). These multilevel problems are classified according to the number of levels and the type of their cost functions and variables: if the problem has two levels, where both cost functions are affine functions and the variables are continuous, the problem is classified as a linear bilevel programming problem (BLPP); if at least one of these functions has a quadratic expression, it is a quadratic BLPP; adding uncertainty to the formulations results in a BLPP with uncertainty; on the other hand, if binary and continuous variables coexist in the same bilevel problem formulation, it corresponds to a mixed integer BLPP.

Recently, Pistikopoulos and co-workers (Dua and Pistikopoulos, 2000; Dua *et al.*, 2002) have proposed novel solution algorithms which open the possibility of using a general framework to address general classes of bilevel and multilevel programming problems. These algorithms are based on parametric programming theory (Acevedo and Pistikopoulos, 1997; Dua, 2000) and use of the Basic Sensitivity Theorem (Fiacco, 1976, 1983). This approach can be classified as a *Reformulation Technique* (Visweswaran *et al.*, 1996) since the bilevel problem is transformed into a number of quadratic or linear problems. The main idea is to divide the follower's feasible area into different rational reaction sets, and search for the global optimum of a simple quadratic (or linear) programming problem in each area.

2.1.1. Global optimum of a bilevel programming problem

While for an optimal control problem (one-player problem) there is a well-defined concept for optimality, the same is not always true for multi-person games (Başar and Olsder, 1982).

In the case of bilevel programming, Vicente (1992), Visweswaran *et al.* (1996), Shimizu *et al.* (1997), Floudas *et al.* (1999), Floudas (2000) and Dempe *et al.* (2005) interpret the optimisation problem as a leader's problem, F, and search for the global minimum of F. The solution point obtained for the follower's problem, f, will respect the stationary (KKT) conditions and hence it can be any stationary point.

Obviously, this solution strategy is acceptable when the player in the upper level of the hierarchy is in the most "powerful" position, and the other levels just react to the decision of their leader. Such approach is sensible in many engineering applications of bilevel programming (for instance, see Clark and Westerberg, 1990; Clark, 1990). It is also a valid strategy for the cases of decentralised manufacturing and financial structures, when the leader has a full insight and control of the overall objectives and strategy of the corporation, while the follower does not.

However, this is not always the case. For example, using the *feedback Stackelberg solution*, where at every level of play a Stackelberg equilibrium point is searched, the commitment of the leader for his/her decision increases with the number of players involved. Cao and Chen (2006) present an example where the sacrifice of the leader's objective on behalf of the followers results in a better solution for both levels. Similar solution strategies have also been studied (Tabucanon, 1988; Lai, 1996; Shih *et al.*, 1996).

Theorem 2.1 *(Vicente, 1992) If for each $x \in X$, f and g are twice continuously differentiable functions for every $y \in C(x)$, f is strictly convex for every $y \in C(x)$ and $C(x)$ is a convex and compact set, then $M(\cdot)$ is a real-valued function, continuous and closed.*

If Theorem 2.1 applies and assuming that $M(x)$ is non-empty, then $M(x)$ will have only one element, which is $y(x)$. Thus, Equation (2.1) can be reformulated as:

$$
\begin{aligned}
& \min_{x,y} F(x, y(x)), \\
s.t.\quad & G(x, y(x)) \leq 0, \\
& x \in C_{rf}, \\
& C_{rf} = \{x \in X : \exists y \in Y, g(x, y) \leq 0\} \ .
\end{aligned}
\tag{2.6}
$$

Considering that f is a strictly convex real function, the function $y(x)$ can be computed as a linear conditional function based on multi-parametric programming theory, as follows (Dua *et al.*, 2002):

$$y(x) = \begin{cases} m^1 + n^1 x, & if \quad H^1 x \leq h^1, \\ m^2 + n^2 x, & if \quad H^2 x \leq h^2, \\ \vdots & \\ m^k + n^k x, & if \quad H^k x \leq h^k, \\ \vdots & \\ m^K + n^K x, & if \quad H^K x \leq h^K, \end{cases} \tag{2.7}$$

where, n^k, m^k and h^k are real vectors and H^k is a real matrix.

Theorem 2.2 *(Vicente, 1992) If the assumptions of Theorem 2.1 hold, F is a real continuous function, X and the set defined by G(x,y) are compact, and if* $\{\exists x \in X : G(x, y(x)) \leq 0\}$*, then there is a global solution for Problem (2.1).*

Since an explicit expression for y can be computed, if the assumptions of Theorem 2.2 hold, and the two players have convex functions to optimise, then the global optimum for Problem (2.1) can be obtained via the parametric programming approach.

The advantage of using this approach is that the final solution will consider the possibility of existence of other global minima, which could correspond to better solutions for the follower. Moreover, the parametric nature of the leader's problem is preserved.

Regarding computational complexity, a number of authors have shown that bilevel programming problems are $\mathcal{NP}$-Hard (Hansen *et al.*, 1992; Deng, 1998). Furthermore, Vicente *et al.* (1994) proved that even checking for a local optimum is a $\mathcal{NP}$-Hard problem.

The goal in this chapter is to describe a parametric programming framework which can solve different classes of multilevel programming problems to global optimality. Section 2.2 presents the fundamental developments for the quadratic bilevel programming case. The theory is extended, in Section 2.3, to cover the existence of RHS uncertainty, and Section 2.4 addresses mixed integer bilevel programming. At last, Section 2.5 presents an application to the plant selection problem.

2.2. Quadratic bilevel programming

Consider the following general quadratic BLPP:

$$\begin{aligned} &\min_{x,y} F(x,y) = L_1 + L_2x + L_3y + \frac{1}{2}x^TL_4x + y^TL_5x + \frac{1}{2}y^TL_6y, \\ s.t.\ \ &G_1x + G_2y + G_3 \leq 0, \\ &\min_{y} f(x,y) = l_1 + l_2x + l_3y + \frac{1}{2}x^Tl_4x + y^Tl_5x + \frac{1}{2}y^Tl_6y, \\ &s.t\ \ g_1x + g_2y + g_3 \leq 0\,, \end{aligned} \tag{2.8}$$

where x and y are the optimisation variables, $x \in X \subseteq \mathbb{R}^{nx}$ and $y \in Y \subseteq \mathbb{R}^{ny}$. $[L_2]_{1xnx}$, $[L_3]_{1xny}$, $[L_4]_{nxxnx}$, $[L_5]_{nyxnx}$, $[L_6]_{nyxny}$, $[l_2]_{1xnx}$, $[l_3]_{1xny}$, $[l_4]_{nxxnx}$, $[l_5]_{nyxnx}$ and $[l_6]_{nyxny}$ are matrices defined in the real space. The matrices $[G_1]_{nuxnx}$, $[G_2]_{nuxny}$, $[G_3]_{nux1}$, $[g_1]_{nuxnx}$, $[g_2]_{nuxny}$, $[g_3]_{nux1}$ correspond to the constraints, also defined in the real space.

Focusing the attention on the follower's optimisation problem, considering x as a parameter vector and operating a variable change ($z = y + l_6^{-1}l_5x$), it can be rewritten as the following multi-parametric quadratic programming (mp-QP) problem:

$$\begin{aligned} &\min_{z} f'(x,z) = l'_1 + l'_2x + \frac{1}{2}x^Tl'_4x + \{l'_3z + \frac{1}{2}z^Tl'_6z\}, \\ s.t\ \ &g'_2z \leq g'_3 + g'_1x\,, \end{aligned} \tag{2.9}$$

where: $l'_1 = l_1$; $l'_2 = l_2 - l_3l_6^{-1}l_5$; $l'_3 = l_3$; $l'_4 = l_4 - l_5^Tl_6^{-1}l_5$; $l'_5 = 0$; $l'_6 = l_6$; $g'_1 = -(g_1 - g_2l_6^{-1}l_5)$; $g'_2 = g_2$; $g'_3 = -g_3$.

The mp-QP Problem (2.9) can be solved applying the algorithm of Dua *et al.* (2002). As a result, a set of rational reaction sets (Definition 2.4) is obtained for different regions of x:

$$z^k = m^k + n^k x; \;\; H^k x \leq h^k, \qquad k = 1, 2, \ldots, K \;. \tag{2.10}$$

By incorporating the expressions (2.10) into Problem (2.8) results in the following K quadratic problems:

$$\begin{aligned} \min_x F'(x) &= L_1'^k + L_2'^k x + \frac{1}{2} x^T L_4'^k x, \\ s.t \;\; G_1'^k x &\leq G_3'^k \,, \end{aligned} \tag{2.11}$$

with:

$L_1'^k = L_1 + L_3 m^k + \frac{1}{2} m^{k^T} L_6 m^k;$
$L_2'^k = L_2 + L_3 n^k - L_3 l_6^{-1} l_5 + m^{k^T} L_5 + m^{k^T} L_6 n^k - m^{k^T} L_6 l_6^{-1} l_5;$
$L_4'^k = L_4 + 2n^k L_5 - 2 l_5^T l_6^{-1} L_5 + n^{k^T} L_6 n^k - 2 n^{k^T} L_6 l_6^{-1} l_5 + l_5^T l_6^{-1} L_6 l_6^{-1} l_5;$
$G_1' = G_1 + G_2 n^k - G_2 l_6^{-1} l_5;$
$G_3' = -(G_3 + G_2 m^k);$
$G_1'^k = [G_1' | H^k]^T_{(nx)\times(nu+n_{hk})};$
$G_3'^k = [G_3' | h^k]^T_{(1)\times(nu+n_{hk})}.$

Clearly, the solution of the BLLP Problem (2.8) is the minimum along the K solutions of Problem (2.11).

Remark 2.1 *The artificial variable, z, introduced in Problem (2.9) is only necessary if $l_5 \neq \underline{0}$. In all other cases the multi-parametric problem can be easily formulated through algebraic manipulations.*

Remark 2.2 *When one of the matrices l_6', $L_4'^k$ is null the optimisation problem where these are involved becomes linear. Particularly, if $l_6' = \underline{0}$, Problem (2.9) is transformed into a mp-LP; on the other hand, if $L_4'^k = \underline{0}$, Problem (2.11) becomes a LP problem. In both cases, the solution procedure is not affected, due to the fact that the Basic Sensitivity Theorem (Fiacco, 1976, 1983) also applies to the mp-LP problem.*

Remark 2.3 *The expression for the artificial variable introduced, z, is only valid*

Table 2.1.: Parametric Programming approach for a BLPP.

Step	*Description*
1	Recast the inner problem as a multi-parametric programming problem, with the leader's variables being the parameters (2.9);
2	Solve the resulting problem using the suitable multi-parametric programming algorithm;
3	Substitute each of the K solutions in the leader's problem, and formulate the K one level optimisation problems;
4	Compare the K optimum points and select the best one.

when l_6 is symmetric. If not, with the following transformation:

$$\bar{l}_6 = \left\{ \frac{l_6 + l_6^T}{2} \right\},$$

the resulting matrix is non-singular. If the resulting matrix is singular the expression for the artificial variable should be given by:

$$\begin{aligned} z &= y + Ax, \\ & \textit{where } A \textit{ should satisfy: } \left\{ A \in \mathbb{R}^{nx \times nx} : l_5 - (\tfrac{1}{2} l_6 + \tfrac{1}{2} l_6^T) A = 0 \right\} . \end{aligned}$$

In this case, several solutions for the system above can exist. However, as long as the bilinear terms are eliminated in Problem (2.9) any solution can be selected.

Remark 2.4 *This technique is not valid when at the same time:(i) f is a pure quadratic cost function; (ii) f involves bilinear terms and (iii) matrix $\bar{l}_6$ is singular.*

Observing Formulation (2.11) we can conclude that the parametric programming approach (Table 2.1), transforms the original quadratic bilevel programming problem into simple quadratic problems, for which a global optimum can be reached.

In the following subsections, examples are presented for LP | LP, LP | QP and QP | QP bilevel programming problems.

2.2.1. LP|LP Bilevel programming problem

Consider the following linear BLPP (Bard and Falk, 1982):

$$\begin{aligned}
\min_{x,y} \quad & F(x,y) = -8x_1 - 4x_2 + 4y_1 - 40y_2 + 4y_3, \\
s.t. \quad \min_{y} \quad & f(x,y) = x_1 + 2x_2 + y_1 + y_2 + 2y_3, \\
s.t. \quad & -y_1 + y_2 + y_3 \leq 1, \\
& 2x_1 - y_1 + 2y_2 - 0.5y_3 \leq 1, \\
& 2x_2 + 2y_1 - y_2 - 0.5y_3 \leq 1, \\
& y \geq 0, \\
& x \geq 0 \ .
\end{aligned} \tag{2.12}$$

Problem (2.12) was solved using the steps described in Table (2.1):

Step 1 Formulate a mp-LP problem for the lower level:

$$\begin{aligned}
& \min_{y} f(x,y) = x_1 + 2x_2 + y_1 + y_2 + 2y_3, \\
s.t. \quad & -y_1 + y_2 + y_3 \leq 1, \\
& -y_1 + 2y_2 - 0.5y_3 \leq 1 - 2x_1, \\
& 2y_1 - y_2 - 0.5y_3 \leq 1 - 2x_, \\
& y \geq 0, \\
& x \geq 0 \, .
\end{aligned} \tag{2.13}$$

Step 2 The application of the mp-LP algorithm to the lower level results in the following five rational reaction sets in Table (2.2).

Step 3 Substituting each of the sets obtained into the leader's problem, five linear programming problems result (Table 2.3).

Step 4 Observing the best values achieved for each region (Table 2.3), the global solution is obtained for: $x_1 = 0; x_2 = 0.9; y_1 = 0; y_2 = 0.6; y_3 = 0.4$ ($F = -26; f = 3.2$).

Table 2.2.: Rational reaction sets (Step 2).

k	$y^k(x) = m^k + (n_k - l_6^{-1} l_5)x$	$H^k x \leq h_i^k$
1	$y_1(x) = 0$ $y_2(x) = 0$ $y_3(x) = 0$	$0 \leq x \leq \frac{1}{2}$
2	$y_1(x) = -1 + 2x_1$ $y_2(x) = 0$ $y_3(x) = 0$	$0 \leq x_2$ $4x_1 + 2x_2 \leq 3$ $\frac{1}{2} \leq x_1$
3	$y_1(x) = \frac{2}{3}x_1 - \frac{2}{3}x_2$ $y_2(x) = 0$ $y_3(x) = -2 + \frac{8}{3}x_1 + \frac{4}{3}x_2$	$0 \leq x_2$ $2x_1 + 2x_2 \leq 3$ $-x_1 + x_2 \leq 0$ $-4x_1 - 2x_2 \leq -3$
4	$y_1(x) = 0$ $y_2(x) = -\frac{2}{3}x_1 + \frac{2}{3}x_2$ $y_3(x) = -2 + \frac{4}{3}x_1 + \frac{8}{3}x_2$	$\frac{2}{3}x_1 + \frac{10}{3}x_2 \leq 3$ $x_1 - x_2 \leq 0$ $0 \leq x_1$ $-2x_1 - 4x_2 \leq -3$
5	$y_1(x) = 0$ $y_2(x) = -1 + 2x_2$ $y_3(x) = 0$	$0 \leq x_1$ $2x_1 + 4x_2 \leq 3$ $\frac{1}{2} \leq x_2$

Table 2.3.: Formulation of new problems (Step 3 and Step 4).

k	Optimisation Problem	Optimised variables	Function values
1	$\min_x F = -8x_1 - 4x_2$ s.t. $0 \leq x \leq \frac{1}{2}$	$x_1 = 0.5; x_2 = 0.5$ $y_1 = 0; y_2 = 0; y_3 = 0$	$F = -6$ $f = 1.5$
2	$\min_x F = -4x_2 - 4$ s.t. $0 \leq x_2$ $4x_1 + 2x_2 \leq 3$ $0.5 \leq x_1$	$x_1 = 0.5; x_2 = 0.5$ $y_1 = 0; y_2 = 0; y_3 = 0$	$F = -6$ $f = 1.5$
3	$\min_x F = \frac{16}{3}x_1 - \frac{4}{3}x_2 - 4$ s.t. $0 \leq x_2$ $2x_1 + 2x_2 \leq 3$ $-x_1 + x_2 \leq 0$ $-4x_1 - 2x_2 \leq -3$	$x_1 = 0.5; x_2 = 0.5$ $y_1 = 0; y_2 = 0; y_3 = 0$	$F = -6$ $f = 1.5$
4	$\min_x F = 24x_1 - 20x_2 - 8$ s.t. $\frac{2}{3}x_1 + \frac{10}{3}x_2 \leq 3$ $x_1 - x_2 \leq 3$ $x_1 - x_2 \leq 0$ $0 \leq x_1$ $-2x_1 - 4x_2 \leq -3$	$x_1 = 0; x_2 = 0.9$ $y_1 = 0; y_2 = 0.6; y_3 = 0.4$	$F = -26$ $f = 3.2$
5	$\min_x F = -8x_1 - 84x_2 + 40$ s.t. $0 \leq x_1$ $2x_1 + 4x_2 \leq 3$ $0.5 \leq x_2$	$x_1 = 0; x_2 = 0.75$ $y_1 = 0; y_2 = 0.5; y_3 = 0$	$F = -23$ $f = 2$

The global minimum obtained is the same as the one reported by Floudas *et al.* (1999). Here, only the solution of a single mp-LP and five LP was required

Table 2.4.: Different solutions for Problem (2.14).

Solution	F	f	x_1	x_2	y_1	y_2
Aiyoshi and Shimizu (1981)	5	100	25	30	5	10
Visweswaran *et al.* (1996)	0	200	0	0	-10	-10
Solution 1	0	100	0	30	-10	10
Solution 2	0	200	0	0	-10	-10

to obtain the global minimum; whereas for the same problem Shimizu *et al.* (1997) report that their strategy requires the solution of ten sub-problems. Clearly, the computational efficiency of the proposed procedure depends on the performance of the underlying multi-parametric programming algorithm (which, in independent studies, has been reported as robust and efficient: Dua and Pistikopoulos, 2000; Dua *et al.*, 2002; Sakizlis *et al.*, 2003; Sakizlis *et al.*, 2004*a*; Sakizlis *et al.*, 2004*b*).

2.2.2. LP|QP Bilevel programming problem

Consider a linear cost function at the leader's level and a quadratic at the lower level:

$$\begin{aligned}
&\min_{x,y} \quad F(x,y) = 2x_1 + 2x_2 - 3y_1 - 3y_2 - 60, \\
s.t.\quad & x_1 + \quad x_2 + y_1 - 2y_2 - 40 \le 0, \\
&\min_{y} \quad f(x,y) = (y_1 - x_1 + 20)^2 + (y_2 - x_2 + 20)^2, \\
&s.t. \quad -x_1 + 2y_1 \le -10, \\
&\qquad -x_2 + 2y_2 \le -10, \\
&\qquad 0 \le x \le 50, -10 \le y \le 20\,.
\end{aligned} \tag{2.14}$$

The solutions found for this problem (Solution 1 and Solution 2) are compared to solutions reported in the literature (Aiyoshi and Shimizu, 1981; Visweswaran *et al.*, 1996), as shown in Table 2.4.

It is interesting to note: (i) Solutions 1 and 2 have the same (global) solution for the leader's problem, $F = 0$. However, they differ in the solution of

Table 2.5.: Different solutions for Problem (2.15).

Solution	F	f	x_1	x_2	y_1	y_2	y_3
Muu and Quy (2003)	0.6426	1.671	0.609	0.391	0	0	1.828
Solution 1	0.6384	1.6799	0.6111	0.3889	0	0	1.8333

the follower's problem; (ii) Solution 2 is identical to the solution reported in Visweswaran *et al.* (1996); (iii) Solution 1 is the global solution (as discussed in 2.1.1), where both the leader and follower's cost functions are optimised. Thus, this comparison enhances a singular property of the framework developed in this work, which is the lower level optimisation; whereas most of the *Reformulation Techniques* just satisfy the requirement of having a stationary point, KKT optimality conditions, for the lower level, with this approach the decision maker optimises firstly the leader cost function but has the opportunity to optimise the lower level as well.

2.2.3. QP|QP Bilevel programming problem

Consider the following problem, introduced by Muu and Quy (2003), which has quadratic functions in both levels:

$$\begin{aligned}
&\min_{x,y} F(x,y) = y_1^2 + y_3^2 - y_1 y_3 - 4y_2 - 7x_1 + 4x_2,\\
s.t.\ \ & x_1 + x_2 \leq 1,\\
&\min_{y} f(x,y) = y_1^2 + \frac{1}{2}y_2^2 + \frac{1}{2}y_3^2 + y_1 y_2 + (1-3x_1)y_1 + (1+x_2)y_2,\\
&s.t.\ \ 2y_1 + y_2 - y_3 + x_1 - 2x_2 + 2 \leq 0,\\
&\qquad x \geq 0,\\
&\qquad y \geq 0\,.
\end{aligned} \tag{2.15}$$

Muu and Quy (2003) have solved this oligopolistic market example to find an ϵ-global minimum ($\epsilon = 0.01$). The global minimum computed in the present work is compared to the former and presented in Table 2.5.

It is interesting to note here that: (i) the solution obtained is in full agreement with the one reported in Muu and Quy (2003); (ii) the solution of one mp-QP and one QP were required to arrive at the global solution.

2.3. Bilevel programming with uncertainty

Evans (1984) highlighted the importance of considering uncertainty/risk (e.g. prices, technological attributes, etc.) in the solution of decentralised decision makers. The presence of uncertainty in bilevel problems has been addressed before for the linear case (Ryu *et al.*, 2004). Uncertainty is considered unstructured, taking any value between its bounds. In the present work, we present an extension of our earlier work to the quadratic case.

We address the following quadratic BLPP with right-hand side uncertainty, θ:

$$\begin{aligned} &\min_{x,y} F(x,y,\theta) = L_1 + L_2x + L_3y + \frac{1}{2}x^T L_4 x + y^T L_5 x + \frac{1}{2}y^T L_6 y, \\ s.t.\quad & G_1x + G_2y + G_3 \le G_4\theta, \\ &\min_{y} f(x,y,\theta) = l_1 + l_2x + l_3y + \frac{1}{2}x^T l_4 x + y^T l_5 x + \frac{1}{2}y^T l_6 y, \\ &s.t.\quad g_1x + g_2y + g_3 \le g_4\theta\ , \end{aligned} \tag{2.16}$$

The steps for solving (2.16) are as follows:

1. Recast the inner problem as a mp-QP, with parameters being both x and θ. The solution obtained is similar to (2.10):

$$z^k = m^k + n_b^k x + \bar{n}_c^k\theta\ ;\ \ H^k x + \bar{H}^k\theta \le h^k, \quad k = 1,2,\ldots,K\ . \tag{2.17}$$

2. Incorporate expressions (2.17) in (2.16) to formulate K mp-QPs, with parameters being the uncertainty θ:

$$\begin{aligned} &\min_{x} F'(x,\theta) = \bar{L}_1'^k + \bar{L}_2'^k x + \frac{1}{2}x^T \bar{L}_4'^k x, \\ s.t.\quad & \bar{G}_1'^k x \le \bar{G}_3'^k + \bar{G}_4'^k\theta\ , \end{aligned} \tag{2.18}$$

where $\bar{L}_1^{\prime k}$, $\bar{L}_2^{\prime k}$, $\bar{L}_4^{\prime k}$, $\bar{G}_1^{\prime k}$, $\bar{G}_2^{\prime k}$, $\bar{G}_4^{\prime k}$ are appropriate matrices derived by algebraic manipulations.

We will illustrate the proposed procedure by revisiting example (2.15) with the addition of two uncertain parameters (θ_1,θ_2) as follows:

$$
\begin{aligned}
&\min_{x,y} F(x,y) = y_1^2 + y_3^2 - y_1y_3 - 4y_2 - 7x_1 + 4x_2,\\
s.t.\quad & x_1 + x_2 \le 1 + \theta_1,\\
&\min_{y} f(x,y) = y_1^2 + \frac{1}{2}y_2^2 + \frac{1}{2}y_3^2 + y_1y_2 + (1-3x_1)y_1 + (1+x_2)y_2,\\
&s.t.\quad 2y_1 + y_2 - y_3 + x_1 - 2x_2 + 2 \le \theta_2,\\
&\qquad x \ge 0,\\
&\qquad y \ge 0,\\
&\qquad 0 \le \theta_1 \le 0.25,\\
&\qquad 0 \le \theta_2 \le 0.5\,.
\end{aligned}
\tag{2.19}
$$

The solution of the inner mp-QP problem of Step 1 results in a single critical region, with the following parametric expressions:

$$
\begin{aligned}
&y_1 = 0,\\
&y_2 = 0,\\
&y_3 = x_1 - 2x_2 - \theta_2 + 2,\\
&x_1 + x_2 \le 1 + \theta_1,\\
&-x_1 + 2x_2 \le 2 - \theta_2,\\
&x \ge 0,\\
&0 \le \theta_1 \le 0.25,\\
&0 \le \theta_2 \le 0.5\,.
\end{aligned}
\tag{2.20}
$$

Then, Step 2 involves (i) the substitution of the expressions in (2.20) into the leader's problem and (ii) formulation and solution of the outer mp-QP problem, based on which the following results were obtained:

$$\begin{aligned} x_1 &= 0.444\theta_1 + 0.556\theta_2 + 0.611, \\ x_2 &= 0.556\theta_1 - 0.556\theta_2 + 0.389, \\ 0 &\le \theta_1 \le 0.25, \\ 0 &\le \theta_2 \le 0.5. \end{aligned} \tag{2.21}$$

For the limiting case, when $\theta_1 = 0$ and $\theta_2 = 0$, the results obtained in (2.21) correspond to the results obtained in section 2.2.3, Table 2.5.

In this example, Step 1 results in a single critical region (2.20). However, it is possible that, by the end, different parametric expressions are computed to the same critical region. We overcome this redundancy by keeping the best solution and discarding the others through the formal comparison procedure proposed by Acevedo and Pistikopoulos (1997).

2.4. Mixed integer bilevel programming

In many real systems, the leader may have to take "yes-no" decisions (Wen and Yang, 1990). This type of decisions can be described by the introduction of binary variables in the model. Assuming that the optimisation variables are separable and appear in linear relations, the following mixed integer bilevel programming problem is derived (Shimizu *et al.*, 1997):

$$\begin{aligned} &\min_{x_1,x_2,y_1,y_2} F(x_1,x_2,y_1,y_2) = L_1 + L_2^T x_1 + L_3^T y_1 + L_4^T x_2 + L_5^T y_2, \\ s.t.\ & G_1 x_1 + G_2 y_1 + G_3 x_2 + G_4 y_2 + G_5 \le 0, \\ &\min_{y_1,y_2} f(x_1,x_2,y_1,y_2) = l_1 + l_2^T x_1 + l_3^T y_1 + l_4^T x_2 + l_5^T y_2, \\ &s.t.\ g_1 x_1 + g_2 y_1 + g_3 x_2 + g_4 y_2 + g_5 \le 0, \end{aligned} \tag{2.22}$$

where x_1, x_2, y_1 and y_2 are the optimisation variables, $x_1 \in X_1 \subseteq \mathbb{R}^{nx_1}$, $x_2 \in \{0,1\}^{nx_2}$, $y_1 \in Y_1 \subseteq \mathbb{R}^{ny_1}$, $y_2 \in \{0,1\}^{ny_2}$. $[L_2]_{nx_1}$, $[L_3]_{ny_1}$, $[L_4]_{nx_2}$, $[L_5]_{ny_2}$, $[l_2]_{nx_1}$, $[l_3]_{ny_1}$, $[l_4]_{nx_2}$, $[l_5]_{ny_2}$ are vectors defined in the real space. The matrices $[G_1]_{nuxnx_1}$, $[G_2]_{nuxny_1}$, $[G_3]_{nuxnx_2}$, $[G_4]_{nuxny_2}$, $[G_5]_{nux1}$, $[g_1]_{nuxnx_1}$, $[g_2]_{nuxny_1}$, $[g_3]_{nuxnx_2}$, $[g_4]_{nuxny_2}$, $[g_5]_{nux1}$ correspond to the constraints, also defined in the real space.

If the integrality conditions, with respect to x_2, are moved to the upper level, a multi-parametric mixed integer linear programming (mp-MILP), with x_1 and x_2 being the parameters, can be formulated as follows (Formulation 2.23):

$$\begin{aligned} &\min_{y_1,y_2} f(x,y_1,y_2) = l_1' + l_2'^T x + l_3'^T y_1 + l_5'^T y_2, \\ &s.t. \quad g_2' y_1 + g_4' y_2 \leq g_5' + g_1' x, \end{aligned} \tag{2.23}$$

where: $x = [x_1|x_2]^T$; $l_1' = l_1$; $l_2' = [l_2|l_4]^T$; $l_3' = l_3$; $l_5' = l_5$; $g_1' = -[g_1|g_3]^T$; $g_2' = g_2$; $g_4' = g_4$; $g_5' = -g_5$.

The mp-MILP problem in (2.23) can be solved applying the algorithm of Dua and Pistikopoulos (2000), from which the following group of K solutions are obtained:

$$\begin{cases} y_2^k = \bar{y}_2^k \\ y_1^k = m^k + n^k x \Leftrightarrow y_1^k = m^k + n_1^k x_1 + n_2^k x_2 \;, \\ H^k x \leq h^k \Leftrightarrow H_1^k x_1 + H_2^k x_2 \leq h^k \end{cases} \qquad k = 1,2,\ldots,K\,. \tag{2.24}$$

Introducing these expressions in (2.22), a set of K independent MILPs is obtained:

$$\begin{aligned} F(x_1,x_2) \; &= \min_{x_1,x_2}\{L_1'^k + L_2'^{k^T} x_1 + L_4'^{k^T} x_2\}, \\ s.t. \qquad & G_1'^k x_1 + G_3'^k x_2 \leq G_5'^k, \end{aligned} \tag{2.25}$$

with: $L_1'^k = L_1 + L_3 m^k + L_5 \bar{y}_2^k$; $L_2'^k = L_2 + L_3 n_1^k$; $L_4'^k = L_4 + L_3 n_2^k$;

$G_1' = G_1 + G_2 n_1^k$;

$G_3' = G_3 + G_2 n_2^k$;

$G_5' = -(G_4 \bar{y}_2^k + G_5 + G_2 m^k)$;

$G_1'^k = [G_1'|H_1^k]^T$; $G_3'^k = [G_3'|H_2^k]^T$; $G_5'^k = [G_3'|h^k]^T$.

The solution of the K MILPs in (2.25) results in the selection of the global optimum by direct comparison.

The proposed strategy will be illustrated by the following MILP | LP bilevel programming problem introduced by (Wen and Yang, 1990):

$$\begin{aligned}
\min_{x,y} \quad & F(x,y) = -(20x_1 + 60x_2 + 30x_3 + 50x_4 + 15y_1 + 10y_2 + 7y_3), \\
s.t. \quad \min_{y} \quad & f(x,y) = -(20y_1 + 60y_2 + 8y_3), \\
s.t. \quad & 5x_1 + 10x_2 + 30x_3 + 5x_4 + 8y_1 + 2y_2 + 3y_3 \leq 230, \\
& 20x_1 + 5x_2 + 10x_3 + 10x_4 + 4y_1 + 3y_2 \leq 240, \\
& 5x_1 + 5x_2 + 10x_3 + 5x_4 + 2y_1 + y_3 \leq 90, \\
& x \in \{0,1\}, \\
& y \geq 0.
\end{aligned} \tag{2.26}$$

Moving the integrality constraint to the outer level, the inner problem can be rewritten as a mp-LP with x being the parameter. Its solution results in the following parametric expressions (single critical region):

$$\begin{aligned}
& y_1 = 0, \\
& y_2 = -6.667x_1 - 1.667x_2 - 3.333x_3 - 3.333x_4 + 80, \\
& y_3 = 2.778x_1 - 2.222x_2 - 7.778x_3 + 0.5556x_4 + 23.33, \\
& 0 \leq x_1, x_2, x_3, x_4 \leq 1 .
\end{aligned} \tag{2.27}$$

Introducing these expressions in the leader's problem, and taking into account the binary nature of x, the following MILP problem is obtained:

$$\begin{aligned}
\min_{x} \quad & F = -(20x_1 + 60x_2 + 30x_3 + 50x_4 + 15y_1 + 10y_2 + 7y_3), \\
s.t. \quad & y_1 = 0, \\
& y_2 = -6.667x_1 - 1.667x_2 - 3.333x_3 - 3.333x_4 + 80.00, \\
& y_3 = 2.778x_1 - 2.222x_2 - 7.778x_3 + 0.5556x_4 + 23.33, \\
& x \in \{0,1\}, \\
& y \geq 0.
\end{aligned} \tag{2.28}$$

Table 2.6 presents the solution for Problem (2.28), and consequently for

Table 2.6.: Solution for Problem (2.26).

$F =$	- 1011.67
$f =$	- 4673.34
$y_1 =$	0.00
$y_2 =$	75.00
$y_3 =$	21.67
$x_1 =$	0
$x_2 =$	1
$x_3 =$	0
$x_4 =$	1

Problem (2.26).

The result obtained is identical to the one obtained by Wen and Yang (1990).

2.5. An application to the plant selection problem

The plant selection problem is inherently a mixed integer bilevel programming problem; with two distinctive layers of decision. The headquarters take strategic decisions, i.e. logical decisions, whereas the manufacturing plants take operational decisions, e.g. optimise steam, raw material and fuel consumption. The fundamental issue is the coordination between the two layers of decision, Figure 2.1.

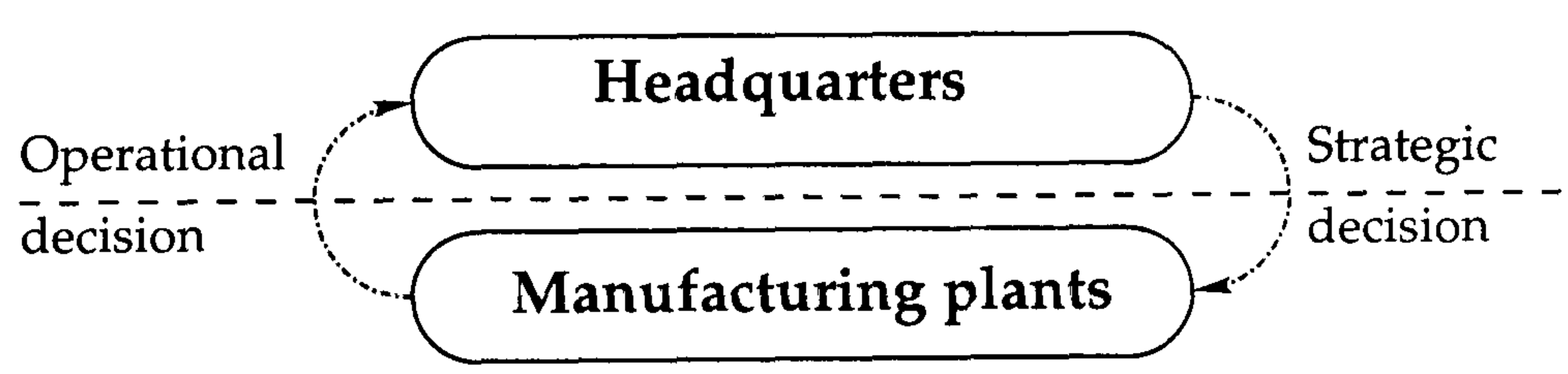


Figure 2.1.: Plant selection problem.

Due to its major impact on multinational companies, the plant selection problem has attracted considerable attention, and consequently, several formulations have been proposed (Ertogral and Wu, 2000; Canel and Khumawala, 2001; Ghiani *et al.*, 2002). Here, we follow a non-monolithic model, which evidences the bilevel nature of the plant selection problem (Cao and Chen, 2006):

$$
\begin{aligned}
&\min_{x\in S_{xy}, y\in S_{xy}} F = \sum_{i=1}^{m} f_i y_i + \sum_{i=1}^{m} p_i \left(\mathrm{Cap}_i\, y_i + \sum_{j\in \mathrm{IS}_i} d_j a_{ij} x_{ij} \right), \\
&s.t.\ \min_{x\in S_{xy}} f = \sum_{i=1}^{m} w_i \left(\sum_{j\in \mathrm{IS}_i} d_j a_{ij} x_{ij} \right) + \sum_{i=1}^{m} \sum_{j\in \mathrm{IS}_i} d_j R_{ij} x_{ij}, \\
&\quad s.t.\ \sum_{i\in \mathrm{JS}_j} x_{ij} = 1, \quad j = 1,\ldots,n, \\
&\quad \sum_{j\in \mathrm{IS}_i} d_j a_{ij} x_{ij} \le \mathrm{Cap}_i \cdot y_i, \quad i = 1,\ldots,m, \\
&\quad \sum_{j\in \mathrm{IS}_i} x_{ij} \le n \cdot y_i, \quad i = 1,\ldots,m, \\
&\quad x_{ij} \ge 0, \ \ y_i \in \{0,1\}, \ \ i = 1,\ldots,m, \ \ j \in \mathrm{IS}_i,
\end{aligned}
\tag{2.29}
$$

where,

- m number of potential plants;
- n number of product types;
- p_i opportunity cost for unused production capacity of plant i after it is opened;
- d_j costumer demand of product j;
- a_{ij} capacity consumption ratio for processing product j in plant i;
- w_i cost to use production capacity in plant i;
- Cap_i available production capacity in plant i;
- IS_i the group of products that can be produced in plant i;
- JS_j the set of plants that can produce product j;
- R_{ij} transportation cost for transferring product j from the principal firm to plant i;
- f_i Opening cost for plant i.

In this section, we consider 6 possible locations for the manufacturing plants, and a portfolio of 8 products to manage - Table 2.7 and Table 2.8.

Table 2.7.: Data for the plant selection problem.

Plant i	1	2	3	4	5	6
f_i	400	265	265	200	150	100
p_i	1.0	0.55	0.55	0.5	0.45	0.40
w_i	1.0	0.55	0.55	0.5	0.45	0.40
Cap_i	450	250	250	200	150	100

Table 2.8.: Data for the plant selection problem.

Product j	1	2	3	4	5	6	7	8
d_i	60	60	60	60	60	60	60	60
a_{ij}								
$i = 1$	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
$i = 2$	1.0	1.0	1.0	1.0	•	•	•	•
$i = 3$	•	•	•	•	1.0	1.0	1.0	1.0
$i = 4$	•	•	•	0.8	0.8	0.8	•	•
$i = 5$	1.1	1.1	•	•	•	•	1.1	1.1
$i = 6$	•	•	1.2	•	1.2	•	•	•
R_{ij}								
$i = 1$	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1
$i = 2$	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1
$i = 3$	0.08	0.08	0.08	0.08	0.08	0.08	0.08	0.08
$i = 4$	0.09	0.09	0.09	0.09	0.09	0.09	0.09	0.09
$i = 5$	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05
$i = 6$	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05

Therefore, Problem (2.29) is a mixed integer bilevel program with 6 integer variables and 48 continuous variables. The solution steps are as follows:

STEP 1. Solve the follower's problem, with leader's optimisation variables being the parameters:

$$\begin{aligned} \min_{x \in S_{xy}} f = & \sum_{i=1}^{m} w_i \left(\sum_{j \in IS_i} d_j a_{ij} x_{ij} \right) + \sum_{i=1}^{m} \sum_{j \in \mathrm{IS}_i} d_j R_{ij} x_{ij}, \\ s.t \ & \sum_{i \in \mathrm{JS}_j} x_{ij} = 1, \quad j = 1, \ldots, n, \\ & \sum_{j \in \mathrm{IS}_i} d_j a_{ij} x_{ij} \leq \mathrm{Cap}_i \cdot y_i, \quad i = 1, \ldots, m, \\ & \sum_{j \in \mathrm{IS}_i} x_{ij} \leq n \cdot y_i, \quad i = 1, \ldots, m, \\ & x_{ij} \geq 0, \ \ 0 \leq y_i \leq 1, \ \ i = 1, \ldots, m, \ \ j \in \mathrm{IS}_i. \end{aligned} \tag{2.30}$$

The solution is given by a set of piece-wise linear expressions:

$$\begin{cases} x^k = m^k + n^k \cdot y \\ H^k x \leq h^k \end{cases}, k = 1, \ldots, 5,$$

for instance, when k=5:

$$\begin{cases} x_{1,1} = 1 - 1.0417y_2 - 0.5682y_5, x_{1,2} = 1 - 1.0417y_2 - 0.5682y_5, x_{1,6} = -1.0417y_3, \\ x_{1,3} = 1 - 1.0417y_2 - 0.6944y_6, x_{1,4} = -1.0417y_2, x_{1,5} = -1.0417y_3 - 0.6944y_6, \\ x_{1,7} = 1 - 1.0417y_3 - 0.5682y_5, x_{1,8} = 1 - 1.0417y_3 - 0.5682y_5, x_{2,1} = 1.0417y_2, \\ x_{2,2} = 1.0417y_2, x_{2,3} = 1.0417y_2, x_{2,4} = 1.0417y_2, x_{3,5} = 1.0417y_3, x_{3,6} = 1.0417y_3, \\ x_{3,7} = 1.0417y_3, x_{3,8} = 1.0417y_3, x_{4,4} = 1, x_{4,5} = 1, x_{4,6} = 1, x_{5,1} = 0.5682y_5, \\ x_{5,2} = 0.5682y_5, x_{5,7} = 0.5682y_5, x_{5,8} = 0.5682y_5, x_{6,3} = 0.6944y_6, x_{6,5} = 0.6944y_6, \\ \\ 0 \leq y_1 \leq 0, 0 \leq y_2 \leq 0.96, 0 \leq y_3 \leq 0.96, 0.72 \leq y_4 \leq 1, 0 \leq y_5 \leq 1, 0 \leq y_6 \leq 1, \\ -y1 - 0.5556\mathrm{x}y2 - 0.5556\mathrm{x}y3 - 0.3030\mathrm{x}y5 - 0.1852\mathrm{x}y6 \leq -0.6667. \end{cases}$$

STEP 2. Introducing these expressions in (2.29), 5 independent MILPs are obtained. The solutions are listed in Table 2.9.

From Table 2.9, we conclude that *Solution 3* is the optimal.

STEP 3. Therefore, the optimal decision is to open one plant in position 1 and one plant in position 2. And, assign the whole production of products 1,2,3,4 to Plant 1, and products 5,6,7,8 to Plant 2.

Table 2.9.: Solution of the MILPs.

Solution	Cost
1	553
2	880
3	**541**
4	880
5	778

2.6. Concluding remarks

We have described the foundations of a novel global optimisation strategy for the solution of general classes of bilevel programming based on our recent developments in multi-parametric programming. It was shown that bilevel linear, quadratic and mixed-integer linear programs, also involving uncertainty, can be effectively solved. It was further shown that issues related to global optimality for both levels of the bilevel program can be addressed.

In Chapter 3, we further extend this approach to address general multilevel programming problems (Ruan *et al.*, 2004) and Stackelberg-Nash equilibrium type of problems (Liu, 1998*b*); the application to hierarchical control structures (Stephanopoulos and Ng, 2000) is also described.

3. Multilevel hierarchical and decentralised optimisation problems

Here, we outline the foundations of a general global optimisation strategy for the solution of multilevel hierarchical and general decentralised multilevel problems, based on our recent developments on multi-parametric programming and control theory, Chapter 2. The core idea is to recast each optimisation subproblem, present in the hierarchy, as a multi-parametric programming problem, with parameters being the optimisation variables belonging to the remaining subproblems. This then transforms the multilevel problem into single-level linear/convex optimisation problems. For decentralised systems, where more than one optimisation problem is present at each level of the hierarchy, Nash equilibrium is considered. A three person dynamic optimisation problem is presented to illustrate the mathematical developments.

3.1. Introduction

The development of a general theory to solve multi-person objective decision problems is of great importance for decision making and control theory (Başar, 1975). Multi-person objective decision problems have attracted numerous investigations (Başar, 1975, 1978; Tolwinski, 1981; Başar and Olsder, 1982; Anandalingman, 1988; Liu, 1998*a*; Li *et al.*, 2002; Shih *et al.*, 2004), with diverse applications in engineering (Morari *et al.*, 1980; Clark, 1983; Stephanopoulos and Ng, 2000), financial problems (Anandalingman, 1988; Nie *et al.*, 2006) and other areas.

Hereto, we focus on multilevel decentralised optimisation problems, where the objectives (optimisation subproblems) are organised in a hierarchy of decisions. In this hierarchy, each optimisation subproblem controls a subset of the full set of optimisation variables; the latter is completely controlled by the unique optimisation problem positioned at the top level.

The multi-layer nature in such problems results in non-linearities and non-convexities (Vicente and Calamai, 1994); hence, it is not surprising that general solution strategies for solving such complex problems are rather limited. Moreover, the possible presence of logical decisions further increases the problems' complexity. Therefore, it is widely accepted that a global optimisation approach is needed for the solution of such multilevel problems (Floudas, 2000).

Recently, Pistikopoulos and co-workers have been developing a general theory, algorithms and computation tools for the solution of general classes of multi-parametric programming problems (Pistikopoulos *et al.*, 2007*a*) and multi-parametric control (Pistikopoulos *et al.*, 2007*b*). The application of parametric programming theory to multi-level problems makes possible the development of an unified strategy for their solution to global optimality. The core idea behind this approach is to recast each optimisation subproblem as a multi-parametric programming problem. Computing the rational reaction set for each subproblem in the entire feasible space, and subsequently, computing the corresponding equilibria within the hierarchical network, reduces the complexity of the original problem. For instance, in an optimisation level with two subproblems or more, these explicit expressions are used to compute the Nash equilibrium between them. In Chapter 2 (Faísca *et al.*, 2007*b*;

Pistikopoulos *et al.*, 2007*a*) we have addressed the bilevel programming problem, a hierarchy of two optimisation subproblems organised in two levels. As aforementioned, in this chapter we extend the methodology proposed in Chapter 2 to cope with multilevel decentralised optimisation problems. Furthermore, the methodology is applied to an optimal control problem of multi-level nature, where the foundations of a general theory for multi-level hierarchical and decentralised problems are established.

Chapter 3 is organised as follows. Section 3.2 introduces the multi-level mathematical formulation, which is used throughout the chapter, and respective definitions of feasible and rational reaction set. It also briefly introduces the relevant multi-parametric programming theory and algorithms. The proposed multi-parametric programming approach for the solution of tri-level programming problems and bilevel programming with multi-followers problems is then described in detail in Section 3.3, and illustrated with example problems. Section 3.4 outlines the application of the proposed approach to multilevel optimal control of dynamic systems.

3.2. Preliminaries

3.2.1. Problem formulation

The general multilevel decentralised optimisation problem can be described as follows:

$$
\begin{aligned}
&\min_{x,y_1^i,y_2^k,\ldots,y_m^l} f_1(x,y_1^i,y_2^k,\ldots,y_m^l), && (1^{st} level)\\
&\quad \text{s.t. } g_1(x,y_1^i,y_2^k,\ldots,y_m^l) \leq 0,\\
&\qquad \text{where } \left[y_1^i,y_2^k,\ldots,y_m^l\right] \text{ solve,}\\
&\quad \ldots, \min_{y_1^i,y_2^k,\ldots,y_m^l} f_2^i(x,y_1^i,y_2^k,\ldots,y_m^l),\ldots && (2^{nd} level)\\
&\qquad \text{s.t. } g_2^i(x,y_1^i,y_2^k,\ldots,y_m^l), \leq 0, && (3.1)\\
&\qquad\quad \text{where } \left[y_2^k,\ldots,y_m^l\right] \text{ solve,}\\
&\qquad\quad \vdots\\
&\qquad\quad \ldots,\min_{y_m^l} f_m^l(x,y_1^i,y_2^k,\ldots,y_m^l),\ldots && (m^{th} level)\\
&\qquad\quad \text{s.t. } g_m^l(x,y_1^i,y_2^k,\ldots,y_m^l) \leq 0,
\end{aligned}
$$

where, f are real convex functions, g are vectorial real functions defining convex sets and x, y are sets of variables belonging to the group of real numbers; $i \in \{1,2,\ldots,I\}, k \in \{1,2,\ldots,K\}, l \in \{1,2,\ldots,L\}$, implying that ($2^{nd}$ *level*) has I optimisation subproblems, (3^{rd} *level*) K optimisation subproblems and (m^{th} *level*) has L optimisation subproblems, respectively.

For the sake of simplicity and without loss of generality, we analyse the relations in Problem (3.1) using two particular classes of multilevel programming problems: the tri-level programming problem, which organises vertically in three levels, and the bilevel programming problem with multifollowers, in a horizontal structure at the second level.

◇ Tri-level programming

The tri-level programming problem can be stated as follows:

$$\begin{aligned}
&\min_{x,y_1,y_2} f_1(x,y_1,y_2), && (1^{st}level)\\
&\quad \text{s.t. } g_1(x,y_1,y_2) \le 0,\\
&\qquad \text{where } [y_1,y_2] \text{ solve},\\
&\qquad \min_{y_1,y_2} f_2(x,y_1,y_2), && (2^{nd}level)\\
&\qquad \text{s.t. } g_2(x,y_1,y_2), \le 0,\\
&\qquad\quad \text{where } [y_2] \text{ solve},\\
&\qquad\quad \min_{y_2} f_3(x,y_1,y_2), && (3^{rd}level)\\
&\qquad\quad \text{s.t. } g_3(x,y_1,y_2) \le 0,
\end{aligned} \tag{3.2}$$

with the following definitions:

- Feasible set for the third level,

$$\Omega_2(x,y_1) = \{y_2 \in Y_2 : g_3(x,y_1,y_2) \le 0\}, \tag{3.3}$$

- Rational reaction set for the third level,

$$\phi_2(x,y_1) = \{y_2 \in Y_2 : y_2 \in \arg\min\{f_2(x,y_1,y_2) : y_2 \in \Omega_2(x,y_1)\}\}, \tag{3.4}$$

- Feasible set for the second level,

$$\Omega_1(x) = \{(y_1,y_2) \in Y_1 \mathrm{x} Y_2 : g_2(x,y_1,y_2) \le 0, g_3(x,y_1,y_2) \le 0\}, \tag{3.5}$$

- Rational reaction set for the second level,

$$\begin{aligned}\phi_1(x) = \{(y_1,y_2) \in Y_1 \mathrm{x} Y_2 : y_1 \in \arg\min\{f_2(x,y_1,y_2) :\\ y_1 \in \Omega_1(x), y_2 \in \phi_2(x,y_1)\}\}.\end{aligned} \tag{3.6}$$

Note the parametric nature of the rational reaction sets, Equation (3.4) and (3.6), which reflects the dependence of the decisions taken at the upper levels on the decisions taken at the lower levels. This in fact, evidences that in multilevel programming problems the relations between the levels differ from the well-known Stackelberg game, where the decisions made by the followers don't affect the decision, already taken by the leader (Vicente, 1992).

◇ Bilevel programming with multi-followers

Bilevel programming problems with multi-followers involve two optimisation levels with several optimisation subproblems at the lower (2^{nd}) level:

$$\begin{aligned} \min_{x,y_1,y_2,\dots,y_m} \quad & F(x, y_1, y_2, \dots, y_m), \qquad\qquad (1^{st} level) \\ s.t. \quad & G(x, y_1, y_2, \dots, y_m) \leq 0, \\ & x \in X, \\ & y_i \in \arg\min\{f_i(x, y_1, y_2, \dots, y_m) : \qquad (2^{nd} level) \\ & \quad g_i(x, y_1, y_2, \dots, y_m) \leq 0, y_i \in Y_i\}, \\ & i \in \{1, 2, \dots, m\}, \end{aligned} \tag{3.7}$$

with the following definitions:

- Feasible set for the i^{th} follower,

$$\Omega_i(x, y_1, y_2, \dots, y_{i-1}, y_{i+1}, \dots, y_m) = \{y_i \in Y_i : g_i(x, x, y_1, y_2, \dots, y_m) \leq 0\}, \tag{3.8}$$

- Rational reaction set for the i^{th} follower,

$$\phi_i(x, y_1, y_2, \dots, y_{i-1}, y_{i+1}, \dots, y_m) = \{y_i \in Y_i : y_i \in \arg\min\{f_i(x, y_1, y_2, \dots, y_m) : y_i \in \Omega_i(x)\}\}. \tag{3.9}$$

Since one assumption is that followers may exchange information, conflicts naturally occur. The Nash equilibrium is often a preferred strategy to coordinate such decentralised systems (Liu, 1998*a*). Consequently, the optimisation subproblems positioned in the lower level reach a Nash equilibrium point, $(x, y_1^*, y_2^*, \ldots, y_m^*)$ (Başar and Olsder, 1982):

$$\begin{cases} f_1(x, y_1^*, y_2^*, \ldots, y_m^*) \leq f_1(x, y_1, y_2^*, \ldots, y_m^*), \forall y_1 \in Y_1, \\ f_2(x, y_1^*, y_2^*, \ldots, y_m^*) \leq f_2(x, y_1^*, y_2, \ldots, y_m^*), \forall y_2 \in Y_2, \\ \vdots \\ f_m(x, y_1^*, y_2^*, \ldots, y_m^*) \leq f_m(x, y_1^*, y_2^*, \ldots, y_m), \forall y_m \in Y_m. \end{cases} \tag{3.10}$$

Once more, observe the parametric nature of the followers' rational reaction set, Equation 3.9. In this case, however, each rational reaction set is a function of both the upper level decision variables and the decision variables of the other subproblems located in the same hierarchical level. Additionally, the priority remains to solve the leader's objective function to global optimality. Thus, we aim to compute the global optimum for the leader and the best possible equilibrium solution for the followers.

3.2.2. Multi-parametric programming

Consider the general multi-parametric non-linear programming problem:

$$\begin{aligned} & \min_x f(x, \theta), \\ s.t. \quad & g_i(x, \theta) \leq 0, \quad \forall \ i = 1, \ldots, p, \\ & h_j(x, \theta) = 0, \quad \forall \ j = 1, \ldots, q, \\ & x \in X \subseteq \mathbb{R}^n, \\ & \theta \in \Theta \subseteq \mathbb{R}^m, \end{aligned} \tag{3.11}$$

where f, g and h are twice continuously differentiable in x and θ. Assume also that f is a convex function and g, h define a convex set, and the linear independence constraint qualification is verified. Therefore, the first-order Karush-Kuhn-Tucker (KKT) optimality conditions for (3.11) are given as follows:

$$\begin{aligned}
&\nabla_x \mathscr{L} = 0, \\
&\lambda_i g_i(x,\theta) = 0, \quad \lambda_i \geq 0, \quad \forall\, i = 1, \ldots, p, \\
&h_j(x,\theta) = 0, \quad \forall\, j = 1, \ldots, q, \\
&\mathscr{L} = f(x,\theta) + \sum_{i=1}^{p} \lambda_i g_i(x,\theta) + \sum_{j=1}^{q} \mu_j h_j(x,\theta).
\end{aligned} \tag{3.12}$$

The main sensitivity result for (3.11) derives directly from system (3.12), as shown in Theorem 3.1.

Theorem 3.1 *Basic Sensitivity Theorem (Fiacco, 1976): Let θ_0 be a vector of parameter values and (x_0, λ_0, μ_0) a KKT triple corresponding to* (3.12)*, where λ_0 is non negative and x_0 is feasible in* (3.11)*. Also assume that (i) strict complementary slackness (SCS) holds, (ii) the binding constraint gradients are linearly independent (LICQ: Linear Independence Constraint Qualification), and (iii) the second-order sufficiency conditions (SOSC) hold. Then, in the neighbourhood of θ_0, there exists a unique, once continuously differentiable function, $z(\theta) = [x(\theta), \lambda(\theta), \mu(\theta)]$, satisfying* (3.12) *with $z(\theta_0) = [x(\theta_0), \lambda(\theta_0), \mu(\theta_0)]$, where $x(\theta)$ is a unique isolated minimiser for* (3.11)*, and*

$$\begin{pmatrix} \frac{dx(\theta_0)}{d\theta} \\ \frac{d\lambda(\theta_0)}{d\theta} \\ \frac{d\mu(\theta_0)}{d\theta} \end{pmatrix} = -(M_0)^{-1} N_0, \tag{3.13}$$

where, M_0 and N_0 are the Jacobian of system (3.12) *with respect to z and θ, respectively, evaluated at θ_0:*

$$M_0 = \begin{pmatrix} \nabla^2_{xx}\mathscr{L} & \nabla_x g_1 & \cdots & \nabla_x g_p & \nabla_x h_1 & \cdots & \nabla_x h_q \\ \lambda_1 \nabla^T_x g_1 & g_1 & & & & & \\ \vdots & & \ddots & & & & \\ \lambda_p \nabla^T_x g_p & & & g_p & & & \\ \nabla^T_x h_1 & & & & & & \\ \vdots & & & & & & \\ \nabla^T_x h_q & & & & & & \end{pmatrix},$$

$$N_0 = (\nabla^2_{\theta x}\mathscr{L}, \lambda_1 \nabla^T_\theta g_1, \ldots, \lambda_p \nabla^T_\theta g_p, \nabla^T_\theta h_1, \ldots, \nabla^T_\theta h_q)^T.$$

□

Proof. See (Fiacco, 1983, pp. 72).

Note that the assumptions stated in the theorem above ensure M_0 is invertible (McCormick, 1976).

Dua *et al.* (2002) has proposed an algorithm to solve Equation (3.13) in the entire range of the varying parameters for general convex problems. This algorithm is based on approximations of the non-linear *optimal* expression, $x = \gamma^*(\theta)$, by a set of first-order approximations (Corollary 3.1).

Corollary 3.1 *First-order estimation of $u(x)$, $\lambda(x)$, $\mu(x)$, near $x = x_0$ (Fiacco, 1983): Under the assumptions of Theorem 3.1, a first-order approximation of $[u(x)$, $\lambda(x)$, $\mu(x)]$ in the neighbourhood of x_0 is,*

$$\begin{bmatrix} x(\theta) \\ \lambda(\theta) \\ \mu(\theta) \end{bmatrix} = \begin{bmatrix} x_0 \\ \lambda_0 \\ \mu_0 \end{bmatrix} - (M_0)^{-1} \cdot N_0 \cdot \theta + O(\|\theta\|), \tag{3.14}$$

where $(x_0, \lambda_0, \mu_0) = [x(\theta_0), \lambda(\theta_0), \mu(\theta_0)]$, $M_0 = M(\theta_0)$, $N_0 = N(\theta_0)$, and $\phi(x) = O(\|\theta\|)$ means that $\phi(\theta)/\|\theta\| \to 0$ as $\theta \to \theta_0$.

Each piecewise linear approximation is confined to regions defined by feasibility and optimality conditions (Dua *et al.*, 2002). If $\check{g}$ corresponds to the non-active constraints, and $\tilde{\lambda}$ to the Lagrangian multipliers of the active constraints:

$$\begin{cases} \breve{g}(x(\theta),\theta) \leq 0 & \rightarrow \text{ Feasibility conditions,} \\ \tilde{\lambda}(\theta) \geq 0 & \rightarrow \text{ Optimality conditions.} \end{cases} \tag{3.15}$$

Consequently, the explicit expressions are given by a conditional piecewise linear function (Dua *et al.*, 2002):

$$\begin{cases} x = \mathbf{C}^1 + \mathbf{K}^1 \cdot \theta, & \forall \theta \in CR^1, \\ x = \mathbf{C}^2 + \mathbf{K}^2 \cdot \theta, & \forall \theta \in CR^2, \\ \vdots & \\ x = \mathbf{C}^L + \mathbf{K}^L \cdot \theta, & \forall \theta \in CR^L, \end{cases} \tag{3.16}$$

where $\mathbf{K}^i$ and $\mathbf{C}^i$ are real matrices, and $CR^i \subset \mathbb{R}^m$.

3.3. Proposed methodology

In this section, we show how we can address tri-level programming and bilevel with multi-followers programming problems and solve them to global optimality through the application of parametric programming.

3.3.1. Tri-level programming problem

Consider the tri-level programming problem with a quadratic objective function and linear constraints:

$$
\begin{array}{lll}
\min\limits_{x,y_1,y_2} f_1 = & L_1^1 + & (1^{st}\ level) \\
& +L_2^1 \cdot x + L_3^1 \cdot y_1 + L_4^1 \cdot y_2 + & \\
& +\frac{1}{2}x^T \cdot L_5^1 \cdot x + \frac{1}{2}y_1^T \cdot L_6^1 \cdot y_1 + \frac{1}{2}y_2^T \cdot L_7^1 \cdot y_2 + & \\
& +x^T \cdot L_8^1 \cdot y_1 + y_2^T \cdot L_9^1 \cdot x + y_2^T \cdot L_{10}^1 \cdot y_1, & \\
s.t. & \left| \begin{array}{lll}
G_1^1 \cdot x + G_2^1 \cdot y_1 + G_3^1 \cdot y_2 \leq 0, & & \\
\min\limits_{y_1,y_2} f_2 = & L_1^2 + & (2^{nd}\ level) \\
& +L_2^2 \cdot x + L_3^2 \cdot y_1 + L_4^2 \cdot y_2 + & \\
& +\frac{1}{2}x^T \cdot L_5^2 \cdot x + \frac{1}{2}y_1^T \cdot L_6^2 \cdot y_1 + \frac{1}{2}y_2^T \cdot L_7^2 \cdot y_2 + & \\
& +x^T \cdot L_8^2 \cdot y_1 + y_2^T \cdot L_9^2 \cdot x + y_2^T \cdot L_{10}^2 \cdot y_1, & \\
s.t. & \left| \begin{array}{lll}
G_1^2 \cdot x + G_2^2 \cdot y_1 + G_3^2 \cdot y_2 \leq 0, & & \\
\min\limits_{y_2} f_3 = & L_1^3 + & (3^{rd}\ level) \\
& +L_2^3 \cdot x + L_3^3 \cdot y_1 + L_4^3 \cdot y_2 + & \\
& +\frac{1}{2}x^T \cdot L_5^3 \cdot x + \frac{1}{2}y_1^T \cdot L_6^3 \cdot y_1 + & \\
& +\frac{1}{2}y_2^T \cdot L_7^3 \cdot y_2 + x^T \cdot L_8^3 \cdot y_1 + & \\
& +y_2^T \cdot L_9^3 \cdot x + y_2^T \cdot L_{10}^3 \cdot y_1, & \\
s.t. & |G_1^3 \cdot x + G_2^3 \cdot y_1 + G_3^3 \cdot y_2 \leq 0. &
\end{array} \right. &
\end{array} \right. &
\end{array}
\tag{3.17}
$$

Problem (3.17) comprises three subproblems, one at each optimisation level. Each optimisation level can be recast as a multi-parametric programming problem, where the optimisation variables corresponding to the upper optimisation levels are classified as parameters. For presentation and computation purposes, (i) we group the parameters in the i^{th}level in a single vector, ω^i and (ii) we introduce an artificial variable, v^i, to eliminate all bilinear terms.

Beginning with the (3^{rd} *level*), and considering a vector,

$$\left[\omega^3\right]^T = [x|y_1],$$

we re-write (3.17) as,

$$\begin{aligned}
\min_{y_2} \ & f_3(y_2, \omega^3) = L_1^3 + \\
& +L_2^{3*} \cdot \omega^3 + L_4^3 \cdot y_2 + \\
& +\tfrac{1}{2}\omega^{3^T} \cdot L_5^{3*} \cdot \omega^3 + \tfrac{1}{2} y_2^T \cdot L_7^3 \cdot y_2 + \\
& +y_2^T \cdot L_8^{3*} \cdot \omega^3, \\
s.t. \ & G_1^{3*} \cdot \omega^3 + G_3^3 \cdot y_2 + G_4^3 \leq 0, \\
& x \in X.
\end{aligned} \tag{3.18}$$

Introducing an artificial variable, $v^3 = y_2 + \Phi \cdot \omega^3$, where Φ is an appropriate matrix, the bilinear terms, represented in (3.18) by matrix L_8^{3*}, are eliminated. Under the right conditions (see Chapter 2), $\Phi = L_7^{3^{-1}} L_8^{3*}$, and (3.18) can be rewritten as follows:

$$\begin{aligned}
\min_{v_3} \ & f_3(v_3, \omega^3) = L_1^3 + \\
& +L_2^{3**} \cdot \omega^3 + \tfrac{1}{2}\omega^{3^T} \cdot L_5^{3**} \cdot \omega^3 \\
& + \min_{v_3} \left\{ L_4^{3**} \cdot v_3 + \frac{1}{2} v_3^T \cdot L_7^{3**} \cdot v_3 \right\}, \\
s.t. \ & G_3^{3**} \cdot v_3 \leq G_4^{3**} + G_1^{3**} \cdot \omega^3, \\
& v \in V,
\end{aligned} \tag{3.19}$$

Problem (3.19) can be solved with a multi-parametric programming algorithm (Dua *et al.*, 2002), resulting in:

$$\begin{aligned}
& v_3^k = m_3^k + n_3^k \cdot \omega^3, \ \ H_3^k \cdot \omega^3 \leq h_3^k, \\
& \text{which can be rewritten as,} \\
& y_2^k = m_3^k + (n_3^k - \Phi) \cdot \omega^3, \ \ H_3^k \cdot \omega^3 \leq h_3^k, \\
& \text{or,} \\
& y_2^k = m_3^k + p_1^k \cdot x + p_2^k \cdot y_1, \ \ H_{31}^k \cdot x + H_{32}^k \cdot y_1 \leq h^k,
\end{aligned} \tag{3.20}$$

where, $k = 1, \ldots, K_2$, with K_2 being the number of critical regions, and consequently, the number of linear approximations done on the optimal rational reaction set $\phi_2(x, y_1)$ (see Corollary 3.1).

The expressions in (3.20) can then be incorporated in the second optimisation level of (3.17). Note that since the expressions in (3.20) are piecewise linear functions of y_2^k, the complexity of the original problem does not increase. Hence, the second level can be reformulated as the following K_2 optimisation problems:

$$
\begin{aligned}
\min_{y_1} \; & f_2 = L_1^{2*} + L_2^{2*} \cdot x + L_3^{2*} \cdot y_1 + \frac{1}{2} x^T \cdot L_4^{2*} \cdot x \\
& + \tfrac{1}{2} y_1^T \cdot L_5^{2*} \cdot y_1 + y_1^T \cdot L_8^{2*} \cdot x, \\
s.t. \; & G_1^{2*} \cdot x + G_2^{2*} \cdot y_1 + G_3^{2*} \leq 0, \\
& x \in X.
\end{aligned}
\tag{3.21}
$$

We can thus proceed with optimisation levels 1 and 2. Following this procedure, tri-level optimisation problems in (3.17) result in K_1 single level convex optimisation problems:

$$
\begin{aligned}
& \min_{x} f_1^*(x, y_1(x), y_2(x, y_1)), \\
s.t. \; & G_1(x, y_1(x), y_2(x, y_1(x))) \leq 0, \\
& x \in C_{rf}, \\
& C_{rf} = \{x \in X : \exists_{y_1, y_2} \in Y_1, Y_2, G_2(x, y_1, y_2) \leq 0, G_3(x, y_1, y_2) \leq 0\} \; .
\end{aligned}
\tag{3.22}
$$

The number of K_1 final convex optimisation problems (3.22) depends on the number of critical regions obtained in each optimisation level. The algorithm is summarised in Table 3.1, and is illustrated with the following example.

Illustrative example 1

Consider the following linear tri-level example (Ruan *et al.*, 2004):

Table 3.1.: Parametric programming algorithm for tri-level programming problems.

Step	Description
1	Recast the third level of the optimisation problem as a multi-parametric programming problem, with parameters being the upper levels optimisation variables, x and y_1 (3.18);
2	Solve the resulting problem using a suitable multi-parametric programming algorithm;
3	Substitute each of the K_2 solutions in the 2^{nd} optimisation level, and formulate K_2 multi-parametric problems with the variables from the leader being the parameters (3.21);
4	Solve the resulting problem using a suitable multi-parametric programming algorithm;
5	Substitute each of the K_1 solutions in the leader's problem, and formulate the K_1 one-level optimisation problems (3.22);
6	Compare the K_1 optima and select the best one.

$$
\begin{aligned}
&\min_{x,y_1,y_2} f_1 = -x - 4 \cdot y_2, \\
&\qquad \text{where } [y_1, y_2] \text{ solve,} \\
&\qquad \min_{y_1,y_2} f_2 = 2 \cdot y_2, \\
&\qquad\quad \text{where } y_2 \text{ solves,} \\
&\qquad\quad \min_{y_2} f_3 = -y_2, \\
&\qquad\quad \text{s.t. } x + y_1 + y_2 \le 2.5, \\
&\qquad\qquad 0 \le x, y_1, y_2 \le 1.
\end{aligned}
\tag{3.23}
$$

Following the steps described in Table 3.1:

Step 1. Recast (3^{rd} *level*) optimisation problem, f_3, as a multi-parametric programming problem, with parameters being x and y_1:

$$
\begin{aligned}
&\min_{y_2} f_3 = -y_2, \\
&\text{s.t. } y_2 \le 2.5 - x - y_1, \\
&\qquad 0 \le x, y_1, y_2 \le 1,
\end{aligned}
\tag{3.24}
$$

solve the resulting problem using a multi-parametric optimisation algorithm (Dua *et al.*, 2002):

$$CR^1\begin{cases} y_2 = 1, \\ 0 \le x, y_1 \le 1, \\ x + y_1 \le 1.5, \end{cases} \quad CR^2\begin{cases} y_2 = -x - y_1 + 2.5, \\ x, y_1 \le 1, \\ -x - y_1 \le -1.5. \end{cases} \tag{3.25}$$

Step 2. Incorporate rational reaction set (3.25) into the optimisation problem corresponding to ($2^{nd} level$);

$$\begin{aligned} &\min_{y_1,y_2} f_2^{CR^1} = 2, \\ &\text{s.t. } 0 \le x \le 1, \\ &0 \le y_1 \le 1, \\ &x + y_1 \le 1.5, \\ &y_2 = 1, \end{aligned} \quad \begin{aligned} &\min_{y_1,y_2} f_2^{CR^2} = -2x - 2y_1 + 5, \\ &\text{s.t. } x, y_1 \le 1, \\ &-x - y_1 \le -1.5 \\ &y_2 = -x - y_1 + 2.5. \end{aligned} \tag{3.26}$$

Step 3. Solve problems (3.26) considering them as multi-parametric programming problems, with x being the parameter;

$$CR^3\begin{cases} y_2 = 1, \\ 0 \le x \le 1, \\ 0 \le y_1 \le 1, \\ x + y_1 \le 1.5, \end{cases} \quad CR^4\begin{cases} y_1 = 1, \\ y_2 = -x + 1.5, \\ 0.5 \le x \le 1. \end{cases} \tag{3.27}$$

Step 4. Incorporate rational reaction set (3.27) into the optimisation problem corresponding to ($1^{st} level$);

$$\begin{aligned} &\min_{x,y_1,y_2} f_1^{CR^3} = -x - 4, \\ &\text{s.t. } 0 \le x \le 1, \\ &0 \le y_1 \le 1, \\ &x + y_1 \le 1.5, \\ &y_2 = 1, \end{aligned} \quad \begin{aligned} &\min_{x,y_1,y_2} f_1^{CR^4} = 3x - 6, \\ &\text{s.t. } 0.5 \le x \le 1 \\ &y_1 = 1, \\ &y_2 = -x + 1.5. \end{aligned} \tag{3.28}$$

Step 5. Solve problems in 3.28;

$$Solution\ 1 \begin{cases} f_1^{CR^3} = -5, \\ x = 1, \\ y_2 = 1, \\ 0 \leq y_1 \leq 0.5, \end{cases} \qquad Solution\ 2 \begin{cases} f_1^{CR^4} = -4.5, \\ x = 0.5, \\ y_1 = 1, \\ y_2 = 1, \end{cases} \tag{3.29}$$

Note that in *Solution 1*, y_1 is represented by an interval. This is due to the fact that the objective function of (2^{nd} *level*) doesn't depend on y_1.

Concluding, two solutions are obtained: *Solution 1* and *Solution 2*, which are compared with the one obtained from the literature (Ruan *et al.*, 2004, *Solution 3*), as shown in Table 3.2.

Table 3.2.: Solutions for Problem (3.23).

	Parametric programming algorithm		(Ruan *et al.*, 2004)
	Solution 1	*Solution 2*	*Solution 3*
f^1	-5	-4.5	-4.5
f^2	2	2	2
f^3	1	1	1
x	1	0.5	-
y_1	0.5	0	-
y_2	1	0	-

From Table 3.2 we conclude that *Solution 1* is the global optimum for this tri-level programming problem.

3.3.2. Bilevel programming problem with multi-followers

Consider the bilevel programming problem with multi-followers, and assume quadratic objective functions, linear constraints and two followers:

$$
\begin{aligned}
\min_{x,y_1,y_2} f_1 = {} & L_1^1 + \\
& +L_2^1 \cdot x + L_3^1 \cdot y_1 + L_4^1 \cdot y_2 + \qquad (1^{st}\ level) \\
& +\tfrac{1}{2}x^T \cdot L_5^1 \cdot x + \tfrac{1}{2}y_1^T \cdot L_6^1 \cdot y_1 + \tfrac{1}{2}y_2^T \cdot L_7^1 \cdot y_2 + \\
& +x^T \cdot L_8^1 \cdot y_1 + y_2^T \cdot L_9^1 \cdot x + y_2^T \cdot L_{10}^1 \cdot y_1, \\
\text{s.t.} \quad & \left|
\begin{aligned}
& G_1^1 \cdot x + G_2^1 \cdot y_1 + G_3^1 \cdot y_2 \le 0, \\
& \qquad (2^{nd}\ level) \\
& \min_{y_1}\ f_2 = L_1^2 + \qquad \underline{Follower\ 1} \\
& \qquad +L_2^2 \cdot x + L_3^2 \cdot y_1 + L_4^2 \cdot y_2 + \\
& \qquad +\tfrac{1}{2}x^T \cdot L_5^2 \cdot x + \tfrac{1}{2}y_1^T \cdot L_6^2 \cdot y_1 + \tfrac{1}{2}y_2^T \cdot L_7^2 \cdot y_2 + \\
& \qquad +x^T \cdot L_8^2 \cdot y_1 + y_2^T \cdot L_9^2 \cdot x + y_2^T \cdot L_{10}^2 \cdot y_1, \\
& \quad \text{s.t. } G_1^2 \cdot x + G_2^2 \cdot y_1 + G_3^2 \cdot y_2 \le 0, \\
& \min_{y_2}\ f_3 = L_1^3 + \qquad \underline{Follower\ 2} \\
& \qquad +L_2^3 \cdot x + L_3^3 \cdot y_1 + L_4^3 \cdot y_2 + \\
& \qquad +\tfrac{1}{2}x^T \cdot L_5^3 \cdot x + \tfrac{1}{2}y_1^T \cdot L_6^3 \cdot y_1 + \tfrac{1}{2}y_2^T \cdot L_7^3 \cdot y_2 + \\
& \qquad +x^T \cdot L_8^3 \cdot y_1 + y_2^T \cdot L_9^3 \cdot x + y_2^T \cdot L_{10}^3 \cdot y_1, \\
& \quad \text{s.t. } G_1^3 \cdot x + G_2^3 \cdot y_1 + G_3^3 \cdot y_2 \le 0.
\end{aligned}
\right.
\end{aligned}
\tag{3.30}
$$

The difference between Problem (3.30) and Problem (3.17) is the existence of two optimisation subproblems in a single level. Accordingly, the concept of Nash equilibrium is introduced.

As in the tri-level programming case, each optimisation subproblem in (2^{nd} *level*) is recast as a multi-parametric programming problem. In this problem, the parameters are all the variables from the optimisation problem at (1^{st} *level*) as well as the optimisation variables of the other subproblems at the same level, *Follower* 1 or *Follower* 2 in this case (3.30). Thus, defining vectors, $\left[\omega^2\right]^T = [x|y_2]$ and $\left[\omega^3\right]^T = [x|y_1]$, we re-write the ($2^{nd}$*level*) optimisation subproblems as,

$$\begin{aligned}\min_{y_1} \ & f_2(y_1, \omega^2) = L_1^2 + \\ & +L_2^{2*} \cdot \omega^2 + L_3^2 \cdot y_1 + \\ & +\tfrac{1}{2}\omega^{2^T} \cdot L_5^{2*} \cdot \omega^2 + \tfrac{1}{2} y_1^T \cdot L_6^2 \cdot y_1 + \\ & +y_1^T \cdot L_8^{2*} \cdot \omega^2, \\ s.t. \ & G_1^{2*} \cdot \omega^2 + G_2^2 \cdot y_1 \leq 0,\end{aligned} \tag{3.31}$$

and,

$$\begin{aligned}\min_{y_2} \ & f_3(y_2, \omega^2) = L_1^3 + \\ & +L_2^{3*} \cdot \omega^3 + L_4^3 \cdot y_2 + \\ & +\tfrac{1}{2}\omega^{3^T} \cdot L_5^{3*} \cdot \omega^{2a} + \tfrac{1}{2} y_2^T \cdot L_7^3 \cdot y_2 + \\ & +y_1^T \cdot L_9^{3*} \cdot \omega^3, \\ s.t. \ & G_1^{3*} \cdot \omega^3 + G_3^3 \cdot y_2 \leq 0,\end{aligned} \tag{3.32}$$

where ω^2 and ω^3 are the vectors of parameters. The bi-linearities can be circumvented using a similar strategy to the one used in the tri-level case. Using a multi-parametric programming algorithm (Dua *et al.*, 2002), problems (3.31) and (3.32) result in the following parametric expressions:

$$\begin{cases} y_1 = \phi_1(x, y_2) & \rightarrow \text{rational reaction set follower 1,} \\ y_2 = \phi_2(x, y_1) & \rightarrow \text{rational reaction set follower 2,} \end{cases} \tag{3.33}$$

which are then used to compute the Nash equilibrium (x, y_1^*, y_2^*):

$$\begin{cases} f_1(x, y_1^*, y_2^*) \leq f_1(x, y_1, y_2^*), \forall y_1 \in Y_1, \\ f_2(x, y_1^*, y_2^*) \leq f_2(x, y_1^*, y_2), \forall y_2 \in Y_2, \end{cases} \tag{3.34}$$

easily computed by direct comparison (Liu, 1998*a*):

$$\phi_1'(x, y_1) = \phi_2(x, y_1), \rightarrow y_1 = \phi_2^*(x), \tag{3.35a}$$

$$\phi_1(x, y_2) = \phi_2'(x, y_2), \rightarrow y_2 = \phi_1^*(x). \tag{3.35b}$$

Finally, substituting the expressions in (3.35) in the leader's optimisation problem, (1^{st} *level*), we end up with a single level convex optimisation problem, involving only the leader's optimisation variables, as follows:

$$
\begin{aligned}
&\min_{x} f_1^*(x, y_1(x, y_2^*(x)), y_2(x, y_1^*(x))), \\
s.t. \quad & G_1(x, y_1(x, y_2^*), y_2(x, y_1^*)) \leq 0, \\
& x \in \{x \in X : \exists_{y_1,y_2} \in Y, Z, G_2(x, y_1, y_2) \leq 0, G_3(x, y_1, y_2) \leq 0\}.
\end{aligned} \tag{3.36}
$$

The algorithm is summarised in Table 3.3 and is illustrated in example 2.

Table 3.3.: Parametric programming algorithm for bi-level programming problems with multi-followers.

Step	*Description*
1	Recast each of the subproblems in the lower level as a multi-parametric programming problem, with the variables out of their control being the parameters (3.31-3.32);
2	Solve the resulting problems using the suitable multi-parametric programming algorithm;
3	Compute a Nash equilibrium point by direct comparison of the rational reaction sets (3.34);
4	Substitute each of the K solutions in the leader's problem, and formulate the K one level optimisation problems;
5	Compare the K optima points and select the best one.

Illustrative example 2

Consider the following linear bilevel programming example involving three followers at the second level (Anandalingman, 1988):

$$
\begin{aligned}
&\min_{x,y_1,y_2,y_3} F(x,y_1,y_2,y_3) = -x - y_1 - 2y_2 - y_3,\\
\text{s.t.}\quad &\min_{y_1} f_1(x,y_1,y_2,y_3) = x - 3y_1 + y_2 + y_3,\\
&\min_{y_2} f_2(x,y_1,y_2,y_3) = x + y_1 - 3y_2 + y_3,\\
&\min_{y_3} f_3(x,y_1,y_2,y_3) = x + y_1 + y_2 - 3y_3,\\
&\text{s.t.}\quad 3x + 3y_1 \le 30,\ \ 2x + y_1 \le 20,\\
&\qquad y_2 \le 10,\ \ y_2 + y_3 \le 15,\\
&\qquad y_3 \le 10,\ \ x + 2y_1 + 2y_2 + y_3 \le 40,\\
&\qquad x, y_1, y_2, y_3 \ge 0.
\end{aligned} \tag{3.37}
$$

Assume that the leader imposes all constraints to all followers. Thus, performing the steps described in Table 3.3:

Step 1. Recast optimisation subproblems $\min_{y_1} f_1$, $\min_{y_2} f_2$ and $\min_{y_3} f_3$ as multi-parametric programming problems, with parameters being the set of variables out of their control;

Step 2. Solve the three multi-parametric programming problems using a suitable algorithm (Dua *et al.*, 2002);

<u>Follower 1</u>

$$
CR_1^1 \begin{cases} y_1 = -x + 10, \\ 0 \le x, y_2, y_3 \le 10, \\ y_2 + y_3 \le 15, \\ -0.5x + y_2 + 0.5y_3 \le 10, \end{cases}
\qquad
CR_1^2 \begin{cases} y_1 = -0.5x - y_2 - 0.5y_3 + 20, \\ 0 \le x, \\ 0.5x - y_2 - 0.5y_3 \le -10, \\ y_2 \le 10, \\ y_2 + y_3 \le 15. \end{cases} \tag{3.38}
$$

<u>Follower 2</u>

$$CR_2^1 \begin{cases} y_2 = 10, \\ 0 \le x, y_1, y_3, \\ x + y_1 \le 10, \\ y_3 \le 5, \\ 0.5x + y_1 + 0.5y_3 \le 10, \end{cases} \quad CR_2^2 \begin{cases} y_2 = -y_3 + 15, \\ 0 \le x, y_1, \\ x + y_1 \le 10, \\ 5 \le y_3 \le 10, \\ 0.5x + y_1 - 0.5y_3 \le 5, \end{cases}$$

$$CR_2^3 \begin{cases} y_2 = -0.5x - y_1 - 0.5y_3 + 20, \\ 0 \le x, \\ x + y_1 \le 10, \\ -0.5x - y_1 + 0.5y_3 \le -5, \\ -0.5x - y_1 - 0.5y_3 \le -10. \end{cases} \tag{3.39}$$

Follower 3

$$CR_3^1 \begin{cases} y_3 = 10, \\ 0 \le x, y_1, y_2, \\ x + y_1 \le 10, \\ y_1 \le 5, \\ 0.5x + y_1 + y_2 \le 15, \end{cases} \quad CR_3^2 \begin{cases} y_3 = -y_1 + 15, \\ 0 \le x, y_2, \\ x + y_1 \le 10, \\ 5 \le y_1, \\ 0.5x + 0.5y_1 + y_2 \le 12.5, \end{cases}$$

$$CR_3^3 \begin{cases} y_3 = -x - 2y_1 - 2y_2 + 40, \\ 0 \le x, y_1, \\ x + y_1 \le 10, \\ -0.5x - 0.5y_1 - y_2 \le -12.5, \\ 0.5x + y_1 + y_2 \le 20, \\ -0.5x - y_1 - y_2 \le -15. \end{cases} \tag{3.40}$$

Step 3. Compute the *Nash* equilibrium point, through direct comparison of the explicit analytical rational reaction sets, (3.38), (3.39) and (3.40). Through this comparison we generate 18 regions, of which 12 have empty feasible sets. After removing empty regions:

$$CR^1 \begin{cases} y_1 = -x + 10, \\ y_2 = 10, \\ y_3 = x, \end{cases} \qquad CR^2 \begin{cases} y_1 = -x + 10, \\ y_2 = -y_3 + 15, \\ y_3 = -x - 2y_1 - 2y_2 + 40, \end{cases}$$

$$CR^3 \begin{cases} y_1 = -x + 10, \\ y_2 = -0.5x - y_1 - 0.5y_3 + 20, \\ y_3 = -x - 2y_1 - 2y_2 + 40, \end{cases} \qquad CR^4 \begin{cases} y_1 = -0.5x - y_2 - 0.5y_3 +, \\ \qquad +20, \\ y_2 = 10, \\ y_3 = -x - 2y_1 - 2y_2 + 40, \end{cases}$$

$$CR^5 \begin{cases} y_1 = -0.5x - y_2 - 0.5y_3 + 20, \\ y_2 = -y_3 + 15, \\ y_3 = -x - 2y_1 - 2y_2 + 40, \end{cases} \qquad CR^6 \begin{cases} y_1 = -0.5x - y_2 - 0.5y_3 + \\ \qquad +20, \\ y_2 = -0.5x - y_1 - 0.5y_3 + \\ \qquad +20, \\ y_3 = -x - 2y_1 - 2y_2 + 40, \end{cases} \tag{3.41}$$

For the sake of brevity we omit here the constraints for each critical region.

Step 4. Incorporate the expressions (3.41) into F, and formulate 6 single level convex optimisation problems. They result in the same unique solution, as follows:

$$F = -35; \; x = 5; \; y_1 = 5; \; y_2 = 10; \; y_3 = 5.$$

The global optimum found is identical to the one reported in Anandalingman (1988).

3.4. An application to optimal control of multilevel systems

An important application of the proposed theory is the hierarchical control of dynamic systems (Başar and Selbuz, 1979), as shown in Figure 3.1.

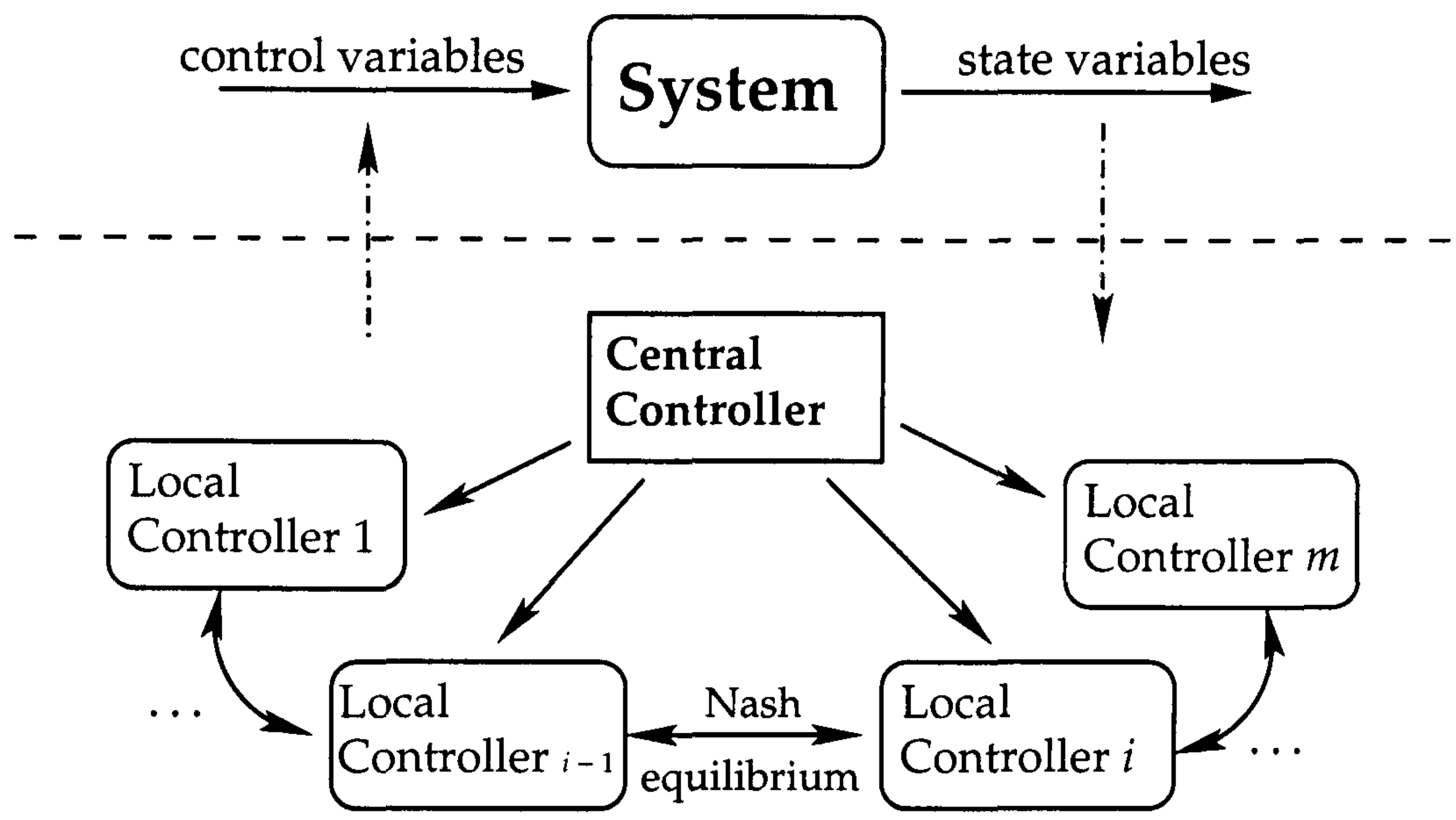


Figure 3.1.: Schematic representation of a hierarchical control configuration for a dynamic system.

In hierarchical control, the performance of a dynamic system is optimised within a complex structure with different objective functions at different levels, for instance, as shown in Figure 3.1 for a control structure involving two levels. In such a system, typically described by a discrete-time dynamic model:

$$x_{n+1} = A_n \cdot x_n + B_n^0 \cdot u_n + \sum_{i=1}^{m} B_n^i \cdot v_n^i, \tag{3.42}$$

we have a central controller, the leader, and m peripheral (local) controllers; x_n is the state vector of the system, u_n is the control vector of the central controller and v_n^i is the control vector of the i^{th} local controller, all at time step n. Each local controller may have its own dynamics, which can be incorporated in Equation (3.42) (Başar and Selbuz, 1979).

The goal is the optimisation of a quadratic objective function corresponding to the central controller:

$$J_0 = (x_N)^T Q_N^0 x_N + \sum_{n=0}^{N-1} \left[(x_n)^T Q_n^0 x_n + (u_n)^T R_n^{00} u_n + \sum_{i=1}^{m} (v_n^i)^T R_n^{0i} v_n^i \right], \tag{3.43}$$

subject to the optimisation of each local controller's objective function:

$$J_i = (x_N)^T Q_N^i x_N + \sum_{n=0}^{N-1} \left[(x_n)^T Q_n^i x_n + (u_n)^T R_n^{i0} u_n + \sum_{k=1}^{m} (v_n^k)^T R_n^{ik} v_n^k \right]. \tag{3.44}$$

Expressions (3.42), (3.43) and (3.44) give rise to a multi-level optimisation problem formulation: the leader, central controller, has control over the complete set of optimisation variables, whereas the local controllers have access to their own optimisation set, v_n^i, and corresponding objective function. The aim is to obtain the global optimum for the central controller and the best optimal strategies for the local controllers. Here, we consider the general case involving constraints (where most previous strategies considered the unconstrained case - see Cruz (1978), Başar and Selbuz (1979), Başar and Olsder (1982)).

We seek a *optimal policy*, as follows:

$$\{u_n\}^* = \{u_0^*, u_1^*, \ldots, u_N^*\} \to \gamma_0^*, \gamma_0^* \in \Gamma^0, \tag{3.45a}$$

$$\{v_n^1\}^* = \{(v_0^1)^*, (v_1^1)^*, \ldots, (v_N^1)^*\} \to \gamma_1^*, \gamma_1^* \in \Gamma_1, \tag{3.45b}$$

$$\vdots$$

$$\{v_n^i\}^* = \{(v_0^i)^*, (v_1^i)^*, \ldots, (v_N^i)^*\} \to \gamma_i^*, \gamma_i^* \in \Gamma_i, \tag{3.45c}$$

$$\vdots$$

$$\{v_n^m\}^* = \{(v_0^m)^*, (v_1^m)^*, \ldots, (v_N^m)^*\} \to \gamma_m^*, \gamma_2^* \in \Gamma_m. \tag{3.45d}$$

Then, the hierarchical control problem can be recast as the following multi-level constrained optimisation problem:

$$
\begin{aligned}
&\min_{\gamma_0,\gamma_1,\ldots,\gamma_m} J_0(\gamma_0,\gamma_1,\ldots,\gamma_m), && \text{(Central controller)},\\
&\qquad \text{s.t. } g_1(\gamma_0,\gamma_1,\ldots,\gamma_m) \le 0,\\
&\qquad\qquad \ldots,\left\{\begin{array}{l} \min\limits_{\gamma_i} J_i(\gamma_0,\gamma_1,\ldots,\gamma_m)\\ \text{s.t. } g_2^i(\gamma_0,\gamma_1,\ldots,\gamma_m), \le 0 \end{array}\right\},\ldots. && (m \text{ local controllers}).
\end{aligned}
\tag{3.46}
$$

Using Equation (3.42) it is possible to express each state variable as a function of the *initial state* and the *control decisions* (Pistikopoulos *et al.*, 2000). Therefore, J_0 and J_i become functions only of the initial state:

$$J_0, J_i = f(x_0, \gamma_1, \gamma_2, \ldots, \gamma_m), \quad \forall i \in \{1, 2, \ldots, m\}.$$

Since in the lower level of this two-level optimisation problem there are multiple optimisation subproblems, and there is the need to coordinate such group, it is fairly natural to assume a Nash equilibrium (Başar and Selbuz, 1979):

$$J_1(\gamma_0^*, \gamma_1^*, \ldots, \gamma_m^*) \le J_1(\gamma_0^*, \gamma_1, \gamma_2^*, \gamma_3^*, \ldots, \gamma_K^*), \forall \gamma_1 \in \Gamma_1, \tag{3.47a}$$

$$J_2(\gamma_0^*, \gamma_1^*, \ldots, \gamma_m^*) \le J_2(\gamma_0^*, \gamma_1^*, \gamma_2, \gamma_3^*, \ldots, \gamma_K^*), \forall \gamma_2 \in \Gamma_2, \tag{3.47b}$$

$$\vdots$$

$$J_m(\gamma_0^*, \gamma_1^*, \ldots, \gamma_m^*) \le J_m(\gamma_0^*, \gamma_1^*, \gamma_2^*, \ldots, \gamma_{m-1}^*, \gamma_m), \forall \gamma_m \in \Gamma_m, \tag{3.47c}$$

where $\forall \gamma_0 \in \Gamma^0$ and $\forall x_0 \in X_0$, with X_0 being the feasible set of the system's initial state.

Problem (3.46) corresponds to a bilevel programming problem with multi-followers; the followers being the local controllers and the leader, the central controller. In contrast to Problem (3.30), the decisions involved in each subproblem are not only parametric relatively to the decisions of the remaining subproblems, but also depend on the initial state of the system. We refer to this class as *multi-level optimisation problems with uncertainty*. The algorithm in Table 3.3 can be directly applied to solve (3.46) only with a modification in *Step 4*, which requires "the formulation and solution of K multi-parametric programming problems".

A similar strategy can also be applied to tri-level optimisation problems. Moreover, if different models are involved in the subproblem, the proposed optimisation strategy is still applicable, with all control subproblems treated in a decentralised fashion. In the next section, a dynamic three person control system is described to illustrate the potential of the proposed approach.

3.4.1. Illustrative example 3

Consider a system which has a discrete dynamic behaviour described by the following linear state transition model (Nie *et al.*, 2006):

$$\begin{aligned} x_{t+1} &= x_t + u_t - 2v_t^1 + v_t^2, \\ y_{t+1}^1 &= y_t^1 + 2v_t^1, \qquad\qquad t = 0, 1, 2, \\ y_{t+1}^2 &= y_t^2 + 2v_t^2, \end{aligned} \tag{3.48}$$

where u, v^1 and v^2 are input variables, and x, y^1 and y^2 output variables. And, with constraints on the input and state variables as follows:

$$\begin{aligned} &-30 \leq v_t^1, v_t^2 \leq 30, \\ &-20 \leq u_t \leq 20, \qquad\qquad t = 0, 1, 2, \\ &-10 \leq x_0, y_0^1, y_0^2 \leq 10. \end{aligned} \tag{3.49}$$

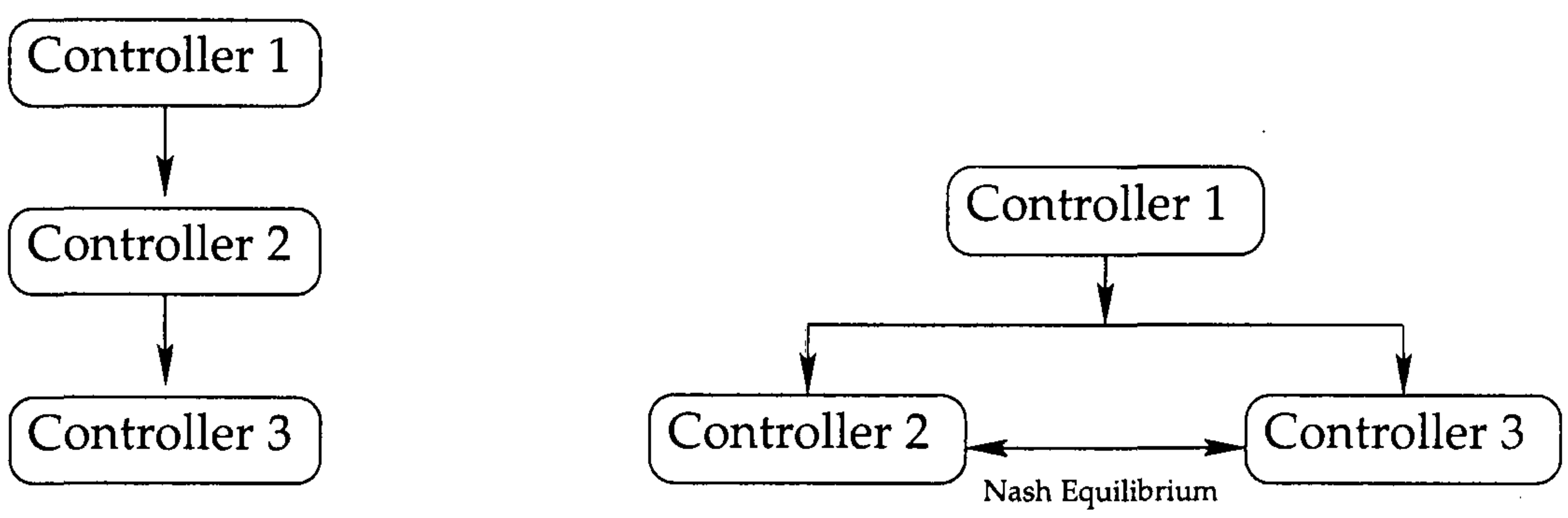
Additionally, consider a three-controller system (Nie *et al.*, 2006):

$$J_1 = \min_{u_0,u_1,u_2} 4x_3 + 3y_3^1 + 2y_3^2 + \sum_{t=0}^{2} \left\{ (u_t)^2 + \left(v_t^1\right)^2 - \left(v_t^2\right)^2 + 2u_t x_t + x_t^2 \right\}, \tag{3.50a}$$

$$J_2 = \min_{v_0^2,v_1^2,v_2^2} 2x_3 + 3y_3^2 + \sum_{t=0}^{2} \left\{ 2 \cdot u_t v_t^2 + \left(v_t^1 + 1\right)^2 + \left(v_t^2 + 1\right)^2 \right\}, \tag{3.50b}$$

$$J_3 = \min_{v_0^1,v_1^1,v_2^1} x_3 + 2y_3^1 - 10y_3^2 + \sum_{t=0}^{2} \left\{ -15u_t + \left(v_t^1 - 1\right)^2 - 2v_t^1 v_t^2 + \left(v_t^2\right)^2 \right\}, \tag{3.50c}$$

where J_1, J_2 and J_3 correspond to Controllers 1,2 and 3, respectively. Figure 3.2 displays two possible configurations for the control structure of the considered system. The objective is then to derive suitable optimal strategies



(a) Three-level controller structure (b) Multifollower controller structure

Figure 3.2.: Three-controller multilevel problem.

for the two controller structures. Case (a) of Figure (3.2) corresponds to a three-level optimisation problem, whereas case (b) refers to a bilevel multi-follower optimisation problem. Therefore, using the proposed methodology we obtain the results summarised in Tables 3.4 and 3.5.

3.5. Concluding remarks

We have described a novel global optimisation strategy for the solution of hierarchical multi-level and decentralised multi-level programs based on our recent developments in multi-parametric programming theory and algorithms, see Chapter 2. The algorithms proposed are suitable for problems involving

Table 3.4.: Solution to the three-level optimisation problem.

Critical Region 1	Critical Region 2
$u_0 = 6.84615 - 0.76928x_0$	$u_0 = -0.333333 - 1.8519x_0$
$u_1 = -20$	$u_1 = -1.33333 + 2.8148x_0$
$u_2 = 15.2308 + 0.15388x_0$	$u_2 = -2 - 2.4444x_0$
$-10 \leq x_0 \leq -6.63161$	$-6.63161 \leq x_0 \leq 7.36377$
Critical Region 3	**Critical Region 4**
$u_0 = -1.53333 - 1.6889x_0$	$u_0 = -9 - 0.72732x_0$
$u_1 = 8.26667 + 1.5111x_0$	$u_1 = 20$
$u_2 = -20$	$u_2 = -20$
$7.36377 \leq x_0 \leq 7.76466$	$7.76466 \leq x_0 \leq 10$
$v_0^1 = v_0^2 = -2 - 0.5u_0;\ v_1^1 = v_1^2 = -2 - 0.5u_1;\ v_2^1 = v_2^2 = -2 - 0.5u_2$	

Table 3.5.: Solution to multi-follower problem.

Critical Region 1
$u_0 = 1 - x_0$
$u_1 = -8 + x_0$
$u_2 = 5 - x_0$
$v_0^1 = v_0^2 = -6 + x_0$
$v_1^1 = v_1^2 = 3 - x_0$
$v_2^1 = v_2^2 = -10 + x_0$
$-10 \leq x_0 \leq 10$

general convex objective functions and convex sets of constraints. Future developments include general non-linear models, for which recent results on global multi-parametric programming (Dua *et al.*, 2004) and explicit non-linear MPC (Sakizlis *et al.*, 2007) can be used; and general dynamic multi-level problems, for which a dynamic programming approach coupled with multi-parametric programming can be applied.

4. Global optimisation of multi-parametric MILP problems

In this chapter, we present a novel global optimisation approach for the general solution of multi-parametric mixed integer linear programs (mp-MILPs). We describe an optimisation procedure which iterates between a (master) mixed integer nonlinear program and a (slave) multi-parametric program. Moreover, we explain how to overcome the presence of bilinearities, responsible for the non-convexity of the multi-parametric program, in two classes of mp-MILPs, with (i) varying parameters in the objective function and (ii) simultaneous presence of varying parameters in the objective function and the right-hand side of the constraints. Examples are provided to illustrate the solution steps.

4.1. Introduction

Numerous investigations are devoted to the development of novel algorithms for the global solution of mixed integer linear programs. The potential embedded in such formulations is attested by the panoply of applications in the systems engineering field. A general mixed integer linear program is posed as follows (Floudas, 1995; Biegler *et al.*, 1997):

$$\min_{x,y}\{c^T x + d^T y : Ax + Ey \leq b, x \in \mathbb{R}^n, y \in \{0,1\}^m\}, \tag{4.1}$$

where $\mathbb{R}$ represents the group of the real numbers, matrices and vectors: A,

E, b, c, d, are constant and have conforming dimensions.

The vast field of applications include synthesis of heat exchanger and general utility networks (Papoulias and Grossman, 1983*a,b*; Biegler *et al.*, 1997), design of multi-purpose and batch plants (Grossman and Sargent, 1979; Shah and Pantelides, 1992; Voudouris and Grossman, 1992), synthesis of distillation sequences (Andrecovich and Westerberg, 1985), integrated design and control (Sakizlis *et al.*, 2003), scheduling (Kondili *et al.*, 1993; Lin *et al.*, 2004; Janak *et al.*, 2007) and planning problems (Liu and Sahinidis, 1996; Sahinidis *et al.*, 1989; Shah and Pantelides, 1991; Iyer and Grossman, 1998), protein identification (DiMaggio and Floudas, 2007), pro-active scheduling (Ryu and Pistikopoulos, 2007) and hybrid control of dynamic systems Sakizlis *et al.* (2002).

Notwithstanding, uncertainties are a common presence in real-life applications. Often, these uncertainties arise from fluctuations in the product demand or resources availability, market prices or from variability in specific data as heat transfer coefficients or kinetics constants. Consequently, Formulation (4.1) is recast to include uncertainty:

$$\begin{aligned} z(\theta) = \min_{x,y} \ & c(\theta)^T \cdot x + d(\theta)^T \cdot y, \\ s.t. \ & A(\theta)x + E(\theta)y \leq b(\theta), \\ & \Gamma(\theta)x + \Phi(\theta)y = \gamma(\theta), \\ & x \in X \subseteq \mathbb{R}^n, y \in \{0,1\}^m, \\ & \theta \in \Theta. \end{aligned} \tag{4.2}$$

where, c, d, A, E, Γ, Φ, b and γ are real matrices linearly dependent on θ, with appropriate dimensions. Formulation (4.2) is solved using one of the following three major options: (i) multi-period/scenario based approaches, where the varying parameters are discretised into a number of deterministic realisations, (ii) stochastic programming, where we assume a probabilistic distribution for the varying parameters, and (iii) multi-parametric programming, where the varying parameters are assumed to be continuous, bounded and unstructured. Multi-parametric programming is selected, because it has the major advantage of obtaining the optimal solution as an explicit continuous function of the varying parameters.

Concurrently, despite multi-parametric programming being a prime strategy (Pistikopoulos *et al.*, 2007*a*,*b*), the presence of integer variables in (4.2) places an extra challenge. Four main strategies have been proposed to address the mixed integer nature of the problem: (a) Enumeration method (Piper and Zoltners, 1976; Roodman, 1974), (b) Bounding method, which explores properties of the objective function (Geoffrion and Nauss, 1977), (c) Cutting plane method, which sequentially introduces constraints that cut regions of the original feasible region where the optimum lies, and (d) Branch and Bound method, which relaxes the integer variables and derives successively upper and lower bounds (Marsten and Morin, 1977; Ohtake and Nisida, 1985). Based on Branch & Bound, Acevedo and Pistikopoulos (1997) proposed the first multi-parametric algorithm to address mixed-integer linear optimisation problems with varying parameters on the right-hand side of the constraints, however, it has been shown to be computationally expensive. To overcome this difficulty, Dua and Pistikopoulos (2000) have proposed an multi-parametric algorithm based on cutting planes, extending the work of Pertsinidis (1992) and Pertsinidis *et al.* (1998) for a single parameter. In this work, we comply with these last developments.

Nevertheless, in the cases aforementioned, the varying parameters are assumed to be isolated on the right-hand side of the constraints. Consequently, many important classes of Problem (4.2) are not considered. Li and Ierapetritou (2007*a*,*b*) present an B&B-based algorithm to address (4.2), however, its performance is reported to be significantly dependent on the non-convexity of the problem.

Based on our recent developments (Dua and Pistikopoulos, 2000), we describe a novel approach to address Problem 4.2. The principal idea is to iterate between a master problem and a slave problem. In the master problem, we solve a mixed integer non-linear programming (MINLP) problem to global optimality, whereas in the slave problem we solve a multi-parametric program, obtained by fixing the integer variables to the previously computed optimal solution. The main challenge consists in addressing the presence of non-convexities in the slave problem. However, in the proposed approach, we circumvent the use of global optimisation tools. A new multi-parametric linear programming (mp-LP) algorithm is developed, which easily handles the bilinearites and frees the slave problem of any global optimisation procedure (Dua *et al.*, 2004). Special focus is given to: (i) multi-parametric MILP problems, involving varying parameters in the objective function (OFC mp-MILP):

$$
\begin{aligned}
z(\theta) = \min_{x,y} \;& c(\theta)^T \cdot x + d(\theta)^T \cdot y, \\
s.t. \;& Ax + Ey \leq b, \\
& \Gamma x + \Phi y = \gamma, \\
& x \in X \subseteq \mathbb{R}^n, y \in \{0,1\}^m, \\
& \theta \in \Theta,
\end{aligned}
\tag{4.3}
$$

and (ii) multi-parametric MILP problems, involving varying parameters in the objective function and on the right-hand side of the constraints (RIM mp-MILP):

$$
\begin{aligned}
z(\theta) = \min_{x,y} \;& c(\theta)^T \cdot x + d(\theta)^T \cdot y, \\
s.t. \;& Ax + Ey \leq b + F(\theta), \\
& \Gamma x + \Phi y = \gamma + \Psi(\theta), \\
& x \in X \subseteq \mathbb{R}^n, y \in \{0,1\}^m, \\
& \theta \in \Theta.
\end{aligned}
\tag{4.4}
$$

Chapter 4 is organised in the following way. In Section 4.2, we describe a two-stage approach to solve OFC mp-MILP problems as in (4.3). Then, we present the master problem and its solution steps, and then, we present a detailed description of the new mp-LP algorithm, establishing the links with our previous work (Pistikopoulos *et al.*, 2007*a*). Section 4.3 describes a general procedure to address RIM mp-MILP problems as in (4.4). Illustrative examples are presented throughout the sections to provide details of the steps of the proposed algorithms.

4.2. Multi-parametric OFC MILP problems

Consider the formulation in (4.3), rewritten in a more compact mathematical form (Kosmidis, 1999):

$$\begin{aligned} z(\theta) = \min_{x,y} \;& (c + H\theta)^T x + (d + L\theta)^T y, \\ s.t. \;& Ax + Ey \leq b, \\ & \Gamma x + \Phi y = \gamma, \\ & x \in X \subseteq \mathbb{R}^n, y \in \{0,1\}^m, \\ & \theta \in \Theta = \{\theta : \theta \in \mathbb{R}^s, G\theta \leq e\}, \end{aligned} \tag{4.5}$$

here, c, d, H and L are real matrices with appropriate dimensions. The presence of parametric uncertainties in the objective function introduces two types of bilinear terms - $\theta^T \cdot H^T \cdot x$ and $\theta^T \cdot H^T \cdot y$ - hence, this is a non-convex objective function. Here, we propose a two-stage global optimisation procedure for the solution of (4.5), described next.

4.2.1. Master problem

In the master problem, we formulate a global optimisation problem considering the varying parameters, θ, as bounded optimisation variables:

$$
\begin{aligned}
z_M(\theta) = \min_{x,y,\theta} \;\; & (c + H\theta)^T x + (d + L\theta)^T y, \\
\text{s.t.} \;\; & Ax + Ey \leq b, \\
& \Gamma x + \Phi y = \gamma, \\
& x \in X \subseteq \mathbb{R}^n, y \in \{0,1\}^m, \\
& \theta \in \Theta = \{\theta : \theta \in \mathbb{R}^s, G\theta \leq e\}.
\end{aligned}
\tag{4.6}
$$

Problem (4.6) is solved using a global optimisation solver (Adjiman *et al.*, 1998*b*,*a*; Floudas, 2000; Smith and Pantelides, 1999). In this work, we use the commercial package GAMS/BARON (Sahinidis, 2000) as our global optimisation solver. From the solution obtained for Problem (4.6), the binary vector is fixed, $y = \bar{y}$, and is an entry data in the slave problem, which is described next.

4.2.2. Slave problem

Fixing $y = \bar{y}$, (4.5) results in the following formulation:

$$
\begin{aligned}
z_S(\theta) = (d + L\theta)^T \bar{y} + \min_{x} \;\; & c^T x + \theta^T H^T x, \\
\text{s.t.} \;\; & Ax \leq b', \\
& \Gamma x = \gamma', \\
& x \in X \subseteq \mathbb{R}^n, \\
& \theta \in \Theta = \{\theta : \theta \in \mathbb{R}^s, G\theta \leq e\},
\end{aligned}
\tag{4.7}
$$

where, $b' = b - E\bar{y}$, and $\gamma' = \gamma - \Phi\bar{y}$. Problem (4.7) involves bilinear terms in the objective function, and hence, it corresponds to a multi-parametric global optimisation problem.

In principle, (4.7) can be addressed by applying the global optimisation algorithm of Dua *et al.* (2004). However, here we explore the structure of (4.7) to design a new mp-LP algorithm suitable for OFC mp-LP problems.

◇ **Algorithm for the OFC mp-LP problem**

The Fritz John first-order conditions state that there exist $p+q+1$ real numbers v, λ, μ, not all zero, such that (Mangasarian and Fromovitz, 1967):

$$\begin{aligned} \mathscr{L}(x,v,\lambda,\mu,\theta) &= vf(x,\theta) + \sum_{i=1}^{p} \lambda_i g_i(x,\theta) + \sum_{j=1}^{q} \mu_j h_j(x,\theta), \\ \nabla_x \mathscr{L}(x,v,\lambda,\mu,\theta) &= 0, \\ \lambda_i g_i(x,\theta) &= 0, \quad \forall \quad i = 1,\dots,p, \\ h_j(x,\theta) &= 0, \quad \forall \quad j = 1,\dots,q, \\ v, \lambda_i, \mu_j &\geq 0, \end{aligned} \tag{4.8}$$

where $\mathscr{L}(x,v,\lambda,\mu)$ is the Lagrangian, $v \in \mathbb{R}, \lambda \in \mathbb{R}^p, \mu \in \mathbb{R}^q$, are the Lagrange multipliers , $f(x,\theta) = c^T x + \theta^T H^T x$, $g(x,\theta) = Ax - b' \leq 0$ and $h(x,\theta) = \Gamma x - \gamma' = 0$. Assuming, we seek a Karush-Kuhn-Tucker (KKT) optimum (Bazaraa and Shetty, 1979) satisfying the linear independence constraint qualification, $v = 1$, (4.8) is rewritten in a more compact form:

$$F(\eta,\theta) = \begin{bmatrix} \nabla_x \mathscr{L} \\ \Lambda g(x,\theta) \\ h(x,\theta) \end{bmatrix} = 0, \tag{4.9}$$

here, $\eta = [x, \lambda, \mu]$ and Λ is a diagonal matrix with $\Lambda_{ii} = \lambda_i$, $i = 1,\dots,p$. Then, differentiating (4.9), $F(\eta(\theta),\theta)$, with respect to θ, we obtain an expression for the optimal solution of (4.7) as an explicit function of θ, as shown next in Theorem 4.1.

Theorem 4.1 *Basic Sensitivity Theorem (Fiacco, 1976): Let θ_0 be a vector of parameter values and (x_0, λ_0, μ_0) a KKT triple corresponding to (4.8), where λ_0 is* *nonnegative and* x_0 *is feasible in* (4.7). *Also assume that (i) strict complementary slackness (SCS) holds, (ii) the binding constraint gradients are linearly independent (LICQ: Linear Independence Constraint Qualification), and (iii) the second-order sufficiency conditions (SOSC) hold. Then, in the neighbourhood of* θ_0*, there exists a unique, once continuously differentiable function,* $\eta(\theta) = [x(\theta), \lambda(\theta), \mu(\theta)]$*, satisfying* (4.8) *with* $\eta(\theta_0) = [x(\theta_0), \lambda(\theta_0), \mu(\theta_0)]$*, where* $x(\theta)$ *is a unique isolated minimiser of* (4.7)*, and*

$$\begin{pmatrix} \frac{dx}{d\theta} \\ \frac{d\lambda}{d\theta} \\ \frac{d\mu}{d\theta} \end{pmatrix} = -\ (M_0)^{-1} N_0, \tag{4.10}$$

where, M_0 *and* N_0 *are the Jacobians of* (4.9) *with respect to* η *and* θ*,*

$$M_0 = \left(\begin{array}{c|ccc|ccc} \nabla^2_{xx}\mathscr{L} & \nabla_x g_1 & \cdots & \nabla_x g_p & \nabla_x h_1 & \cdots & \nabla_x h_q \\ \hline \lambda_1 \nabla^T_x g_1 & g_1 & & & & & \\ \vdots & & \ddots & & & 0 & \\ \lambda_p \nabla^T_x g_p & & & g_p & & & \\ \hline \nabla^T_x h_1 & & & & & & \\ \vdots & & 0 & & & 0 & \\ \nabla^T_x h_q & & & & & & \end{array}\right),$$

$$N_0 = (\nabla^2_{\theta x}\mathscr{L}, \lambda_1 \nabla^T_\theta g_1, \ldots, \lambda_p \nabla^T_\theta g_p, \nabla^T_\theta h_1, \ldots, \nabla^T_\theta h_q)^T.$$

Proof. See (Fiacco, 1983, pp. 72).

Note that the assumptions stated in the theorem above ensure M_0 is nonsingular in the neighbourhood of the solution point (η_0, θ_0), and hence, invertible (McCormick, 1976).

Corollary 4.1 *First-order estimation of* $[x(\theta), \lambda(\theta), \mu(\theta)]$*, near* $\theta = \theta_0$ *(Fiacco, 1983): Under the assumptions of Theorem 4.1, a first-order approximation of* $[x(\theta), \lambda(\theta), \mu(\theta)]$ *in the neighbourhood of* θ_0 *is,*

$$\begin{bmatrix} x(\theta) \\ \lambda(\theta) \\ \mu(\theta) \end{bmatrix} = \begin{bmatrix} x_0 \\ \lambda_0 \\ \mu_0 \end{bmatrix} - (M_0)^{-1} \cdot N_0 \cdot (\theta - \theta_0) + o(\|\theta\|), \tag{4.11}$$

where $(x_0, \lambda_0, \mu_0) = [x(\theta_0), \lambda(\theta_0), \mu(\theta_0)]$, $M_0 = M(\theta_0)$, $N_0 = N(\theta_0)$, *and* $\phi(\theta) = o(\|\theta\|)$ *means that* $\phi(\theta)/\|\theta\| \to 0$ *as* $\theta \to \theta_0$.

From Theorem 4.1, it is obvious that the matrices M_0 and N_0 are independent of θ, i.e. it is equally applicable to (4.7) as it is for the righ-hand-side (RHS) case considered in Dua and Pistikopoulos (2000). Theorem 4.1 clearly states that the first order estimation of the explicit optimal function, (4.11), is the general solution inside the incumbent critical region, where a critical region is defined as a subset of the parameters space inside which the same set of active constraints applies.

The main difference between the RHS case and the OFC mp-LP problem, in (4.7), is the non-null Hessian of the Lagrangian with respect to θ and x, i.e. $\nabla_{\theta x}\mathscr{L} = H^T$. Yet, all matrices in (4.11) are constant, and hence, the explicit expression is indeed valid inside the entire critical region. By substituting the appropriate variables, (4.11) results in the following expression for (4.7):

$$\begin{bmatrix} x(\theta) \\ \lambda(\theta) \\ \mu(\theta) \end{bmatrix} = \begin{bmatrix} x_0 \\ \lambda_0 \\ \mu_0 \end{bmatrix} - \begin{bmatrix} 0 & A^T & \Gamma^T \\ \Lambda A & diag(g) & 0 \\ \Gamma & 0 & 0 \end{bmatrix}_{\eta_0}^{-1} \cdot \begin{bmatrix} H^T \\ 0 \\ 0 \end{bmatrix}_{\eta_0} \cdot (\theta - \theta_0). \tag{4.12}$$

The analytical expressions in (4.12) are used to derive the boundaries of the critical region by checking the conditions stated in the following Theorem 4.2.

Theorem 4.2 *(Poore and Tiahrt, 1987) Let* (η_0, v_0, θ_0) *be a solution to* (4.8). *Additionally, assume that* f, g *and* h *are twice continuously differentiable in a neighbourhood of* (x_0, θ_0), *and define two index sets:* $\mathcal{A}$ *and* $\bar{\mathcal{A}}$, *and a corresponding tangent* *space $\bar{\mathcal{T}}$ by,*

$$\bar{\mathcal{A}} = \{i : 1 \leq i \leq p, g_i(x_0, \theta_0) = 0\}, \tag{4.13a}$$

$$\mathcal{A} = \{i \in \bar{\mathcal{A}} : \lambda_i \neq 0\}, \tag{4.13b}$$

$$\bar{\mathcal{T}} = \{t \in \mathbb{R}^n : [\nabla_x h(x_0, \theta_0)]^T y = 0, [\nabla_x g_i(x_0, \theta_0)]^T y = 0, \forall i \in \bar{\mathcal{A}}\}. \tag{4.13c}$$

Then a necessary and sufficient condition for $\nabla_\eta F$, i.e. M_0, to be non-singular is that, each of the following three conditions hold:

(i) $\bar{\mathcal{A}} = \mathcal{A}$;

(ii) $S \triangleq \{\nabla_x g_i(x_0, \theta_0) \cup \nabla_x h_j(x_0, \theta_0), i \in \bar{\mathcal{A}}, j = 1, \ldots, q\}$ is a linearly independent collection of $q + |\bar{\mathcal{A}}|$ vectors, where $|\cdot|$ denotes cardinality;

(iii) The Hessian of the Lagrangian $\nabla_x \mathcal{L}$ is non-singular on the tangent space $\bar{\mathcal{T}}$.

If $\nabla_\eta F(\eta_0, \theta_0)$ is non-singular, there exist neighbourhoods $\mathcal{B}_1$ of θ_0 and $\mathcal{B}_2$ of (η_0, θ_0) and a function $\phi \in C^1(\mathcal{B}_1)$ such that $F(\phi(\theta), \theta) = 0, \forall_\theta \in \mathcal{B}_1$ and $\phi(\theta_0) = \eta_0$. This solution is locally unique in the sense that if $(\eta, \theta) \in \mathcal{B}_2$ and $F(\eta, \theta) = 0$, then η belongs to the manifold defined by ϕ, i.e., $\eta = \phi(\theta)$. Furthermore, if f, g and h are $C^k (k \geq 2) (C^\infty$ or real analytic) then ϕ is $C^{k-1} (C^\infty$ or real analytic, respectively) on $\mathcal{B}_1$.

Proof. See Poore and Tiahrt (1987).

Essentially, the singular point/surface occurs when at least one of the three conditions enumerated in Theorem 4.2 is violated: (i) Loss of strict complementarity, which is identified by any change of sign or occurrence of zeros in any of the inactive constraints or active inequality Lagrange multiplier; (ii) Violation of the linear independence constraint qualification, identified by a change of sign or the occurrence of a zero in v; and (iii) Singularity of the Hessian of the Lagrangian on the tangent space to the active constraints, which is identified by a change in $\text{in}(\nabla_x^2 \mathscr{L}_T)$, where the operator $\text{in}(\cdot)$ represents the inertia of the matrix. By inertia of a matrix we understand the number of positive, negative and zero eigenvalues (Lundberg and Poore, 1993).

However, since we have a KKT point computed in the master problem, (4.6), and an explicit optimal function, (4.12), the limits for the validity of these explicit expressions can be resumed by the following (Dua *et al.*, 2002):

$$\tilde{\lambda}(x(\theta)) \geq 0, \tag{4.14}$$

$$\breve{g}(x(\theta)) \leq 0, \tag{4.15}$$

where, $\tilde{\lambda}$ represents the set of Lagrange multipliers of the active constraints, $\mathcal{A}$, and $\breve{g}$ the set of inactive constraints.

The parameters' initial area is further explored using the methodology described by Dua and Pistikopoulos (2000) - see Appendix 4.4. At the end, a complete map of all critical regions is obtained. Each critical region is associated with a corresponding analytical expression as in (4.12). By substituting this expression in z_S, (4.7), a valid upper bound is obtained for (4.5).

The OFC mp-LP algorithm was implemented in Matlab.

Remark 4.1 *Note that optimisation variables, x, in* (4.12), *are independent of θ, inside each critical region, $x \neq f(\theta)$; this is expected as* (4.7) *has uncertainty only in the objective function and it is linear with respect to x. Of course, the varying parameters, θ, continue to affect the optimal value function, z^*.*

4.2.3. The algorithm

Between every master-slave iteration, we need to (i) introduce integer and parametric cuts in the master problem (MINLP), in order to avoid already examined 0-1 combinations and cut off worse solutions; and (ii) compare the parametric solution with the solutions obtained in previous iterations. For (i), we introduce the following constraints (Dua and Pistikopoulos, 2000):

$$\sum_{j\in J^{ik}} y_j^{ik} - \sum_{j\in L^{ik}} y_j^{ik} \leq |J^{ik}| - 1, \quad k = 1,\ldots,K^i, \tag{4.16}$$

$$(c + H\theta)^T x + (d + L\theta)^T y \leq z_S(\theta)^i, \tag{4.17}$$

where, $J^{ik} \triangleq \{j : y_j^{ik} = 1\}$ and $L^{ik} \triangleq \{j : y_j^{ik} = 0\}$, the operator $|\cdot|$ corresponds to cardinality and K^i is the number of integer solutions analysed in a specific critical region. Equation (4.16) and Equation (4.17) exclude integer solutions already visited, and integer solutions with higher values than the current upper bound, z_S, respectively.

For (ii), since the optimal value functions for the slave problem are linear, the comparison procedure described in Acevedo and Pistikopoulos (1997) is used.

The algorithm terminates when the master problem, (4.6), is declared infeasible. The algorithmic steps are summarised in Figure 4.1 and Table 4.1.

Next, we apply the steps of the proposed algorithm to an illustrative example.

4.2.4. Example 1

In a chemical engineering company, the decision maker has to choose between Reactor I, which is expensive but has a high rate of conversion, and Reactor II, which is more economic but has a lower rate of conversion, Figure 4.2 (adapted from Biegler *et al.*, 1997).

Due to the presence of uncertainty in the cost coefficients, the multi-parametric OFC MILP problem results:

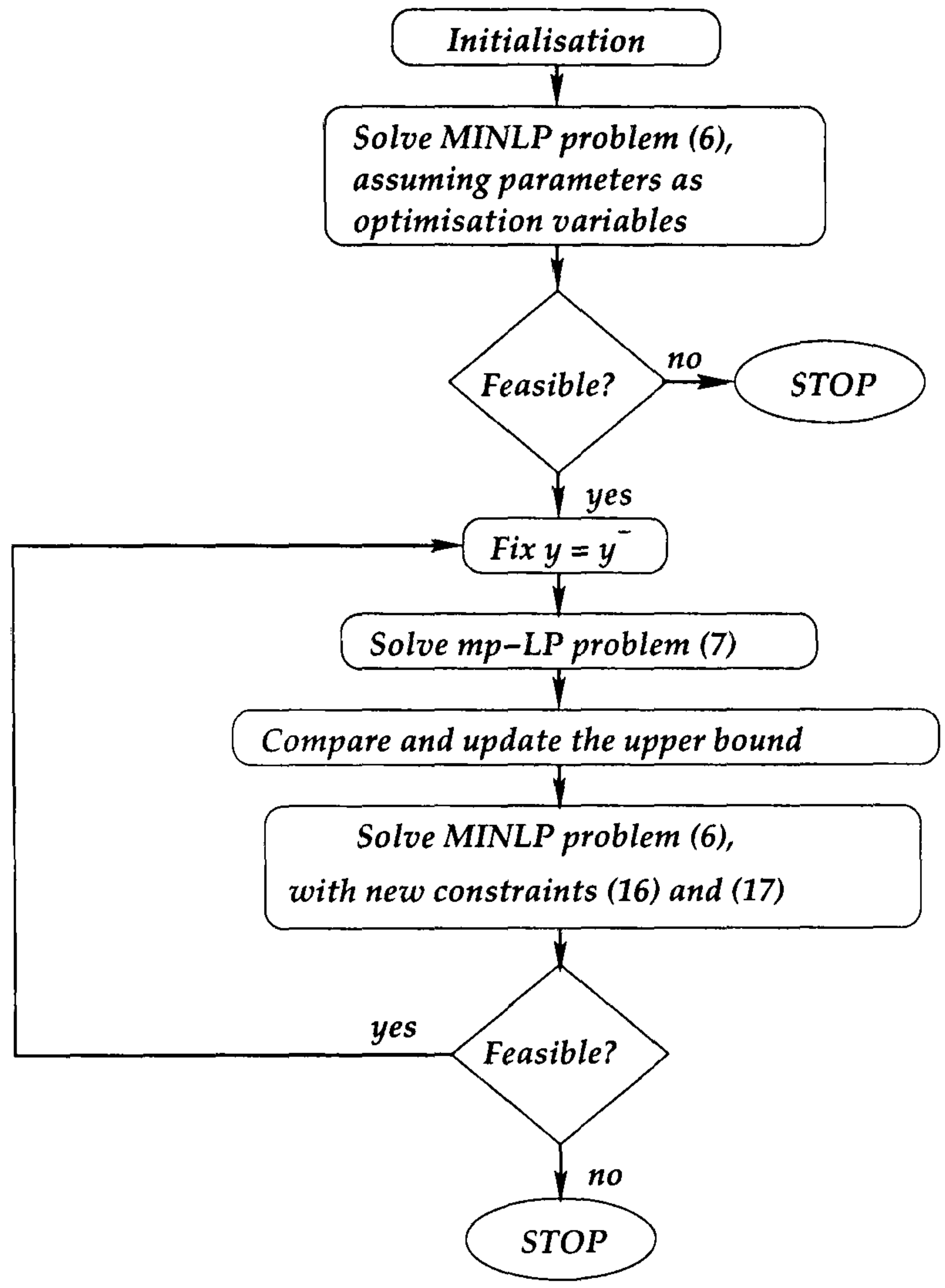


Figure 4.1.: Algorithm for OFC mp-MILP problems.

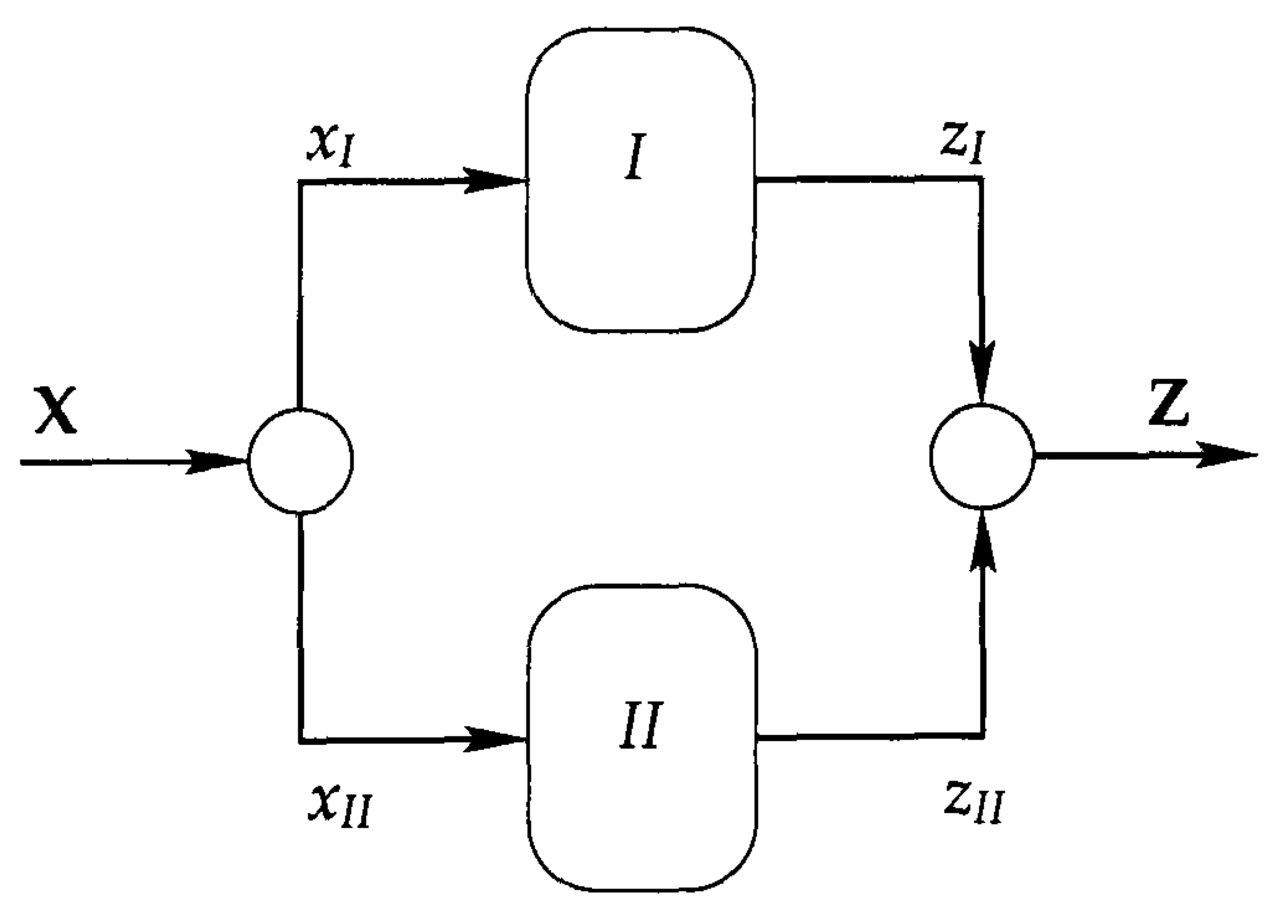


Figure 4.2.: Superstructure of illustrative example 1.

$$\begin{aligned}
\min_{x_1,x_2,y_I,y_{II}} \quad & (6.4 + 0.25\theta_1)x_1 + (6.0 + 0.17\theta_2)x_2 \\
& + (7.5 + 0.3\theta_1)y_I + (5.5 + 0.15\theta_2)y_{II}, \\
s.t. \quad & 0.8 \cdot x_1 + 0.67 \cdot x_2 = 10, \\
& x_1 \leq 20y_I, x_2 \leq 20y_{II}, x_1, x_2 \geq 0, \\
& 0 \leq \theta_1, \theta_2 \leq 20, y_I, y_{II}, \in \{0,1\}.
\end{aligned} \tag{4.18}$$

Table 4.1.: Steps of the algorithm for OFC mp-MILP problems.

Step 0.	(Initialization) Define an initial region of Θ, CR, with best upper bound $\hat{z}^*(\theta) = \infty$, and an initial integer solution, $\bar{y}$.
Step 1.	(Slave subproblem - multiparametric LP problem) For each region with a new integer solution, $\bar{y}$: (a) Solve the mp-LP subproblem (4.7) to obtain a set of parametric upper bounds, $\hat{z}(\theta) = z_S^*$, and the corresponding critical regions *CR*; (b) If $\hat{z}(\theta) \leq z_S^*(\theta)$ for some region of θ, update the best upper bound function, $\hat{z}(\theta)$, and the corresponding integer solutions, y^*; (c) If an infeasibility is found in some region *CR*, go to *Step 2*.
Step 2.	(Master subproblem - MINLP problem) For each region *CR*, formulate and solve to global optimality the MINLP master subproblem, (4.6), (i) treating θ as an optimisation variable, (ii) introducing an integer cut (4.16) and (iii) introducing a parametric cut (4.17). Return to *Step 1* with new integer solutions and corresponding *CR*s.
Step 3.	(Convergence) The algorithm terminates in a region where the solution of the master MINLP subproblem is infeasible. Then, the optimal parametric solution is given by the current upper bounds $\hat{z}^*(\theta)$.

The solution steps of the algorithm in Table 4.1 and Figure 4.1 are listed next.

Step 0.(initialization) Solve Problem (4.18) considering θ as being optimisation variables, $\bar{y} = (1, 0)$.

Step 1.(k=1, slave subproblem) Fix $y = \bar{y}$. The mp-LP problem in (4.7) is formulated as:

$$\begin{aligned} \min_{x_1, x_2} \quad & (6.4 + 0.25\theta_1)x_1 + (6.0 + 0.17\theta_2)x_2 + (7.5 + 0.3\theta_1), \\ s.t. \quad & 0.8 \cdot x_1 + 0.67 \cdot x_2 = 10, \\ & 0 \leq x_1 \leq 20, 0 \leq x_2 \leq 0, \\ & 0 \leq \theta_1, \theta_2 \leq 20. \end{aligned} \tag{4.19}$$

The solution of (4.19) is computed using the OFC mp-LP algorithm:

$$\begin{cases} x_1 = 12.5, \\ x_2 = 0, \\ 0 \le \theta_1 \le 20, \\ 0 \le \theta_2 \le 20. \end{cases}$$

Step 2.(k=1, master subproblem) Solve the master problem in (4.6) with two additional constraints, due to (4.16) and (4.17):

$$y_1 - y_2 \le 0, \tag{4.20}$$

$$(6.4 + 0.25\theta_1)x_1 + (6.0 + 0.17\theta_2)x_2 + (7.5 + 0.3\theta_1)y_I + (5.5 + 0.15\theta_2)y_{II} \le 3.425\theta_1 + 87.5. \tag{4.21}$$

The solution is obtained using the commercial package GAMS/BARON (Sahinidis, 2000): $\bar{y} = (0, 1)$.

Step 1.(k=2, slave subproblem) By fixing $y = (0, 1)$, the solution of (4.7) results in:

$$\begin{cases} x_1 = 0, \\ x_2 = 14.9254, \\ 0 \le \theta_1 \le 20, \\ 0 \le \theta_2 \le 20. \end{cases}$$

Step 1.(k=2, comparison of solutions) Solutions valid in $0 \le \theta_1, \theta_2 \le 20$:

Solution 1	Solution 2
$x_1 = 12.5$	$x_1 = 0$
$x_2 = 0$	$x_2 = 14.9254$
$y_1 = 1$	$y_1 = 0$
$y_2 = 0$	$y_2 = 1$
$z_S^1 = 87.5 + 3.4250\theta_1$	$z_S^2 = 95.0524 + 2.6873\theta_2$

The intersection of the two planes is given by the line:

$$3.4250\theta_1 - 2.6873\theta_2 = 7.524,$$

Table 4.2.: Map of optimal parametric solutions for Example 1.

Region	Solution
$\left\{\begin{array}{c}\theta_1 \geq 0 \\ 0 \leq \theta_2 \leq 20 \\ 3.42500\theta_1 - 2.68730\theta_2 \leq 7.524\end{array}\right\}_{CR1}$	$x_1 = 0$, $x_2 = 14.9254$, $y_1 = 0$, $y_2 = 1$
$\left\{\begin{array}{c}\theta_1 \leq 20 \\ 0 \leq \theta_2 \leq 20 \\ 3.42500\theta_1 - 2.68730\theta_2 \geq 7.524\end{array}\right\}_{CR2}$	$x_1 = 12.5 + 1.25\theta_2$, $x_2 = 0$, $y_1 = 1$, $y_2 = 0$

below which $z_S^1(\theta)$ is valid.

Step 2.(k=2, master problem) Solve Problem (4.6) with 4 additional constraints:

$$y_1 - y_2 \leq 0, \tag{4.22}$$

$$y_2 - y_1 \leq 0, \tag{4.23}$$

$$(6.4 + 0.25\theta_1)x_1 + (6.0 + 0.17\theta_2)x_2 + (7.5 + 0.3\theta_1)y_I + (5.5 + 0.15\theta_2)y_{II} \leq 3.42500\theta_1 + 87.5, \tag{4.24}$$

$$(6.4 + 0.25\theta_1)x_1 + (6.0 + 0.17\theta_2)x_2 + (7.5 + 0.3\theta_1)y_I + (5.5 + 0.15\theta_2)y_{II} \leq 95.0524 + 2.68730\theta_2. \tag{4.25}$$

The resulting problem is infeasible, and hence, the algorithm terminates. The final solution is listed in Table 4.2.

4.3. Multi-parametric RIM MILP problems

In this section, we consider the formulation in (4.4), i.e. the general case with independent varying parameters both in the objective function and the right-hand side of the constraints, rewritten in a more compact mathematical form

(Kosmidis, 1999):

$$
\begin{aligned}
z(\theta) = \min_{x,y} \ & (c + H\theta)^T x + (d + L\theta)^T y, \\
s.t. \ & Ax + Ey \leq b + F\theta, \\
& \Gamma x + \Phi y = \gamma + \Psi\theta, \\
& x \in X \subseteq \mathbb{R}^n, y \in \{0,1\}^q, \\
& \theta \in \Theta = \{\theta : \theta \in \mathbb{R}^s, G\theta \leq e\},
\end{aligned}
\tag{4.26}
$$

where, b, γ, F and Ψ are real matrices with appropriate dimensions. For the solution of (4.26), we present in the following an extension of the algorithm presented in section 4.2, which iterates between two optimisation subproblems, a master MINLP problem and a slave multi-parametric problem. The principal difference is the comparison procedure between two parametric solutions, since in this case the optimal value function is non-linear.

4.3.1. Master problem

By considering the parameters θ as optimisation variables, (4.26) results in the following MINLP formulation (Kosmidis, 1999):

$$
\begin{aligned}
z_M(\theta) = \min_{x,\theta,y} \ & (c + H\theta)^T x + (d + L\theta)^T y, \\
s.t. \ & Ax + Ey \leq b + F\theta, \\
& \Gamma x + \Phi y = \gamma + \Psi\theta, \\
& x \in X \subseteq \mathbb{R}^n, y \in \{0,1\}^q, \\
& \theta \in \Theta = \{\theta : \theta \in \mathbb{R}^s, G\theta \leq e\}.
\end{aligned}
\tag{4.27}
$$

Note that (4.27) involves bilinear terms in the objective function, thereby it is a non-convex problem which requires a global optimisation procedure (Floudas, 2000). The solution of (4.27) returns a new binary vector, $y = \bar{y}$, to the slave problem, which is described next.

4.3.2. Slave problem

By fixing $y = \bar{y}$, the slave problem is formulated in the following way:

$$
\begin{aligned}
z_S(\theta) = (d + L\theta)^T \bar{y} + \min_x \quad & c^T x + \theta^T H^T x, \\
s.t. \quad & Ax \leq b' + F\theta, \\
& \Gamma x \leq \gamma' + \Psi\theta, \\
& x \in X \subseteq \mathbb{R}^n, \\
& \theta \in \Theta = \{\theta : \theta \in \mathbb{R}^s, G\theta \leq e\},
\end{aligned}
\tag{4.28}
$$

where, $b' = (b - E\bar{y})$ and $\gamma' = (\gamma - \Phi\bar{y})$. Once again, Problem (4.28) is solved using a modified version of the original mp-LP algorithm (Dua *et al.*, 2002). As shown before, applying Equation (4.11) to problem (4.28) results in:

$$
\begin{bmatrix} x(\theta) \\ \lambda(\theta) \\ \mu(\theta) \end{bmatrix} = \begin{bmatrix} x_0 \\ \lambda_0 \\ \mu_0 \end{bmatrix} - \begin{bmatrix} 0 & A^T & \Gamma^T \\ \Lambda A & diag(g) & 0 \\ \Gamma & 0 & 0 \end{bmatrix}_{\eta_0}^{-1} \cdot \begin{bmatrix} H^T \\ F \\ \Psi \end{bmatrix}_{\eta_0} \cdot (\theta - \theta_0). \tag{4.29}
$$

The RIM mp-LP algorithm was implemented in Matlab.

Remark 4.2 *Note that in Equation* (4.29), *contrary to* (4.12), N_0 *is a full rank matrix and therefore the explicit expression of the optimisation variables depends on the parameters.*

4.3.3. The algorithm

Between every master-slave iteration, we need to (i) introduce integer and parametric cuts in the master MINLP problem (Equation 4.16 and Equation 4.17, respectively), and (ii) compare the parametric solutions obtained in the slave problem in order to retain the best. While the cuts are identical to the OFC problem, the comparison of different solutions of the slave problem is itself a global optimisation problem, since in this case the optimal value functions are non-linear. Here, we address this issue by storing all different optimal solutions valid inside overlapping regions and computing the optimum solution online by direct value comparisons (enclosure of all solutions - see Dua *et al.*, 2002).

The algorithm terminates when the master problem is infeasible.

The algorithmic steps are summarised in Table 4.3, and are described in detailed in two illustrative problems, shown next.

Table 4.3.: Steps of the algorithm for RIM mp-MILP problems.

Step 0.	(Initialization) Define an initial region of Θ, CR, with best upper bound $\hat{z}^*(\theta) = \infty$, and an initial integer solution, $\bar{y}$.
Step 1.	(Slave subproblem - multiparametric LP problem) For each region with a new integer solution, $\bar{y}$: (a) Solve the mp-LP subproblem (4.28) to obtain a set of parametric upper bounds, $\hat{z}(\theta) = z_S^*$, and the corresponding critical regions *CR*; (b) If $\hat{z}(\theta) \leq z_S^*(\theta)$ for some region of θ, update the best upper bound function, $\hat{z}(\theta)$, and the corresponding integer solutions, y*; (c) If an infeasibility is found in some region *CR*, go to *Step 2*.
Step 2.	(Master subproblem - MINLP problem) For each region *CR*, formulate and solve to global optimality the MINLP master subproblem, (4.27), (i) treating θ as an optimisation variable, (ii) introducing an integer cut (4.16) and (iii) introducing a parametric cut (4.17). Return to *Step 1* with new integer solutions and corresponding *CR*s.
Step 3.	(Convergence) The algorithm terminates in a region where the solution of the master MINLP subproblem is infeasible. Then, the optimal parametric solution is given by the current upper bounds $\hat{z}^*(\theta)$.

4.3.4. Example 2

Consider again Example 1 in Figure 4.2, but now with uncertainty involving both the customer's demand and the objective function, as follows:

$$\begin{aligned}
\min_{x_1,x_2,y_I,y_{II}} \quad & (6.4 + 0.25\theta_1)x_1 + 6.0x_2 + (7.5 + 0.3\theta_1)y_I + 5.5y_{II}, \\
s.t. \quad & 0.8 \cdot x_1 + 0.67 \cdot x_2 \geq 10 + \theta_2, \\
& x_1 \leq 40y_I, \\
& x_2 \leq 40y_{II}, \\
& x_1, x_2 \geq 0, \\
& 0 \leq \theta_1 \leq 20, \\
& 0 \leq \theta_2 \leq 10, \\
& y_I, y_{II}, \in \{0, 1\}.
\end{aligned} \tag{4.30}$$

We apply the solution steps of the proposed algorithm in Table 4.3.

Step 0.(initialisation) Solve Problem (4.30) considering θ as being optimisation variables, $\bar{y} = (1, 0)$.

Step 1.(k=1, slave subproblem) Fix $y = \bar{y}$. The RIM mp-LP problem is formulated as:

$$\begin{aligned}
\min_{x_1,x_2} \quad & (6.4 + 0.25\theta_1)x_1 + 6.0x_2 + (7.5 + 0.3\theta_1), \\
s.t. \quad & 0.8 \cdot x_1 + 0.67 \cdot x_2 \geq 10 + \theta_2, \\
& 0 \leq x_1 \leq 40, 0 \leq x_2 \leq 0, \\
& 0 \leq \theta_1 \leq 20, 0 \leq \theta_2 \leq 10.
\end{aligned} \tag{4.31}$$

The solution of (4.31) is computed using the RIM mp-LP algorithm:

$$\begin{cases}
x_1 = 12.5 + 1.25 \cdot \theta_2, \\
x_2 = 0, \\
0 \leq \theta_1 \leq 20, \\
0 \leq \theta_2 \leq 10.
\end{cases}$$

Step 2.(k=1, master subproblem) Solve the master problem in (4.27) with two additional constraints, due to (4.16) and (4.17):

$$y_1 - y_2 \leq 0, \tag{4.32}$$

$$(6.4 + 0.25\theta_1)x_1 + 6.0x_2 + (7.5 + 0.3\theta_1)y_I$$
$$+ 5.5y_{II} \leq 87.5 + 3.425\theta_1 - 0.3125\theta_1\theta_2 + 8\theta_2. \tag{4.33}$$

The solution is obtained using the commercial package GAMS/BARON (Sahinidis, 2000): $\bar{y} = (0, 1)$.

Step 1.(k=2, slave problem) By fixing $y = \bar{y}$, the solution of (4.28) is:

$$\begin{cases} x_1 = 0, \\ x_2 = 14.9254 + 1.49254\theta_2, \\ 0 \leq \theta_1 \leq 20, 0 \leq \theta_2 \leq 10. \end{cases}$$

Step 1.(k=2, comparison of solutions) Solutions valid in $0 \leq \theta_1 \leq 20, 0 \leq \theta_2 \leq 10$:

Solution 1	Solution 2
$x_1 = 12.5 + 1.25\theta_2$	$x_1 = 0$
$x_2 = 0$	$x_2 = 14.9254 + 1.492540\theta_2$
$y_1 = 1$	$y_1 = 0$
$y_2 = 0$	$y_2 = 1$
$z = 87.5 + 3.425\theta_1 - 0.3125\theta_1\theta_2 + 8\theta_2$	$z = 8.955\theta_2 + 95.0524$

In this specific case, we can compute the intersection of the two solutions:

$$-0.955\theta_2 - 0.3125\theta_1\theta_2 - 7.5524 + 3.425\theta_1 = 0.$$

Otherwise, we store all parametric solutions of the slave problems and compute on-line the best decision.

Step 2.(k=2, master problem) Solve Problem (4.27) with 4 additional constraints:

$$y_1 - y_2 \leq 0, \tag{4.34}$$

$$y_2 - y_1 \leq 0, \tag{4.35}$$

$$(6.4 + 0.25\theta_1)x_1 + 6.0x_2 + (7.5 + 0.3\theta_1)y_I$$
$$+ 5.5y_{II} \leq 87.5 + 3.425\theta_1 - 0.3125\theta_1\theta_2 + 8\theta_2, \tag{4.36}$$

$$(6.4 + 0.25\theta_1)x_1 + 6.0x_2 + (7.5 + 0.3\theta_1)y_I$$
$$+ 5.5y_{II} \leq 8.955\theta_2 + 95.0524. \tag{4.37}$$

The resulting problem is infeasible, and thus, the algorithm terminates. The final solution is listed in Table 4.4.

Table 4.4.: Map of critical regions for Problem (4.30).

Region	Solution
$\left\{\begin{array}{c} 0 \leq \theta_1 \leq 20 \\ 0 \leq \theta_2 \leq 10 \\ -0.955\theta_2 - 0.3125\theta_1\theta_2 + 3.425\theta_1 \leq 7.5524 \end{array}\right\}_{CR1}$	$x_1 = 12.5 + 1.25\theta_2$, $x_2 = 0$, $y_1 = 1$, $y_2 = 0$
$\left\{\begin{array}{c} 0 \leq \theta_1 \leq 20 \\ 0 \leq \theta_2 \leq 10 \\ 0.955\theta_2 + 0.3125\theta_1\theta_2 - 3.425\theta_1 \leq -7.5524 \end{array}\right\}_{CR2}$	$x_1 = 0$, $x_2 = 14.9254 + 1.4925\theta_2$, $y_1 = 0$, $y_2 = 1$

4.3.5. Example 3

This example is a variant of a process synthesis problem described by Biegler *et al.* (1997), shown in Figure 4.3. A chemical product C is produced using either process unit *II* or *III*, both of which use chemical B as raw material; on the other hand, B can either be directly purchased or manufactured using process *I* and purchasing raw material A.

Moreover, the decision is subject to uncertainty in the operation cost and

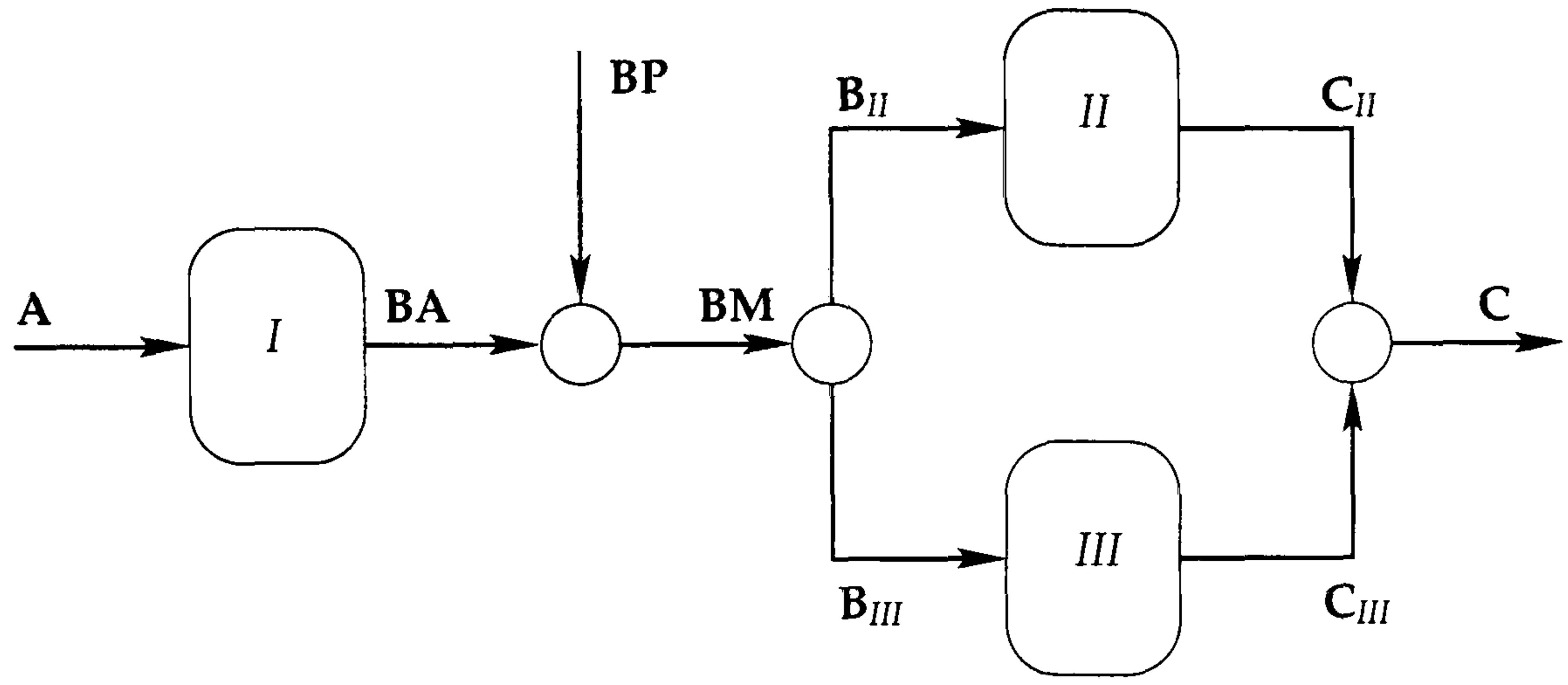


Figure 4.3.: Superstructure of the illustrative example 3.

product demand. The uncertainty - θ_1, θ_2 - is assumed to be unstructured and bounded. The multi-parametric RIM MILP problem is posed as follows:

$$\begin{aligned}
\min \quad & -18 \cdot C + (10 \cdot y_I + 15 \cdot y_{II} + 20 \cdot y_{III}) \\
& + (2.5 \cdot A \cdot y_I + (4 + \theta_1) \cdot B_{II} \cdot y_{II} + 5.5 \cdot B_{III} \cdot y_{III}), \\
s.t. \quad & C = 0.82 \cdot B_{II} + 0.95 \cdot B_{III}, \\
& 2 \leq C \leq 5 + \theta_2, A \leq 16 \cdot y_I, \\
& y_{II} + yIII \geq 1, B_{II} - 30 \cdot y_{II} \leq 0, \\
& B_{III} - 30 \cdot y_{III} \leq 0, B_{II} + B_{III} - BP - 0.9 \cdot A = 0, \\
& 0 \leq \theta_1 \leq 5, 0 \leq \theta_2 \leq 10, y_I, y_{II}, y_{III} \in \{0, 1\}, \\
& C, A, BP, B_{II}, B_{III} \geq 0.
\end{aligned} \tag{4.38}$$

The final solution is depicted in Figure 4.4 and Table 4.5.

Table 4.5.: Solution of Problem 4.38 ($x = \Omega\,\theta + \omega$).

Region		Integer solution $\bar{y}$	Continuous solution Ω	ω
⋮				
CR7	$2.42 \le \theta_1 \le 5$ $8.68 \le \theta_2 \le 10$	$\{1,0,1\}$	$\begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 1.0526 \\ 0 & 0 \\ 0 & 1.0526 \end{bmatrix}$	$\begin{bmatrix} 5 \\ 16 \\ -9.1368 \\ 0 \\ 5.2632 \end{bmatrix}$
CR8	$2.42 \le \theta_1 \le 5$ $6.81 \le \theta_2 \le 8.68$	$\{1,0,1\}$	$\begin{bmatrix} 0 & 1 \\ 0 & 1.1696 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1.0526 \end{bmatrix}$	$\begin{bmatrix} 5 \\ 5.8480 \\ 0 \\ 0 \\ 5.2632 \end{bmatrix}$
CR9	$2.42 \le \theta_1 \le 5$ $0 \le \theta_2 \le 6.80$	$\{1,0,1\}$	$\begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 1.0526 \\ 0 & 0 \\ 0 & 1.0526 \end{bmatrix}$	$\begin{bmatrix} 5 \\ 16 \\ -9.1368 \\ 0 \\ 5.2632 \end{bmatrix}$
		$\{0,0,1\}$	$\begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 1.0526 \\ 0 & 0 \\ 0 & 1.0526 \end{bmatrix}$	$\begin{bmatrix} 5 \\ 0 \\ 5.2632 \\ 0 \\ 5.2632 \end{bmatrix}$

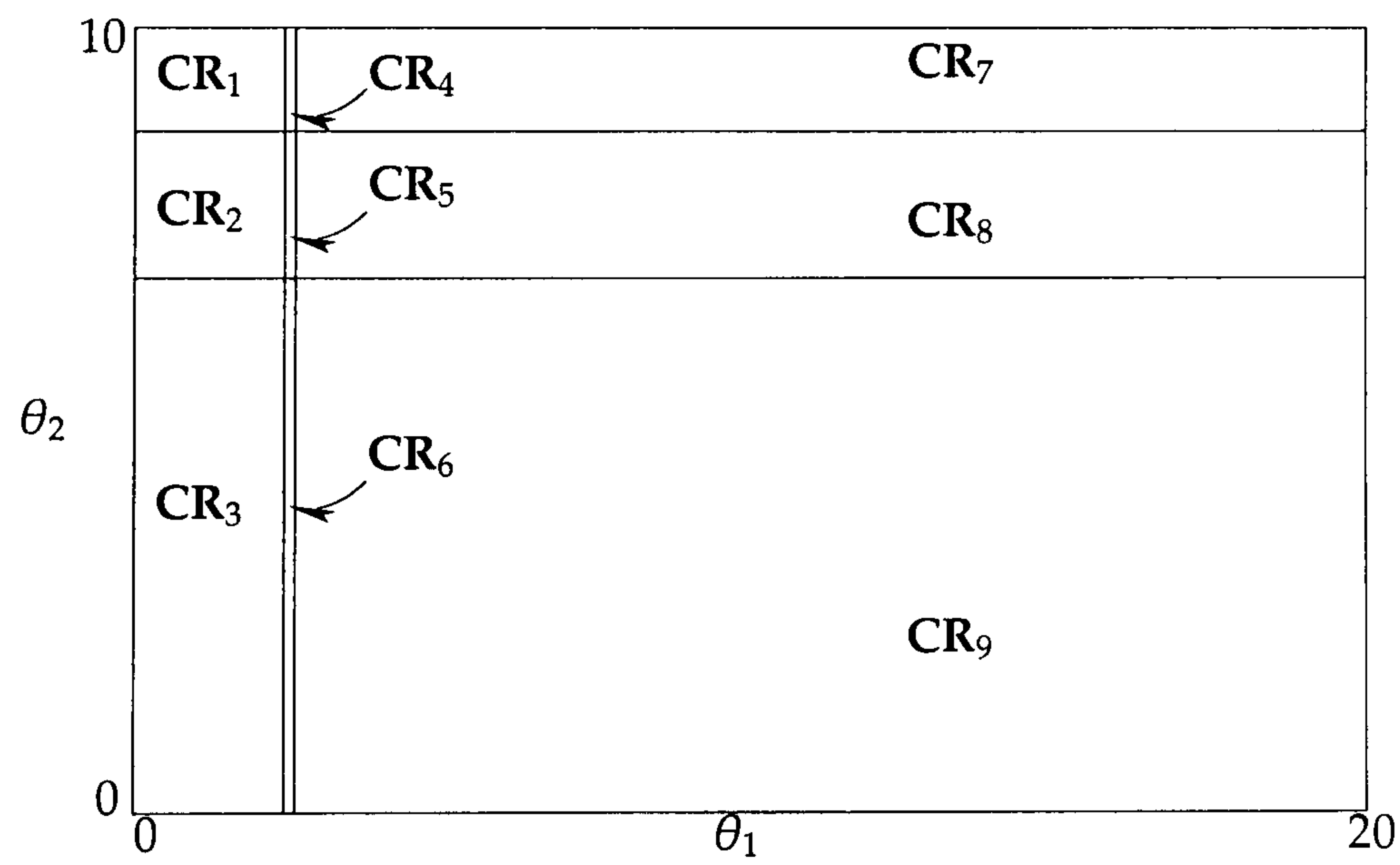


Figure 4.4.: Map of critical regions of Problem (4.38).

Remark 4.3 *Although we focus on OFC and RIM classes of mp-MILP problems, the procedure is still valid when matrices E, Φ also depend linearly on the parameters, as* *follows:*

$$
\begin{aligned}
z(\theta) = \min_{x,y} \quad & (c + H\theta)^T x + (d + L\theta)^T y, \\
\text{s.t.} \quad & Ax + (e_1 + E_2\theta)y \leq b + F\theta, \\
& \Gamma x + (\phi_1 + \Phi_2\theta)y = \gamma + \Psi\theta, \\
& x \in X \subseteq \mathbb{R}^n, y \in \{0,1\}^q, \\
& \theta \in \Theta,
\end{aligned} \tag{4.39}
$$

because, fixing the binary vector to the solution obtained in the master subproblem, $y = \bar{y}$, (4.39) *is rewritten as a RIM mp-LP problem:*

$$
\begin{aligned}
z_S(\theta) = (d + L\theta)^T \bar{y} + \min_{x} \quad & c^T x + \theta^T H^T x, \\
\text{s.t.} \quad & Ax \leq b' + F'\theta, \\
& \Gamma x \leq \gamma' + \Psi'\theta, \\
& x \in X \subseteq \mathbb{R}^n, y \in \{0,1\}^m, \\
& \theta \in \Theta,
\end{aligned} \tag{4.40}
$$

where, $b' = (b - e_1\bar{y})$, $\gamma' = (\gamma - \phi_1\bar{y})$, $F' = (F - E_2\bar{y})$ *and* $\Psi' = (\Psi - \Phi_2\bar{y})$.

4.4. Concluding remarks

We have presented a novel optimisation framework for the global solution of general mp-MILP problems, involving uncertainty in the objective function and the right-hand side of the constraints. Based on our previous work on multi-parametric programming (Dua and Pistikopoulos, 2000; Dua *et al.*, 2004; Pistikopoulos *et al.*, 2007*a*), a novel mp-LP algorithm was developed, which overcomes the presence of the non-convexities due to bilinear terms. This is then used in an efficient procedure, which iterates between a master MINLP subproblem, solved to global optimality, and a slave mp-LP subproblem. A number of examples are also presented.

The proposed approach has many applications in hybrid and robust control - a topic which is currently being investigated and will be introduced in Chapter 6.

A. Definition of Rest of the Region

Given an initial region, CR^{IG} and a region of optimality, CR^{Q} such that $CR^{Q} \subseteq CR^{IG}$, a procedure is described in this section to define the rest of the region, $CR^{rest} = CR^{IG} - CR^{Q}$. For the sake of simplifying the explanation of the procedure, consider the case when only two parameters, θ_1 and θ_2, are present (see Figure 4.5), where CR^{IG} is defined by the inequalities: $\{\theta_1^L \leq \theta_1 \leq \theta_1^U, \theta_2^L \leq \theta_2 \leq \theta_2^U\}$ and CR^{Q} is defined by the inequalities: $\{C1 \leq 0, C2 \leq 0, C3 \leq 0\}$ where $C1, C2$ and $C3$ are linear in θ. The procedure consists of considering one by one the inequalities which define CR^{Q}. Considering, for example, the inequality $C1 \leq 0$, the rest of the region is given by, $CR_1^{rest} : \{C1 \geq 0, \theta_1^L \leq \theta_1, \theta_2 \leq \theta_2^U\}$, which is obtained by reversing the sign of inequality $C1 \leq 0$ and removing redundant constraints in CR^{IG} (see Figure 4.6). Thus, by considering the rest of the inequalities, the complete rest of the region is given by: $CR^{rest} = \{CR_1^{rest} \cup CR_2^{rest} \cup CR_3^{rest}\}$, where CR_1^{rest}, CR_2^{rest} and CR_3^{rest} are given in Table 4.6 and are graphically depicted in Figure 4.7. Note that for the case when CR^{IG} is unbounded, simply suppress the inequalities involving CR^{IG} in Table 4.6.

Table 4.6.: Definition of rest of the regions.

Region	Inequalities
CR_1^{rest}	$C1 \geq 0, \theta_1^L \leq \theta_1, \theta_2 \leq \theta_2^U$
CR_2^{rest}	$C1 \leq 0, C2 \geq 0, \theta_1 \leq \theta_1^U, \theta_2 \leq \theta_2^U$
CR_3^{rest}	$C1 \leq 0, C2 \leq 0, C3 \geq 0, \theta_1^L \leq \theta_1 \leq \theta_1^U, \theta_2^L \leq \theta_2$

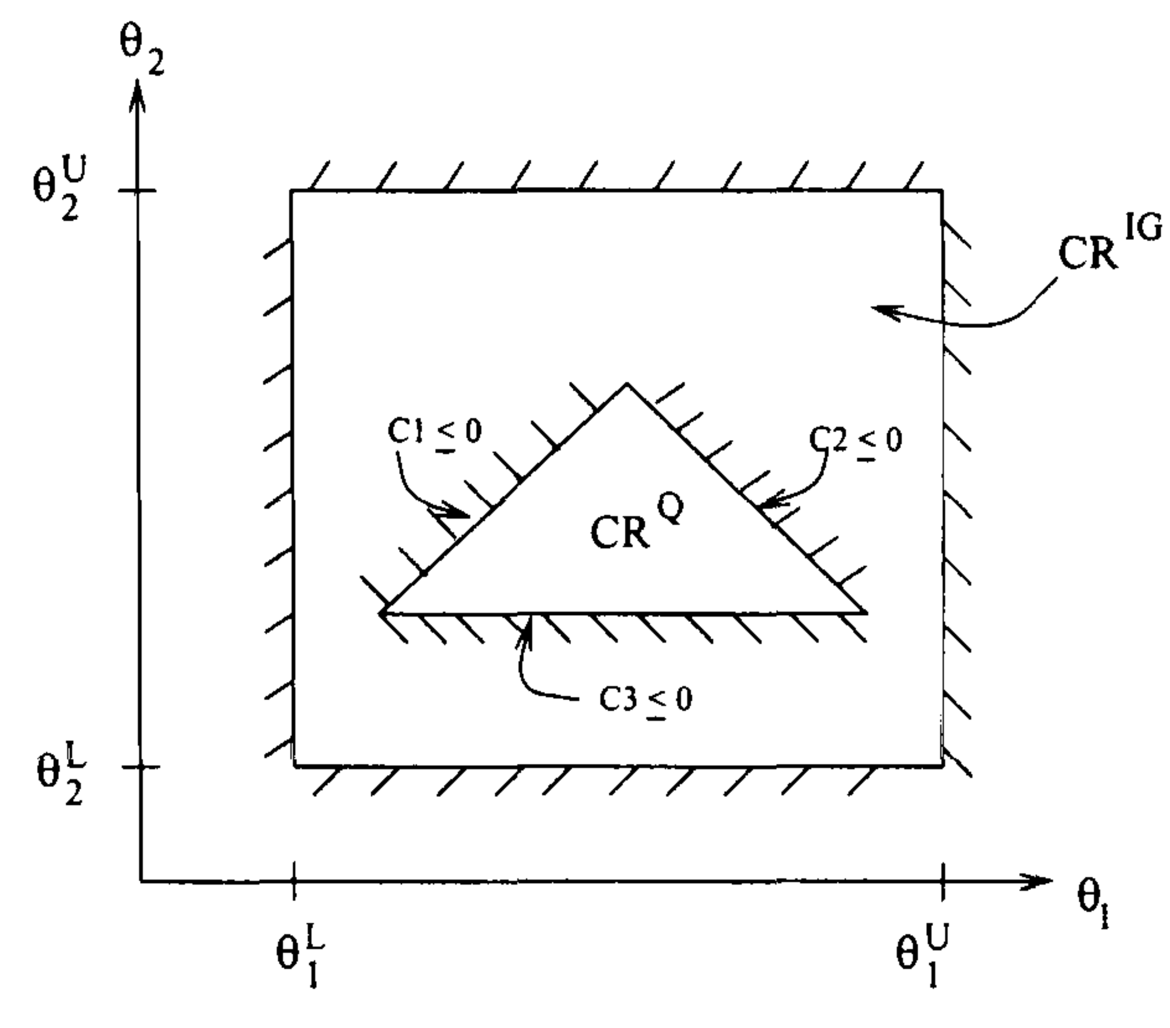


Figure 4.5.: Critical regions, CR^{IG} and CR^{Q}.

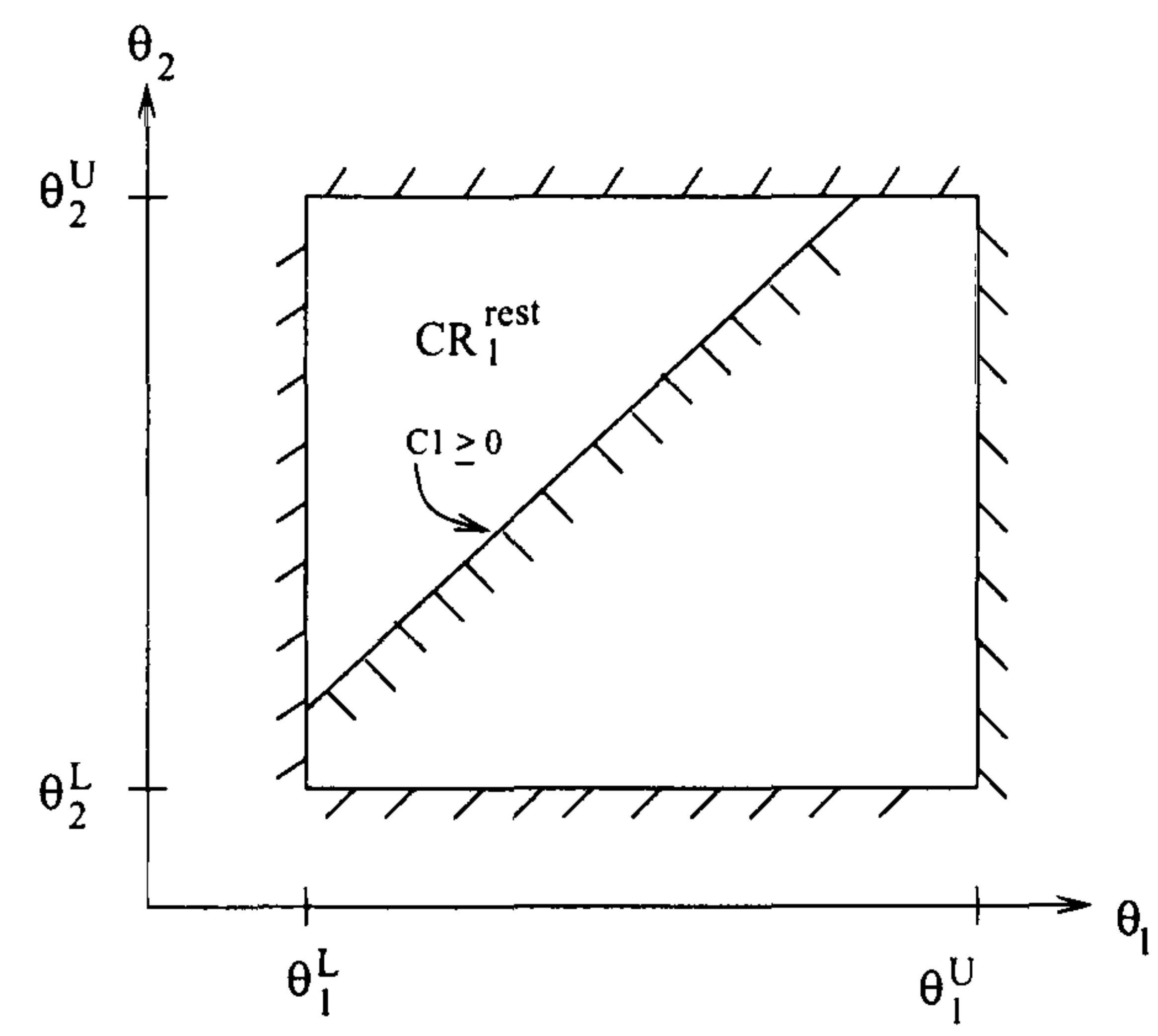


Figure 4.6.: Division of critical regions - Step 1.

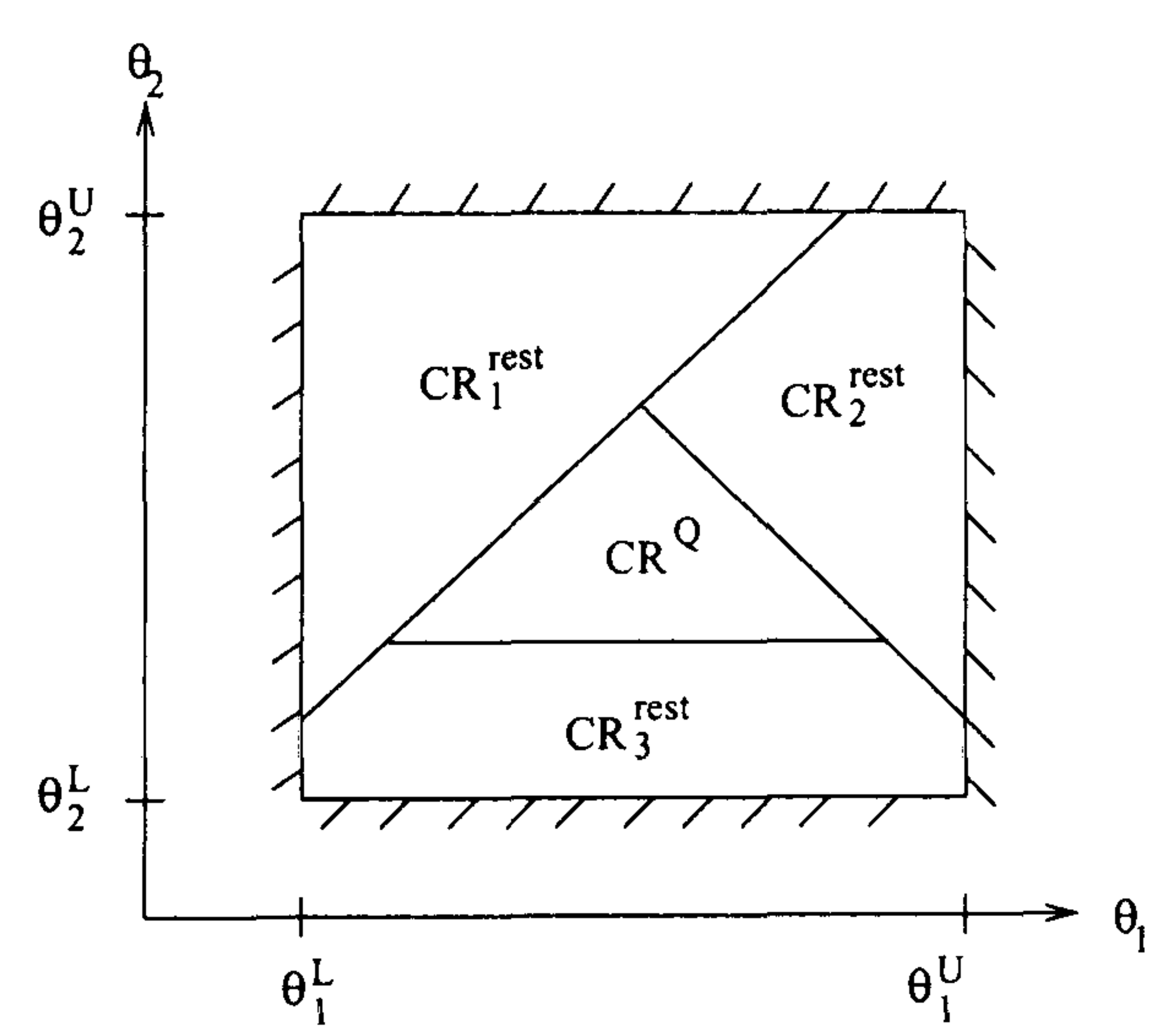


Figure 4.7.: Division of critical regions - rest of the regions.

Part II.

Advances in robust optimisation & control

5. Constrained dynamic programming problems

In this chapter, we present a new algorithm for solving complex multi-stage optimisation problems involving hard constraints and uncertainties, based on dynamic and multi-parametric programming techniques. Each echelon of the dynamic programming procedure, typically employed in the context of multi-stage optimisation models, is interpreted as a multi-parametric optimisation problem, with the present states and future decision variables being the parameters, while the present decisions the corresponding optimisation variables. This re-formulation significantly reduces the dimension of the original problem, essentially to a set of lower dimensional multi-parametric programs, which are sequentially solved. Furthermore, the use of sensitivity analysis circumvents non-convexities that naturally arise in constrained dynamic programming problems. The potential application of the proposed novel framework to robust constrained optimal control is highlighted.

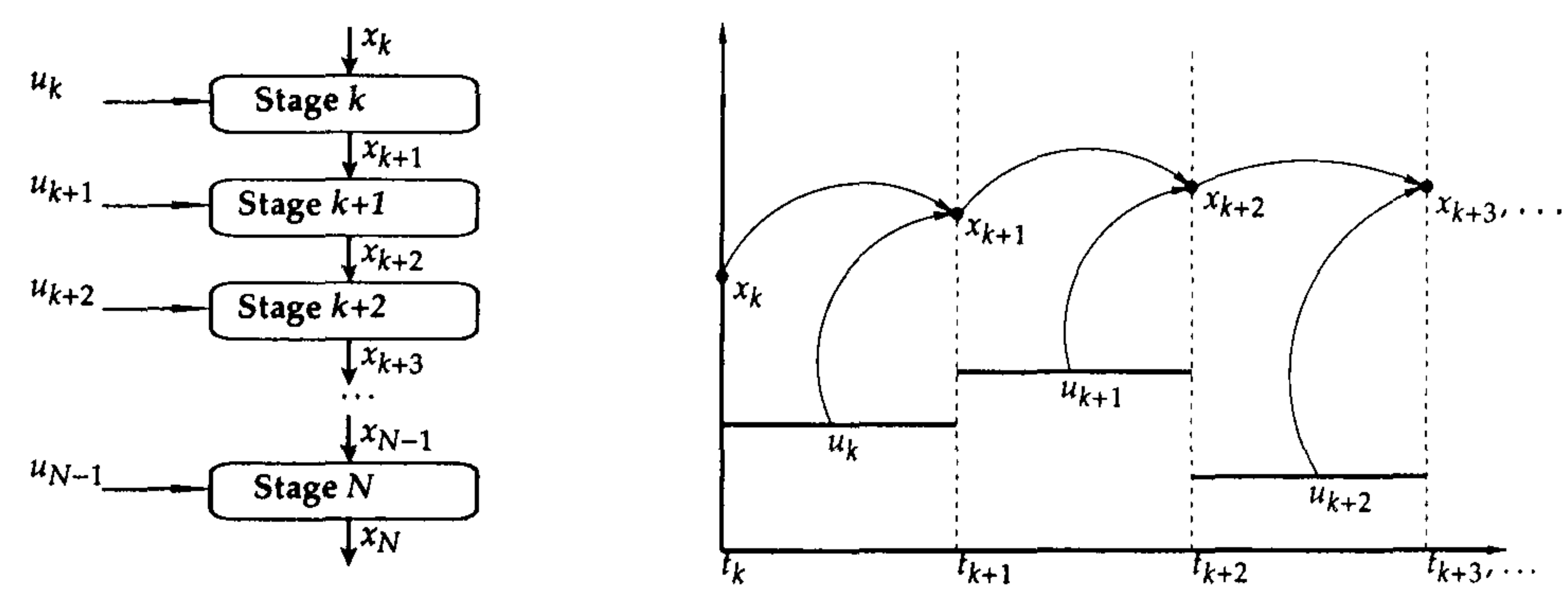
5.1. Introduction

Multi-stage decision processes have attracted considerable attention in the open literature. With many applications in engineering, economics and finances, theory and algorithms for multi-stage decision problems have been presented in the open literature (Bellman, 2003; Bertsekas, 2005). A typical multi-stage decision making process, involving a discrete-time model and a convex stage-additive cost function, can be posed as follows (Başar and Olsder, 1982; Bertsekas, 2005):

$$x_{k+1} = f_k(x_k, u_k), \quad x_k \in \mathcal{X}, u_k \in \mathcal{U}_k, \quad k \in \{0, 1, \ldots, N-1\}, \tag{5.1a}$$

$$J(U) = g_N(x_N) + \sum_{k=0}^{N-1} g_k(x_k, u_k), \tag{5.1b}$$

where, k is the index of the time, x_k is the state of the system at time k, $\mathcal{X} \subseteq \mathbb{R}^n$, u_k denotes the optimisation (decision) variable at time k, $U \triangleq \{u_0, u_1, \ldots, u_{N-1}\}$, $\mathcal{U}_k \subseteq \mathbb{R}^m$, f_k describes the dynamic behaviour of the system and g_k is the cost occurred at time k. Based on a sequence of stage-wise optimal decisions, the system transforms from its original state, x_0, into a final state, x_N (as shown in Figure 5.1). The set of optimal decisions, $\{u_0^*, u_1^*, \ldots, u_{N-1}^*\}$, and the corresponding path, $\{x_1^*, x_2^*, \ldots, x_N^*\}$, optimise a pre-assigned cost function (5.1b). In other words, if the sequence of decisions is optimal the reward is maximum.



(a) Multi-stage process. (b) Time horizon of actions.

Figure 5.1.: Multi-stage decision process.

Dynamic Programming is well-documented (Bellman, 2003) as being a powerful tool to solve this class of optimisation problems. Based on the optimality principle, the original problem disassembles into a set of problems of lower dimensionality, thereby significantly reducing the complexity of obtaining the solution. The *value function* for a general multi-stage optimisation problem, as in (5.1), is given by:

$$V_k(x_k) = \min_{\mu_k,\ldots,\mu_N} \left[g_N(x_N) + \sum_{i=k}^{N-1} g_i(u_i, x_i) \right], \tag{5.2}$$

where $u_i = \mu_i(x_i) \in \mathcal{U}_i$, and $\mu_i(\cdot)$ is an admissible policy. Applying the optimality principle to Equation (5.2) results in the following recursive equation (Başar and Olsder, 1982):

$$V_k(x_k) = \min_{u_k \in \mathcal{U}_k} \left[g_k(u_k, x_k) + V_{k+1}(x_{k+1}) \right]. \tag{5.3}$$

From (5.3) we conclude that incumbent cost functions are a compound of all future cost functions, previously optimised, and the cost corresponding to the decision taken at the present time. Bellman (2003) proved that this methodology solves the original problem to global optimality. The obvious advantage is that at each time step/stage the decision maker only takes decisions corresponding to it, provided that all future stages are optimised up to the incumbent stage.

Although dynamic programming is a well-established methodology, a number of limitations can be identified, especially in the presence of hard constraints. As an example, consider the application of dynamic programming to unconstrained linear-quadratic regulator control problems (Rawlings, 1999; Mayne *et al.*, 2006): $u_0 = K_0 \cdot x_0, u_1 = K_1 \cdot x_1, \ldots, u_{N-1} = K_{N-1} \cdot x_{N-1}$, where the control action is set to be admissible, $u_k \in \mathcal{U}_k$, and K_i are real matrices. In the presence of (hard) inequality constraints, non-linear decision laws result, which introduce non-convexities and hence significantly increase the complexity of the implementation as the use of specialised global optimisation techniques is required.

Borrelli *et al.* (2005) presented an approach to address hard constrained multi-stage problems in a dynamic programming fashion. Based on multi-parametric programming theory (Pistikopoulos *et al.*, 2007*a*) and Bellman's optimality principle, the authors compute, for each stage, the corresponding control law, $u_k = \mu_k(x_k)$, using multi-parametric programming algorithms (Dua *et al.*, 2002; Pistikopoulos *et al.*, 2007*a*). The key idea is to incorporate the conditional piecewise linear function in the cost function of the previous stage, reducing it to a function of only the incumbent stage variables, u_{k-1} and x_{k-1}. However, as the objective function at each stage is a piecewise quadratic function of $\{x_k, u_k\}$, overlapping critical regions result, and a parametric global optimisation procedure is thus required to obtain the explicit solution.

In this chapter, we present a novel algorithm for the solution of constrained dynamic programs which effectively avoids the need for any global optimisation procedure. The algorithm combines the principles of multi-parametric programming (Pistikopoulos *et al.*, 2007*a*) and dynamic programming, and can readily be extended to handle uncertainty in the model data (El-Ghaoui and Lebret, 1997; Ben-Tal and Nemirovski, 2000; Lin *et al.*, 2004; Janak *et al.*, 2007). These developements are described in the following sections.

5.2. Constrained dynamic programming

Consider the last stage of the decision chain depicted in Figure 5.1(a), ***Stage** N*, and its corresponding optimisation problem:

$$\begin{aligned} &\min_{u_{N-1}} J_N(u_{N-1}, x_{N-1}) = g_N(x_N) + g_{N-1}(x_{N-1}, u_{N-1}), \\ s.t.\ \ & u_{N-1}^{min} \leq u_{N-1} \leq u_{N-1}^{max}, \\ & x_N^{min} \leq x_N \leq x_N^{max}, \\ & x_N = f_{N-1}(x_{N-1}, u_{N-1}), \end{aligned} \tag{5.4}$$

where, $x_N, x_{N-1} \in \mathbb{R}^n$ and $u_{N-1} \in \mathbb{R}^m$. A Karush-Kuhn-Tucker (KKT) point for (5.4) satisfies the linear independence constraint qualification and the following system of equations (Bazaraa and Shetty, 1979):

$$
\begin{aligned}
&\nabla\mathscr{L}(u_{N-1},\lambda,\mu,x_{N-1}) = 0,\\
&\lambda_i\psi_i(u_{N-1},x_{N-1}) = 0, \quad \forall \;\; i = 1,\ldots,2m+2n,\\
&\omega_j(u_{N-1},x_{N-1}) = 0, \quad \forall \;\; j = 1,\ldots,n,\\
&\mathscr{L} = J_N(u_{N-1},x_{N-1}) + \sum_{i=1}^{2m+2n} \lambda_i\psi_i(u_{N-1},x_{N-1}) + \sum_{j=1}^{n} \mu_j\omega_j(u_{N-1}),
\end{aligned}
\tag{5.5}
$$

where λ and μ are vectors of the Lagrange multipliers of the inequalities and equalities of (5.4), respectively. Since x_{N-1} is a varying parameter, the solution of (5.5), u^*_{N-1}, instead of being a point, corresponds to an optimal function, $u^*_{N-1} = \mu^*_{N-1}(x_{N-1})$. The existence of such function depends on the conditions stated in Theorem 5.1, where we set $x = x_{N-1}$ and $u = u_{N-1}$.

Theorem 5.1 *Basic Sensitivity Theorem (Fiacco, 1976): Let x_0 be a vector of parameter values and (u_0, λ_0, μ_0) a KKT triple corresponding to (5.5), where λ_0 is nonnegative and u_0 is feasible in (5.4). Also assume that (i) strict complementary slackness (SCS) holds, (ii) the binding constraint gradients are linearly independent (LICQ: Linear Independence Constraint Qualification), and (iii) the second-order sufficiency conditions (SOSC) hold. Then, in neighbourhood of x_0, there exists a unique, once continuously differentiable function, $z(x) = [u(x), \lambda(x), \mu(x)]$, satisfying (5.5) with $z(x_0) = [u(x_0), \lambda(x_0), \mu(x_0)]$, where $u(x)$ is a unique isolated minimiser for (5.4), and*

$$
\begin{pmatrix} \frac{du(x_0)}{dx} \\ \frac{d\lambda(x_0)}{dx} \\ \frac{d\mu(x_0)}{dx} \end{pmatrix} = -(M_0)^{-1}N_0,
\tag{5.6}
$$

where, M_0 and N_0 are the jacobian of system (5.5) with respect to z and x:

$$M_0 = \begin{pmatrix} \nabla^2\mathscr{L} & \nabla\psi_1 & \cdots & \nabla\psi_p & \nabla\omega_1 & \cdots & \nabla\omega_q \\ \lambda_1\nabla^T\psi_1 & \psi_1 & & & & & \\ \vdots & & \ddots & & & & \\ \lambda_p\nabla^T\psi_p & & & \psi_p & & & \\ \nabla^T\omega_1 & & & & & & \\ \vdots & & & & & & \\ \nabla^T\omega_q & & & & & & \end{pmatrix},$$

$$N_0 = (\nabla^2_{xu}\mathscr{L}, \lambda_1\nabla^T_x\psi_1, \ldots, \lambda_p\nabla^T_x\psi_p, \nabla^T_x\omega_1, \ldots, \nabla^T_x\omega_q)^T.$$

Note that M_0 is always invertible (non-singular) because the solution for the homogeneous system of M_0 is always zero (Fiacco, 1983, pp. 80-81). From (3.13) it is possible to derive a general analytic expression for u_{N-1}, however, this is obviously limited to all but simplest cases. Nonetheless, Dua *et al.* (2002) have recently proposed an algorithm to solve equation (3.13) for general convex problems in the entire range of the varying parameters. The algorithm is based on approximations of the non-linear optimal expression by a set of first-order expansions (Corollary 5.1) valid for different combinations of active constraints.

Corollary 5.1 *First-order estimation of $u(x)$, $\lambda(x)$, $\mu(x)$, near $x = x_0$ (Fiacco, 1983): Under the assumptions of Theorem 5.1, a first-order approximation of $[u(x), \lambda(x), \mu(x)]$ in a neighbourhood of x_0 is,*

$$\begin{bmatrix} u(x) \\ \lambda(x) \\ \mu(x) \end{bmatrix} = \begin{bmatrix} u_0 \\ \lambda_0 \\ \mu_0 \end{bmatrix} - (M_0)^{-1} \cdot N_0 \cdot x + o(\|x\|), \tag{5.7}$$

where $(u_0, \lambda_0, \mu_0) = [u(x_0), \lambda(x_0), \mu(x_0)]$, $M_0 = M(x_0)$, $N_0 = N(x_0)$, and $\phi(x) = o(\|x\|)$ means that $\phi(x)/\|x\| \to 0$ as $x \to x_0$.

Each piecewise linear approximation, (5.7), is confined to regions defined by the feasibility and optimality conditions (Dua *et al.*, 2002). If $\tilde{\psi}$ corresponds to the non-active constraints, and $\tilde{\lambda}$ corresponds to the active constraints:

$$\begin{cases} \tilde{\psi}(u(x_{N-1}), x_{N-1}) \leq 0 & \rightarrow \text{ Feasibility conditions,} \\ \tilde{\lambda}(x_{N-1}) \geq 0 & \rightarrow \text{ Optimality conditions.} \end{cases} \tag{5.8}$$

Consequently, the explicit expression is given by a conditional piecewise linear function:

$$u_{N-1} = \mu^*_{N-1}(x_{N-1}), \tag{5.9}$$

or,

$$\begin{cases} u_{N-1} = \mathbf{K}^1_{N-1} \cdot x_{N-1} + \mathbf{C}^1_{N-1} & \forall x_{N-1} \in CR^1_{N-1}, \\ u_{N-1} = \mathbf{K}^2_{N-1} \cdot x_{N-1} + \mathbf{C}^2_{N-1} & \forall x_{N-1} \in CR^2_{N-1}, \\ \vdots & \\ u_{N-1} = \mathbf{K}^{L_{N-1}}_{N-1} \cdot x_{N-1} + \mathbf{C}^{L_{N-1}}_{N-1} & \forall x_{N-1} \in CR^{L_{N-1}}_{N-1}, \end{cases} \tag{5.10}$$

where, $\mathbf{K}^i_{N-1}$ and $\mathbf{C}^i_{N-1}$ are real matrices, and $CR^i_{N-1} \subset \mathbb{R}^n$. Note that similarly to Bellman's procedure (Bellman, 2003) we obtain a piecewise linear function for the incumbent states; the difference being the fact that we compute different decision laws in different regions of the states, CR^i_{N-1}, because of the presence of hard constraints and as a result different combinations of active constraints.

Consider then ***Stage N-1***, and its corresponding optimisation problem:

$$\begin{aligned} \min_{u_{N-2}} \; & J_{N-1}(u_{N-2}, x_{N-2}) = g_N(x_N) \\ & + g_{N-1}(x_{N-1}, u_{N-1}) + g_{N-2}(x_{N-2}, u_{N-2}), \\ s.t. \; & x_N = f_{N-1}(x_{N-1}, u_{N-1}), \\ & x_{N-1} = f_{N-2}(x_{N-2}, u_{N-2}), \\ & u^{min}_{N-2} \leq u_{N-2} \leq u^{max}_{N-2}, \\ & x^{min}_{N-1} \leq x_{N-1} \leq x^{max}_{N-1}, \\ & x_N, x_{N-1}, x_{N-2} \in \mathbb{R}^n, u_{N-1}, u_{N-2} \in \mathbb{R}^m. \end{aligned} \tag{5.11}$$

Proceeding as in the conventional dynamic programming procedure, we can incorporate the expression in (5.9) into (5.11), integrate the model information: $x_{k+1} = A \cdot x_k + B \cdot u_k$, express the incumbent cost function in terms of only $\{x_{N-2}, u_{N-2}\}$, and then optimise manipulating u_{N-2}. Note that since (5.9) is a nonlinear function of x_{N-1}, it is also a nonlinear function of x_{N-2} and u_{N-2}; i.e., $u_{N-1} = \mu^*_{N-1}(f_{N-2}(x_{N-2}, u_{N-2}))$, therefore, the resulting problem, formulation (5.11), corresponds to a global optimisation problem, which is obviously undesirable. However, note that (5.11) is convex with respect to u_{N-1}, u_{N-2} and x_{N-2}. We then take advantage of the following Lemma.

Lemma 5.1 *If a dynamic system is described by a convex function* (5.1a) *and we aim to minimise a convex stage-additive cost function* (5.1b), *then the dynamic programming recursive formula for the value function at stage k,*

$$V_k(x_k) = \min_{u_k \in \mathcal{U}_k} \left[g_k(u_k, x_k) + V_{k+1}(x_{k+1}) \right], \tag{5.12}$$

implies that the solution computed, $\left(x_k, u^*_k(x_k),\ u^*_{k+1}(x_{k+1}), \ldots,\ u^*_{N-1}(x_{N-1})\right)$, *satisfies the following inequalities,*

$$\begin{aligned} &V_k(x_k, u^*_k(x_k), u^*_{k+1}(x_{k+1}), \ldots, u^*_{N-1}(x_{N-1})), \\ &\quad \leq V_k(x_k, u_k(x_k), u^*_{k+1}(x_{k+1}), \ldots, u^*_{N-1}(x_{N-1})), \end{aligned} \tag{5.13}$$

$$\begin{aligned} &V_{k+1}(\overbrace{x_k, u^*_k}^{x_{k+1}}(x_k), u^*_{k+1}(x_{k+1}), \ldots, u^*_{N-1}(x_{N-1})), \\ &\quad \leq V_{k+1}(\underbrace{x_k, u^*_k(x_k)}_{x_{k+1}}, u_{k+1}(x_{k+1}), \ldots, u^*_{N-1}(x_{N-1})), \end{aligned} \tag{5.14}$$

where, u^*_i, $i \in \{k, k+1, \ldots, N-1\}$, *is the optimal value of the optimisation variable for the incumbent stage.*

Proof. Since the cost functions at each stage are convex the proof is obvious.

Since V_{k+1} and V_k are convex functions, their interception will be unique. Thus, one concludes that the optimum at stage k may be obtained comparing the optimal function for stage k and stage $k+1$, with u_{k+1} being a varying parameter at the k^{th} stage optimisation problem. In other words, similarly to Problem (5.4), Problem (5.11) is recast as a multi-parametric program (Dua *et al.*, 2002, 2004; Pistikopoulos *et al.*, 2007*a*), with both x_{N-2} and u_{N-1} being varying parameters:

$$u_{N-2} = \mu^*_{N-2}(x_{N-2}, u_{N-1}), \tag{5.15}$$

or,

$$\begin{cases} u_{N-2} = \mathbf{K}^1_{N-2} \cdot x_{N-2} + \mathbf{H}^1_{N-2} \cdot u_{N-1} + \mathbf{C}^1_{N-2}, & x_{N-2}, u_{N-1} \in CR^1_{N-2}, \\ u_{N-2} = \mathbf{K}^2_{N-2} \cdot x_{N-2} + \mathbf{H}^2_{N-2} \cdot u_{N-1} + \mathbf{C}^2_{N-2}, & x_{N-2}, u_{N-1} \in CR^2_{N-2}, \\ \vdots \\ u_{N-2} = \mathbf{K}^{L_{N-2}}_{N-2} \cdot x_{N-2} + \mathbf{H}^{L_{N-2}}_{N-2} \cdot u_{N-1} + \mathbf{C}^{L_{N-2}}_{N-2}, & x_{N-2}, u_{N-1} \in CR^{L_{N-2}}_{N-2}, \end{cases} \tag{5.16}$$

where $\mathbf{K}^j_{N-2}$, $\mathbf{H}^j_{N-2}$ and $\mathbf{C}^j_{N-2}$ are real matrices, and $CR^j_{N-2} \subset \mathbb{R}^{n \cdot m}$. Then, the explicit optimal decision law, $u_{N-2} = \mu^*_{N-2}(x_{N-2})$, for the incumbent stage, Stage $N-1$, is computed by incorporating (5.10) in (5.16). On the other hand, the constraints are also reformulated since we have to consider the propagation of constraints along the horizon of decisions in order to get a *consistent* constraint satisfaction problem (Apt, 2003, see Appendix). Due to this fact, the final critical regions are defined as a union of the inequalities from (5.10) and (5.16), resulting in (i) realisable sets of inequalities and (ii) empty sets of inequalities. Empty sets are regions of the domain for which no feasible solution exists. Consequently, feasibility tests are performed here, after which empty regions are pruned and a compact set of regions is obtained. Note that in this way the need for global optimisation problem is circumvented, as all possible combinations of policies are checked and the ones corresponding to inconsistent paths are systematically pruned. For instance, when incorporat-

Algorithm

Step 1. (*j=1*) Solve the N^{th} stage of the problem, considering it as a multi-parametric optimisation problem, with parameters being the incumbent state-space, x_{N-1};

Step 2. ($j = j + 1$) Solve the $(N - j + 1)^{th}$ stage of the problem, considering it as multi-parametric optimisation problem, with parameters being the incumbent state-space, x_{N-j} and the future optimisation (control) variables, $u_{N-j+1}, \dots, u_{N-1}$;

Step 3. Compute the optimal control action for sample time j, comparing the two sets obtained in the steps before, $u_{N-j+1} = \mu_{N-j+1}(u_{N-j+2}, \dots, u_{N-1}, x_{N-j+1})$, (if $j = 2 \Rightarrow u_{N-1} = \mu_{N-1}(x_{N-1})$), and $u_{N-j} = f_{N-j}(u_{N-j+1}, \dots, u_{N-1}, x_{N-j})$, and compute,$u_{N-j} = \mu_{N-j}(x_{N-j})$;

Step 4. If j = N stop. Else go to Step 1.

Figure 5.2.: Dynamic programming via multi-parametric programming.

ing region i into j the feasibility test is formulated as:

$$
\begin{aligned}
&\min_{x_{N-2},u_{N-2}} \ 0, \\
s.t \ & u_{N-1} = \mathbf{K}^i_{N-1} \cdot x_{N-1} + \mathbf{C}^i_{N-1}, \\
& x_{N-1} = A \cdot x_{N-2} + B \cdot u_{N-2}, \\
& x_{N-1} \in CR^1_{N-1}, x_{N-2}, u_{N-1} \in CR^j_{N-2},
\end{aligned}
\tag{5.17}
$$

If (5.17) does not have a solution, we conclude that it defines an empty region, which is discarded with its corresponding control policy.

Figure 5.2 summarises the steps of the proposed overall algorithm.

Remark 5.1 *Within this methodology we may also assume unknown but bounded uncertainty in the data of matrices A,B of the dynamic model as follows:* $\{A = \bar{A} + \delta_1 A; -\epsilon_1|\bar{A}| \le \delta_1 A \le \epsilon_1|\bar{A}|\}$ *and* $\{B = \bar{B} + \delta_2 B; -\epsilon_2|\bar{B}| \le \delta_2 B \le \epsilon_2|\bar{B}|\}$. *Hence, a previous step is required to the algorithm in Figure 5.2. For the linear model and path constraints, the original optimisation problem is recast by introducing the following constraints, as suggested in (El-Ghaoui and Lebret, 1997; Ben-Tal and Nemirovski,* *2000; Lin* et al., *2004), in order to immunise the solution to uncertainty:*

$$x_{k+1}^{min} \leq Ax_k + Bu_k \leq x_{k+1}^{max}, \tag{5.18}$$

$$A \cdot x_k + \epsilon_1 \cdot |A| \cdot |x_k| + B \cdot u_k + \epsilon_2 \cdot |B| \cdot |u_k| \leq x_{k+1}^{max} + \delta \cdot \max[1, |x_{k+1}^{max}|], \tag{5.19}$$

$$-A \cdot x_k + \epsilon_1 \cdot |-A| \cdot |x_k| + (-B) \cdot u_k + \epsilon_2 \cdot |-B| \cdot |u_k| \leq -x_{k+1}^{min} + \delta \cdot \max[1, |x_{k+1}^{min}|]. \tag{5.20}$$

Details of the application of the proposed algorithm to robust control can be found in Chapter 6.

5.3. Illustrative example

In this section we revisit a popular optimal control problem (Pistikopoulos *et al.*, 2000; Borrelli *et al.*, 2003, 2005; Pistikopoulos *et al.*, 2007*a*):

$$\min_U J\{U, x\} = x_N' \cdot \mathbf{P} \cdot x_N + \sum_{k=0}^{N-1} [x_k' \cdot \mathbf{Q} \cdot x_k + u_k' \cdot \mathbf{R} \cdot u_k],$$

$$s.t. \quad x_{k+1} = A \cdot x_k + B \cdot u_k, \quad k = 0, 1, \ldots, N-1, \tag{5.21a}$$

$$-2 \leq u_k \leq 2, \quad k = 0, 1, \ldots, N-1, \tag{5.21b}$$

where, $x_k \in \mathbb{R}^2, u_k \in \mathbb{R}$,

$$N = 2; A = \begin{bmatrix} 0.7326 & -0.0861 \\ 0.1722 & 0.9909 \end{bmatrix}; B = \begin{bmatrix} 0.0609 \\ 0.0064 \end{bmatrix};$$

$$\mathbf{P} = \begin{bmatrix} 1.8588 & 1.2899 \\ 1.2899 & 6.7864 \end{bmatrix}; \mathbf{Q} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; \mathbf{R} = 0.01.$$

Following the steps of the methodology proposed in Figure 5.2:

Step 1. Second stage - Recast the second stage optimisation problem as a multi-parametric program with x_1 being the parameters:

$$\min_{u_1} J_N = x_2' \cdot P \cdot x_2 + x_1' \cdot Q \cdot x_1 + u_1' \cdot R \cdot u_1, \tag{5.22a}$$

$$s.t. \ \ x_2 = A \cdot x_1 + B \cdot u_1, \tag{5.22b}$$

$$-2 \leq u_1 \leq 2. \tag{5.22c}$$

A suitable multi-parametric programming algorithm (Dua *et al.*, 2002; Pistikopoulos *et al.*, 2007*a*) can be used to obtain its solution, resulting in the decision law: $u_1 = f(x_1)$, which comprises the critical regions listed in Table 5.1 and depicted in Figure 5.3. Substituting the model information, $x_{k+1} = Ax_k + Bu_k$ (for x_1), we obtain the critical regions listed in Table 5.2.

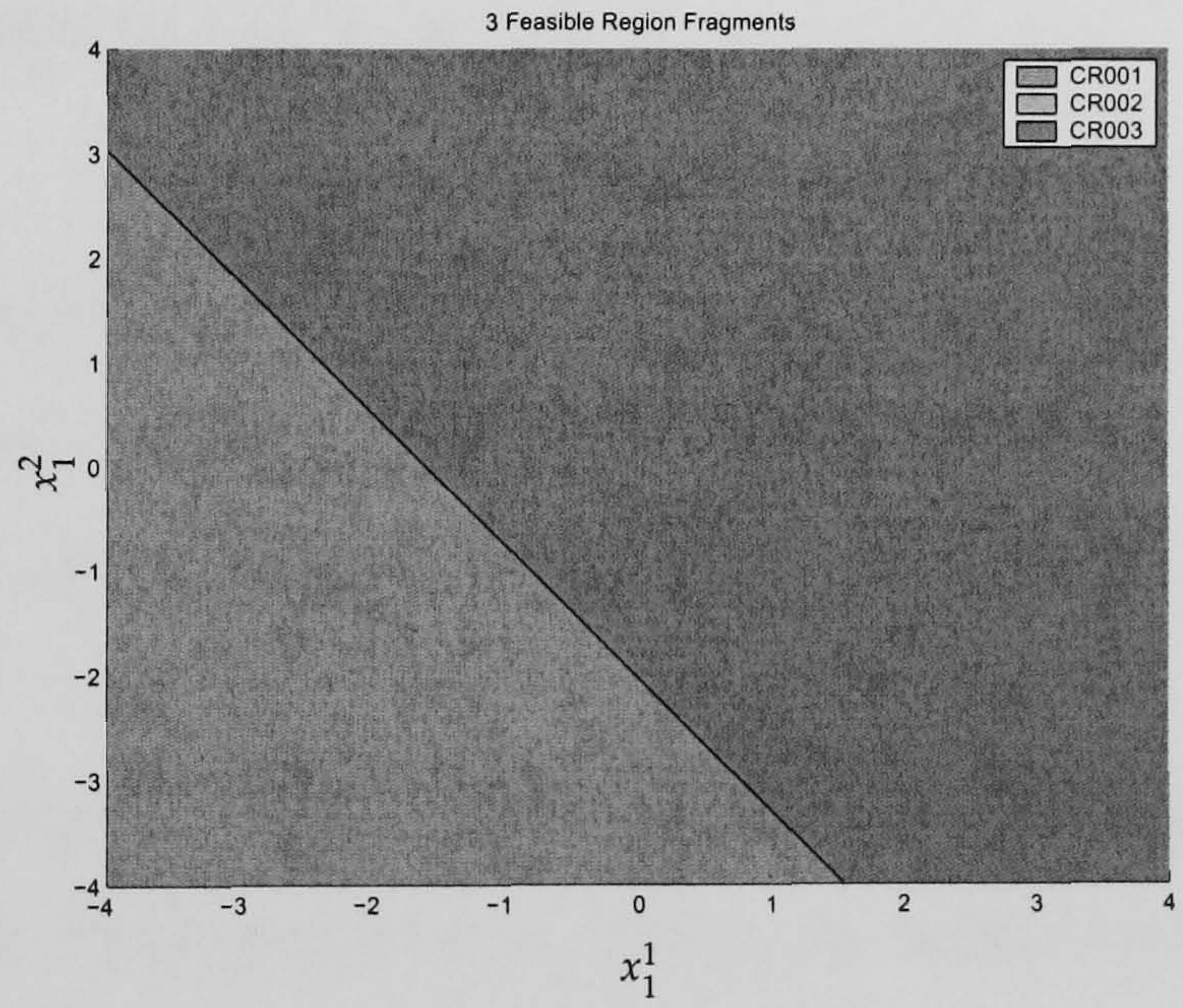


Figure 5.3.: Map of critical regions for the second stage

Step 2. First stage - Recast the first stage optimisation problem as a multi-

Table 5.1.: Explicit solution of $u_1 = f(x_1)$, *Stage* 2.

	Control action	Critical region
1	$u_1 = 2$	$+0.9961 \cdot x_1^1 + x_1^2 \leq -0.3292$
2	$u_1 = -6.051 \cdot x_1^1$ $-6.074 \cdot x_1^2$	$-0.9961 \cdot x_1^1 - x_1^2 \leq 0.3292$ $+0.9961 \cdot x_1^1 + x_1^2 \leq 0.3292$
3	$u_1 = -2$	$-0.9961 \cdot x_1^1 - x_1^2 \leq -0.3292$

Table 5.2.: Explicit solution of $u_1 = f(x_0, u_0)$, *Stage* 2.

	Control action	Critical region
1	$u_1 = 2$	$0.9019 \cdot x_0^1 + 0.9051 \cdot x_0^2 +$ $+0.0671 \cdot u_0 \leq -0.3292$
2	$u_1 = -4.5376 \cdot x_0^1$ $-0.0809 \cdot x_0^2$ $-0.3723 \cdot u_0$	$-0.90194 \cdot x_0^1 - 0.90514 \cdot x_0^2 -$ $-0.067062 \cdot u_0 \leq 0.3292$ $0.90194 \cdot x_0^1 + 0.90514 \cdot x_0^2 +$ $+0.067062 \cdot u_0 \leq 0.3292$
3	$u_1 = -2$	$-0.90194 \cdot x_0^1 - 0.90514 \cdot x_0^2 +$ $-0.067062 \cdot u_0 \leq -0.3292$

parametric program with x_0 and u_1 being the parameters:

$$\min_{u_1} J_N = x_2' \cdot P \cdot x_2 + x_1' \cdot Q \cdot x_1 + u_1' \cdot R \cdot u_1 + + u_0' \cdot R \cdot u_0, \quad (5.23a)$$

$$s.t. \; x_2 = A \cdot x_1 + B \cdot u_1, \quad (5.23b)$$

$$x_1 = A \cdot x_0 + B \cdot u_0, \quad (5.23c)$$

$$-2 \leq u_0 \leq 2. \quad (5.23d)$$

Again, using a suitable multi-parametric programming algorithm (Dua *et al.*, 2002; Pistikopoulos *et al.*, 2007*a*), we obtain the explicit decision law: $u_0 = f(x_0, u_1)$, listed in Table 5.3.

Table 5.3.: Explicit solution of $u_0 = f(x_0, u_1)$, Stage 1.

	Control action	Critical region
1	$u_0 = 2$	$0.9962 \cdot x_0^1 + x_0^2 + 0.0490 \cdot u_1 \leq -0.2805$
2	$u_0 = -7.104 \cdot x_0^1 - 7.131 x_0^2$ $-0.3494 u_1$	$-0.9962 \cdot x_0^1 - x_0^2 - 0.0490 \cdot u_1 \leq 0.2805$ $+.9962 x_0^1 + x_0^2 + 0.0490 u_1 \leq 0.2805$
3	$u_0 = -2$	$-0.9962 x_0^1 - x_0^2 - 0.0490 u_1 \leq -0.2805$

Step 3. Incorporate decisions u_1, Table 5.2, into u_0, Table 5.3, and express u_0 as function only of the incumbent state-space, x_0. After performing feasible tests in each of the 9 generated regions we obtain the results depicted in Figure 5.4 and listed in Table 5.4.

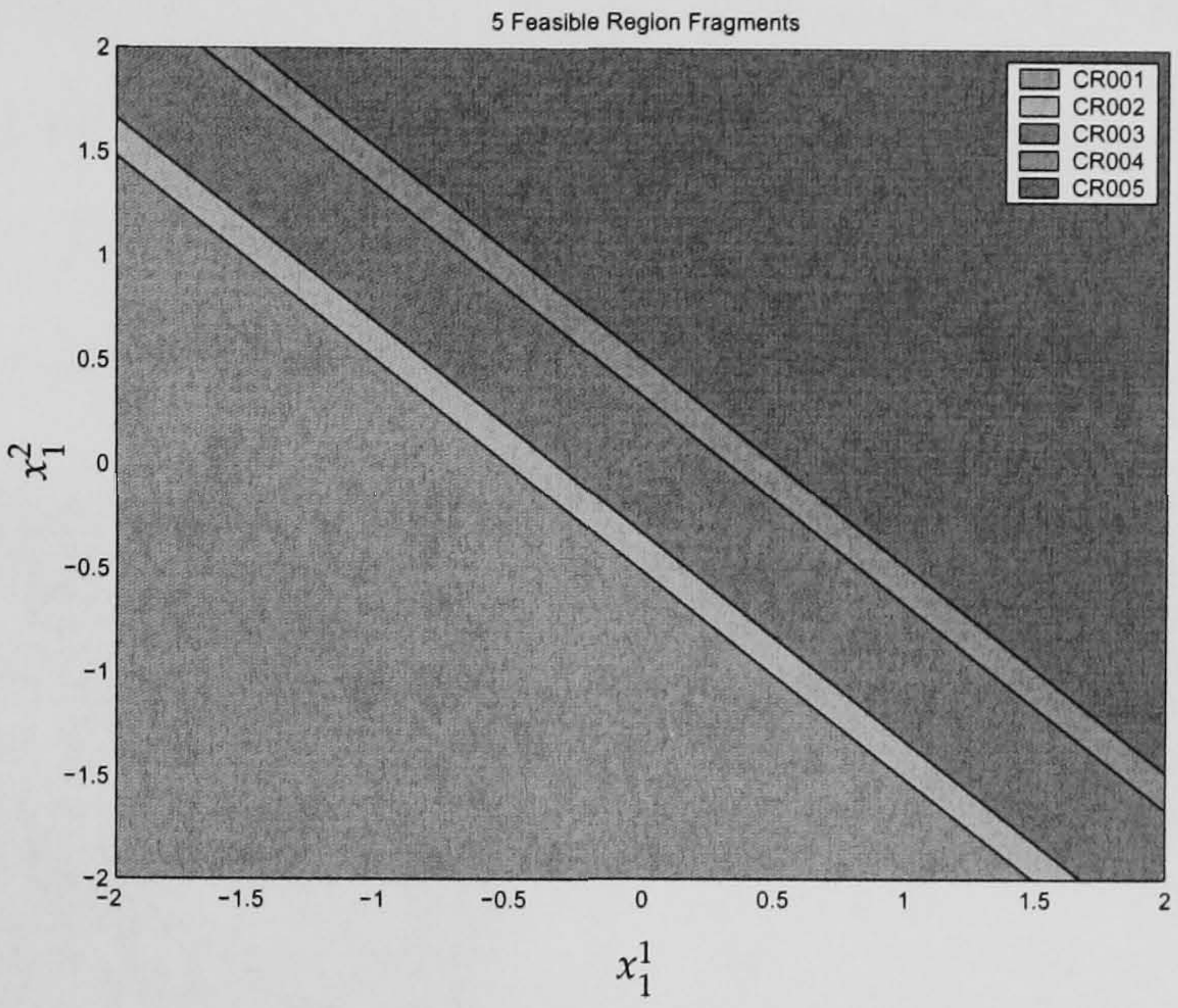


Figure 5.4.: Final map of critical regions.

Table 5.4.: Map of control policies' regions.

	Control action	Critical region
1	$u_0 = 2$	$0.9965 \cdot x_0^1 + x_0^2 \leq -0.5119$
	$u_1 = 2$	$-2 \leq x_0^1, x_0^2$
2	$u_0 = 2$	$-0.9965 \cdot x_0^1 - x_0^2 \leq 0.5119$
	$u_1 = -5.479 \cdot x_0^1$	$0.9961 \cdot x_0^1 + x_0^2 \leq -0.3292$
	$-5.498 \cdot x_0^2 - 0.8147$	$-2 \leq x_0^1, x_0^2$
3	$u_0 = -6.051 \cdot x_0^1$	$0.9961 \cdot x_0^1 + x_0^2 \leq 0.3292$
	$-6.074 \cdot x_0^2$	$-0.9961 \cdot x_0^1 - x_0^2 \leq 0.3292$
	$u_1 = -3.014 \cdot x_0^1$	$-2 \leq x_0^1, x_0^2 \leq 2$
	$-3.024 \cdot x_0^2$	
4	$u_0 = -2$	$+0.9965 \cdot x_0^1 + x_0^2 \leq 0.5119$
	$u_1 = -5.479 \cdot x_0^1$	$-0.9961 \cdot x_0^1 - x_0^2 \leq -0.3292$
	$-5.498 \cdot x_0^2 + 0.8147$	$x_0^1, x_0^2 \leq 2$
5	$u_0 = -2$	$-0.9965 \cdot x_0^1 - x_0^2 \leq -0.5119$
	$u_1 = -2$	$x_0^1, x_0^2 \leq 2$

Note that each critical region in Figure 5.4 corresponds to a different policy, however, many regions may have the same identical first-stage optimal decision, u_0. In the example above, Table 5.4, only 3 different first-stage optimal decisions were identified. The implication of this in a closed-loop control implementation strategy, where only the first-stage decisions are updated, is that a very significant reduction of the number of critical regions (control laws) can take place, by merging the adjacent regions with identical first-stage control actions.

In order to test the algorithm in different situations we have performed two independent modifications in the original illustrative example: (i) we make $N = 100$ and (ii) we introduce a path constraint in the first stage (Pistikopoulos *et al.*, 2000): $-0.5 \leq x_1^1, x_1^2$. In both cases the methodology was successfully applied, and the results are depicted in Figure 5.5 and Figure 5.6, respectively.

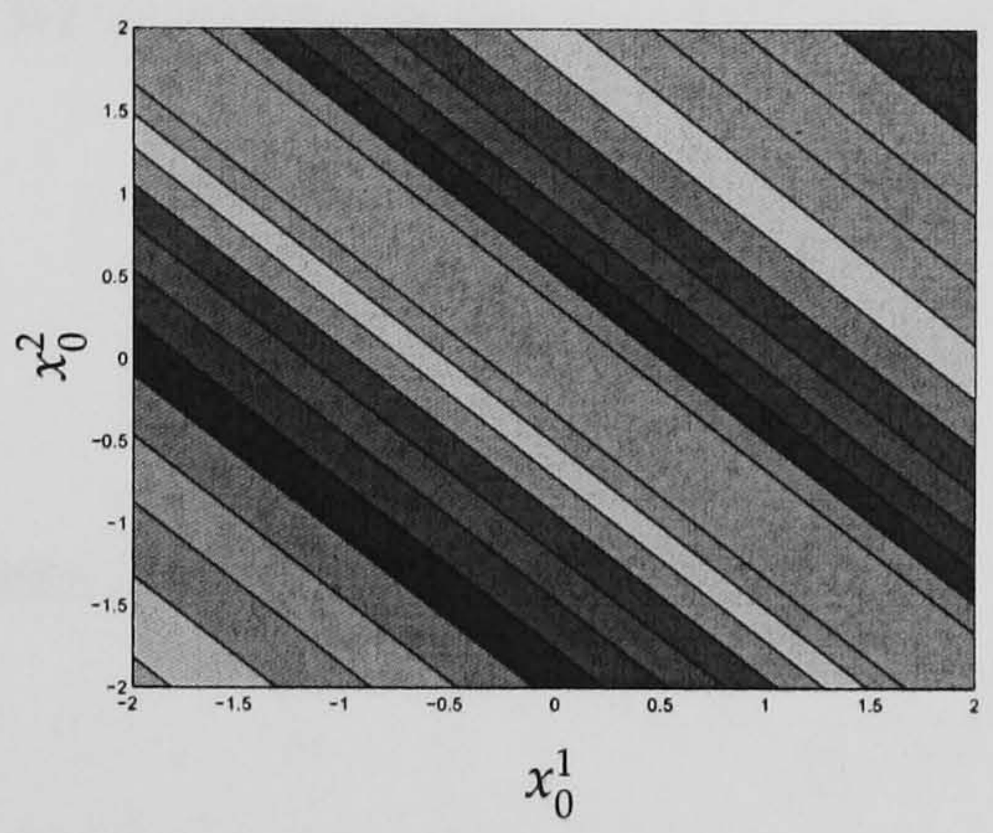


Figure 5.5.: Maps of feasible regions.

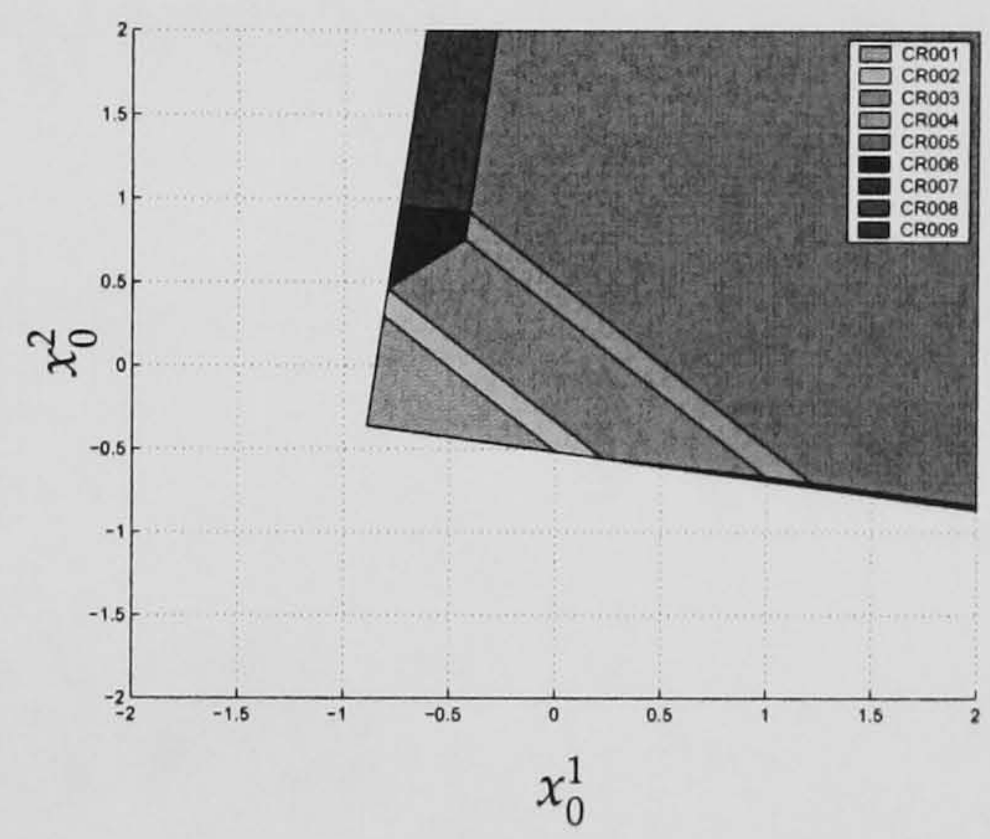


Figure 5.6.: Maps of feasible regions.

5.4. Computational complexity

In this section, an analysis of the computational complexity of the proposed algorithm is presented and comparisons are performed with the direct solution of a multi-parametric programming problem.

We assume that all input and output variables, at each step, are bounded. We first consider the solution of the multi-stage problem as a single, large scale multi-parametric program. For a problem with m optimisation variables and N stages, the total number of inequality constraints, p_{QP}, is:

$$p_{QP} = N \cdot (2 \cdot m + 2 \cdot n), \tag{5.24}$$

Table 5.5.: Complexity comparison analysis for *Example 1.*

Algorithm	mp-QP	mp-DP
Example 1	*Horizon Length, N=2*	
# mp problems	1	2
# optimisation variables	2	1
# inequalities	4	2
	Horizon Length, N=100	
# mp problems	1	100
# optimisation variables	100	1
# inequalities	200	2

The maximum number of regions, η_r, is bounded by (Dua *et al.*, 2002):

$$\eta_r \leq \sum_{k=0}^{\eta-1} k! N^k \cdot (2 \cdot m + 2 \cdot n)^k, \tag{5.25}$$

where

$$\eta \triangleq \frac{(2 \cdot m \cdot N + 2 \cdot n \cdot N)!}{(2 \cdot m \cdot N + 2 \cdot n \cdot N - i)! i!}. \tag{5.26}$$

This can result in a rapid increase in the number of regions as N increases. However, in our proposed algorithm the dependence on N is eliminated, i.e.,

$$p_{DP} = 2 \cdot m + 2 \cdot n \Rightarrow \eta_r \leq \sum_{k=0}^{\eta-1} k!(2 \cdot m + 2 \cdot n)^k, \tag{5.27}$$

where

$$\eta \triangleq \frac{(2 \cdot m + 2 \cdot n)!}{(2 \cdot m + 2 \cdot n - i)! i!}. \tag{5.28}$$

In other words, the large-scale multi-parametric program in our case is disassembled into a set of smaller problems, with the extra requirement of additional, albeit almost negligible computational cost, algebraic manipulations and feasibility tests. Note also that the computational performance of the proposed algorithm can be further improved by performing parallel computations.

Table 5.5 illustrates these points. While for small-scale problems ($N = 2$) the computational advantages are not clear, for large-scale problems ($N = 100$)

the computational savings are of several orders of magnitude.

Comparisons of the proposed algorithm to the one presented in Borrelli *et al.* (2003) are not meaningful, as their work requires solution of global optimisation problems and involves a larger number of intermediate multi-parametric programs, naturally resulting in much higher computational requirements.

5.5. Concluding remarks

Chapter 5 presents the main steps of a novel multi-parametric programming approach for the solution of general, constrained convex multi-stage problems. Through a literature example of optimal control problems, we highlighted how (i) we can use recently proposed multi-parametric programming theory and algorithms (Pistikopoulos *et al.*, 2007*a*) to efficiently address constrained dynamic programming procedures, used in the context of multi-stage formulations, and (ii) we can avoid any need for global optimisation methods by carefully posing and conducting feasibility tests, based on sensitivity analysis of the obtained parametric solutions. The work presented here establishes the foundations towards a comprehensive general theory for robust optimal control, which is described in detailed in Chapter 6.

Appendix

Definition 5.1 *Consistency of a Constraint Satisfaction Problem (CSP) (Apt, 2003): Consider a finite sequence of variables* $X := x_1, \ldots, x_n$ *with respective domains* $D_1, \ldots, D_n$*, together with a finite set* C *of constraints, each on a sub sequence of* X*. We write such CSP as* $\langle C; \mathcal{DE} \rangle$*, where* $\mathcal{DE} := x_1 \in D_1, \ldots, x_n \in D_n$ *and call each construct of the form* $x \in D$ *a domain expression. We now define the crucial notion of a solution to a CSP. Intuitively, a solution to a CSP is a sequence of legal values for all of its variables such that all its constraints are satisfied. More precisely,* *consider a CSP $\langle \mathcal{C}; \mathcal{DE} \rangle$ with $\mathcal{DE} := x_1 \in D_1, \ldots, x_n \in D_n$. We say that n-tuple $(d_1, \ldots, d_n) \in D_1 \times \ldots \times D_n$ satisfies a constraint $C \in \mathcal{C}$ on the variables $x_{i_1}, \ldots, x_{i_m}$ if $D_1 \times \ldots \times D_n$ is a solution to $\langle \mathcal{C}; \mathcal{DE} \rangle$ if it satisfies every constraint $C \in \mathcal{C}$. If a CSP has solution, we say that it is* ***consistent*** *and otherwise we say that it is* ***inconsistent***.

6. Robust optimal control of discrete linear systems

Chapter 6 describes the foundations of a novel optimisation framework for the solution of the linear quadratic regulation problem of parametric uncertain systems. Based on dynamic and multi-parametric programming techniques, the procedure recast the original problem into a robust formulation considering the worst-case variation in the system's dynamic model. Moreover, we describe how the robust formulation, which preserves the original linear-quadratic program, is solved using the multi-parametric dynamic programming algorithm for linear time-invariant systems, developed in Chapter 5. The solution steps are illustrated with the double integrator example.

6.1. Introduction

Model-based predictive control is a celebrated control strategy. Based on an optimisation formulation, model-based predictive control (MPC) handles efficiently the complexity of multi-variable systems subject to constraints, in the presence of uncertainties and/or disturbances. A typical MPC formulation includes (i) path and input constraints and (ii) the explicit mathematical model representing the dynamic behaviour of the system; it corresponds to a finite horizon open-loop constrained optimal control problem. The on-line solution of this problem is a sequence of control decisions (policy), based on the measurement of the current state (or output) and on the predictions of the future states. To ensure a closed-loop control strategy, the problem is solved recursively at each sampling time with only the first control action being implemented.

The on-line (implicit) MPC implementation strategies dominate the panorama of both industrial applications and theoretical developments. Yet, the numerical methods involved in the solution of the resulting optimisation problems have a high computational cost. Consequently, many applications are prohibitive. Recently, Pistikopoulos *et al.* (2000) and Bemporad *et al.* (2002) reported a new approach which moves the demanding computations to a off-line procedure. In this approach, the optimisation problem is regarded as a multi-parametric program where the optimal control actions are obtained as an explicit map of the system's states (parameters). The resulting closed-loop controller is known as explicit MPC control, multi-parametric control or ParOS control (ParOS, 2007).

Most of the multi-parametric control theory available is for linear time-invariant (LTI) systems with no uncertainty (Pistikopoulos *et al.*, 2000; Bemporad *et al.*, 2002; Pistikopoulos *et al.*, 2007*b*). But, relatively little attention has been given to the worst-case design of multi-parametric controllers for parametric uncertain systems, in other words, the design of controllers when we have worst-case variations in the available model (Witsenhausen, 1968*a*,*b*; Bertsekas and Rhodes, 1973). Sakizlis *et al.* (2004*c*) described a multi-parametric algorithm to address the explicit MPC problem with additive disturbances in the system's dynamic model and quadratic cost functions, minimising the expectation of the objective function over the uncertainty space or the nominal value function. On the other hand, Bemporad *et al.* (2003) described a min-max approach to address the explicit MPC problem for parametric uncertain system, such as (6.1); however, it is limited for linear cost functions only.

In this chapter, we attempt for the first time, to our knowledge, to address the challenging constrained Linear Quadratic Regulation (LQR) problem of parametric uncertain systems (6.1). The proposed approach recasts the original problem into a robust formulation, considering the worst-case of variation in the model (Ben-Tal and Nemirovski, 2000; Lin *et al.*, 2004; Janak *et al.*, 2007), which is then solved using a multi-parametric dynamic programming algorithm. These development are described in the following sections.

6.2. Problem definition

Consider the discrete time linear system:

$$x_{t+1} = f(x_t, u_t) := Ax_t + Bu_t, \tag{6.1}$$

$$x \in \mathcal{X}, \ \ \mathcal{X} := \{x \in \mathbb{R}^n \ : \ Gx \leq w\}, \tag{6.2}$$

$$u \in \mathcal{U}, \ \ \mathcal{U} := \{u \in \mathbb{R}^m \ : \ Mu \leq \mu\}, \tag{6.3}$$

here, $\mathcal{X}, \mathcal{U}$ are the sets of the state and input constraints which are assumed to be compact and non-empty polytopic sets; $G \in \mathbb{R}^{n_g \times n}$, $w \in \mathbb{R}^{n_g}$, $M \in \mathbb{R}^{m_g \times m}$ and $\mu \in \mathbb{R}^{m_g}$. And, A and B are uncertain matrices defined as:

$$A = A_0 + \Delta A \ , \ \Delta A \in \mathcal{A}, \tag{6.4}$$

$$\mathcal{A} = \{\Delta A \in \mathbb{R}^{n \times n} : -\epsilon|A_0| \leq \Delta A \leq \epsilon|A_0|\}, \tag{6.5}$$

$$B = B_0 + \Delta B \ , \ \Delta B \in \mathcal{B}, \tag{6.6}$$

$$\mathcal{B} = \{\Delta B \in \mathbb{R}^{n \times m} : -\epsilon|B_0| \leq \Delta B \leq \epsilon|B_0|\}. \tag{6.7}$$

The matrices A_0 and B_0 denote the nominal parts of A and B, respectively; in other words, A_0 and B_0 correspond to the measurements or calculations, while ΔA and ΔB denote the uncertainty in the system matrices. Note that, from (6.4)-(6.7) we conclude that the uncertainty is bounded between an upper and a lower bound defined using an auxiliary parameter ϵ, which corresponds to a percentage of the nominal values, $\epsilon \in [0, 1)$. In the limiting case, $\epsilon = 0$, (6.1) is a linear time-invariant (LTI) dynamic system, $A = A_0$ and $B = B_0$, and the MPC problem for (6.1) is formulated as follows ($\epsilon = 0$):

$$V^*(x) = \min J(x, \mathbf{U}) = \min_{\mathbf{U}} \sum_{t=0}^{N-1} \ell(x_t, u_t) + \phi(x_N), \tag{6.8}$$

$$\text{s.t. } x_{t+1} = f(x_t, u_t) := Ax_t + Bu_t, \tag{6.9}$$

$$x_t, x_{t+1} \in \mathcal{X}, \tag{6.10}$$

$$u_t \in \mathcal{U}, \tag{6.11}$$

$$x_0 = x, \tag{6.12}$$

$$\mathbf{U} = \{u_0, \ldots, u_{N-1}\} \in \mathcal{U}^N, \tag{6.13}$$

where, N is the prediction horizon, $\ell(x_t, u_t) := x_t' Q x_t + u_t' R u_t$, $\phi(x_N) := x_N' P x_N$, $Q \succeq 0$ is a positive semidefinite matrix, $R \succ 0$ is a positive definite matrix and $P \succeq 0$ is a positive semidefinite matrix such that ensures stability to the MPC controller (Rawlings, 1999; Mayne *et al.*, 2006) (usually P is obtained by solving the discrete-time Riccatti equation for (6.1)). The open-loop control problem, (6.8)–(6.13), is solved at each sampling instant, $t \geq 0$, to obtain the control policy, $\mathbf{U} = [u_0, u_1, \ldots, u_{N-1}]$, with only the first control action, u_0, being implemented. This strategy establishes a feedback control policy.

The dynamic programming recursive formula for Problem (6.8)–(6.13) is as follows:

$$V_t(x_t) = \min_{u_t \in \mathcal{U}} \sum_{i=t}^{N-1} \ell(x_i, u_i) + \phi(x_N), \tag{6.14}$$

$$\text{s.t } x_{i+1} = Ax_i + Bu_i\,,\ i = t, \ldots, N-1, \tag{6.15}$$

$$x_t \in \mathcal{X}, \tag{6.16}$$

$$u_t \in \mathcal{U}. \tag{6.17}$$

In Chapter 5, we have described an algorithm to address the class of problems defined in 6.14-6.17, however, for completeness in the presentation of the problem, we highlight the main steps.

In (6.14) the optimisation variable is the current control u_t, and only the constraints on the current control and state are considered. Furthermore, in the conventional dynamic programming approach, all stages from $i = t+1, \ldots, N-1$, are substituted by the next stage value function, $V_{t+1}(x_{t+1})$. However, given the fact that the problem at each time t is solved as a multi-parametric quadratic program, $V_{t+1}(x_{t+1})$ is a piecewise quadratic function; thus, incorporating this solution in the current optimisation problem results in a global optimisation problem that cannot be solved via the known multi-parametric methods (Borrelli *et al.*, 2005; Faísca *et al.*, 2008).

As described in Chapter 5, the multi-parametric dynamic programming approach takes advantage of the fact that the objective function in (6.14)–(6.17) is convex with respect to $x_t, u_t, u_{t+1}, \ldots, u_{N-1}$, as shown next in Lemma 6.1 (Faísca *et al.*, 2008).

Lemma 6.1 *Consider the optimisation* (6.8)–(6.13) *and assume that* x_t, $u_t^*(x_t)$, $u_{t+1}^*(x_{t+1})$, $\ldots$, $u_{N-1}^*(x_{N-1})$ *is the solution to the multi–stage problem. It holds that,*

$$\begin{aligned} & V_t(x_t, u_t^*(x_t), u_{t+1}^*(x_{t+1}), \ldots, u_{N-1}^*(x_{N-1})) \\ & \leq V_t(x_t, u_t(x_t), u_{t+1}^*(x_{t+1}), \ldots, u_{N-1}^*(x_{N-1})) \end{aligned} \tag{6.18}$$

$$\begin{aligned} & V_{t+1}(\overbrace{x_t, u_t^*(x_t)}^{x_{t+1}}, u_{t+1}^*(x_{t+1}), \ldots, u_{N-1}^*(x_{N-1})) \\ & \leq V_{t+1}(\underbrace{x_t, u_t^*(x_t)}_{x_{t+1}}, u_{t+1}(x_{t+1}), \ldots, u_{N-1}^*(x_{N-1})) \end{aligned} \tag{6.19}$$

where $V_t(\cdot)$ *and* $V_{t+1}(x_{t+1})$ *are the objective functions of* (6.14) *at times* t *and* $t+1$.

Proof 6.1 *The proof is straightforward and is a result of the convexity of* $V_t(x_t)$ *and* $V_{t+1}(x_{t+1})$ *(see Faísca* et al. *(2008)). Note also that* $V_{t+1}(\cdot)$ *is a function of* x_{t+1}, $u_{t+1}^*(x_{t+1})$, $u_{t+2}^*(x_{t+2})$, $\ldots$, $u_{N-1}^*(x_{N-1})$, *and however* x_{t+1} *is a function of* x_t, u_t *so is* $V_{t+1}(\cdot)$.

The lemma above states that if a feasible solution to (6.14) exists, the solution is unique. This implies that the solution for the multi-stage problem (6.14)-(6.17) is also unique. The main idea behind this approach is to solve (6.14)-(6.17) as a mp-QP problem, where:

- u_t is the optimisation variable;
- $x_t, u_{t+1}, u_{t+2}, \ldots, u_{N-1}$ are the parameters;
- the state constraints apply only on the current state, x_t;
- the input constraints apply only on the current control input u_t.

Replacing $x_i = A^{i-t}x_t + \sum_{j=0}^{i-t-1} A^j B u_{i-1-j}$, for all $t+1 \leq i \leq N$, in the objective function (6.14), $x_{t+1} = Ax_t + Bu_t$ in the state constraints (6.16) and $\mathcal{X}$, $\mathcal{U}$ by (6.2) and (6.3) respectively, the optimal control problem (6.14)–(6.17) can be re-written as a mp-QP problem:

$$V_t(x_t) = \min_{u_t \in \mathcal{U}} \left\{ \frac{1}{2} u_t' H u_t + \theta(t)' F u_t \right\} + \frac{1}{2} \theta(t)' Y \theta(t), \tag{6.20}$$

$$\text{s.t. } GAx_t + GBu_t \leq w, \tag{6.21}$$

$$Mu_t \leq \mu, \tag{6.22}$$

where, $\theta(t)$ is the vector of parameters, $\theta(t) = \left[x_t' \; u_{t+1}' \; \cdots \; u_{N-1}' \right]$, and u_t is the optimisation variable (current control input). Problem (6.20)-(6.22) can be re-written in a more compact mathematical form, as follows:

$$V_t(x_t) = \min_{u_t \in \mathcal{U}} \left\{ \frac{1}{2} u_t' H u_t + \theta(t)' F u_t \right\}, \tag{6.23}$$

$$\text{s.t. } \bar{G} u_t \leq \bar{W} + \bar{E} \theta(t), \tag{6.24}$$

where,

$$\bar{G} = \begin{bmatrix} GB \\ M \end{bmatrix}, \; \bar{W} = \begin{bmatrix} w \\ \mu \end{bmatrix} \text{ and } \bar{E} = \begin{bmatrix} GA & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \end{bmatrix}.$$

The term $1/2\theta(t)' Y \theta(t)$ is neglected from the objective function since it does not influence the minimisation problem. Problem (6.23)–(6.24) is a multi-

Algorithm
Step 1. (j=1) Solve the N^{th} stage of the problem, considering it as a multi-parametric optimisation problem, with parameters being the incumbent state-space, x_{N-1};
Step 2. ($j = j + 1$) Solve the $(N - j + 1)^{th}$ stage of the problem, considering it as multi-parametric optimisation problem, with parameters being the incumbent state-space, x_{N-j} and the future optimisation (control) variables, $u_{N-j+1}, \ldots, u_{N-1}$;
Step 3. Compute the optimal control action for sample time j, comparing the two sets obtained in the steps before, $u_{N-j+1} = \mu_{N-j+1}(u_{N-j+2}, \ldots, u_{N-1}, x_{N-j+1})$,
(if $j = 2 \Rightarrow u_{N-1} = \mu_{N-1}(x_{N-1})$), and $u_{N-j} = f_{N-j}(u_{N-j+1}, \ldots, u_{N-1}, x_{N-j})$, and compute,$u_{N-j} = \mu_{N-j}(x_{N-j})$;
Step 4. If j = N stop. Else go to Step 1.

Figure 6.1.: Dynamic programming via multi-parametric programming.

parametric quadratic programming problem with u_t being the vector of optimisation variables and $\theta(t)$ being the vector of parameter, and thus, can be addressed using a mp-QP algorithm (Pistikopoulos *et al.*, 2000, 2007*a*).

Remark 6.1 *Note that* (6.23)-(6.24) *is the re-formulation of the* (6.14)-(6.17) *at time t. At other times $t + j$ the formulation is similar where the optimisation variable is the control u_{t+j} at that time, but the parameter is the vector $\theta(t + j) = \begin{bmatrix} x'_{t+j} & u'_{t+j+1} & \cdots & u_{N-1} \end{bmatrix}$, which has smaller dimension than $\theta(t)$.*

Remark 6.2 *The multi-parametric program* (6.23)–(6.24) *is solved at all times $t = 0, \ldots, N - 1$, resulting each time in a multi-parametric quadratic program in which the optimisation variable has the same dimensions but the parameter vector's $\theta(t)$ dimension decreases as t increases (Remark 6.1).*

The solution steps of the dynamic programming algorithm for constrained multi-stage optimisation problems is depicted in Figure 6.1 (as in Chapter 5).

6.3. Robust multi-parametric control

In this section we focus on the case: $\epsilon \geq 0$. This is the case when system matrices A,B are uncertain, (6.4)-(6.7). If (6.8) were defined using 1-norm or

∞-norm costs (linear cost functions), Problem (6.23)–(6.24) could be reformulated as min-max problems to account for the influence of the uncertainty in the optimal solution and guarantee satisfaction of the constraints for all admissible values of the disturbance (Bemporad *et al.*, 2003). Obviously, the effectiveness of the method is due to the linearity of the objective function, since the maximisation term of the min-max problem can be substituted as an extra linear constraint in the optimisation problem, and hence, resulting in a multi-parametric linear program. Obviously, in the presence of quadratic cost functions, this procedure is not applicable since the substitution of the maximisation term in the min-max formulation by a quadratic constraint, results in a nonlinear problem for which global optimisation methods are required.

Here, we propose a new procedure to solve the linear quadratic regulation problem of parametric uncertain systems, considering the worst-case variation in the system dynamics. We assume that (i) the cost function, (6.8), is formulated only for the nominal system, i.e. the effect of the uncertainty in A and B is not considered in the cost function; and, (ii) the constraints are reformulated taking into account the uncertainty in the system dynamics, to ensure that none of the constraints are violated for all possible values of the uncertainty.

In order to re-formulate (6.23)–(6.24) to account for the presence of the uncertainties, we proceed as follows:

1. The objective function (6.23) is formulated only including the nominal system dynamics ($A = A_0, B = B_0$),

$$V_t(x_t) = \min_{u_t \in \mathcal{U}} \left\{ \frac{1}{2} u_t' H u_t + \theta(t)' F u_t \right\}, \tag{6.25}$$

   where $H = H' \succ 0, F$ are obtained from Q, R, A_0, B_0 after substituting $x_i = A_0^{i-t} x_t + \sum_{j=0}^{i-t-1} A_0^j B_0 u_{i-1-j}$ in (6.14);

2. Since the effect of the uncertainty on the constraints has to be considered in the constraints, (6.24) has to be reformulated to account for all admissible values of the uncertainty. Since the uncertain matrices appear only in the constraint (6.21), this constraint is re-written as,

$$GAx_t + GBu_t \leq w, \tag{6.26}$$
$$A = A_0 + \Delta A, B = B_0 + \Delta B, \;\; \forall\, (\Delta A, \Delta B) \in (\mathcal{A}, \mathcal{B}),$$

or,

$$GA_0x_t + G\Delta Ax_t + GB_0u_t + G\Delta Bu_t \leq w, \;\; \forall\, (\Delta A, \Delta B) \in (\mathcal{A}, \mathcal{B}). \tag{6.27}$$

Hence, the optimal control problem (6.23)–(6.24) becomes a robust optimisation problem,

$$V_t(x_t) = \min_{u_t \in \mathcal{U}} \left\{ \frac{1}{2} u_t' H u_t + \theta(t)' F u_t \right\}, \tag{6.28}$$
$$\text{s.t.} \;\; GA_0x_t + G\Delta Ax_t + GB_0u_t + G\Delta Bu_t \leq w,$$
$$\forall\, (\Delta A, \Delta B) \in (\mathcal{A}, \mathcal{B}), \tag{6.29}$$
$$Mu_t \leq \mu, \tag{6.30}$$

in which, the coefficients of the linear inequality (6.29) (or some of the coefficients) are uncertain. Since the cost function is strictly convex ($H = H' \succ 0$), (6.28)–(6.30) is a *Robust Multi-parametric Quadratic Program* (Robust mp-QP) (Pistikopoulos *et al.*, 2007*b*), where u_t is the optimisation variable and $\theta(t)$ is the vector of parameters.

Remark 6.3 *The problem of robust solution of Linear Programming Problems with a linear objective function and linear inequalities with uncertain coefficients (Robust Linear Programming) has been studied in (Ben-Tal and Nemirovski, 2000), where no parameters were included in the optimisation problem. An initial study on robust* *solution of multi–parametric optimisation problems is given in (Pistikopoulos* et al., *2007b).*

Remark 6.4 *The main idea of the methodology described in (Ben-Tal and Nemirovski, 2000; Lin* et al., *2004; Pistikopoulos* et al., *2007b) is based on deriving a* reliable *(or feasible) solution to* (6.28)–(6.30), *i.e., a solution that satisfies the constraints for all* $(\Delta A, \Delta B) \in (\mathcal{A}, \mathcal{B})$. *This requires the re-formulation of the robust optimisation problem into its interval robust counterpart (IRC), for which the constraints consider the worst-case uncertainty realisation.*

The following definition of *reliable solution* for (6.28)–(6.30) is necessary before proceeding.

Definition 6.1 *A solution* $u_t(\theta(t))$ *to the Robust mp-QP problem* (6.28)–(6.30) *is called reliable if it is feasible (i.e. satisfies the constraints* (6.29) *and* (6.30)*) both for the nominal problem* ($A = A_0, B = B_0$) *and the uncertain problem.*

Our objective is to obtain a reliable or robust solution for (6.28)-(6.30). We follow a methodology similar to the one described in (Ben-Tal and Nemirovski, 2000; Lin *et al.*, 2004; Janak *et al.*, 2007). It is clear from Definition 6.1 that a solution to (6.28)-(6.30) is reliable if the constraints are satisfied for all admissible values of $(\Delta A, \Delta B)$, i.e. for all possible values of the uncertain coefficients in (6.29). It is also clear that the uncertain coefficients in (6.29) are the coefficients of the matrices $G\Delta A$ and $G\Delta B$. Although the coefficients of $G\Delta A, G\Delta B$ are uncertain, it is easy to obtain their bounds; since $-\epsilon|A_0| \leq \Delta A \leq \epsilon|A_0|$ and $-\epsilon|B_0| \leq \Delta B \leq \epsilon|B_0|$, it follows that:

$$-\epsilon|G||A_0| \leq G\Delta A \leq \epsilon|G||A_0|, \tag{6.31}$$

$$-\epsilon|G||B_0| \leq G\Delta B \leq \epsilon|G||B_0|. \tag{6.32}$$

Take for example the ij-th coefficient $(G\Delta A)_{ij}$ of matrix $G\Delta A$ given by:

$$(G\Delta A)_{ij} = \sum_{k=1}^{n} g_{ik}\delta A_{kj},$$

where g_{ik} is the ik-th entry of G and δA_{kj} is the kj-th entry of ΔA. Then, since g_{ik} is fixed and $-\epsilon|A_{0kj}| \leq |\delta A_{kj}| \leq \epsilon|A_{0kj}|$, where A_{0kj} is the kj-th entry of A_0, it follows that the lower bound of $(G\Delta A)_{ij}$ is as follows:

$$(G\Delta A)_{ij} = \sum_{k=1}^{n} g_{ik}\delta A_{kj} \geq -\epsilon \sum_{k=1}^{n} |g_{ik}||A_{0kj}|$$

Following a similar procedure for all the upper and lower bounds of $G\Delta A$ and $G\Delta B$ we obtain (6.31) and (6.32). For u_t^* to be a reliable solution for (6.28)–(6.30), it has to satisfy all constraints for all admissible $(\Delta A, \Delta B)$, especially (6.29) where the uncertain coefficients of $G\Delta A$ and $G\Delta B$ appear. Hence, a reliable solution u_t will have to satisfy each of the constraints in (6.29) even for the worst case realisation of the uncertain coefficients of $G\Delta A$ and $G\Delta B$. Then, the worst-case realisation of the constraints (6.29) is given by:

$$GA_0x_t + \epsilon|G||A_0||x_t| + GB_0u_t + \epsilon|G||B_0||u_t| \leq w. \tag{6.33}$$

Problem (6.28)-(6.30) can then be re-written as:

$$V_t(x_t) = \min_{u_t \in \mathcal{U}} \left\{ \frac{1}{2}u_t'Hu_t + \theta(t)'Fu_t \right\}, \tag{6.34}$$

$$\text{s.t. } GA_0x_t + \epsilon|G||A_0||x_t|$$
$$+ GB_0u_t + \epsilon|G||B_0||u_t| \leq w, \tag{6.35}$$
$$Mu_t \leq \mu. \tag{6.36}$$

An optimal solution to (6.34)-(6.36), if it exists, is a reliable solution to (6.28)–(6.30).

Lemma 6.2 *An optimal solution for* (6.34)–(6.36) *is a reliable solution for* (6.28)–

(6.30).

Proof 6.2 *An optimal solution (also a feasible solution)* $u_t^*(x_t)$ *to* (6.34)–(6.36) *will satisfy the constraint* (6.35)–(6.36) *and hence the constraints* (6.29) *for all admissible values of* $(G\Delta A, G\Delta B)$ *and* (6.30). *Therefore, it is a reliable solution to* (6.28)–(6.30).

It is obvious that solving problem (6.34)–(6.36) we can obtain a reliable solution to our initial robust optimisation problem (6.28)–(6.30). Furthermore, (6.34)–(6.36) can be recast as a multi-parametric programming problem where u_t and $\theta(t)$ are again the optimisation variables and the parameters, respectively. However, the constraints (6.35) are not any more linear due to the presence of $|u_t|$ and $|x_t|$ and hence the known multi-parametric programming methods cannot be applied to solve explicitly (6.34)–(6.36). Nevertheless, we overcome this problem by introducing the following relaxation of the constraints (6.35) (Ben-Tal and Nemirovski, 2000; Kouramas *et al.*, 2008*b*; Pistikopoulos *et al.*, 2007*b*):

$$V_t(x_t) = \min_{u_t \in \mathcal{U}} \left\{ \frac{1}{2} u_t' H u_t + \theta(t)' F u_t \right\}, \tag{6.37}$$

$$\text{s.t. } GA_0 x_t + \epsilon |G||A_0| w_t$$

$$+ GB_0 u_t + \epsilon |G||B_0| z_t \leq w, \tag{6.38}$$

$$- z_t \leq u_t \leq z_t, \tag{6.39}$$

$$- w_t \leq x_t \leq w_t, \tag{6.40}$$

$$M u_t \leq \mu, \tag{6.41}$$

or,

$$V_t(x_t) = \min_{u_t \in \mathcal{U}} \left\{ \frac{1}{2} u_t' H u_t + \theta(t)' F u_t \right\}, \tag{6.42}$$

$$\text{s.t. } GBu_t + \epsilon|G||B_0|z_t + \epsilon|G||A_0|w_t \leq w - GA_0 x_t, \tag{6.43}$$

$$-z_t \leq u_t \leq z_t, \tag{6.44}$$

$$-w_t \leq x_t \leq w_t, \tag{6.45}$$

$$Mu_t \leq \mu. \tag{6.46}$$

Problem (6.42)-(6.46) is the *Interval Robust Counterpart* (IRC) of the uncertain problem (6.28)-(6.30) (Ben-Tal and Nemirovski, 2000).

In this new formulation two variables z_t, w_t and four new inequalities (6.39)-(6.40) have been added to transform the nonlinear inequalities, (6.35), into linear inequalities, (6.43)-(6.45). The optimisation variables in (6.37)-(6.41) are u_t, z_t, w_t and the parameters are $\theta(t)$. Hence, in (6.37)-(6.41) we introduce two extra optimisation variables and four new inequalities, whereas the number of parameters remains the same.

Lemma 6.3 *An optimal solution for* (6.37)-(6.41) *is also a reliable solution for* (6.28)-(6.30).

Proof 6.3 *The proof is simple and is based on the fact that an optimal solution* (6.37)-(6.41) *is also a feasible solution for* (6.34)-(6.36) *and hence a reliable solution for* (6.28)-(6.30).

Therefore, solving (6.37)-(6.41), an optimal control action can be obtained to ensure that all admissible values of the uncertainty satisfy the constraints, at time t. Hence, (6.37)-(6.41) are recast as a mp-QP problem, where u_t, z_t, w_t are the optimisation variables and $\theta(t)$ is the vector of the parameters. However, the objective function is not anymore strictly convex (although it remains convex) with respect to the optimisation variables. The method can be further refined by adding an extra term in the objective function, (6.37), which is strictly convex with respect to z_t, w_t:

$$V_t(x_t) = \min_{u_t \in \mathcal{U}} \left\{ \frac{1}{2} u_t' H u_t + \theta(t)' F u_t + h(z_t, w_t) \right\}, \tag{6.47}$$

$$\text{s.t. } GAx_t + \epsilon |G||A_0| w_t$$

$$+ GBu_t + \epsilon |G||B_0| z_t \leq w, \tag{6.48}$$

$$-z_t \leq u_t \leq z_t, \tag{6.49}$$

$$-w_t \leq x_t \leq w_t, \tag{6.50}$$

$$Mu_t \leq \mu. \tag{6.51}$$

where $h(z_t, w_t)$ is a strictly convex function of z_t, w_t.

Remark 6.5 *There are many candidate functions $h(z_t, w_t)$ that are strictly convex with respect to z_t, w_t. A simple and obvious choice is the quadratic function*

$$h(z, w) := z' Q_z z + w' Q_w w,$$

where $Q_z = Q_z' \succ 0$ and $Q_w = Q_w' \succ 0$.

6.3.1. The case of polytopic parametric model uncertainty

A different way to describe the uncertainty in system (6.1) is by the polytopic parametric uncertain model (Boyd *et al.*, 1994):

$$\Delta A \in \mathrm{Co}\{A_i\},\ \Delta B \in \mathrm{Co}\{B_i\},\ i = 1, \ldots, s, \tag{6.52}$$

where, $\mathrm{Co}\{\cdot\}$ denotes the convex hull of its entries, $A_i \in \mathbb{R}^{n \times n}$ and $B_i \in \mathbb{R}^{n \times m}$, $i = 1, \ldots, s$. In this way it is assumed that matrices A and B are given as linear combinations of A_i and B_i, respectively:

$$A = A_0 + \sum_{i=1}^{s} \lambda_i A_i,\ B = B_0 + \sum_{i=1}^{s} \lambda_i B_i, \tag{6.53}$$
$$\sum_{i=1}^{s} \lambda_i = 1,\ \lambda_i \geq 0.$$

The above description of $(\Delta A, \Delta B)$ is more general than the one considered in (6.1)–(6.7) and can be used to describe a wider class of linear uncertain systems (Boyd *et al.*, 1994).

The procedure described in the previous section can be extended and (6.28)–(6.30) can be re–formulated to accommodate this type of uncertain systems. Consider the constraint (6.29) in which the uncertain matrices $(\Delta A, \Delta B)$ appear. Then, this constraint is always satisfied for all admissible values of $(\Delta A, \Delta B)$ if

$$GA_0 x_t + GA_i x_t + GB_0 u_t + GB_i u_t \leq w,\ i = 1, \ldots, s. \tag{6.54}$$

This is a consequence of the convexity of the constraints in (6.28)–(6.30) and of the convex description of A, B in (6.53).

The robust optimisation problem (6.28)-(6.30) can then be re-formulated as follows:

$$V_t(x_t) = \min_{u_t \in \mathcal{U}} \left\{ \frac{1}{2} u_t' H u_t + \theta(t)' F u_t \right\}, \tag{6.55}$$
$$\text{s.t. } GA_0 x_t + GA_i x_t + GB_0 u_t + GB_i u_t \leq w, \tag{6.56}$$
$$i = 1, \ldots, s,$$
$$Mu_t \leq \mu, \tag{6.57}$$

which is re-written as,

$$V_t(x_t) = \min_{u_t \in \mathcal{U}} \left\{ \frac{1}{2} u_t' H u_t + \theta(t)' F u_t \right\}, \tag{6.58}$$

$$\text{s.t. } GB_0 u_t + GB_i u_t \leq w - GA_0 x_t - GA_i x_t, \tag{6.59}$$

$$i = 1, \ldots, s,$$

$$M u_t \leq \mu. \tag{6.60}$$

Problem (6.58)-(6.60) is a mp-QP problem with the same optimisation variables, u_t, and same parameters, $\theta(t)$, as in (6.28)-(6.30). A set of s extra constraints are included corrresponding to each different realisation A_i, B_i of the uncertain matrices $(\Delta A, \Delta B)$. It is obvious that if a feasible solution of (6.58)-(6.60) exists, u_t, then this is also a reliable solution for (6.28)-(6.30), due to the convexity of A, B and to the fact that (6.59) is satisfied. It follows:

$$\begin{aligned} & GAx_t + GBu_t = \\ & GA_0 x_t + G \sum_{i=1}^{s} \lambda_i A_i x_t + GB_0 u_t + G \sum_{i=1}^{s} \lambda_i B_i u_t = \\ & \sum_{i=1}^{s} \lambda_i (GA_0 x_t + GA_i x_t + GB_0 u_t + GB_i u_t) \leq \sum_{i=1}^{s} \lambda_i w = w. \end{aligned}$$

6.3.2. Deriving a robust optimal control policy

It is obvious that both problems (6.42)-(6.46) and (6.58)-(6.60) are mp-QPs similar to the one described in (6.23)–(6.24). Therefore, a similar strategy to the one depicted in Figure 6.1, is used to compute robust control policies for parametric uncertain systems, Figure 6.2.

6.4. Illustrative example

The main steps of our approach are summarised in Figure 6.2. Here, we will illustrate in detail how the algorithm can be applied in the context of robust

Algorithm
Step 0. (j=1) Reformulate the original problem, introducing the constraints (6.48)-(6.50);
Step 1. (j=1) Solve the N^{th} stage of the problem, considering it as a multi-parametric optimisation problem, with parameters being the incumbent state-space, x_{N-1};
Step 2. ($j = j + 1$) Solve the $(N - j + 1)^{th}$ stage of the problem, considering it as multi-parametric optimisation problem, with parameters being the incumbent state-space, x_{N-j} and the future optimisation (control) variables, $u_{N-j+1}, \ldots, u_{N-1}$;
Step 3. Compute the optimal control action for sample time j, comparing the two sets obtained in the steps before, $u_{N-j+1} = \mu_{N-j+1}(u_{N-j+2}, \ldots, u_{N-1}, x_{N-j+1})$,
(if $j = 2 \Rightarrow u_{N-1} = \mu_{N-1}(x_{N-1})$), and $u_{N-j} = f_{N-j}(u_{N-j+1}, \ldots, u_{N-1}, x_{N-j})$, and compute, $u_{N-j} = \mu_{N-j}(x_{N-j})$;
Step 4. If j = N stop. Else go to Step 1.

Figure 6.2.: Robust multi-parametric programming.

optimal control, by revisiting a popular control example problem (Pistikopoulos *et al.*, 2000; Borrelli *et al.*, 2005):

$$\min_{U} J\{U, x\} = x_N' \cdot \mathbf{P} \cdot x_N + \sum_{k=0}^{N-1} [x_k' \cdot \mathbf{Q} \cdot x_k + u_k' \cdot \mathbf{R} \cdot u_k], \tag{6.61a}$$

$$s.t. \;\; x_{k+1} = A \cdot x_k + B \cdot u_k, \tag{6.61b}$$

$$-1 \le u_k \le 1, \qquad k = 0, 1, \ldots, N-1, \tag{6.61c}$$

$$-10 \le x_k \le 10, \qquad k = 0, 1, \ldots, N, \tag{6.61d}$$

where, $x_k \in \mathbb{R}^2$, $u_k \in \mathbb{R}$,

$$N = 3; A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}; B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}; P = \begin{bmatrix} 2.6005 & 2.0810 \\ 2.0810 & 3.3306 \end{bmatrix}; Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}; R = 1.$$

The solution steps are as follows.

Step 0. For the linear model, (6.61b), and path constraints, (6.61d), the following constraints are introduced to obtain a solution immune to uncertainty:

$$x_{k+1}^{min} \leq Ax_k + Bu_k \leq x_{k+1}^{max}, \tag{6.62}$$

$$A \cdot x_k + \epsilon_1 \cdot |A| \cdot |x_k| + B \cdot u_k + \\ +\epsilon_2 \cdot |B| \cdot |u_k| \leq x_{k+1}^{max}, \tag{6.63}$$

$$-A \cdot x_k + \epsilon_1 \cdot |-A| \cdot |x_k| + (-B) \cdot u_k + \\ +\epsilon_2 \cdot |-B| \cdot |u_k| \leq -x_{k+1}^{min}, \tag{6.64}$$

resulting in the following robust optimal control formulation:

$$\min_{U} J\{U, x\} = x_N' \cdot \mathbf{P} \cdot x_N + \sum_{k=0}^{N-1} [x_k' \cdot \mathbf{Q} \cdot x_k + u_k' \cdot \mathbf{R} \cdot u_k], \tag{6.65a}$$

$$s.t.\ x_{k+1} = A \cdot x_k + B \cdot u_k, \tag{6.65b}$$

$$A \cdot x_k + \epsilon_1 \cdot |A| \cdot y_k + B \cdot u_k + \epsilon_2 \cdot |B| \cdot \omega_k \leq 10, \tag{6.65c}$$

$$-A \cdot x_k + \epsilon_1 \cdot |-A| \cdot y_k + (-B) \cdot u_k + \epsilon_2 \cdot |-B| \cdot \omega_k \leq 10, \tag{6.65d}$$

$$-y_k \leq x_k \leq y_k, \tag{6.65e}$$

$$-\omega_k \leq u_k \leq \omega_k, \tag{6.65f}$$

$$-1 \leq u_k \leq 1, \qquad k = 0, 1, \ldots, N-1, \tag{6.65g}$$

$$-10 \leq x_k \leq 10, \qquad k = 0, 1, \ldots, N. \tag{6.65h}$$

Step 1. Third stage - Recast the third stage optimisation problem as a multi-parametric program with x_2 being the parameters:

$$\min_{u_2,\omega_2,y_2} J = x_3' \cdot \mathbf{P} \cdot x_3 + u_2' \cdot \mathbf{R} \cdot u_2, \tag{6.66a}$$

$$s.t.\ x_3 = A \cdot x_2 + B \cdot u_2, \tag{6.66b}$$

$$+ A \cdot x_2 + \epsilon_1 \cdot |A| \cdot y_2 + B \cdot u_2 + \epsilon_2 \cdot |B| \cdot \omega_2 \leq 10, \tag{6.66c}$$

$$- A \cdot x_2 + \epsilon_1 \cdot | - A| \cdot y_2 + (-B) \cdot u_2 + \epsilon_2 \cdot | - B| \cdot \omega_2 \leq 10, \tag{6.66d}$$

$$- y_2 \leq x_2 \leq y_2, -\omega_2 \leq u_2 \leq \omega_2, -1 \leq u_2 \leq 1, -10 \leq x_3 \leq 10. \tag{6.66e}$$

A suitable multi-parametric programming algorithm Dua *et al.* (2002) can be used to obtain its solution, resulting in the decision law: $(u_2, \omega_2, y_2) = f(x_2)$, which comprises 12 critical regions;

Step 2. Incorporate the model information, $x_{k+1} = Ax_k + Bu_k$ (for x_2), in each critical region. For instance, in critical region #8:

Critical region #8, $f(x_2)$:	$\Rightarrow$	**Critical region #8, $f(x_1, u_1)$:**
$-0.385x_2^1 - x_2^2 \leq 0.800,$		$-0.385x_1^1 - 1.385x_1^2 - u_1 \leq 0.800,$
$x_2^1 + 0.980x_2^2 \leq 9.90,$		$x_1^1 + 1.980x_1^2 + 0.980u_1 \leq 9.90,$
$0.385x_2^1 + x_2^2 \leq 0,$		$0.385x_1^1 + 1.385x_1^2 + u_1 \leq 0,$
$-x_2^1 \leq 0,$		$-x_1^1 - x_1^2 \leq 0,$
Optimal decision law $u_2 = f(x_2)$	$\Rightarrow$	*Optimal decision law* $u_2 = f(x_1, u_1)$
$u_2 = -0.481x_2^1 - 1.25x_2^2,$		$u_2 = -0.481x_1^1 - 1.73x_1^2 - 1.25u_1,$ (6.67)
$\omega_2 = -0.481x_2^1 - 1.25x_2^2,$		$\omega_2 = -0.481x_1^1 - 1.73x_1^2 - 1.25u_1,$ (6.68)
$y_2^1 = x_2^1,$		$y_2^1 = x_1^1 + x_1^2,$ (6.69)
$y_2^2 = -x_2^2,$		$y_2^2 = -x_1^2 - u_1;$ (6.70)

Step 3. Second stage - Recast the second stage optimisation problem as a multi-parametric programming problem, with x_1 and u_2 being the pa-

rameters:

$$\min_{u_1,\omega_1,y_1} J = x_3' \cdot \mathbf{P} \cdot x_3 + u_2' \cdot \mathbf{R} \cdot u_2 + x_2' \cdot \mathbf{Q} \cdot x_2 + u_1' \cdot \mathbf{R} \cdot u_1, \tag{6.71a}$$

$$s.t.\ \ x_2 = A \cdot x_1 + B \cdot u_1, \tag{6.71b}$$

$$+ A \cdot x_1 + \epsilon_1 \cdot |A| \cdot y_1 + B \cdot u_1 + \epsilon_2 \cdot |B| \cdot \omega_1 \le 10, \tag{6.71c}$$

$$- A \cdot x_1 + \epsilon_1 \cdot | - A| \cdot y_1 + (-B) \cdot u_1 + \epsilon_2 \cdot | - B| \cdot \omega_1 \le 10, \tag{6.71d}$$

$$- y_1 \le x_1 \le y_1, -\omega_1 \le u_1 \le \omega_1, -1 \le u_1 \le 1, -10 \le x_2 \le 10. \tag{6.71e}$$

The solution of (6.71) can be obtained by multi-parametric programming, resulting in explicit expressions, $u_1 = f(x_1, u_2)$, in 22 critical regions;

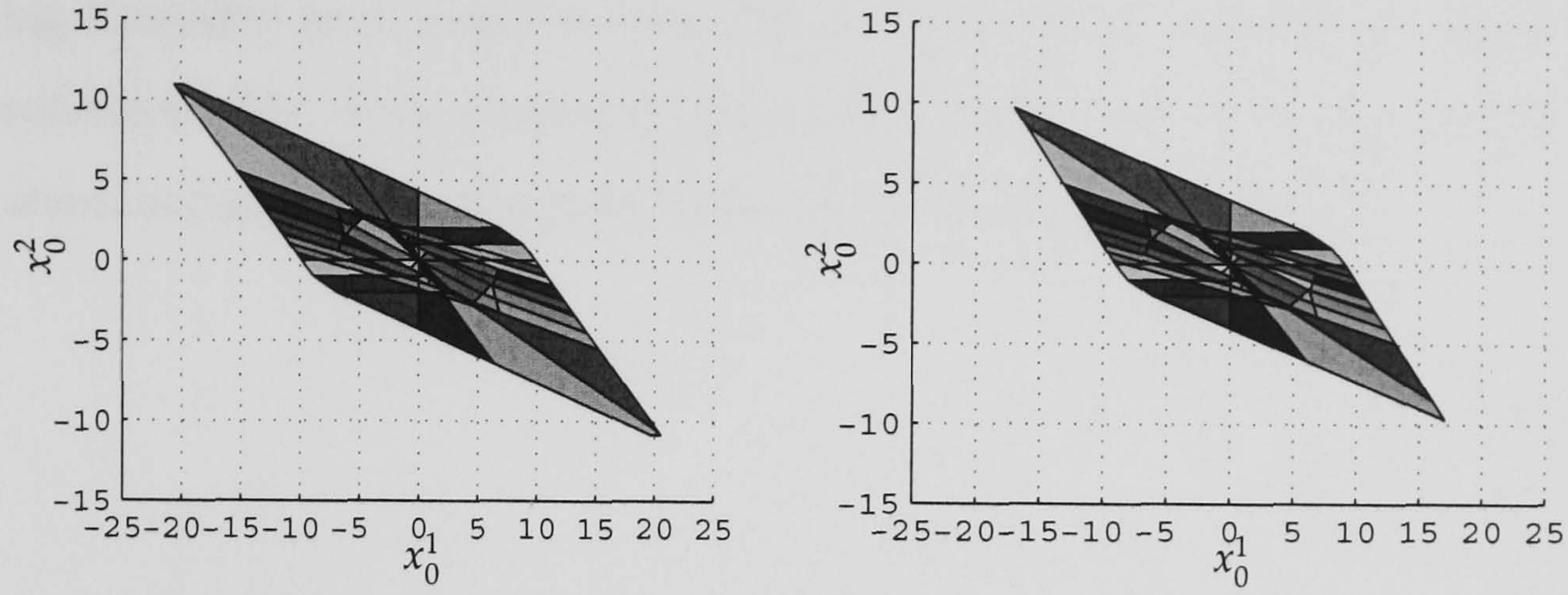
Step 4. We then incorporate the future decision, $(u_2, \omega_2, y_2) = f(x_1, u_1)$, in the current decisions, $u_1 = f(x_1, u_2)$, by which we obtain expressions: $u_1 = f(x_1)$. Note that we need to incorporate the 12 regions obtained in *Step 2* in each one of the 22 regions obtained in *Step 3*, i.e. we generate 264 critical regions. Feasibility tests are performed here (see Chapter), with which infeasible critical regions are eliminated and a compact set of regions is obtained, resulting in only 80 regions to examine further. For example, incorporating (6.67-6.70) (critical region #8 in *Step 2*) in one of the regions obtained in *Step 3*, results in the following critical region #14:

$$
\begin{array}{rcl}
\textbf{Critical region \#14, } f(x_1, u_2): & \Rightarrow & \textbf{Critical region \#14, } f(x_1): \\
x_1^2 \le 8.90, & & x_1^2 \le 8.90, \\
x_1^1 + 0.980x_1^2 \le 9.90, & & x_1^1 + 0.980x_1^2, \\
-0.980x_1^1 - x_1^2 \le 9.90, & & -0.980x_1^1 - x_1^2 \le 9.90, \\
-x_1^2 \le 9.90, & & -x_1^2 \le 9.90, \\
-x_1^1 \le 0, & & -x_1^1 \le 0, \\
-0.865x_1^1 - 0.680x_1^2 - u_2 \le -20.3, & & -1.346x_1^1 - 0.199x_1^2 \le -7.90, \\
\textit{Optimal decision law } u_1 = f(x_1, u_2) & \Rightarrow & \textit{Optimal decision law } u_1 = f(x_1) \\
u_1 = -x_1^2 - 9.90, & & u_1 = -x_1^2 - 9.90, \\
\omega_1 = x_1^2 + 9.90, & & \omega_1 = x_1^2 + 9.90, \\
y_1^1 = 0, & & y_1^1 = 0, \\
y_1^2 = 0, & & y_1^2 = 0.
\end{array}
$$

Note also that constraints belonging to future stages are not considered in (6.71), as future constraints satisfaction is implicitly guaranteed by the definition of the present map of critical regions. Hence, the use of a global optimisation procedure is not required;

Step 5. First stage - Similarly, we can obtain the final map of critical regions, i.e. all feasible solutions, as depicted in Figure 6.3 (for different values of $\{\epsilon_1, \epsilon_2\}$) involving 464 critical regions.

Each critical region in Figure 6.3 corresponds to a different policy, however, many regions may have the same identical first-stage optimal decision, u_0. In the example above, only 20 different first-stage optimal decisions were identified (i.e. a potential reduction over 95%). The implication of this in a closed-loop robust control implementation strategy, where only the first-stage decisions are updated, is that a very significant reduction of the number of critical regions (control laws) can take place, by merging the adjacent regions with identical first-stage control actions.



(a) $\epsilon_1, \epsilon_2 = 0\%$ - nominal case

(b) $\epsilon_1, \epsilon_2 = 10\%$

Figure 6.3.: Maps of critical regions - feasible solutions.

From 6.3 we observe that the space of feasible initial states, map of critical regions, corresponding to the robust solution ($\epsilon_1, \epsilon_2 = 10\%$) is smaller than the nominal system's case solution. This observation complies with the discussion in Pistikopoulos *et al.* (2007*b*), where it is further shown that the nominal solution cannot guarantee robustness in the presence of uncertainty, and the nominal system control trajectory may result in constraint violation; opposed to the robust controller, which retains the trajectory in the set of feasible states.

6.5. Concluding remarks

We have presented a novel multi-parametric programming approach for the linear quadratic regulator problem of parametric uncertain systems. We have highlighted how: (i) we can reformulate the original multi-stage optimal control problem involving polytopic uncertainties into its robust equivalent, while preserving the original model structure and features, (ii) we can use recently proposed multi-parametric programming theory and algorithms to efficiently address constrained dynamic programming procedures, used in the context of multi-stage optimal control formulations, (iii) we can avoid any need for global optimisation methods by carefully posing and conducting feasibility tests, based on sensitivity analysis of the obtained parametric solutions. The work presented here clearly establishes the foundations towards a comprehensive general theory for robust optimal control.

7. Conclusions and future work

Que vous importe, lecteur, ma chétive individualité?
Proudhon

In this thesis we have developed novel theory and algorithms for the solution of many classes of global optimisation problems. Part I has focused on the advances in global optimisation using multi-parametric programming tools. In Chapter 2, we have described the foundations of a novel global optimisation strategy for the solution of general classes of bilevel programming based on our recent developments in multi-parametric programming. It has been shown that bilevel linear, quadratic and mixed-integer linear programs, also involving uncertainty, can be effectively solved. It was further shown that issues related to global optimality for both levels of the bilevel program can be addressed. In Chapter 3, the approach has been extended to address general multi-level programming problems. A promising novel global optimisation strategy has been described for the solution of hierarchical multi-level and decentralised multi-level programs. The algorithms proposed are suitable for problems involving general convex objective functions and convex sets of constraints. Stackelberg-Nash Equilibrium type of problems (Liu, 1998*b*), as well as the application to hierarchical control structures (Stephanopoulos and Ng, 2000), have been discussed.

In Chapter 4, we have presented a novel optimisation framework for the global solution of general mp-MILP problems, involving uncertainty in the objective function and the right-hand side of the constraints. Based on our previous work on multi-parametric programming (Dua and Pistikopoulos, 2000; Dua *et al.*, 2004; Pistikopoulos *et al.*, 2007*a*), a novel mp-LP algorithm was developed, which overcomes the presence of the non-convexities due to bilinear terms. This is then used in an efficient procedure, which iterates between a master MINLP subproblem, solved to global optimality, and a slave mp-LP subproblem. A number of examples are also presented. The proposed approach has many applications in hybrid and robust control - a topic which is currently being investigated and introduced in Chapter 6.

Part II has focused on the advances in robust optimisation and control. Chapter 5 has described the main steps of a novel multi-parametric programming approach for the solution of general, constrained convex multi-stage problems. Through a literature example of optimal control problems, we have highlighted how (i) we can use recently proposed multi-parametric programming theory and algorithms (Pistikopoulos *et al.*, 2007*a*) to efficiently address constrained dynamic programming procedures, used in the context of multi-stage formulations, and (ii) we can avoid any need for global optimisation methods by carefully posing and conducting feasibility tests, based on sensitivity analysis of the obtained parametric solutions. Summing-up, the work presented in this chapter establishes the foundations towards a comprehensive general theory for robust optimal control, for which Chapter 6 is a step further. In Chapter 6 we have presented a novel multi-parametric programming approach for the linear quadratic regulator problem of parametric uncertain systems. We have highlighted how: (i) we can reformulate the original multi-stage optimal control problem involving polytopic uncertainties into its robust equivalent, while preserving the original model structure and features, (ii) we can use recently proposed multi-parametric programming theory and algorithms to efficiently address constrained dynamic programming procedures, used in the context of multi-stage optimal control formulations, (iii) we can avoid any need for global optimisation methods by carefully posing and conducting feasibility tests, based on sensitivity analysis of the obtained parametric solutions. The work presented in this chapter clearly establishes the foundations towards a comprehensive general theory for robust optimal control.

7.1. Key contributions

The key contributions of this thesis can be summarised as follows:

- a novel optimisation approach for the global solution of different classes of bilevel programming, e.g. bilevel linear programming, bilevel quadratic programming, mixed integer bilevel linear programming and bilevel programming involving uncertainty;
- a new global optimisation approach for the solution of general multi-level programming problems, namely ones involving hierarchical and decentralised optimisation structures;
- a novel optimisation strategy for the global solution of MILP problems, involving uncertainty in the cost function and/or right-hand side of the constraints;
- a multi-parametric programming approach for dynamic programming problems in the presence of hard contraints;
- the foundations of a general theory for robust optimal control. The robust optimisation framework was fully implemented in Matlab®.

7.2. Future work

In this section, suggestions for future work are discussed.

7.2.1. Theoretical and algorithm developments

◇ **Bilevel programming with non-convex functions;**

In Chapter 2, a general framework has been developed for the global solution of several classes of bilevel programming, involving convex functions. To further extend the framework to non-convex functions, we may utilise multi-parametric global optimisation algorithms (Dua *et al.*, 2004). A bilevel program involving non-convex functions is posed as:

$$\begin{aligned} &\min_{x,y} F(x,y), \\ s.t.\quad &G(x,y) \leq 0, \\ &x \in X, \\ &y \in \text{argmin}\{f(x,y) : g(x,y) \leq 0, y \in Y\}, \end{aligned} \tag{7.1}$$

where $X \subseteq \mathbb{R}^{nx}$ and $Y \subseteq \mathbb{R}^{ny}$ are both compact sets; F and f are real functions: $\mathbb{R}^{(nx+ny)} \rightarrow \mathbb{R}$; G and g are vectorial real functions, $G : \mathbb{R}^{(nx+ny)} \rightarrow \mathbb{R}^{nu}$ and $g : \mathbb{R}^{(nx+ny)} \rightarrow \mathbb{R}^{nl}$; $nx,\ ny \in \mathbb{N}$ and $nu,\ nl \in \mathbb{N} \cup \{0\}$.

Obviously, the hierarchical decentralised optimisation framework - Chapter 3 - possibly addresses also general classes of non-convex cost functions.

◇ **Global optimisation algorithms for mp-MILP;**

Chapter 4 has described an efficient algorithm to solve OFC and RIM mp-MILP problems. Furthermore, in Remark 4.3 the algorithm has been shown to address other classes of mp-MILP problems, as the general class of mp-

MILP problems:

$$
\begin{aligned}
z(\theta) = \min_{x,y} \quad & (c + H\theta)^T x + (d + L\theta)^T y, \\
s.t. \quad & A(\theta)x + (e_1 + E_2\theta)y \leq b + F\theta, \\
& \Gamma(\theta)x + (\phi_1 + \Phi_2\theta)y = \gamma + \Psi\theta, \\
& x \in X \subseteq \mathbb{R}^n, y \in \{0,1\}^q, \\
& \theta \in \Theta.
\end{aligned} \tag{7.2}
$$

In the master problem we obtain an integer solution, $y = \bar{y}$. Then, the resulting slave multi-parametric problem is posed as follows:

$$
\begin{aligned}
(d + L\theta)^T \bar{y} + \min_{x} \quad & c^T x + \theta^T H^T x, \\
s.t. \quad & A(\theta)x \leq b' + F'\theta, \\
& \Gamma(\theta)x = \gamma' + \Psi'\theta, \\
& x \in X \subseteq \mathbb{R}^n, y \in \{0,1\}^m, \\
& \theta \in \Theta,
\end{aligned} \tag{7.3}
$$

where, $b' = (b - e_1\bar{y})$, $\gamma' = (\gamma - \phi\bar{y})$, $F' = (F - E_2\bar{y})$ and $\Psi' = (\Psi - \Phi_2\bar{y})$. Performing a robust re-formulation of (7.3), as described in Chapter 6, the slave problem is:

$$
\begin{aligned}
(d + L\theta)^T \bar{y} + \min_{x} \quad & c^T x + \theta^T H^T x, \\
s.t. \quad & Ax \leq b' + F'\theta, \\
& A \cdot x + \epsilon_1 \cdot |A| \cdot |x| \leq b' + F'\theta, \\
& \Gamma \cdot x \leq \gamma' + \Psi'\theta, \\
& \Gamma \cdot x + \epsilon_2 \cdot |\Gamma| \cdot |x| \leq \gamma' + \Psi'\theta, \\
& -\Gamma \cdot x \leq -\gamma' - \Psi'\theta, \\
& -\Gamma \cdot x + \epsilon_2 \cdot |-\Gamma| \cdot |x| \leq -\gamma' - \Psi'\theta, \\
& x \in X \subseteq \mathbb{R}^n, y \in \{0,1\}^m, \\
& \theta \in \Theta,
\end{aligned}
\tag{7.4}
$$

where, ϵ_1, ϵ_2 define the uncertainty affecting matrices A and Γ, respectively.

◇ **Advances in constrained dynamic programming;**

In Chapter 5 we have proposed a new algorithm for multi-stage optimisation problems. Possible developments in this promising algorithm are listed as follows:

- The cost function at each recursive step is computed as a compound of all future cost functions, previously optimised, and the cost corresponding to the decision taken at the present time. Since in the proposed approach the future actions are considered as parameters, we might consider only the control decision ahead dismissing all other future decisions. For instance, consider Stage $N-2$, Figure 7.1.

  If it is proven to give the optimal solution, the *curse of dimensionality* (Bellman, 2003) is solved, because the vector of parameters at each stage has the same dimension, and therefore, regardless the dimension of the receding horizon the problem has always the same dimension;

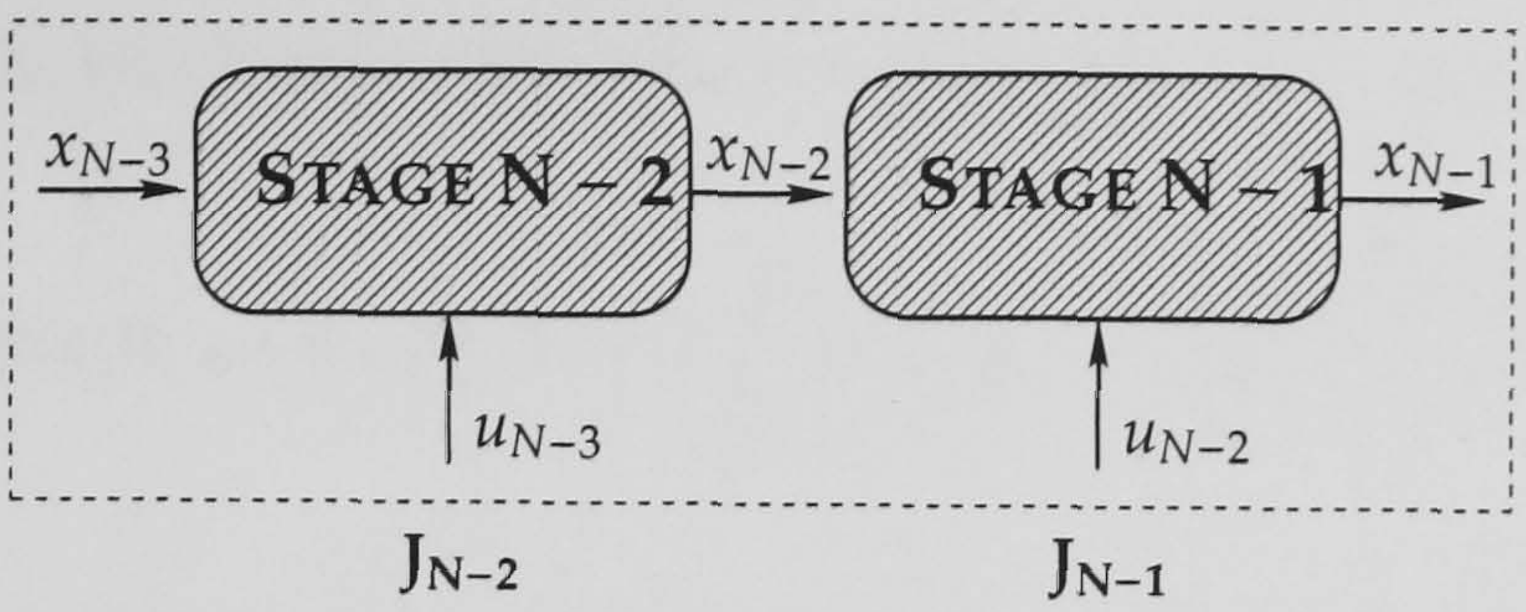


Figure 7.1.: Multi-stage optimisation problem - Stage $N-2$.

- Instead incorporating the maps obtained in each other and obtaining a final map of regions for the initial states, if we store the different maps we improve storage demand and we might accelerate the identification of the critical region. The challenge is to derive an efficient algorithm to locate on-line the valid critical regions.

◇ **Distinction between hard and soft constraints;**

In Chapter 6, we have studied the application of multi-parametric programing to model-based predictive control (Santos, 2001). The original control problem was recast to incorporate robustness and solved using a multi-parametric programming approach. However, the performance of the algorithm was observed to be high dependent on the number of constraints.

Oliveira (1994) and Oliveira and Biegler (1994) describe the benefits of handling the soft constraints within a merit function; which might reduce significantly the number of constraints affecting the final map of regions. The decrease in the number of constraints, is therefore, a high motivation for constructing the merit function. Moreover, approaches addressing non-linear multi-parametric programs with merit functions have been reported. Regarding the perturbed optimality conditions as follows:

$$\nabla_x \mathscr{L}(x,\theta,\lambda,\mu) = 0, \tag{7.5}$$

$$\lambda_i g_i(x,\theta) = r, \qquad i = 1,\dots,p, \tag{7.6}$$

$$h_j(x,\theta) = \mu_j r, \qquad j = 1,\dots,q, \tag{7.7}$$

Armacost (1974, 1976) derive the following logarithm quadratic loss function:

$$W(x,\theta,r_k) = f(x,\theta) + r_k \sum_{i=1}^{m} \ln g_i(x,\theta) + \sum_{j=1}^{p} \frac{h_j^2(x,\theta)}{r_k}, \tag{7.8}$$

where, r is an artificial parameter which controls the relaxation of $W(x,\theta,r_k)$. The strategy is similar to a interior-point strategy, or other homotopy-based algorithm, with $x \to x^*$ as $r_k \to 0$.

◇ **Fractional programming;**

Fractional programming considers optimisation problems of one or more ratios of functions, $Q_k(x) = \frac{N_k(x)}{D_k(x)}$, subject to constraints. A fractional program with $k = 1$, is posed in the following way:

$$\min_x \left\{ \frac{N(x)}{D(x)} : x \in S \right\}. \tag{7.9}$$

Assume:

- E^n is an Euclidean space of dimension n;
- S is a compact and connected subset of E^n;
- $N(x)$,$D(x)$ are continuous and real-valued functions of $x \in S$;
- $D(x) > 0, \forall x \in S$.

Then, Problem (7.9) has the same solution as the following parametric programming (Dinkelbach, 1967):

$$\min_x \left\{ N(x) - \theta \cdot D(x) : x \in S, \theta \in E^1 \right\}. \tag{7.10}$$

Consequently, the algorithms developed through out this thesis address certain classes of Problem (7.10). Plus, we can further explore fractional programs involving integer variables and the multi-ratio case:

$$\min_{x} \left\{ \sum_{k=1}^{\gamma} Q_i(x) : x \in S \right\}. \tag{7.11}$$

Real-life applications of Formulation (7.11) include, among others, genomics (Leber *et al.*, 2005), agricultural systems (Lara and Stancu-Minasian, 1999), energy systems (Kobayashi *et al.*, 2002) and finances (Goedhart and Spronk, 1995).

◇ **Model reduction techniques;**

Model reduction techniques are very popular techniques within optimisation and control communities. Because, it is extremely useful to compact the information described by large scale and complex mathematical models. Although the model is compacted, it still captures the essential dynamic behaviour of the system. In the context of multi-parametric programming and control, the use of these techniques is considered very promising (Narciso and Pistikopoulos, 2008).

◇ **Novel class of global multiparametric programming algorithms;**

A general multi-parametric program with non-convex functions is posed in the following way:

$$\min_{x \in \chi} \{ f(x, \theta) : h(x, \theta) = 0, g(x, \theta) \leq 0, \theta \in \Theta \}, \tag{7.12}$$

where, $\chi \subseteq \mathbb{R}^n$, $\Theta \subseteq \mathbb{R}^m$, $f : \mathbb{R}^{n+m} \rightarrow \mathbb{R}$, $h : \mathbb{R}^{n+m} \rightarrow \mathbb{R}^q$ and $g : \mathbb{R}^{n+m} \rightarrow \mathbb{R}^p$ are assumed to be at least twice continuously differentiable. Furthermore, it is assumed that all minima are isolated minima and that the varying parameters are unstructured.

The advantages of solving (7.12) using multi-parametric programming (Pistikopoulos *et al.*, 2007*a*), rather than a standard optimisation algorithm, is two-folded: (i) there is a qualitative characterisation of the parameter space, Figure 7.2 and 7.3; and (ii) the optimal decision is an analytical function of

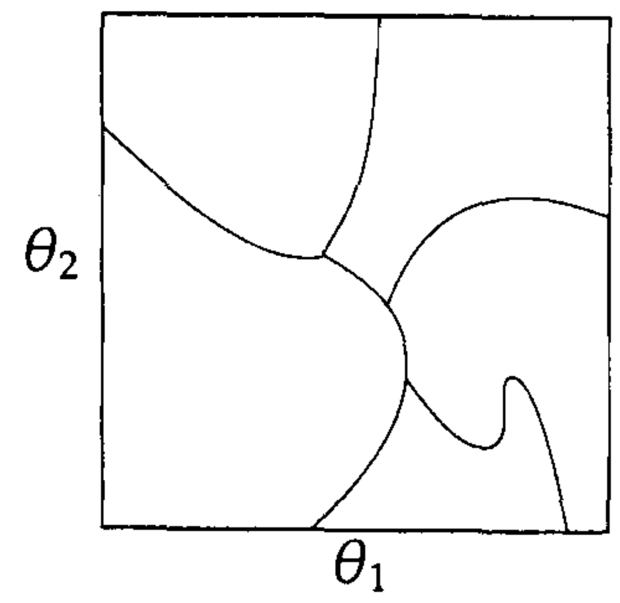


Figure 7.2.: A map of critical regions, $\Theta \subset \mathbb{R}^2$.

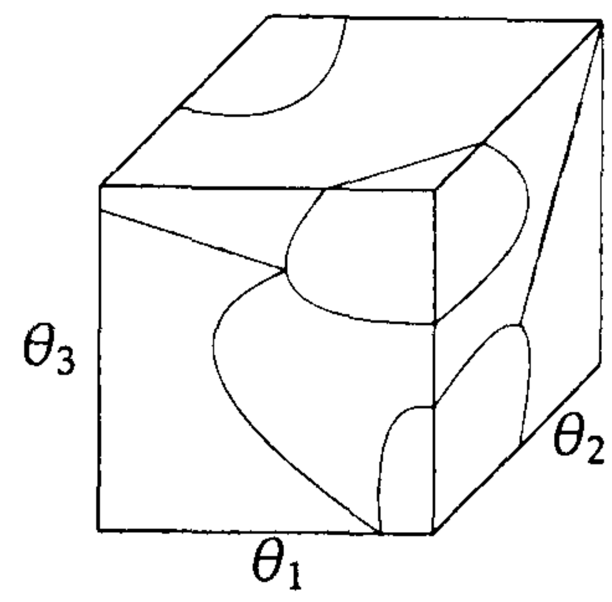


Figure 7.3.: A map of critical regions, $\Theta \subset \mathbb{R}^3$.

the parameters, and therefore, on-line computation is a very fast procedure.

However, in the presence of non-convex functions the performance of the actual algorithms (Dua *et al.*, 2004) decreases. The algorithms face an explosion of critical regions, due to the numerous linear approximations. The implications are a high storage demand, sub-optimal solutions and the on-line computation of the optimal decision slows down. Therefore, the two main advantages of multi-parametric programming are lost, because: (i) the critical regions no longer correspond to qualitative changes, but to validity of the approximations; (ii) the on-line computation becomes computationally expensive. Nevertheless, recent investigations into singularity theory (Poore and Tiahrt, 1987; Tiahrt and Poore, 1990*a*,*b*; Lundberg and Poore, 1993; Hasan and Poore, 1996*a*,*b*; Allgower and Georg, 2003), unveiled some important properties of Problem (7.12).

The singularity theory provides a solid framework of conditions to derive the analytical limits of the critical regions. Then, this data accelerates the numerous available traditional global optimisation algorithms. Therefore, the new class of multi-parametric programming algorithms should deploy the complexity in two steps: (i) Off-line, determine the analytical limits of the critical regions, and (ii) On-line, compute the global optimum with the input of the parameters' value and respective active set of constraints.

Step 1 - Determine the map of critical regions (off-line)

First, we identify a closed system of equations for (7.12). Since f, h and g are non-convex we consider Fritz-John (FJ) first-order necessary conditions (Mangasarian and Fromovitz, 1967):

$$\nu \cdot \nabla_x f(x,\theta) + \sum_{i=1}^{p} \lambda_i \cdot \nabla_x g_i(x,\theta) + \sum_{j=1}^{q} \mu_j \cdot \nabla_x h_j(x,\theta) = 0, \tag{7.13a}$$

$$\lambda_i g_i(x,\theta) = 0, \qquad i = 1,\dots,p, \tag{7.13b}$$

$$h_j(x,\theta) = 0, \qquad j = 1,\dots,q, \tag{7.13c}$$

$$g_i(x,\theta) \leq 0, \qquad i = 1,\dots,p, \tag{7.13d}$$

$$\lambda_i, \nu \geq 0, \qquad i = 1,\dots,p. \tag{7.13e}$$

Note that equations (7.13a) - (7.13c) correspond to $n + q + p$ equations, and there are $n + q + p + 1$ unknowns, $x \in \mathbb{R}^n$, $\mu \in \mathbb{R}^q$, $\lambda \in \mathbb{R}^p$ and $\nu \in \mathbb{R}$. Thence, we formulate an extra equation (Tiahrt and Poore, 1990*b*):

$$\nu^2 + \mu^T \mu + \lambda^T \lambda - \beta_0^2 = 0, \tag{7.14}$$

with β_0 being a fixed positive real number, which normalises the Lagrange multipliers. Therefore, the original problem, Problem (7.12), is recast as a closed system of non-linear equations:

$$F(z,\theta) = \begin{bmatrix} \nabla_x \mathscr{L}(z,\theta) \\ \Lambda g(x,\theta) \\ h(x,\theta) \\ \nu^2 + \mu^T\mu + \lambda^T\lambda - \beta_0^2 \end{bmatrix} = 0, \quad \textit{where } z = \begin{bmatrix} x \\ \nu \\ \lambda \\ \mu \end{bmatrix}, \tag{7.15}$$

where, $\nabla_x \mathscr{L}(x,\theta)$ is the gradient of the Lagrangian, Equation (7.13a), and Λ is a diagonal matrix, $\Lambda = diag(\lambda_i)$; And, $F : \mathbb{R}^{n+p+q+1} \to \mathbb{R}^{n+p+q+1}$, is assumed to be a smooth mapping, which means it has as many continuous derivatives as the discussion requires (Allgower and Georg, 2003). Moreover, note that it has been assumed that all minima are isolated minima, i.e. $\nabla^2 F(x)$ is positive definite at every minimum.

The solution of (7.15) includes: minima, maxima, saddle and infeasible points, and thus, all KKT and FJ points. To be classified as a FJ point, a solution to System (7.15) has to be either a solution to Equations (7.13d) and (7.13e), additionally, if $\nu > 0$, the KKT conditions are also satisfied (Bazaraa and Shetty, 1979). Henceforth, we refer to any solution to System (7.15) as a *critical point*, and refer to a *singular point* and to a *regular point* to distinguish between singular and non-singular *critical points*, respectively. It follows naturally that the group of all *critical points*, corresponding to a specific active set, is called *critical region*.

The limits of a critical region are defined by points for which the Fréchet derivative is singular, $\mathcal{D}_z F(z_0, \theta_0)$ (Tiahrt, 1986). In multidimensional finite spaces, this is equivalent to state that the Jacobian matrix has to be singular. As we have seen in Theorem 4.2, these conditions resume to:

(i) Loss of strict complementarity;

(ii) Violation of the linear independence constraint qualification;

(iii) Singularity of the Hessian of the Lagrangian on the tangent space to the active constraints.

Yet, the classes of problems which have been fully addressed in the open literature (Pistikopoulos *et al.*, 2007*a*,*b*) exhibit a very important property: the manifold, with respect to the parameters, is isotropic inside each active set. In an isotropic manifold, the Jacobian matrices are constant in the neighbourhood of the optimum point, i.e. $M, N \neq f(z, \theta)$. From (7.15):

$$F(z(\theta), \theta) = 0, \tag{7.16}$$

deriving with respect to θ,

$$\left\{ \underbrace{\frac{\partial F(z,\theta)}{\partial \theta}}_{N(z,\theta)} + \underbrace{\frac{\partial F(z,\theta)}{\partial z}}_{M(z,\theta)} \frac{\partial z}{\partial \theta} \right\} = 0 \Leftrightarrow \frac{\mathrm{d}z}{\mathrm{d}\theta} = -M^{-1} \cdot N. \tag{7.17}$$

since M and N are constant,

$$\int_{z_0}^{z} \mathrm{d}z = \int_{\theta_0}^{\theta} -M^{-1} \cdot N\mathrm{d}\theta \Leftrightarrow z = -M^{-1} \cdot N\,(\theta - \theta_0) + z_0. \tag{7.18}$$

Therefore, inside the respective *critical region*, the Taylor expansion will be the exact optimal solution for $z(\theta)$. Inherently, it also corresponds to the exact representation of the optimum value function, $f^*(\theta)$. Consequently, since we have an explicit analytical expression for $x(\theta)$, $\lambda(\theta)$ and $\mu(\theta)$, it is trivial to check the singularity conditions stated in Theorem 4.2.

Remark 7.1 *A manifold is a topological space that is connected and locally Euclidean.*

Remark 7.2 *Note that in this case the neighbourhood of the optimum point is defined by the non-singularity condition of matrix M.*

Remark 7.3 *The set of constraints defining the frontier of a* critical region *is a completely different identity from the set of active constraints inside a* critical region. *For instance, we can have hundreds of constraints defining the* critical region *and just a couple of active constraints.*

Therefore, the challenge is how to check for singularities within a non-isotropic manifold, i.e. $M, N = f(z, \theta)$. Since, it is complex to compute an analytical expression for $z(\theta)$, valid inside the whole *critical region*. Many investigations have been using continuation methods (Henderson, 2002, 2005;

Brodzik, 1996, 1998), however, they rely on exploring the space using triangles or simplices; which has a predictably poor performance, when the size of parameters' dimensionality exceeds $\mathbb{R}^2$ (Hedengrena and Edgar, 2008).

Concurrently, further studies should be pursued in a gradient projection method along geodesics (Luenberg, 1972). Following the geodesics of the function, finite points of the critical region limit (singular points) are identified. However, conditions to connect efficiently these points and define a closed critical region are still not available.

Another pathway should be the use of second-order approximation of the optimum value function. As in (7.17):

$$\frac{\partial}{\partial\theta}\left\{N(z,\theta)+M(z,\theta)\cdot\frac{\partial z}{\partial\theta}\right\}=0\Leftrightarrow\frac{\partial}{\partial\theta}\{N(z,\theta)\}+\frac{\partial}{\partial\theta}\left\{M(z,\theta)\frac{\mathrm{d}z}{\mathrm{d}\theta}\right\}=0\Leftrightarrow$$
$$\Leftrightarrow\left\{\frac{\partial N(z,\theta)}{\partial\theta}+\frac{\partial N(z,\theta)}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}\theta}\right\}+\frac{\partial}{\partial\theta}\{M(z,\theta)\}\cdot\frac{\mathrm{d}z}{\mathrm{d}\theta}+M(z,\theta)\cdot\underbrace{\frac{\mathrm{d}^2z}{\mathrm{d}\theta^2}}_{second-order}=0.$$

The challenge is the manipulation of tensors of rank-3, e.g. $\frac{\partial N(z,\theta)}{\partial\theta}$.

Step 2 - Global optimisation (on-line)

The bibliography in dynamic and global optimisation is vast (Vassiliadis *et al.*, 1994*a*,*b*; Adjiman *et al.*, 1998*b*,*a*; Smith and Pantelides, 1999; Floudas, 2000; Forsgren *et al.*, 2002; Gertz and Gill, 2004; Akrotirianakis and Rustem, 2005), and an algorithm should be selected after clarifying the off-line step.

7.2.2. Applications

◇ **Energy market;**

The energy market is a very dynamic multi-agent environment with strong competition between rival companies. In this non-cooperative environment, each company has to set a price for the energy production, efficiently manage its energy production, meet the obligations agreed contractually with the customers, and to maximise profit by selling or buying energy from a competitor, in the spot market or using forward contracts. The mathematical formulation of such problems has constraints. For example, distribution companies inescapably have to supply to the customers the energy contractually agreed. Hence, designing an appropriate optimisation framework to address planning issues in the energy market relies to a great extent on the correct formulation and solution of constrained multi-agent problems.

In multi-agent optimisation the individual reward is dependent on the decisions of the remaining agents. Interactions between agents can be modelled as straightforward competition with or without dominant player(s). Dominant agents models are characterised by Stackelberg equilibria. If all agents have comparable authority, or dominance, relative to each other, Nash equilibria are considered. Game theory models become complicated in presence of constraints that affect more than one agent. Therefore, developing a game strategy/algorithm to resolve constrained problems is important for managing the interactions between networks of agents within the energy market.

In Chapter 3, theory and algorithms have been developed which can be used in the solution of this class of problems.

◇ **Unmanned aircraft systems;**

Unmanned aircraft systems (UAS) have attracted considerable attention in the last decade, primarily because they have important civil and defence applications. Examples of which are rescue missions, fire extinction, identification of hazardous materials, oceanographic/geological surveys, border inspection and surveillance for traffic control (Swaroop and Hedrick, 1999; Burns *et al.*, 2000; Girard *et al.*, 2003).

The mathematical problem underlying UAS is a typical hierarchical decentralised control problem. Compounded by a central controller (leader) and multiple local controllers (vehicles), UAS have a complex dynamic behaviour.

Which is expected, since there are combinatorial interactions between leader, vehicles and environment. Therefore, the efficient coordination between the embedded guidance schemes (vehicles) and the control schemes (leader) is a crucial piece to guarantee a good and verifiable dynamic performance. Obviously, the application of UAS to safety-critical missions is dependent upon the design of a robust and reliable control structure.

In Chapters 3,5 and 6, theory and algorithms have been developed for these type of interconnected dynamic systems, where individual dynamics constrain the leader's performance.

7.3. Publications from this thesis

◇ **Chapters in books**

- Pistikopoulos, E.N., **N.P. Faísca**, P.M. Saraiva and B. Rustem (2007). A bilevel programming framework for enterprise-wide process networks under uncertainty. In Christodoulos A. Floudas and Panos M. Pardalos (Eds.), *Encyclopedia of Optimization, 2^{nd} Edition*. New York: Kluwer Academic Publishers.

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