An analysis of spending behaviour under liquidity constraints with an application to financial hedging.

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Abstract

For counter-parties to an options contract, the method employed to hedge their exposures depends on the agents' risk preferences and the underlying asset. The thesis begins with a rudimentary model exploring the effects of sequential decisions to an investment problem. The insights gained from this analysis are discussed, as are the refinements necessary for more detailed investigation of the problem under consideration.

Incorporating these into the rudimentary model leads to the main model for the thesis, one which describes agents' behaviour in a dynamically constrained optimisation framework, and holds similarities to models of markets for storable commodities. When combined with fundamental results about the quadratic variation of stochastic price processes, the model provides a link to option pricing. The analysis thus facilitates the development of a financial instrument that can assist agents in reducing the risk associated with volatility of corporate cash flows. The instrument can be priced using fundamental (financial statement) analysis and market prices, both of which also provide information related to corporate creditworthiness, thus allowing corroboration of the results from the option pricing model.

Under a variety of assumptions related to this model, the behaviour of a set of inter-connected agents is simulated. The results are suggestive of structures in market prices whose characteristics of which can be described in terms of price volatility. This allows the calibration of option prices by means of wavelet-based price processes, which are especially suitable for modelling quadratic variation. Our model generates related time series for unit prices as well as quantity flows. Selected aspects of this are elaborated upon to illustrate areas for future application of the concepts developed in the thesis.

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Chapter 1

Introduction and Literature Review

The fields of finance and economics utilise mathematical techniques to formalise the analysis of the problems of production, exchange, allocation and consumption. This approach has been very successful, delivering significant leaps in our understanding of scarce resource allocation and risk, but it has its limitations as succinctly described by a quote in Coase (1937) [37]:

> "Mrs. Robinson has said that "the two questions to be asked of a set of assumptions in economics are: Are they tractable? and: Do they correspond with the real world?" Though, as Mrs. Robinson points out, "more often one set will be manageable and the other realistic," yet there may well be branches of theory where assumptions may be both manageable and realistic."

As mathematical methods have developed, it has become possible to use assumptions that more closely approximate real activities. Notable mathematical tools include the formalised treatment of chance through probability theory as well as Newtonian and stochastic calculus. In recent decades, increases in computing power have gradually opened up the possibility of using simulation to learn about economic activity. This is done by comparing the results of simulation and predictions of analytical methods against observations in life.

The development of large corporations and banks,[1] along with international financial markets large enough to accommodate them, has created a focal point for applying these techniques. The widespread adoption of mathematical and computational tools has allowed more efficient methods of financing economic activity and dealing with risk. A common feature of negotiations in the financial world is the desire of each party to maximise its financial flexibility and minimise the risk to which it is exposed. The issue of flexibility is central to this thesis and its effect on economic behaviour is investigated in several ways.

Financial flexibility is clearly an important issue for businesses and individuals. They have traditionally used loans as a principal source of financing their activities and this service, principally supplied by banks, has retained its importance even as the financial markets have grown in scale and scope and reduced the role of banks as intermediary in various economic activities.

Bank loans exist in various forms, tailored to suit the specific circumstances of various kinds of borrower. For example, many borrowers do not need a lump sum of money for immediate use but want the certainty of being able to borrow money as and when their own circumstances dictate. Chapter 4 looks at a financial instrument that can, in some circumstances, take the place of this type of bank facility and provides a methodology for pricing the instrument. The key features of this type of bank lending are as follows.

The bank commits to providing a loan facility that the borrower may draw on at any time. The unpredictability of cash flows in the borrower's business is thereby passed from the borrower to the bank, which now needs to have enough cash ready to lend for as long as the loan facility is contracted to be available. This creates a cost for the bank which is related to the bank's ability to match the sources and uses of its own funds. Consequently, banks will often propose restrictions on when and how the borrower can withdraw

[1] See Chandler (1990) [33] and Morrison & Wilhelm (2004) [153] for accounts.

cash, in order to be able to offer competitive interest rates on the funds they lend.

A company may want to draw on a loan in response to fluctuations in its operations resulting from predictable factors such as seasonality of sales as well as some unpredictable factors. However, in addition to internal operational reasons, a company's cash flow is dependent on the prices for goods and services related to it. If the company has insufficient cash to purchase or provide these, it may adversely affect its ability to operate smoothly and reduce the value of the company. These situations are avoided by maintaining financial and/or operational flexibility. Both forms come at some cost and it is not always clear which is more attractive, given that there are many ways of achieving both sorts of flexibility.

Operational flexibility reflects the ability of the company to change its method of doing things, either temporarily or permanently, and continue making a profit. The manifestations of this flexibility can be found in all scales of analysing a business: from the intricacies of changing the process used on a production line, to the global strategy for the company, its business partners and its shareholders. The analysis of operational flexibility often falls under the scope of 'real options' theory, which has had some success in analysing the economic rationale behind some working practices as well as guiding companies on future decisions. The models developed in chapter 2 are the most closely linked to analysis of operational flexibility and were suggested by real options literature.

As far as financial flexibility is concerned, a committed loan facility from a bank is often a cheaper solution than holding cash reserves in a bank account. Both of these solutions tackle shortfalls arising from any cause, be it operational or from external market events. However, it may be more efficient for the company to use market instruments to hedge financial risk arising from market events, instead of using the static approaches of holding cash or paying for a loan facility, or even resorting to diverting funds from other parts of its own operations. Chapters 2 and 3 investigate quantitative methods to implement the idea and, in the process, develop a model to approximate various economic activities. Its predictions are investigated using both analytical and computational methods. The remainder of this chapter introduces the concepts and previous works that motivate the choice of tools in the thesis.

1.1 Scarcity of information and of control

Economics is the study of decisions. The work of Black & Scholes (1973) [15] and Merton (1973) [149] significantly reduced the uncertainty about the valuation of choice embodied in financial option contracts. Their observations allowed researchers to focus on other areas of uncertainty, leading to better understanding and modelling of the volatility of prices and also to dealing with problems like transactions costs, jumps in prices and intermittent trading opportunities. The advances in financial option pricing have not however solved the problem of decision-making in general.[2]

The principal consideration is that decisions have to be made under physical constraints of varying degree, including informational constraints since information is retrieved through physical means. In large scale activities, finite information processing power imposes a limit on what can be gleaned from numerous and detailed observations. Meanwhile, small scale observations are constrained by the accuracy of measuring tools (in addition to the intrinsic impossibility to determine all the properties of an object being observed).

Given that choices appear real, whether free will exists or not, and decisions must be made without all the information, it is necessary to use guesses and estimates based on prior assumptions. As described by Debreu (1959) [47] (p.37), the economic

"agent is characterized by the limitations on his choice, and by

[2] Although its applications are being broadened through real option valuation theory, see Dixit & Pindyck (1994) [53] for an introduction.

his choice criterion."

The forms of the limitations and of the criterion are now discussed.

Nature dictates people have consumption requirements that motivate them to avoid the undesirable consequences of not satisfying them. Invariably, things perceived as requirements are hard to distinguish from more peripheral wants, leading to a possibly unending list of aspirations. Both needs and wants steer the direction of choices in daily decisions. The limitations imposed by nature translate into the fundamental choice criteria.

The perception that choices have to be made suggests that people are able to control their surroundings to a sufficient degree for achieving some of their goals. There are many concurrent choices to be made in any situation and it is not feasible to consider each one consciously. Instead, rules of thumb develop through habit formation as well as social and individual learning.

Choices are not isolated and, while it is possible to analyse sequential decisions under some circumstances, most choices are subject to complex interactions[3] making it difficult to identify the applicable rule correctly. Furthermore, learning from the consequences of previous decisions has limited scope, resulting in rule-of-thumb behaviour that is frequently sub-optimal.[4]

Adam Smith (1776) [193] divided the factors of production into three segments: labour, capital and land, with the latter constantly compared against labour. While wages, profit and rent are the rewards to each kind of stock, it appears that wages reflect effort expended rather than some other measure of labour. As such, it is reasonable to include mental analysis and learning as tasks that contribute to wages. This is akin to the notion of research that leads to human capital and technological progress in endogenous growth theory.[5]

[3]See Carr (1988) [24] for an example of analytical treatment of sequential decisions and Trigeorgis (1991) [202] for examples of complex interactions in real option theory.

[4]This is a common issue in statistics and artificial neural networks, where calibrating or training the system on one data set can reduce its performance with regard to other data.

[5]Romer (1996) [174] Section 3.9 provides a good discussion of models.

Viewing basic consumption and assets as ideal types suggests the following definitions. Ideal-type consumption is a purely transient good that quickly reaches satiation, turning from an economic good into a bad beyond a certain rate of consumption. The satiation point is dictated by physics, nature, society, etc. An asset is an infinitely storable good, which yields no consumable flow by and of itself and can be accumulated and improved ad infinitum. In practice, goods form a spectrum with varying degrees of storability and consumability, with neither consumption goods nor assets existing as ideal types.

Ownership of an asset is usually defined as a relationship whereby the costs and benefits inherent in owning the asset accrue to the owner. It is implicit that the owner has control of the asset. Accordingly, the option to buy or sell an asset is the ability to create or dissolve the relationship and change the costs and benefits to which the owner has access.

Assets are traditionally viewed as having limited liability, but the preceding arguments suggest that the relationships between assets and their effect on the owner can blur the limit. Limited liability implicitly deals only with assets that can be disposed of at no cost, which is an embedded option to sell for a zero price. This embedded put option contributes to the value of the asset and guarantees a non-negative value.

Usually, economic agents only have limited control over the allocation of scarce resources. Relaxing the assumption of an embedded put option, to reflect an aspect of the lack of control, means that non-negative asset value is not necessarily certain. Assets do include other types of optionality but, given the informational constraints, it is not certain that the owner will be able to use them to replicate the put option and ensure limited liability.

Following this argument, the model in section 2.1 views the consumable yield and value of an asset as arising solely from the ability of the owner to understand and exploit embedded options rather than as an intrinsic property of the asset. It explores the nature of the interplay between an ideal-type asset and ideal-type consumption in a scenario where the embedded optionality of the capital asset is limited and subject to some uncertainty.

It considers an economic agent seeking to maximise consumption power at a future point in time following a sequence of decisions. The agent begins with an endowment of ideal-type assets, that have the embedded option to be exchanged for a varying amount of consumption goods. Each period, the agent chooses between exchanging the assets for consumption goods, or increasing assets held by giving away consumption goods. Increasing the amount of assets held increases the magnitude of subsequent decisions, whereas accumulating consumption goods makes direct progress towards the goal of accumulating consumption power. The terms of forthcoming options are not clear in advance, so the agent has to make non-trivial decisions. The results show that the agent takes the high risk strategy of investing heavily in assets before selling as much as possible in a small number of large trades before the terminal period of the model. By comparing this with other models of portfolio selection and investment behaviour, the distorted picture of real behaviour can be identified as being due to the equal emphasis placed by the agent on incremental losses and incremental successes. It corresponds to unlimited ambition (no satiation constraint) and no fear of loss (no minimum constraint). In practice, the same physical and informational constraints mentioned earlier prevent the agent from consuming an unlimited amount in any given time period. If he wishes to consume without limit, the agent will still be aware of the limitations. Acting in accordance with those limitations is a necessary part of rational behaviour, so where behaviour along the lines of the high-risk strategy is observed, it can be attributed to bounded rationality arising from limited information.[6]

While describing the role of metals as currency, Smith [193] writes,

> "These qualities of utility, beauty and scarcity, are the original foundation of the high price of those metals, or of the great quan-

[6]See Arrow (1986) [4] and two review articles by Conlisk (1996) [39] and Vriend (1996) [205] that discuss the nature of rationality and optimality in the context learning and decisions that have to be made with restricted resources.

tity of other goods for which they can everywhere be exchanged."

In present-day economic theory, utility – something that can be more easily measured and compared across people – and beauty – that can be measured only indirectly if at all and may not be comparable between different people – are usually combined into one mathematical function. The function then represents the preferences of the person or group for whom it was calibrated.

The axiomatic analysis of economic equilibrium requires a complete pre-ordering of preferences with the properties of insatiability, continuity and convexity. Preferences are taken to describe only the individual's consumption behaviour and are not affected by the resale value of the goods. In addition, preferences are taken as fixed over time (since the pre-ordering includes different locations and times). It can be argued that real behaviour is not consistent with objective utility because of differences in perception and that behaviour and the apparent utility function would change if different information and other necessary resources were available to everyone. The view will not be incorporated because, to quote Vriend (1996) [205] (p. 269):[7]

> "If preferences were flexible, then the concept of *self-interest* would no longer be defined." ... "Pushing the logic of economics to its limits, and following its line of argument consistently into every conceivable corner of social event, [the Chicago School has] demonstrated that the rationality postulate is necessarily constrained to be essentially contentless in economics."

In addition to interacting directly with nature, people in society interact with each other. It is often possible to fulfil the imposed commitments to nature through social means. Social commitments can be used to reduce the uncertainty around the hard constraints of nature. Contracts are the formalisation of the process and, although meant to be honoured, they are not enforced as unequivocally as natural laws and do not cover every possible

[7] Italics by the Vriend.

eventuality, resulting in some residual uncertainty and allowing agents to apply discretion.

Historically, it has been more difficult to perform verifiable experiments with pure consumption than with (physical) assets[8] so models of the economy from an asset-based approach are more easily testable than from a consumption approach. Almost all economic activity is viewed as the pursuit of indirect goals, which provide only the means of meeting basic consumption requirements. Consequently, the majority of economic activity is governed by soft constraints. A similar discussion by Hull (2003) (p.41) [97] describes consequences of immediate consumption requirements on the behaviour of markets, although it implies that contracts and corporate operational planning entail hard, rather than soft, constraints.

The second model of the thesis, in section 2.2, introduces soft constraints and risk aversion into the behaviour of agents, implicitly dealing with investment assets rather than dealing directly with consumption goods. The following section introduces related models and compares their features.

1.2 Prices in the economy

Scarcity of a valuable resource increases its value, but value is only imperfectly measured by the amount of money, corn, or other items for which it can be exchanged. Taking the view that the majority of economic decisions concern the management of assets that indirectly generate and control consumable goods, the price of an asset reflects its usefulness as a tool rather than its direct consumption value.

The time, effort and other scarce resources that need to be spent on achieving an aim are not easily measured because of the wide variety of forms of the resources and the indirectness of aims. It is also difficult to measure the improvement of welfare that a resource brings, even if the resource-cost

[8]As the economy develops and the role of intangible assets grows, the difficulty of measuring assets gains prominence and impairs the ability to corroborate financial accounts. See Lev (2001) [127] for discussion.

is measured, especially in areas where innovation is rapid.[9]

Smith's [193] approach to measurement is to value resources in terms of labour (p. 133): "The value of any commodity, [...] is equal to the quantity of labour which it enables him to purchase or command." The method extends to the rent of land and profit of capital, each viewed as the price of the portion of labour that they can command (p.153): "Labour measures the value not only of that part of price which resolves itself into labour, but of that part which resolves itself into rent, and that which resolves itself into profit." The effort that goes into acquiring – investing in – an asset depends on the perceived benefits that it will yield.

Investment theory describes the decisions made in this manner and only suggests the price at which an asset should be bought and sold. The way in which people interact with each other in managing assets, formalised in various market-clearing mechanisms, determines the actual, observable price. The next paragraphs describe the salient points of investment theory and then discuss some market-clearing mechanisms considered in the thesis.

The naïve accelerator model of investment considers a producer that maintains a certain stock of capital as part of the production of goods. It states that investment – changes in the stock level – is equal to the change of output, implying that the producer-investor reacts instantaneously to exogenous changes in output. This is represented as[10]

$$I_t = \Delta K_t = v\Delta Y_t, \tag{1.1}$$

where I_t is the investment between time periods $t-1$ and t, K_t is the level of capital stock at time t, v is a constant and Y_t is the level of output.

The flexible accelerator model, which includes lagged output changes as explanatory variables of investment, suggests a limited supply rate of capital goods or behaviour arising from adaptive expectations formation. However, it is not possible to determine if the cause of lagged effects is due to physical supply issues, expectation formation or some combination of the two.

[9] See an example by P. Krugman at http://web.mit.edu/krugman/www/viagra.html.

[10] Notation in this section follows Precious (1987) [166].

The model of investment building on the work of Jorgenson (1963) [108] begins to investigate the effect of particular decision-making mechanisms. By declaring its assumptions more precisely than the accelerator models, it becomes possible to test hypotheses regarding the structure of the economy through its effect. Specific examples are discussed in Precious (1987) [166].

First, a simple investment model assumes a perfect market for the supply and disposal of a homogeneous capital good (all units of which have the same quality which does not change over time). The assumption can be violated by the presence of adjustment costs, transaction costs, a lack of continuous clearing of the market by other means and a changing nature of capital goods over time. These have been investigated in detail using models of q theory and convex adjustment costs, costs of searching, auction models and putty-clay models of investment where the ex-ante production function provides more flexibility than the corresponding ex-post function of investment.[11]

Second, given a well-behaved production function, it is possible to calculate the rate of investment at any moment from relative prices in the economy at the time. For example, with a Cobb-Douglas production function, $Y_t = AK_t^\alpha L_t^\beta$ where L_t is the labour employed at time t and A, α and β are constants, the first order condition $\frac{\partial Y_t}{\partial K_t} = \frac{q_t(r_t+\delta_t)-\dot{q}_t}{p_t}$ results in

$$I_t = \frac{dK_t}{dt} = \alpha \frac{d}{dt}\left(\frac{Y_t p_t}{q_t(r_t+\delta_t)-\dot{q}_t}\right), \tag{1.2}$$

where q_t is the price of capital goods with the $\dot{q}_t$ being its instantaneous rate of change, p_t is the price of output, r_t is the unique interest rate and δ_t is the rate of depreciation of capital stock.

It implies that a rational investor can base decisions on only the concurrent data, without considering the future demand for his output or the relative prices of inputs. The 'myopic' optimal behaviour depends on the

[11] See Hayashi (1982) [87], Abel *et. al.* (1996) [1] for discussion of q theory and adjustment costs, Stigler (1961) [197] and others for search theoretical approaches, Smith *et. al.* (2003) [194] for a recent model of market clearing mechanisms. Parallel work starting with Coase (1937) [37] introduces transaction costs to the structure and behaviour of the firm.

known price of capital goods, which may not necessarily be assumed given the previous point. More interestingly, the result emphasises the effects on investment behaviour of relative prices, interest rates and tax regimes rather than demand constraints. The firm may produce as much as it wishes within the limits of a production function and budget constraint. A demand constraint can be implied by assuming a cost-minimising (rather than profit-maximising) firm and several approaches are described in [166].

Following the approach leads to a third observation, that the rational behaviour of the investor and the informational structure of each model are crucial determinants of the predictions. Apparent deviations from rational behaviour and imperfections in the flow of information are observed in practice and have been investigated. The work of Lucas (1976) [137] provides a critique of the application of rational expectations in macroeconomics and policy formation. Works following that of Keynes (1936) [118] and of Kahneman & Tversky (1979) [110][12] suggest that 'rules of thumb' can be appropriate ways of characterising behaviour of real people. Review articles such as those by Conlisk (1996) [39] and Vriend (1996) [205] discuss the rationality of such behaviour given costs of acquiring information and making decisions, while Cochrane (1989) [38] compares the properties of near-rational models of behaviour to rational ones.

A last point to note is that the firm is assumed to be able to react instantaneously to changes in economic variables. A problem with this assumption is that step changes in the environment can cause an infinite investment rate, unless other, potentially unrealistic, assumptions are made about the way that the variables can change.

The thesis models an economy in which a stock of cash is moved between agents, all of whom prefer the circulation of larger volumes, all other things being equal. While the model creates a growing trend in the level of economic activity, the economy can show modes of behaviour where the flow of cash falls due to a perceived lack of sufficient cash to act as a buffer against

[12] See Hansen & Sargent (2001) [83] for a recent discussion.

fluctuations.

Most economic literature is concerned with the competitive aspects of the economy. The analytical models discussed so far share that focus. A good example of such an approach in computational finance is the Minority Game,[13] in which objectives of agents are conflicting. These forms of model have been used to investigate the conditions under which a large economy with many agents converges to perfect market models.[14]

This thesis investigates the effect of co-ordination and expectation in an economy made up of self-interested agents with imperfect information. In contrast to the minority game, the agents are not in direct competition with each other for a scarce resource.

The agents in sections 2.2 and 2.3 assume that the economy is in one of two particular growth states. Each agent therefore estimates this state variable in addition to any others. It could try to estimate the properties of the two growth states and of the transition probabilities between them, but the depth of estimation has to be truncated at some level and is a strong argument for the use rules of thumb.

It is useful also to compare the model in this thesis to the recent work of Beaudry & Portier (2004) [10], who show that changes in expectations in a neoclassical framework can generate business cycles, defined as the positive co-movement of consumption, investment and employment. However, as they point out, it is necessary to create a multi-sector economy with interactions between the sectors subject to economies of scope[15] to generate the result. The model shown in section 2.3 of this thesis achieves a similar result without explicitly defining separate sectors within the economy and allowing for a form of market non-clearing.

The market clearing mechanism is an important determinant of the behaviour of prices. The model of section 2.3 introduces an imperfect informa-

[13]See Jefferies and Johnson (2002) [104] and references therein for a recent overview.

[14]See for example Sabourian (1999) [176].

[15]See Chandler (1990) [33].

tion structure similar to that of an unobserved aggregate price level[16] while also creating dynamics resembling those of an economy with staggered price adjustment with state-dependent pricing.[17] It allows for perceived surpluses as each agent spends enough to equate his perceived marginal prices of saving and investment. Since the spent money is dispersed between many agents, no single market price is visible or defined.

Laroque (1989) [124] shows that, when prices are not set through a process of tâtonnement, inventory behaviour can generate cycles between two unstable stationary competitive equilibria. Trades may take place at a non-competitive price and the aggregate quantity of money is constant over time. But in contrast to this thesis, Laroque's distinguishes between (overlapping-generations) consumers and (non-profit making) firms. It deals with three types of goods: labour, money and output, and the latter two can be saved as cash and inventories. As such, the paper presents a very different model to the one described here.

Related work in Sabourian (1999) [176] shows that a form of measurement-noise or uncertainty is an important requirement for the convergence in behaviour of an economy with a finite number of agents to that of a perfect market. This is present in all the models (sections 2.1, 2.2 and 2.3), with risk aversion being introduced in going from 2.1 to 2.2 and uncertainty about a global state variable being added to the last model.

Assumptions about market structure and agent behaviour can be used to generate one or more prices that change over time. Such time series can be recreated by following each step of the process generating the prices in a numerical simulation. Under some sets of assumptions, the characteristics of the price changes are simple enough that the time series can be used directly and the underlying mechanism overlooked. The simplest time series arises in an informationally efficient and frictionless market, yielding the random walk hypothesis for financial markets. The concept was used in finance as early as Bachelier (1900) [5] with a rigorous exposition of the hypothesis in

[16]For example as described in Lucas (1972) [136] and Phelps (1970) [164].

[17]See Caplin and Spulber (1987) [22] and Akerlof (1969) is an important earlier paper.

Samuelson (1965b) [179].

Studies of the random walk hypothesis and the efficient markets hypothesis are presented in Fama (1970, 1991) [63], [64]. It is noted that tests for market efficiency involve supporting hypotheses so that no clear consensus has been reached on its validity. The random walk hypothesis is more easily tested and, while it is generally accepted in the weak form for a wide variety of markets, studies taking different approaches have questioned the validity of the strong form of the hypothesis.[18] This is particularly the case for variables such as household saving and consumption, which do not follow the random walk as implied by the permanent income hypothesis of Hall (1978) [81]. Nevertheless, the random walk approximation has allowed the development of tractable models, notably the option pricing theory of Black & Scholes (1973) [15] and Merton (1973) [149].

The random walk is a discrete time stochastic process, the continuous time limit of which is the Brownian motion that has well studied properties.[19] Brownian motion can be viewed as a subset of stable stochastic processes, which are in turn a subset of infinitely divisible stochastic processes that can be represented as the sum of i.i.d. random variables.[20] Brownian motion is the only stable process with probability distributions of finite variance describing increments over finite periods of time and is the only continuous process with stationary increments. It also has infinite linear variation, meaning that the up and down increments over any finite time period sum to infinity. The behaviour is reconciled with financial markets by Harrison *et. al.* (1984) [85], noting that prices with bounded variation would be inconsistent with an idealised (frictionless) market.[21] Quadratic variation, the notional sum of squared increments, increases at a constant rate. This non-random property makes analysing and transforming Brownian motion easier, notably through Itô's Lemma.

[18] See for example Lo & MacKinlay (2001) [131].

[19] For examples, see Freedman (1971) [71] and Harrison (1985) [86].

[20] See Mantegna & Stanley (2000) [145].

[21] Also implicit is the assumption of costless information, as discussed earlier.

1.3 Replication Methods

Financial options exist as traded contracts and many forms can be priced because it is relatively easy to specify the costs and benefits of ownership of a financial security. One way of pricing, based on an elementary trading strategy, shows the theoretical link between an option's price and particular properties of the underlying price process. Quadratic variation, the key property of Brownian motion, can be analysed and synthesised by wavelet methods in an elegant and intuitive way, preparing the way to price options based on a broad class of wavelet-based price processes.

Two common methods for determining the price of an option are:

- the expected amount the seller of the option is going to gain or lose;
- the cost of recreating by other means the payoff created by the contract.

The standard Black & Scholes (1973) [15] and Merton (1973) [150] analysis takes the latter approach, trading (continuously) in the underlying asset and riskless bonds. The quantities of underlying asset and bond that are required in the portfolio change smoothly with respect to the price of the underlying and other variables (such as time to expiration). As mentioned in [15] (p. 642):

> Thus the risk in the hedged position is zero if the short position in the option is adjusted continuously. If the position is not adjusted continuously, the risk is small, and consists entirely of risk that can be diversified away by forming a portfolio of a large number of such hedged positions.

If trading is done in continuous time, as is assumed in the model, no costs are incurred during the lifetime of the hedge – the strategy is self-financing. The price of the option is the cost of setting up the replication portfolio by borrowing (using bonds) in order to finance buying the required amount of the underlying asset. This cost is known with certainty.

An alternative replication strategy was proposed that seems to provide, with certainty, prices that conflict with Black-Scholes-Merton.[22] The main element of the strategy, a stop-loss order, is abundant in markets. Since all trades are supposed to take place at the strike price, the strategy appears to be self-financing. This paradox was resolved in a series of papers, see Seidenverg (1988) [187], Omberg (1989) [157] and Carr & Jarrow (1990) [25].

The intuitive solution arises from identifying the false assumption that all trades can be made at the strike price. The mathematical solution, which draws the link to local time, is due to the result that Brownian motion and its common variations in the form of semi-martingales have unbounded variation (Harrison *et. al.* (1984) [85]) and bounded quadratic variation. Chapter 4.2 describes how the result can be used to develop a trading strategy for general variance-based derivatives. Chapter 4.3 develops pricing methodologies and shows applications.

[22]The issue arose from discussion of delivery options, see Livingston (1987) [130] and Kane & Marcus (1988) [114].

Chapter 2

Models for Generating Price Processes

Any intertemporal economic model can generate price processes in some form. Neoclassical models as well as models with incomplete nominal price adjustment, typified by Keynesian theory, generate equilibrium prices that vary over time in response to exogenous shocks. The time required to reach equilibrium can result in different forms of behaviour and different equilibria depending on the starting conditions.[1]

This chapter will investigate two parsimonious models which generate prices that change over time. Unlike many economic frameworks, prices of goods and wages to labour are not treated separately. This abstraction from complexity of the observed economy highlights the rich features that can be obtained from simple assumptions.

The first section (2.1) investigates the dependency of price on the nature of current and future opportunities to exchange one good for another. By maintaining an abstract form, it emphasises the role of choice in creating value. The model is placed in a multi-period but finite-horizon setting with each period reflecting one choice and the nature of choices described by a

[1] More complete comparisons of economic theory can be found in most textbooks, e.g. Romer (1996) [174].

limited number of variables. The choice available at each period is a binary one: the agent is offerred the ability either to exchange a limited amount of one good for another or else to remain in his current position. The price derived is a ratio at which the agent will be indifferent to carrying out the exchange. The price or value depends on coordinated timing of favourable exogenously offered terms of exchange, which implies natural flexibility or a strong bargaining position and these are aspects that have not been modelled.

By attributing the choice available at each moment to one of the two goods, such that a larger quantity of it provides a larger magnitude of choice, the model attributes value to the good with which it is linked. As such, the choice is embedded in the good itself and is not a characteristic of the agent making the decisions and links with literature on compound options (notably papers by Geske (1979a) [75]). Referring again to flexibility and bargaining power, it is therefore the asset with the embedded option that implicitly provides these to the agent that owns them.

Successful selection of the correct direction for each decision does depend on the agent. The decision is based on the characteristics of the current and future choices. In this regard, the model maintains ambiguity regarding information structure in order to show the clear distinction between embedded optionality and information. The price in the model can be equally well be considered as absolute or based on the perception or beliefs of the agent. In light of these restrictions, the purpose of the model is to show the value of flexibility in an idealised form.

The remainder of the chapter (starting with section 2.2) changes the basis of analysis to one bearing closer resemblance to real situations, but still keeping the intention of showing the role of flexibility and its limitations on the measure of value. The sections gradually introduce more features but at each stage the basis of the model remains that of an agent with periodic choices which are constrained to a greater or lesser extent depending on the amount of cash (or other good representing flexibility) that has been 'purchased' up to that point. The principle of section 2.1 is maintained, although the perspective moves from that of absolute and exogenously given flexibility to one of variable and discretionary efforts to avoid constraints.

The more advanced versions of the model in this chapter begin to discuss the nature of information and uncertainty in relation to decision making and their influence on behaviour of individuals and groups. While the modelling assumptions and structure become more involved, the variables can still be taken to represent a variety of situations. For example, the model can be used to describe some commodity storage decisions, from which literature some aspects of the model have been taken. Similarly, other sources of the approach in this chapter are models that have been developed in business cycle theory in order to describe consumption and saving behaviour in the broader economy.

2.1 A Simple Exchange Model

Consider a firm initially endowed with capital assets and (liquid) funds. At a set of points in time, the firm can exchange a proportion, positive or negative, of its capital for a quantity of funds. The quantity of funds paid or received in the exchange depends on both the quantity of capital and an exogenous exchange rate, or unit price of capital. The random variables affecting each exchange opportunity are independent of each other and of variables at other dates. The realisation of each random variable is not determined until the date of the exchange opportunity that it describes.

The firm wishes to maximise expected wealth, or accumulated profits, at a given terminal date, defined as any liquid funds remaining at the terminal date. All remaining capital is valueless and can be disposed of without cost. There is no other way for the firm to alter the amount of either asset and this is the only activity in which the firm is engaged. We wish to find the value of the firm, given initial endowments of capital and funds and probability distribution functions for the random variables.

Exchange opportunities occur sequentially and are proportional to the amount of capital held at the time. This creates interactions related to compound options, described in detail in Trigeorgis (1991) [202] and Abel *et. al.* (1996) [1], which do not have straight-forward valuation methods.

The problem can represent various situations.

1. valuation of a portfolio of two securities in a finite liquidity setting;
2. modelling the production possibilities set of a firm that is engaged solely in transformation of one asset into another; or
3. valuing a project with a large number of interacting expansion/contraction real options with uncertain properties.

It is worth noting that the framework presented here avoids assuming an external continuous price process and of continuous, unrestricted trading.

This section develops a recursive algorithm for valuing the firm, given distributions of a set of random variables characterising future exchange opportunities. The adopted method avoids complexity that would arise from approaching the problem as in [202] and [1]. Section 2.1.1 introduces the model and notation. Section 2.1.2 derives the valuation result for a general case. Section 2.1.3 discusses a numerical implementation of the model and section 2.1.4 discusses its features and shortcomings.

2.1.1 The model

The firm's endowment of infinitely divisible funds and capital are denoted by x_0 and y_0 respectively and at subsequent time steps by x_t and y_t, where $t \in [0, T]$ and t and T are positive integers. The objective of the firm is to maximise its expectation of x_T.

We assume that the quantity of assets that the firm can exchange at time t is proportional to y_t, where the proportionality is via an exogenous random variable c_t. If the exchange is carried out, we have

$$y_{t+1} = (1 + c_t) y_t, \tag{2.1}$$

and we can think of c_t as a growth rate.

The exchange opportunity occurring at time t has an exchange rate, λ_t, determining the quantity of funds that must be paid in exchange for one unit of capital. In general, the exchange rate can be any finite real number. For any exercise date, only one exchange rate and one growth rate apply. Funds do not affect the quantity of assets that can be traded and can be thought of as being passive. Therefore we write the effect of an exchange on the quantity of funds by

$$x_{t+1} = x_t - \lambda_t c_t y_t, \tag{2.2}$$

which can be rewritten with λ_t as the subject:

$$\lambda_t = -\frac{(x_{t+1} - x_t)}{(y_{t+1} - y_t)}. \tag{2.3}$$

The change in quantity of funds is affected by the quantity of capital and it is clearly possible for x_t to become negative. In contrast, it is reasonable to define c_t such that it is never less than -100%, with the result that y_t will not change sign – a realistic assumption when interpreting y_t as installed capital.[2]

The dynamics described above are equivalent to the firm holding a portfolio consisting of y_t units of capital where each unit has associated with it an option with the payoff function

$$\max\left[c_t \text{Value}_t - c_t\lambda_t, 0\right], \tag{2.4}$$

where Value_t is the value of capital relative to funds, to be determined.

When $c_t = 0$, there are no options. $c_t > 0$ corresponds to a call option such that $c_t = 1$ is an option to double the amount of installed capital. $c_t < 0$ corresponds to an option to put $|c_t y_t|$ units of capital at unit price λ_t. $c_t = -1$ is then an option to abandon with salvage value $\lambda_t y_t$. Note that it is assumed that the option cannot be partially exercised — either $|c_t y_t|$ units of capital are traded or none. Lastly, if the option is not exercised then $x_{t+1} = x_t$ and $y_{t+1} = y_t$.

[2]The restriction on c_t can be relaxed in a continuous-time problem to $\{c_t\}_{t\geq 0} \in \mathcal{L}^2$.

To summarise, the activity of the firm consists of selecting an action $a_t \in \{0, 1\}$ at every time step, determining the portfolio evolution

$$x_{t+1} = x_t - a_t \lambda_t c_t y_t \tag{2.5}$$

$$y_{t+1} = (1 + a_t c_t) y_t. \tag{2.6}$$

As suggested above, $c_t \in [-1, \infty)$ and $y_0 \geq 0$ in order to ensure that y_t is always non-negative. Since x_t and y_t have to be greater than zero, we have

$$c_t > -1 \qquad \text{and} \qquad 1 - \frac{c_t \lambda_t}{r_t} \geq 0 \tag{2.7}$$

where $r_t = \frac{x_t}{y_t}$. The shaded area in figure 2.1 shows the region of valid combinations of c_t and λ_t.

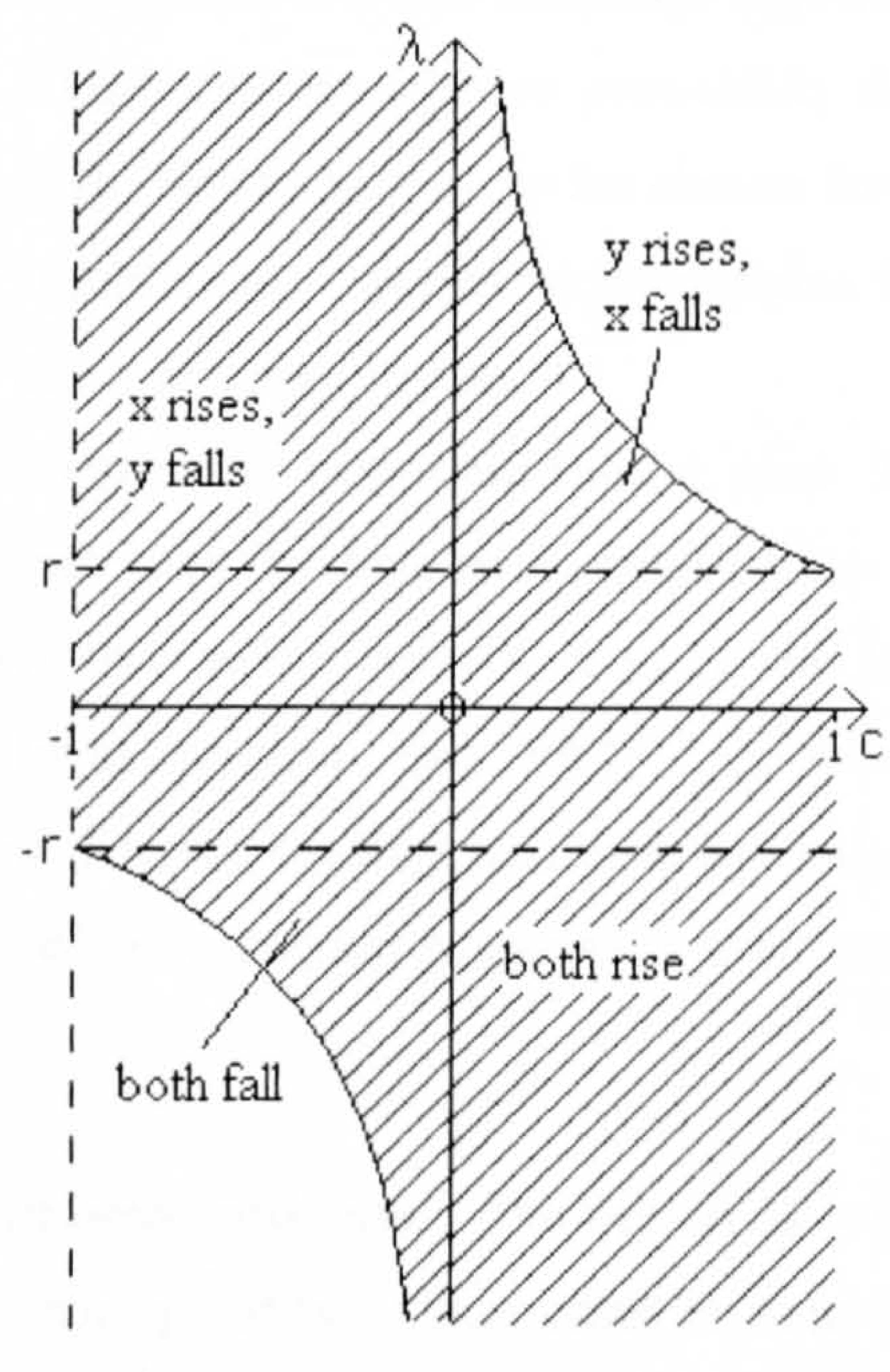


Figure 2.1: Effect of an exchange being carried out

Other things being equal, exercising a call-type option at date s increases the amount of capital at all dates $t > s$ and gives rise to a corresponding increase in the number of future exchange options. Equivalently, the exercise of a call-type option purchases a quantity of capital plus a portfolio of its embedded options in return for a quantity of asset x. This highlights the compound character of the options and that there may be interaction between exercise decisions.

The uncertainty concerning c_t and λ_t is assumed to disappear (only) at the exercise date and to be constant at other times. The first part ensures that the properties of each expiring option will be known by the firm so it can make its exercise decision with regard to the estimated effect on subsequent options. The second part of the assumption regarding uncertainty will be explained later.

The model can be used to describe notions of liquidity and reversibility as follows. A liquid market is one where exchange opportunities are dense on the timeline with a continuous cumulative probability distribution function for c_t and where $a_t \in \mathbb{R}$. Restricting a_t to be chosen from $\{0, 1\}$ introduces irreversibility while liquidity in a weaker sense remains because exchange of some sort can take place at every instance.

The existence of exchange opportunities at time T or beyond is not relevant due to the fact that the firm is maximising x_T and that exchanges at T and beyond will only affect x_{T+1} and y_{T+1} and later points in time. It is equivalent to the assumption that, $\forall t \geq T$, $c_t \equiv 0$. Similarly, the use of a finite number of discrete trading points is equivalent to a model with more trading points where a certain proportion of intermediate points have $c_t = 0$.

Comparison to financial options The region $\lambda_t > 0$ contains exchange opportunities where the quantity of one asset is decreased if the quantity of the other is increased. The region with $\lambda_t < 0$ involves simultaneous increases (or decreases) of both assets, suggesting creation (or destruction) of assets. For $c_t < 0$, $\lambda_t \leq 0$, standard financial options would never be exercised since (x_t, y_t) dominates (x_{t+1}, y_{t+1}) – assuming a positive Value_t function for y_t. For $c_t > 0$, $\lambda_t < 0$ all financial options will be exercised since the point (x_t, y_t) is dominated. Situations with $\lambda_t \leq 0$ can be interpreted as the destruction of assets, receiving of gifts or the effect of some other externality and are not wholly unrealistic outside the financial markets.

2.1.2 Valuation

The value of all future exchange opportunities is valued using a backwards induction argument. Since c_t and λ_t are independent from x_t and y_t as well as from other c_s and λ_s, $(s \neq t)$, the option at time $t = T-1$ is exercised if it will increase the quantity of x_T. Since the realisations of c_{T-1} and λ_{T-1} are known at $t = T-1$, the exercise decision is determined. At any other date $s < T-1$, the estimated probability of exercising the final option is given by:

$$\begin{aligned} \mathbb{P}[a_{T-1} = 1] &= \mathbb{P}[y_{T-1}c_{T-1}\lambda_{T-1} < 0|\mathcal{F}_s] \\ &= \mathbb{P}[c_{T-1}\lambda_{T-1} < 0|\mathcal{F}_s]. \end{aligned}$$

since $y_t > 0$ by design. This probability will only change in response to changes in the degrees of belief about c_{T-1} and λ_{T-1}. Particularly, the exercise decision will be made only on the realisations of c_{T-1} and λ_{T-1} and on the value of capital at the next time step T, which is known to be zero. For times more than one step before termination, it is not immediately clear which variables form the criteria for the exercise decision.

The quantity x_T can be written explicitly in terms of x_{T-n} and y_{T-n} and subsequent exercise decisions, as follows.

$$\begin{aligned} x_T &= x_{T-1} - a_{T-1}c_{T-1}\lambda_{T-1}y_{T-1} \\ &= x_{T-1} - a_{T-1}c_{T-1}\lambda_{T-1}y_{T-2}(1 + a_{T-2}c_{T-2}) \\ &= x_{T-2} - a_{T-2}c_{T-2}\lambda_{T-2}y_{T-2} \\ &\quad -a_{T-1}c_{T-1}\lambda_{T-1}y_{T-2}(1 + a_{T-2}c_{T-2}) \\ &= x_{T-2} - a_{T-2}c_{T-2}\lambda_{T-2}y_{T-3}(1 + a_{T-3}c_{T-3}) \\ &\quad -a_{T-1}c_{T-1}\lambda_{T-1}y_{T-3}(1 + a_{T-3}c_{T-3})(1 + a_{T-2}c_{T-2}) \\ &\vdots \end{aligned}$$

$$
\begin{aligned}
&= x_t - \sum_{v=t}^{T-1} \left\{ a_v c_v \lambda_v y_t \prod_{u=t}^{v-1} (1 + a_u c_u) \right\} \\
&= x_t - y_t \sum_{v=t}^{T-1} \left\{ a_v c_v \lambda_v \prod_{u=t}^{v-1} (1 + a_u c_u) \right\},
\end{aligned}
$$

using the conventions that a sum over an empty set is 0 and a product over an empty set is 1.

Consider the choice of a_t at some arbitrary time. If a_t is chosen to maximise $\mathbb{E}_t[x_T]$, then $a_t = 1$ if and only if

$$
\mathbb{E}_t[x_T | a_t = 1] > \mathbb{E}_t[x_T | a_t = 0].
$$

Inserting the expression for x_T yields

$$
\begin{aligned}
&\mathbb{E}_t \left[x_t - y_t c_t \lambda_t - y_t(1 + c_t) \sum_{v=t+1}^{T-1} \left\{ a_v c_v \lambda_v \prod_{u=t+1}^{v-1} (1 + a_u c_u) \right\} \right] \\
> \; &\mathbb{E}_t \left[x_t - y_t \sum_{v=t+1}^{T-1} \left\{ a_v c_v \lambda_v \prod_{u=t+1}^{v-1} (1 + a_u c_u) \right\} \right].
\end{aligned}
$$

By assumption, c_t and λ_t are known at time t and are independent random variables before. x_t and y_t are known and $y_t > 0$ by design, so

$$
c_t \left(\lambda_t + \mathbb{E}_t \left[\sum_{v=t+1}^{T-1} \left\{ a_v c_v \lambda_v \prod_{u=t+1}^{v-1} (1 + a_u c_u) \right\} \right] \right) < 0.
$$

Note that this criterion holds for any objective function that is monotonically increasing in x_T and that does not depend on capital y_t. Introducing new variables z_t (a random variable) and $\hat{z}_t$ such that

$$
\hat{z}_t = \mathbb{E}_{t-1}[z_t] = \mathbb{E}_{t-1} \left[\sum_{v=t}^{T-1} \left\{ a_v c_v \lambda_v \prod_{u=t}^{v-1} (1 + a_u c_u) \right\} \right],
$$

allows the following simplifications:

$$
c_t(\lambda_t + \hat{z}_{t+1}) < 0, \tag{2.8}
$$

$$
a_t = 1_{\{c_t(\lambda_t + \hat{z}_{t+1}) < 0\}}, \tag{2.9}
$$

$$
\mathbb{E}_t[x_T] = x_t - y_t \hat{z}_t. \tag{2.10}
$$

It is possible to express z_t as a recursive formula

$$\begin{aligned} z_t &= \sum_{v=t}^{T-1}\left\{a_v c_v \lambda_v \prod_{u=t}^{v-1}(1+a_u c_u)\right\} \\ &= a_t c_t \lambda_t + (1+a_t c_t)\sum_{v=t+1}^{T-1}\left\{a_v c_v \lambda_v \prod_{u=t+1}^{v-1}(1+a_u c_u)\right\} \\ &= a_t c_t \lambda_t + (1+a_t c_t) z_{t+1} \\ &= z_{t+1} + a_t c_t(\lambda_t + z_{t+1}). \end{aligned}$$

Note that z_t is independent of x_t and y_t. It depends on exogenous random variables c_v and λ_v for $v \geq t$ and, through a_t, on their expectations at t. Inspection of a_t shows that it is not z_{t+1} that determines each exercise decision but rather the expectation $\hat{z}_t$.

It has been assumed that, $\forall s < t$, the density functions for c_t and λ_t are unchanging with respect to s. The following notation can therefore be used:

$$\begin{aligned} \mathbb{E}[c_t|\mathcal{F}_s] &= \mu_{c_t}, \\ \mathbb{E}[\lambda_t|\mathcal{F}_s] &= \mu_{\lambda_t}. \end{aligned}$$

The effect on $z_{T-1} = c_{T-1}\lambda_{T-1}a_{T-1}$ is that it is also unchanging, leading to the notation $\mathbb{E}[\hat{z}_{T-1}|\mathcal{F}_s](=\mu_{\hat{z}_{T-1}}) = z_{T-1}$. By separating z_t into its mean and a deviation term $z_t = \hat{z}_t + \epsilon_t$, the deviation term has zero expectation at all times $s < t$. For $t = T-1$,

$$\epsilon_{T-1} = a_{T-1}c_{T-1}\lambda_{T-1} - \hat{z}_{T-1}$$

where $\hat{z}_{T-1} = \mathbb{E}_s[c_{T-1}\lambda_{T-1}|a_{T-1}=1]\cdot\mathbb{P}_s[a_{T-1}=1]$. Similarly for $t = T-2$,

$$\begin{aligned} z_{T-2} &= z_{T-1} + a_{T-2}c_{T-2}(\lambda_{T-2} + z_{T-1}) \\ &= \hat{z}_{T-1} + \epsilon_{T-1} + a_{T-2}c_{T-2}(\lambda_{T-2} + \hat{z}_{T-1} + \epsilon_{T-1}). \\ \Rightarrow \mathbb{E}_s[z_{T-2}] &= \hat{z}_{T-1} + \mathbb{E}_s[a_{T-2}c_{T-2}(\lambda_{T-2} + \hat{z}_{T-1})] + \mathbb{E}_s[a_{T-2}c_{T-2}\epsilon_{T-1}]. \end{aligned}$$

The last term is zero because ϵ_{T-1} is independent of the other terms so that $\mathbb{E}_s[a_{T-2}c_{T-2}\epsilon_{T-1}] = \mathbb{E}_s[a_{T-2}c_{T-2}]\mathbb{E}_s[\epsilon_{T-1}]$ and, by construction, $\mathbb{E}_s[\epsilon_{T-1}] = 0$. So

$$\hat{z}_{T-2} = \hat{z}_{T-1} + \mathbb{E}_s[a_{T-2}c_{T-2}(\lambda_{T-2} + \hat{z}_{T-1})] \tag{2.11}$$

and

$$\begin{aligned}\epsilon_{T-2} &= \hat{z}_{T-1} + a_{T-2}c_{T-2}(\lambda_{T-2} + \hat{z}_{T-1}) + (1 + a_{T-2}c_{T-2})\epsilon_{T-1} \\ &\quad -\hat{z}_{T-1} - \mathbb{E}_s[a_{T-2}c_{T-2}(\lambda_{T-2} + \hat{z}_{T-1})]\end{aligned}$$

$$\Rightarrow \epsilon_{T-2} = \epsilon'_{T-2} + (1 + a_{T-2}c_{T-2})\epsilon_{T-1}, \tag{2.12}$$

where $\epsilon'_{T-2} = a_{T-2}c_{T-2}(\lambda_{T-2} + \hat{z}_{T-1}) - \mathbb{E}_s[a_{T-2}c_{T-2}(\lambda_{T-2} + \hat{z}_{T-1})]$. It can be proven by backward induction that the relationships 2.11 and 2.12 hold for all $0 < t < T$. Note that if $\hat{z}_t$ can be established for any t, then formula 2.10 for $\mathbb{E}_t[x_T]$ can be applied.

$$\begin{aligned}\mathbb{E}_t[x_T] &= \mathbb{E}_t[\mathbb{E}_{t+1}[x_T]] \\ &= \mathbb{E}_t[x_{t+1}] - \mathbb{E}_t[y_{t+1}\hat{z}_{t+1}] \\ &= x_t - a_tc_t\lambda_t y_t - (1 + a_tc_t)y_t\mathbb{E}_t[\hat{z}_{t+1}] \\ &= x_t - a_tc_t\lambda_t y_t - (1 + a_tc_t)y_t\hat{z}_{t+1} \\ &= x_t - y_t\left\{\hat{z}_{t+1} + a_tc_t(\lambda_t + \hat{z}_{t+1})\right\}.\end{aligned}$$

It is known that $\hat{z}_{t+1}$ depends only on the expectation of exogenous random variables c_u and λ_u in the future $u \geq t$ (we have shown this to be the case for $T-2$ and will need to show that it holds earlier). Since $a_t = 1$ if and only if

$$\begin{aligned}\mathbb{E}_t[x_T|a_t = 0] &< \mathbb{E}_t[x_T|a_t = 1], \\ x_t - y_t\hat{z}_{t+1} &< x_t - y_t\left\{\hat{z}_{t+1} + c_t(\lambda_t + \hat{z}_{t+1})\right\} \\ c_t(\lambda_t + \hat{z}_{t+1}) &< 0.\end{aligned}$$

Therefore, a_t depends only on expectations of exogenous random variables c_t, λ_t and on $\hat{z}_{t+1}$. Since $\hat{z}_{t+1}$ depends only on the expectations of exogenous random variables c_u, λ_u further in future ($u > t$) and the expectations at $s < t$ of variables at t do not change with s, then a_t depends only on $\{\mathbb{E}_t[c_u]\}_{t\leq u<T}$ and $\{\mathbb{E}_t[\lambda_u]\}_{t\leq u<T}$. Taking the expectation of $\hat{z}_t$ at $s < t$ yields the recursion relationship as required.

To summarise, the following system of equations applies for $0 \le t < T$:

$$\begin{aligned} x_{t+1} &= x_t - a_t \lambda_t c_t y_t && (2.13) \\ y_{t+1} &= (1 + a_t c_t) y_t && (2.14) \\ \hat{z}_t &= \hat{z}_{t+1} + \mathbb{E}[a_t c_t (\lambda_t + \hat{z}_{t+1}) | \mathcal{F}_0] && (2.15) \\ \hat{z}_T &= 0 && (2.16) \\ a_t &= \mathbf{1}_{\{c_t(\lambda_t + \hat{z}_{t+1}) < 0\}} && (2.17) \\ \mathbb{E}_t[x_T] &= x_t - y_t \hat{z}_t. && (2.18) \end{aligned}$$

$\hat{z}_t$ would only change if beliefs about c_u and λ_u ($t \le u < T$) change over time. As it is necessary to assume that such change is not possible, it becomes apparent that the model cannot describe situations where gradual resolution of uncertainty about individual variables is significant.

Due to the inherent restrictions present in trading opportunities, x_t and y_t will change in an unknown manner and will depend on the particular realisations of c_t and λ_t. However, since we know a formula for $\mathbb{E}_t[x_T]$ valid for all t including zero, there is a formula for the initial value of the investment opportunities:

$$\mathbb{E}_0[x_T] = x_0 - y_0 \hat{z}_0.$$

z_t is a random variable that is only known at T (when all z_t for all t become known simultaneously). By assuming that the decision maker is maximising $\mathbb{E}_t[x_T]$, the uncertainty regarding z_t is not taken into account. The existence of the recursive formula for ϵ_t allows, at least in theory, the determination of moments and other risk measures such as value at risk, if given sufficient detail about the distributions of c_t and λ_t. As such, it is possible to extend this valuation technique to a framework with other risk-preferences and is left for future investigation.

Inspecting equation 2.17 shows that the decision maker compares $\hat{z}_{t+1}$ directly to λ_t at each decision point. We interpret the variable z_t as indicating the value of capital at time t. Its estimated value $\hat{z}_t$ is the Value_t

function referred to earlier and λ_t is the price (or cost) of capital. The intuitive as well as mathematical definition of $\hat{z}_t$ is very close to that of marginal q in investment theory. However, the added property of $\hat{z}_t$ is that it has been derived under the less idealised assumptions of uncertain investment criteria and that capital does not, of itself, provide a stream of cash flows.

2.1.3 Numerical implementation

Of the system of equations 2.13 – 2.18, equation 2.15 can be easily implemented for hypothetical situations with many time steps. This was done on a spreadsheet with a model for 50 time steps (i.e. maximising $\mathbb{E}_0[x_{51}]$). It shows the dependence of the value of capital $\hat{z}_t$ on the underlying parameters of the model and how it evolves over the investment period.

The formula for $\hat{z}_t$ can be written explicitly as

$$\begin{aligned}
\hat{z}_t &= \hat{z}_{t+1} \\
&+ \mathbb{E}_0[c_t(\lambda_t + \hat{z}_{t+1})\,|c_t(\lambda_t + \hat{z}_{t+1}]) < 0] \cdot \mathbb{P}_0[c_t(\lambda_t + \hat{z}_{t+1}]) < 0] \\
&= \hat{z}_{t+1} + \int_{c'_t=0}^{\infty} \int_{\lambda'_t=-\infty}^{-\hat{z}_{t+1}} c'_t(\lambda'_t + \hat{z}_{t+1})\, f(c'_t, \lambda'_t) d\lambda'_t dc'_t \\
&+ \int_{c'_t=-1}^{0} \int_{\lambda'_t=-\hat{z}_{t+1}}^{\infty} c'_t(\lambda'_t + \hat{z}_{t+1})\, f(c'_t, \lambda'_t) d\lambda'_t dc'_t.
\end{aligned}$$

where $f(c_t, \lambda_t)$ denotes the joint probability distribution. Note that the formulation is still valid for cases where, as suggested earlier, the marginal probability density function of c_t includes a delta function at zero.

λ_t and c_t were modelled as independent distributions to make the joint distribution function $f(c_t, \lambda_t)$ separable. The λ are distributed normally with variance σ_{λ_t}, so each has a density function

$$f_1(\lambda_t|\mathcal{F}_0) \equiv \frac{1}{\sqrt{2\pi\sigma_{\lambda_t}}} \exp\left\{-\frac{(\lambda_t - \mu_{\lambda_t})^2}{2\sigma_{\lambda_t}}\right\}.$$

Next, specifying a log-normal distribution for $c_t + 1$ constrains c_t within $[-1, \infty)$ so the density function, f_2, becomes

$$f_2(c_t|\mathcal{F}_0) \equiv \frac{1}{(c_t+1)\sqrt{2\pi\sigma_{c_t}}} \exp\left\{-\frac{(\ln(c_t+1) - \ln(\mu_{c_t}+1))}{2\sigma_{c_t}}\right\}.$$

Note that the log-normal distribution is skewed and also imposes the unnecessary restriction of zero probability density at $c_t = -1$. Given $y_0 > 0$, it follows that $\mathbb{P}[y_t \leq 0] = 0 \forall t$.

The results indicate that the value of capital can be consistently non-zero for a wide range of parameter values. Two effects were commonly observed. The first effect is that $\hat{z}_t$ decays to zero as we approach the termination date, which is consistent with the observation that the value of the options embedded in capital is compounded.

Second, $\hat{z}_t$ is typically very large (and negative as expected) for $0 \ll t \ll T$. This is due to the number of options being strictly proportional to the amount of capital and due to the lack of a minimum funds constraint (x_t may be negative without limit and typically will be so). The large valuation placed on capital for $0 \ll t \ll T$ suggests that the firm should 'borrow' as much as possible. It should initially exercise every option that buys capital even if that means $x_t < 0$. The strategy maximises the quantities of assets that can be traded for any given set of $\{c_t\}_{t\in[0,T)}$ and $\{\lambda_t\}_{t\in[0,T)}$. Given that trades which the decision maker thinks are probably detrimental can be avoided, the strategy will maximise profits.

2.1.4 Discussion

The basic version of the consumption-based capital asset pricing model describes the nature of the interplay between the two ideal types under certain conditions, such as the asset being exchangeable for money at a price that changes over time in a regular fashion.

In Merton (1973) [149] develops "...an equilibrium model of the capital market which (i) has the simplicity and empirical tractability of the capital asset pricing model; (ii) is consistent with the expected utility maximization and the limited liability of assets; and (iii) provides a specification of the relationship among yields that is more consistent with empirical evidence." but goes on to say that it cannot be constructed without assumptions like homogeneous expectations, which "make the new model subject to some of the same criticisms."

The model developed in the chapter above uses transformations to represent real options, occasions when the firm has the ability to increase or decrease the scale of a project, or amount of installed capital, conditional upon a corresponding injection or release of funds. Alternatively, the opportunities can represent the possibility to exchange with another party some quantity of one asset for some quantity of the other.

As compared to portfolio theory, the assets are not tradable except at certain dates corresponding to the transformation dates. As compared to real options theory, we have introduced an element of uncertainty to the properties of future exchange opportunities: the quantities of assets that can be traded are not known with certainty until the date at which the exchange can take place. In both cases, the lack of continuous tradability means that there is no explicit market price for the assets. The model proposes a valuation despite such 'market imperfections', although the valuation depends entirely upon subjective beliefs.

Particular cases of the model's solutions are realistic optimal strategies and are robust to variation in the parameters. When there is uncertainty about both the exchange ratio and quantity of options, which is the most realistic of the scenarios, the model's valuation is unrealistic. The main source of the problem is the strict scaling of trading opportunities with the amount of capital. This is, however, necessary to ensure non-negativity of capital and is a close approximation of many real situations.

Maximising expectation implies that the investor does not care more about negative increments of the terminal cash balance than positive ones, encouraging the extreme borrowing strategy. Such behaviour is not realistic in situations of constrained borrowing and for risk-averse economic agents[3], or where conversion of cash to consumption is not a linearly scalable process. Various formulations of solvency constraint are possible, reflecting the vari-

[3]The term 'risk' meaning either the probability of loss or the spread of possible outcomes.

ous legal solvency measures and the results of debtor-creditor bargaining.

So far, the model is of an agent interacting with nature. A market economy populated with interacting agents could be used, with one desirable aim being to endogenise the distributions of the exogenous variables in terms of other parameters. The variables $\mathbb{E}[|c_t|]$ and λ_t may take on meanings closer to those of market liquidity and spot price respectively.

While Appendix A points to several direct extensions of the framework, Section 2.2 develops a different model to include solvency constraints in a more natural manner. It is subsequently extended to incorporate interaction between agents.

2.2 Recursive Optimisation Models

The previous model required characterisation of all future exchange opportunities. In the following analysis, the decision at each time period is allowed to be implemented in degrees, rather than as a boolean choice, which takes away from the idealisation of the previous model and brings the analysis closer to a wider variety of economic situations thereby addressing some of the questions raised in section 2.1.4. The following sections gradually work towards a model of an economy populated by agents making choices subject to broadly similar aims and constraints on choice.

The initial versions follow easily tractable models that characterise small scale or isolated behaviour of individuals under a set environment (which is subject to some uncertainties nevertheless). Once some of the properties of the environment are endogenised and interaction between agents is included, the agents can in principle have much greater control of their environment. Concurrently, given that the agents have limited means of communication and gathering information, their degree of control is limited again to a level that is commonly observed in the global economy. Although the two conflicting drivers maintain an overall balance, the types of behaviour possible become much broader and the analysis is more easily done by computer simulation. The chapter ends by summarising the results of investigation of the simulation model, showing the varied forms of individual and systemic behaviour that can be created by changing parameters of the model.

2.2.1 Sequential Storage Decisions

Smith (p. 207) writes,

> "The constancy or inconstancy of employment cannot affect ordinary profits of stock in any particular trade. Whether the stock is or is not constantly employed depends, not upon the trade, but the trader."

Although the ordinary profits are not an easily identifiable quantity in many markets, the statement nevertheless makes the point that participants in the economy spend effort on using available resources in the best way possible.

The reasons for holding stock are most commonly cited as

- a transactions demand depending on the volume of trade and that is dictated by the nature of the business and its processes,
- a contingency stock held to act as a buffer against unforseen events that can have a negative impact upon the flow of resources,
- stock held in anticipation of changes in the value of stock in the future, including seasonal changes of demand as well as speculative holding of stock that will not necessarily be used by the business.

In all cases, holding stock can be considered as providing the trader with flexibility and give a greater choice but not an obligation to perform certain actions. The notion of stock here therefore holds the same meaning as the asset in the previous section. In the same vein as before, the following analysis shows how the trader may plan employment of stock.

A profit-maximising business aims to make profitable investments. If it has almost unlimited access to resources, it can invest in all available projects that are in some sense profitable. In practice, there are many limitations to the choices the firm can make and this section considers limitations that can be attributed to scarcity of resources.

A business uses a large number of different resources, ranging from tangible examples such as money, employees and machinery to intangibles such as brand loyalty, trade relationships and corporate culture. In general, the resources can be used by the business to realise profit from its investments. The fact that investments do frequently lose money is attributed to arising from imperfect control of resources and measurement error. Examples are apparent in the issues of incomplete contracting, bargaining and principal-agent interactions.

The nature of scarce resources and the interactions between them means that some are seen as assets while others can become liabilities. The following analysis will treat money as separate from other resources in order to create a framework that uses familiar tools of economic theory.

A business usually has access to a finite quantity of funds at any time, making them a scarce resource. Consequently, businesses need to make some form of decision about how much of the available funds to commit to investments. By entering into financial contracts or arranging its non-financial projects suitably, the business can protect itself against situations when exogenous influences cause liquidity problems. The reduction of risk means the firm can control its financial flexibility more easily and reap the benefits of its improved risk characteristics.

As an initial approximation, exogenous influences can be treated as a single source of noise and the company's resulting financial flexibility modelled using a stochastic process. In practice, the various accounting measures of financial flexibility[4] would need to be represented by a set of related stochastic processes. The model developed in this section, with a single source of noise, provides an estimate of the probability of running into financial distress and gives a direct link to hedging and pricing the insurance contract[5]. However,

[4]Measures such as the cash conversion cycle, the quick ratio and debt service cover ratio are used to measure flexibility at various levels of corporate activity.

[5]as shown in Section 4.2

this method is not well suited to dealing with non-stationarities present in the environment. It also does not allow for information and assumptions held by the business that affect its plans and actions. It is therefore desirable to compare the method to a model of the firm's behaviour in response to exogenous influences and calibrate prices using both methods.

It is customary to classify investment as the sum of a maintenance part, needed to keep capital stock constant, and a part that represents increases in the stock level. Due to the fact that the model will use business plans that frequently contain non-stationary time series, the classification used here will be slightly different.[6] Investment by the firm consists of a voluntary investment component that increases the firm's quantity of installed capital faster than the business plan and of a maintenance component necessary for continuity of business (again, according to the business plan).

Cash returns from the investment part of capital expenditure may take place many months or years after the cash outflow, whereas the returns on maintenance spending are assumed to be relatively short-term. As an initial approximation to the complicated path-dependent behaviour of cash flows, it is therefore assumed that the growth of average profits from expected investments is already contained in the business plan and that any significant cash effects due to deviations from the business plan occur after the planning horizon. This ensures that the framework of expected profit maximisation remains applicable.

The model described here draws on two papers and applies the results in the context of behaviour of a firm. The main parameters discussed here are the amount of voluntary investment[7] and the quantity of liquid funds held by the firm. The first paper, Deaton (1991) [44] shows a rational expectations equilibrium model of the saving behaviour of liquidity-constrained consumers. The second, Routledge *et. al.* (2000) [175] develops a link re-

[6]Similarly, terminology is borrowed from accounting throughout the thesis but does not necessarily adhere to accounting conventions, the definitions being driven instead by the models developed herein.

[7]The term 'voluntary' will be implicit unless stated otherwise.

quired to price derivatives but in the context of commodity pricing. The fundamental parameters in the thesis necessarily have different meanings than in either of the papers above, however the structure of the models are closely related.[8] Despite the similarities, a few differences were introduced to the model developed here for the following reasons.

Both the commodity market and consumer behaviour models assume strictly positive income that must be redistributed between immediate use and storage, given an objective function that has a form resembling discounted expected utility. The commodity market model assumes the existence of a well-behaved invertible demand function that gives the market-clearing price of the commodity for any given level of immediate use. The market discount rate is given by the cost of holding inventory (assumed to be proportional to the quantity stored) per unit time. The consumer behaviour model uses utility functions, yielding a marginal utility of consumption at each point in time. Future utility of consumption is discounted at the consumer's personal time discount rate.

The firm here is assumed to maximise expected profits and to have no time-preference or risk-aversion apart from that implied by its cost of funds. Consequently, the firm maximises the sum of expected profits discounted at the appropriate cost of funds for each project. The firm's funding mix is left undefined and is only altered by the effect of the choice between investing current profits now or adding them to an inventory of liquid funds.

In contrast to the other models in the references above, it is necessary for the firm to be subject to a risk of some form of loss – of being forced to make negative 'investment'. Consumer saving and commodity market models typically maintain strictly positive consumption (or immediate use) when the liquidity constraint becomes binding. The feature is appropriate for those models, where negative consumption does not have a physical counterpart and because income (or supply) is assumed to be strictly positive.

[8]Routledge *et. al.* (2000) [175] also derive convenience yield as an endogenous result. It is interesting to consider the meaning of the equivalent concept when applied to the internal 'market' for cash within the firm.

Businesses, however, have financial obligations and face costly bankruptcy, liquidation or closure. The risk of loss can be introduced by allowing cash flow and investment to be negative. The profit function (corresponding to the inverse demand and utility functions) needs to become negative at some point, although not necessarily as soon as voluntary investment becomes negative (as this only represents investment becoming less than the business plan).

Under normal circumstances, a firm that reduces costs is still investing since each action increases the expected profits. While in some circumstances, a firm may sell assets because it can sell them at a profit (the price exceeds the profits they would make for the firm), in other circumstances assets may be sold in a fire sale situation. While the firm is still aiming to maximise profit, the assets may be sold for less than the profit they would generate for the firm on a standalone basis. This is an indication that the short-term value to the firm of an incremental quantity of money is higher than the profitable investment it sells. The fact that it had previously started the investment suggests that money was more easily available and that the firm has encountered financial distress.

Lastly, when used for commodity price modelling, the cited papers use market price histories to calibrate many unobserved parameters. In contrast, many of the fundamental parameters here are observable or more easily estimated by the firm than the shadow price of marginal funds, leading to a different procedure of calibration.[9]

2.2.2 Objective and Constraints

The firm earns a low, riskless real return (possibly negative) on its cash balances whereas physical investments are risky. In each period, it must use gross cash income to cover operating costs. The remaining cash flow, which

[9]The cost of funds in some cases can be estimated using the cost of standby liquidity facilities, commercial paper programs and other short-term, dynamic financing mechanisms. This can be useful to check the assertion above.

may be negative in some instances, is hereafter referred to simply as cash flow and denoted a_t. It is assumed that the firm will keep liquidity (the sum of undrawn borrowing facilities, marketable securities and cash) if doing so raises the discounted expected value of the firm.

The firm can use the cash flow in each period plus any liquidity remaining from previous periods to make net investments (capital expenditure needed to maintain its capital stock is included in operating costs), pay dividends or raise its cash balance. All of these expenditures are assumed voluntary and are denoted by c_t. Implicit in this assumption is that the firm can, to an extent, control some of its cash flows (change capital expenditure, accumulate or run down work in progress, change the timing of transactions with trade debtors and creditors, etc.). From here on, the differences between net investment in operating projects and payment of dividends are ignored. It is assumed that the value of the firm's equity corresponds exactly to the value of investments, so maximising shareholder value is equivalent to maximising returns on investments.

Changing the level of investment can be thought of as changing the size and constitution of its portfolio of active projects, attacking the lowest-return projects first. Such actions are accompanied by costs or benefits (such as increased revenues, redundancy payments, etc.) that will be realised at various points in the future.[10] The future benefits of such deviations will be summarised only by a decrease in the firm's current marginal return on investment. The marginal return function is finite, invertible and decreasing in the current voluntary investment level.

The model will use projections of cash flows that are subject to two risks, both of which increase with time into the future:

- Forecast error. Overall profitability y_t is assumed to grow along a geometric path in most business models so that y_t is not stationary. The growth $g_t = y_t/y_{t-1} - 1$ may be higher or lower than assumed in

[10] It is assumed that there are no hysteresis effects from raising and lowering investment, i.e. from starting up and shutting down marginal projects.

the budget and such deviations from the forecast tend to be persistent. If the forecast growth is denoted μ_t, then $g_t - \mu_t$ can be modeled as an autocorrelated series of random variables.

- Short-term fluctuations. Receipts and payments in any one period follow seasonalities due to periodic payments to and from customers and suppliers. These seasonalities might not be treated explicitly in the business plan (and therefore not appear in μ_t). In addition, cash flow in each period may be larger or smaller than the long-term average because of uncertain timing of cash flows, resulting in short-term fluctuations of the cash balance. These fluctuations will be summarised by a negatively autocorrelated zero-mean series of random variables ϵ_t, such that $a_t = y_t(1 + \epsilon_t)$.

These cash flow shocks are driven by exogenous random variables and are ultimately what determine the behaviour of prices.

Both supply and demand for cash are stochastic, whereas the model assumes that investment is a deterministic function of the income state (y_t), so the variables g_t and ϵ_t can be seen as including the stochastic part of investment as well as of income. The chosen framework means that it is necessary to use three variables, g_t, ϵ_t and x_t, in order to describe the state of the system fully, where the new variable x_t represents the amount of cash available to spend in period t and is described below.

If there is no storage, each period's cash income, a_t, must be spent, so that the equation $c_t = a_t$ would apply in every period. Introducing storage (cash deposits and equivalents), $q_t \geq 0$, means that the cash at hand, $x_t := q_t + a_t$, in any period is the sum of deposits remaining from the previous period (plus interest accrued) and the current cash flow:

$$x_{t+1} = (1 + r)(x_t - c_t) + a_{t+1}, \qquad (2.19)$$

where r is the (riskless) real return on cash deposits. Note that expenditures do not have to equal income x_t and can be negative. The firm's decision variable each period is (only) the level of investment. The return on investment is assumed equivalent to the increase in value of equity, denoted by $v(c_t)$ and takes the form of a von Neumann-Morgenstern utility function. The firm's value function is therefore shareholder value, given by

$$V_0 = \mathbb{E}_0 \left[\sum_{t=0}^{T} \left\{ \frac{v(c_t)}{(1+\delta)^t} \right\} \right]. \tag{2.20}$$

A well-known result (see e.g. [46]) states that the convexity of the profit function is sufficient to introduce prudence (holding a non-zero quantity liquid reserves) into the firm's behaviour.

Saving in one period makes additional funds available in the next period that can be used to increase investment – the firm has the option to sell cash to its future operations at the return on marginal investment. The expected gain from the marginal unit of liquidity is, therefore, given by:

$$\mathbb{E}_t[\pi_{t+1}] = \frac{1+r}{1+\delta}\mathbb{E}_t[\lambda(c_{t+1})] - \lambda(c_t), \tag{2.21}$$

where the notation $\lambda(c_t) \equiv \frac{\partial v(c_t)}{\partial c_t}$ has been introduced. It is by now apparent that the setup is following the dynamic programming approach and that equation 2.21 is the Euler equation. A profit-maximising firm (that is therefore also maximising shareholder value) will increase or decrease current investment until $\mathbb{E}_t[\pi_{t+1}] = 0$.

Introducing the constraint, $c_t \leq x_t \ \forall t \in \{0, 1, ..., T\}$, to the firm's maximisation problem means that it cannot choose its investment level without bound. Note that this constraint is equivalent to a non-negativity constraint on cash balances, $q_t \geq 0$.[11] This borrowing constraint is maintained here. However, if cash flow a_t is negative and q_t is small enough, x_t may be negative and, in such instances, investment c_t will be negative.

The finite-life firm's maximisation problem can be decomposed into a stepwise backward induction process. It can be represented in terms of a value function maximisation or in terms of the price of marginal investment,

[11] A result of the cited models is that the degree of prudence is increased by the borrowing constraint, which does not obviously carry over to this paper.

both of which follow. The solutions are linked by the envelope property $p_t = V_t'$.

$$V_t(x_t, g_t, \epsilon_t) = \max_{c_t \leq x_t} \left\{ v(c_t) + \frac{1}{1+\delta} \mathbb{E}_t \left[V_{t+1}(x_{t+1}, g_{t+1}, \epsilon_{t+1}) \right] \right\} \quad (2.22)$$

$$p_t(x_t, g_t, \epsilon_t) = \max \left(\lambda(x_t), \frac{1+r}{1+\delta} \mathbb{E}_t \left[p_{t+1}(x_{t+1}, g_{t+1}, \epsilon_{t+1}) \right] \right) \quad (2.23)$$

The first term on the right hand side of the second equation is the price of funds if all available funds (x_t) are used for investment in the current period, leaving no cash deposits for future periods. The second term is the expected marginal return on investment in the next period discounted at the firm's cost of funds.

2.2.3 Non-Stationarity and Equilibrium

The existing literature (e.g. Deaton and Laroque (1992) [45] and Routledge *et. al.* (2000) [175]) proves existence of a stationary rational expectations equilibrium (SREE) in the form of a set of functions that describe the optimal level of investment and corresponding marginal price at all times as a function of the amount at hand x_t, assuming the firm's planning horizon T in equation 2.20 is infinite and that the state of the system evolves in a stationary manner (in this case ensured by making $\{a_t\}_{t\geq 0}$ i.i.d. random variables). The solution and proof use an iterative algorithm starting with a straightforward finite-life solution, letting $T \to \infty$ and showing that on successive iterations, the time-zero optimal function converges to a unique solution.[12]

It is straightforward to apply the same solution technique to models with more state variables, by defining the equilibrium functions as functions of all the state variables. The number of computations required to solve the enlarged problem is kept tractable by approximating the (continuous) state variables by a finite number of states.

[12] In general, the infinitely lived agent setup is formally equivalent to one in which agents live only a finite number of periods themselves, provided they derive utility from the utility of their descendants (a bequest motive). The argument is detailed in Barro (1974) [6].

The behaviour of the firm during the business plan cannot be found in terms of a stationary solution.[13] Instead, a finite-life utility maximisation must be performed, with an appropriate choice of terminal boundary condition for the iterative solution algorithm. The boundary condition determines the firm's behaviour in the final period. A firm that really has a finite life can be thought of as investing all remaining funds in the final period (and reaping the returns subsequently). This boundary condition is used to find the stationary equilibrium solution, however it does not approximate the behaviour of a going concern at the final date covered by its current business plan. The SREE model needs to be used to solve for the firm's optimal behaviour beyond the business plan where, in line with conventional valuation methods, the model's parameters can be assumed constant over time. This creates a model yielding optimal behaviour that can change during the current planning horizon in response to anticipated fluctuations of the business environment.

In order to be able to reach a stationary equilibrium solution with non-stationary income, the variables for investment, income and the quantity of cash at hand are scaled by y_t, so that the equilibrium functions are defined in terms of ratios (see [44], section 2.1). The new variables are

$$\begin{aligned} w_t &= x_t/y_t = (q_t + a_t)/y_t \\ \theta_t &= c_t/y_t, \end{aligned}$$

and the other state variables, income growth and cash flow noise, appear as

$$\begin{aligned} 1 + \epsilon_t &= a_t/y_t \\ 1 + g_{t+1} &= y_{t+1}/y_t. \end{aligned}$$

The equation describing the evolution of the amount at hand ratio becomes

$$w_{t+1} = 1 + \epsilon_{t+1} + \frac{(1+r)}{(1+g_{t+1})}(w_t - \theta_t) \tag{2.24}$$

[13]The business plan can be characterised as a non-stationary, non-random change in the mean.

and the price equation linking investment across periods of time becomes

$$p_t(w_t, g_t, \epsilon_t) = \max\left(\lambda(w_t), \frac{1+r}{1+\delta}\mathrm{E}_t\left[p_{t+1}(w_{t+1}, g_{t+1}, \epsilon_{t+1})\right]\right). \quad (2.25)$$

The subscript of the variable θ_t is omitted if it is the stationary solution, reflecting the fact that it will not depend on the time period (only on the other three state variables g_t, ϵ_t and w_t). The terminal condition for the finite-horizon solution is $p_T = \lambda(x_T)$. The iterative solution method, whether stationary or not, keeps the following relationship between θ and p:

$$p_t(x_t, g_t, \epsilon_t) \equiv \lambda\left(\theta_t(x_t, g_t, \epsilon_t)\right), \quad (2.26)$$

since both quantities are the rational expectations optimal solutions for that state.

As a result of allowing negative income and using ratios in deriving stationary equilibrium, the choice of profit function is important. References [44] and [175] choose the iso-elastic function, but it is not appropriate for negative investment values. The exponential function

$$\lambda(c) = be^{-\rho c} \quad \text{and} \quad v(c) = \frac{b}{\rho}\left\{k - e^{-\rho c}\right\} \quad (2.27)$$

with $b > 0$, $\rho > 0$ has several properties that warrant exploration. First, λ is convex over the entire real line, corresponding to diminishing marginal returns to investment. Second, v is bounded above by bk/ρ, indicating that even though an infinite amount of money can be invested profitably, there is a finite availability of profits from investments. This is a reasonable approximation of a firm with a finite set of investment opportunities.

If $k = 1$ then v preserves the sign of its parameter c. This corresponds to a situation where any disinvestment is unprofitable (and reduces the overall value of the firm's equity). Higher values of k indicate that a degree of disinvestment can take place that still makes an overall profit. This corresponds to the assumption that the firm's normal operations (business plan) generate a stream of profits. Usefully, this choice is arbitrary since optimal investment behaviour is derived by equating marginal returns over time.[14]

[14]The choice of k does affect the growth rate of an economy populated with many agents and is not arbitrary if using the framework to value the firm.

In order to facilitate finding the SREE function θ, the profit function needs to take θ as its parameter rather than c. Making the change yields

$$\lambda(\theta) = be^{-\rho\theta} \tag{2.28}$$

$$v(\theta) = \int_{c=\bar{c}}^{c_t} \lambda\left(\frac{c}{y_t}\right) dc = \frac{by_t}{\rho}\left\{k - e^{-\rho\theta}\right\}. \tag{2.29}$$

Interpreting $v(\theta)$ rather than $v(c)$ as the actual profits from investment creates a situation where the profits available to the firm are bounded above but the boundary scales with the size of the firm's average cash income y_t. As a result, larger firms have access to more investment opportunities while all firms have access to identical returns on investment and so firms of all sizes face similar restrictions to their ability to expand suddenly.

The incentive to invest and expand is included in the model, implicit in the utility function, but a period of large positive investment does not directly raise the model's average cash flow in later periods. In a version of the model to be explored later, high spending by one firm does raise the cash flow of its neighbours in the next period, so an indirect effect can take place. Consequently, the model should only be applied to short horizon plans.

2.2.4 Investment Asset or Consumption Good

Routledge *et. al.* (2000) [175] classify two ways in which a commodity is priced: as an asset or consumption good. If it is optimal for the commodity to be consumed in the current period, it is priced as a consumption good. If it is optimal for the commodity to be stored for future use, it is priced as an asset. A consumption good reaches satiation point at a finite value, whereas an investment asset does not.

The form of the (sub)profit function and the resulting indirect felicity (referring to the utility-like function determining the agent's choice criterion) and value functions also give rise to two measures. First, the expected sum of discounted total return on future investments is equal to the level of the value function, giving rise to the valuation of the firm as an asset. Second, the gradient of the value function can represent a demand function, so that it corresponds to the price of marginal investment or consumption.[15]

The asset value of the firm rises with expected investment levels and has a fairly straightforward interpretation. The marginal price aspect is a direct model of the marginal value of short-term funds to the firm - i.e. a short-rate model of interest. Each aspect has interesting consequences for existing theories.

The asset value aspect emphasises the difference between obligations and options. It suggests that debt financing cash flows should be considered as part of the company's operational, rather than financing, costs because of the necessity to service debt in order to continue business as usual.

The recursive optimisation model of the firm's discretionary cash flows provides, through methods very similar to well-established financial models, a single model of the first three observations. The recursive optimisation model of the firm's discretionary cash flows may be able to illustrate links between the 'market' rate of interest as well as individual credit spreads. This is the topic of the next section.

2.2.5 Forward prices for liquidity

Following Routledge *et. al.* (2000) [175], the marginal return on investment in each period (and its variability with respect to the state variables) can be used to derive forward value of investment and some dynamics of the forward price. Although the model in [175] is intended for pricing commodity derivatives, the concept can be applied to the current context. The methodology makes it possible in principle to price options and other derivatives on the company's liquidity state by using the standard deviation and other

[15]The appropriateness of each measure depends on the situation being modeled and on the parameters of the model. Hull (2003) [97] refers to the relative abundance of financial investors and (non-financial) consumers in the market as a determinant of the nature of the market, due to liquidity effects arising from their behaviour. This description is close to the model of the section, with different definitions of the wealth-to-noise ratio.

properties of the forward prices.[16]

Since the marginal prices for liquidity are calculated at discrete intervals of time, the resulting forward prices will not allow a continuous time specification of the demand for liquidity. The inter-period price changes are directly modelled but intra-period price changes are not. This issue is addressed by interpolation between time periods and is now described.

First, a measure of the magnitude of intra-period fluctuations can be derived by extending the analysis to more than one agent and measuring the dispersion of marginal prices for liquidity across different agents at the same time. Second, the measure of short-term volatility can be used to determine the quadratic variation of a Brownian path (bridge) between the start and end points of the time period, where the forward price has been determined. The Brownian bridge can be simulated using the wavelet method, which is described in Chapter 3. The analysis required to generate these time series is described in section 2.3.

2.3 Building Up Model Complexity

This section will introduce more complexity to a basic recursive optimisation model in an attempt to add more realism. The models will create time series that can be likened to those of a market. Importantly, Walsrasian equilibruim will not be assumed to exist at each point in time. Neither will it be assumed that a market-maker or other explicit market-order matching mechanism operates.

Beginning with a model similar to that of Deaton (1991) [44], the following changes are made:

- There is a market consisting of N agents. All agents assume that future cash flows follow the same behaviour as in the basic recursive

[16]The resulting predictions may need to be checked for features such as the effect of parallel existence of spot and forward-contract markets. See e.g. Carlton (1979) [23] for a model without storage.

optimisation model and do not learn about the model parameters (or the model structure) or subject it to any confidence testing. I also assume that each agent's cash flows are generated in an exogenous manner, with the same global variable y_t applying to all agents. In addition to the Deaton [44] model, each agent i has to infer the states g_t and $\epsilon_{i,t}$ (and therefore of y_t) from the history $a_{i,t}$, which they know exactly. It is important that agents do not know y_t exactly (or even approximately) except by learning from the observed time series $a_{i,t}$. In the first version of the model, agents begin the time series with identical prior beliefs regarding the initial state $(y_0, g_0, \epsilon_{i,0})$. This will be relaxed in later versions.

Under these assumptions, agents may make different estimates of the regime due to their independent cash flow noise variables.[17] The sets of asset values and commodity prices generated over time in this way yield time-series for the market.

- The agents' behaviour is as the previous case. I assume that the market is the entire economy and I define its structure and define each agent's idiosyncratic cash flow noise and the global income y_t, thereby allowing the finite-size economy to be deterministic. I will compare the time-series generated in this way with those generated from the simpler model.

- Agents are aware of the mechanisms operating in the economy, but do not know all the state variables of the economy. This necessitates their continued use of the stochastic regime-switching model for the economic macro-state, the validity of which they do not, however, question. They use their knowledge of the mechanics to estimate the

[17]This model may represent a market populated by agents that are a small subset of the economy and whose cash flow incomes are dominated by factors outside the market model. Under a commodity market view, the model represents agents with access to a costly storage technology and who use a commonly assumed market demand curve to make inventory decisions.

beliefs and consequent behaviour of other agents, in order to refine their own estimates of the transition probabilities between the assumed states.

- Each agent uses observations in order to adapt its estimates of the stochastically stable growth rates in each regime as well as adapt its estimates of transition probabilities. The agents still formally assume a regime-switching model but, if the two stochastically stable states coincide and the noise processes are not regime-dependent, there is in effect no regime switching.

2.3.1 Simplest market model — no interaction

The market consists of N agents, indexed by i, with identical profit functions $v_i(\theta_i) = v(\theta_i)$ who seek to maximise the sum of expected discounted profits as before. The average income quantity y_t applies to all agents, therefore so does the growth rate $g_t = \frac{y_t}{y_{t-1}} - 1$. Each agent experiences a cash flow noise $\epsilon_{i,t}$ that is independent from and identically distributed to the cash flow noise for that period for all other agents. Furthermore, $\epsilon_{i,t}$ depends only on $\epsilon_{i,t-1}$. In summary, the market consists of N agents acting independently of each other, who do not affect each other or the market's overall growth rate and whose only shared property is the market size variable y_t and its corresponding growth rate.

The time series are constructed in the same way for all models, as will be described now. At each time period, the effects of the actions of all agents in the previous time period are calculated, yielding global income and income growth rate as well as the individual cash flow noise for each agent. Based on the new amount at hand and estimate of the corresponding growth regime, each agent makes a spending decision. The decision-making process implicitly involves a valuation of expected future profits, or 'asset value', and a marginal value of current funds, or 'commodity price'.

Each agent is assumed to make the decision in turn, in an arbitrary but constant order. The agent's asset value and commodity price are posted in the time series as they make their decisions, but no agent has access to the current-period actions of others. This reflects to some degree the discrete, periodic monitoring employed by most market participants. Once all agents have made their decisions, the calculations for the next time period are made. Decisions in the next period are added to the end of the time series as before. The model does not describe a market-clearing mechanism, but the movement of cash is consistent – i.e. cash is conserved and its flow is traced precisely.

The intra-period time series generated by this method cannot be expected to yield any useful information about market dynamics and only generate a distribution of marginal prices within the period. The inter-period changes of each individual's marginal price and the overall price distribution indicate the characteristics of the market and show features to be compared to empirical market price dynamics.[18]

Some parameters will be more important than others, with some affecting only the scale of certain types of behaviour, while some will determine whether or not a particular mode of behaviour takes place at all. It is expected that the marginal price formula's exponent, which controls the curvature of the marginal price of consumption with respect to the quantity of consumption, is an important variable. To repeat the price formulae above,

$$\begin{aligned}\lambda(\theta) &= be^{-\rho\theta}\\ v(\theta) &= \frac{by}{\rho}\left\{k - e^{-\rho\theta}\right\}\end{aligned}$$

the parameter ρ affects both the sensitivity to changes in consumption as well as the maximum level of profit that can be earned in any one period.

2.3.2 Estimation method used by the agents

The agents estimate the current state and make spending decisions by adapting an algorithm presented by Hamilton (1989) [82]. A change is required

[18] Features such as local or stochastic volatility, leverage effect and regime switching.

in order to compensate for the dependence of a pair of probabilities that are assumed independent in [82]. The procedure is outlined here.

The historical path of growth states s_t from time t_0 to T will be denoted as $s_{T\leftarrow t_0}$. Given a value for y_{t_0-1}, each path defines the income level y_t and for all time steps $t \in \{t_0, t_0+1, \ldots, T\}$. When combined with the observed history of cash flows, it implies a noise state for each time period through the relationship

$$y_t(1+\epsilon_{i,t}) = a_{i,t} \quad \forall i, t.$$

The observed history of cash flows is denoted as $a_{T\leftarrow t_0}$. The combination of $s_{T\leftarrow t_0}$ and $a_{T\leftarrow t_0}$:

$$s_{T\leftarrow t_0} \cap a_{T\leftarrow t_0} \equiv \omega_{T\leftarrow t_0}$$

defines the complete state-space history of the agent. The realisation of any random variable X at time t in a particular path is denoted $X_t(\omega_{T\leftarrow t_0})$ for $t_0 \leq t \leq T$.

Starting with a known income amount y_{t-m-1}, an observed history of $m+1$ cash flows $a_{t\leftarrow t-m}$ and a vector $\mathbb{P}(s_{t\leftarrow t-m}|a_{t\leftarrow t-m})$ of length 2^{m+1} containing the probability for each possible growth state path over $m+1$ time steps we can calculate the probability of the next growth state for each path history as a vector of length 2^{m+2},

$$\mathbb{P}(s_{t+1\leftarrow t-m}|a_{t\leftarrow t-m}) = \mathbb{P}(s_{t+1}|\omega_{t\leftarrow t-m}) \cdot \mathbb{P}(s_{t\leftarrow t-m}|a_{t\leftarrow t-m})$$

where the conditional probability $\mathbb{P}(s_{t+1}|\omega_{t\leftarrow t-m})$ simplifies to $\mathbb{P}(s_{t+1}|s_t)$ due to the assumption that growth state transitions are independent of other variables and governed by a first-order Markov chain.

The cash flow probability density in the next period is given by probability of the income level – assumed to be concentrated at two point masses – superimposed on an independent first-order Markov multiplicative noise distribution. This will create a mixture of distributions that need to be placed in the same vector.

Assuming we have a probability density function $f(y_t|a_{t\leftarrow t-m})$ for the distribution of the income level, we can write as a mixture of distributions

$$\begin{aligned}
f(y_{t+1}|a_{t\leftarrow t-m}) &= f(y_{t+1}\cap s_{t+1}=1|a_{t\leftarrow t-m}) \\
&\quad +f(y_{t+1}\cap s_{t+1}=0|a_{t\leftarrow t-m}) \\
&= f(y_{t+1}|s_{t+1}=1\cap a_{t\leftarrow t-m})\mathbb{P}(s_{t+1}=1|a_{t\leftarrow t-m}) \\
&\quad +f(y_{t+1}|s_{t+1}=0\cap a_{t\leftarrow t-m})\mathbb{P}(s_{t+1}=0|a_{t\leftarrow t-m}) \\
&= f(y_t(1+\acute{g})|a_{t\leftarrow t-m})\mathbb{P}(s_{t+1}=1|a_{t\leftarrow t-m}) \\
&\quad +f(y_t(1+\grave{g})|a_{t\leftarrow t-m})\mathbb{P}(s_{t+1}=0|a_{t\leftarrow t-m}).
\end{aligned}$$

The joint conditional density function for the next period's cash flow and growth state is given by

$$\begin{aligned}
& f(a_{t+1}\cap s_{t+1\leftarrow t-m}|a_{t\leftarrow t-m}) \\
= \ & \mathbb{P}(s_{t+1\leftarrow t-m}|a_{t\leftarrow t-m})\cdot f(a_{t+1}|s_{t+1}\cap \omega_{t\leftarrow t-m}),
\end{aligned}$$

which is a vector with $N_\epsilon \times 2^{m+2}$ elements, N_ϵ being the number of noise states used in the discretisation of the continuous multiplicative noise quantity. The conditional probability is decomposed into a part for the first-order Markov noise term and a part for the implied income level y_{t+1}, which depends on the realised growth state s_{t+1},

$$\begin{aligned}
& f(a_{t+1}|s_{t+1}\cap\omega_{t\leftarrow t-m}) \\
= & \sum_{\omega_{t\leftarrow t-m}} f_\epsilon\left(\frac{a_{t+1}}{y_{t+1}(s_{t+1\leftarrow t-m})}-1|s_{t+1}\cap\omega_{t\leftarrow t-m}\right)\cdot\mathbb{P}(s_{t+1\leftarrow t-m}|\omega_{t\leftarrow t-m}) \\
= & \sum_{\omega_{t\leftarrow t-m}} f_\epsilon\left(\frac{a_{t+1}}{y_{t+1}(s_{t+1\leftarrow t-m})}-1|s_{t+1}\cap\omega_{t\leftarrow t-m}\right)\cdot\mathbb{P}(s_{t+1}|s_t(\omega_{t\leftarrow t-m})) \\
= & \sum_{\omega_{t\leftarrow t-m}} f_\epsilon(\frac{a_{t+1}}{y_{t+1}(s_{t+1\leftarrow t-m})}-1|\frac{a_t}{y_t(s_{t\leftarrow t-m})}-1\cap s_{t+1})\cdot\mathbb{P}(s_{t+1}|s_t(\omega_{t\leftarrow t-m})) \\
= & \sum_{\omega_{t\leftarrow t-m}} f_\epsilon(\epsilon_{t+1}(\omega_{t+1\leftarrow t-m})|\epsilon_t(\omega_{t\leftarrow t-m}))\cdot\mathbb{P}(s_{t+1}|s_t(\omega_{t\leftarrow t-m}))
\end{aligned}$$

and $f_\epsilon(\ldots)$ is the probability density function of noise ϵ_{t+1} conditional on noise ϵ_t in the previous period implicit in the path $\omega_{t\leftarrow t-m}$.

The sum of the joint likelihoods over all possible paths gives the unconditional density for the cash flow in the next period given the observed history of cash flows:

$$f(a_{t+1}|a_{t\leftarrow t-m+1}) = \sum_{n=1}^{2^{m+1}} \left(f(a_{t+1} \cap \text{path}_n|a_{t\leftarrow t-m+1})\right).$$

Once the cash flow a_{t+1} is observed, it can be used to update the likelihood for each path, using Bayes' rule:

$$\mathbb{P}(s_{t+1\leftarrow t-m+1}|a_{t+1\leftarrow t-m+1}) = \frac{f(a_{t+1} \cap s_{t+1\leftarrow t-m+1}|a_{t\leftarrow t-m+1})}{f(a_{t+1}|a_{t\leftarrow t-m+1})}$$

which involves storing a history of $m+1$ cash flow observations. This cannot be maintained by the agent's finite memory, so the probability vector is made smaller by merging together the probability pairs of paths with the same period $t-m+1$ growth state. Note that, since the initial-period income level y_{t-m+1} was taken as certain, the next-period income level y_{t-m+2} must be used in the same way in order to keep the finite memory property. This is again calculated as the expectation:

$$\begin{aligned}
&\frac{y_{t-m+2}}{y_{t-m+1}} \\
= &\ (1+g_{lo})\left\{\sum_{s_{t+1}=0}^{1}\cdots\sum_{s_{t-m+1}=0}^{1} \mathbb{P}(s_{t+1\leftarrow t-m+2} \cap s_{t-m+1} = 0|a_{t+1\leftarrow t-m+1})\right\} \\
+ &\ (1+g_{lo}+g_{add})\left\{\sum_{s_{t+1}=0}^{1}\cdots\sum_{s_{t-m+1}=0}^{1} \mathbb{P}(s_{t+1\leftarrow t-m+2} \cap s_{t-m+1} = 1|a_{t+1\leftarrow t-m+1})\right\}.
\end{aligned}$$

Once all next-period actions have been calculated and estimates have been updated with the new observation of cash flow, the historical paths are one period longer than is permitted under the finite memory bound. The paths are enumerated with a binary form based on the growth state in each period, so shortening involves dropping the least significant bit (the earliest remembered time period in this system) and shifting the remaining bits (dividing the resulting even number by 2). A complication arises in the program due to the shifting of the mean income level implied by each path over time, so that the probability weights used in the discrete cash flow grid dealing with multiplicative noise have to be adjusted. The adjustment procedure can cause rounding errors and result in probability mass 'leaking' between grid points or out of the distribution altogether. Fine tuning of the code has kept this to less than 0.001% of probability in each period, which is deemed acceptable given that the results of the model seem unaffected by rounding errors.

2.3.3 Interacting Agents

While different agents may have different initial endowments of stock and the dynamics are specified as a 'push' mechanism whereby the decision to spend some amount of money is made only by the holder of the money, nothing in principle prevents from its flow between agents being very high, zero or even negative.[19] This is in contrast to models of bargaining where two or more parties must agree on a price and quantity[20].

My model differs from models of markets populated by many agents (with profit-maximisation or other aims) in that it aims to characterise part-localised income generation while the market models aim to replicate the characteristics of quoted market prices. Similarly, price-quotation and market-clearing or market-making mechanisms are not specified in detail.[21]The market mechanism is important because it affects price discovery as well as the way prices are quoted. However, these mechanisms are social and legal constructions designed to facilitate the more fundamental nature of spending of income over time. While the microstructure may have significant unintended macro-economic consequences, the fundamental drivers of behaviour are mostly constant over time and place.

[19]A negative flow corresponds to stealing or defaulting on a loan contract, whether it be performed due to perception or need.

[20]See Bergman & Callen (1991) [11] for an example.

[21]For examples of market microstructure models, see Precious (1987) [166] and Smith *et.al.* (2003) [194].

Model Class Hypothesis We define

$$g_t = \frac{1}{N}\sum_{i=1}^{N}\{v(\theta_{i,t-1})\} \tag{2.30}$$

and

$$\epsilon_{i,t} = \frac{1}{N}\sum_{j=1} N\{\alpha_{i,j}v(\theta_{j,t-1})\} + \eta_{i,t}. \tag{2.31}$$

Agents are identified by the index $i \in \{1, 2, ..., N\}$. All agents have common knowledge of the global variable g_t while each agent i knows the fluctuation variable $\epsilon_{i,t}$ corresponding to its locality. Given that each agent only has limited information, it uses the current and past values of the variables to estimate future values of its income.

The fluctuation variable $\epsilon_{i,t}$ is assumed to be temporary as well as local, while the global growth variable g_t has a permanent effect on the size of the economy y_t. This is a direct consequence of the model's assumptions, as shown by

$$\begin{aligned} a_t &= y_t(1+\epsilon_{i,t}) \\ y_t &= y_{t-1}(1+g_t) \\ \Rightarrow a_t &= a_{t-1}(1+g_t)\cdot\frac{(1+\epsilon_{i,t})}{(1+\epsilon_{i,t-1})} \\ \Rightarrow a_{t+1} &= a_{t-1}(1+g_{t+1})(1+g_t)\cdot\frac{(1+\epsilon_{i,t+1})}{(1+\epsilon_{i,t-1})}, \end{aligned}$$

and so on for $\tau > t$.

2.4 Results

Cyclical economic activity appears in the endogenous model (as well as the exogenous model in which Markov regime shifts are imposed). Under the endogenous model, agents with longer memory increase the periodic length of each cycle but the number of agents does not affect the cycle period or amplitude.

Exogenous noise results in persistent uncertainty and diversity of state among agents. Over time, most initial coordination between agents is lost and there is persistent disagreement between agents' estimates of state variables, but regime estimates stay correlated and cyclicality of the economy remains.

Endogenous noise, which arises purely from different starting values of agents, disappears in each set of conditions examined. Coordination between agents increases but cyclicality of economic activity dies down until the flow of money becomes stable.

The estimates of the growth rate ("est_y" and "lag_est_y") in figure 2.2 show stronger regime-switching behaviour than the global activity growth rate ("true_y"). This suggests that it is possible for a prior regime-switching hypothesis to make the economy appear to its agents to behave in a regime-switching manner.

The next three figures, 2.3, 2.4 and 2.5, show the features existing at the scale of the individual agent. Figure 2.3 separates spend ratios of agents who estimate the economy to be in a growth state from those of agents who estimate the economy to be in decline. It shows that most agents agree with each other as to the state of the economy. However, agents' estimates of their income level are less accurate. In addition, the actual spend ratio achieved (shown in yellow) does not match what each agent is trying to achieve. This is further shown in figure 2.4. The mis-estimation of noise is more systematic, in that the dispersion of the estimate is approximately symmetric around and proportional to the true value, shown in figure 2.5.

As expected, the distribution of marginal prices is highly skewed, with a few having a very high price when nearing liquidity constraints, shown in figure 2.6.

The standard deviation and skew of various measures of perception follow the cyclical pattern of cash flow, as shown in figure 2.7. This suggests that

By defining a negative spend ratio as a default (although the agent would not necessarily be in default of a particular loan, it would nevertheless be in some form of financial distress), the time series shown in figure 2.8 is generated showing that defaults are cyclical. This result is not apparent from

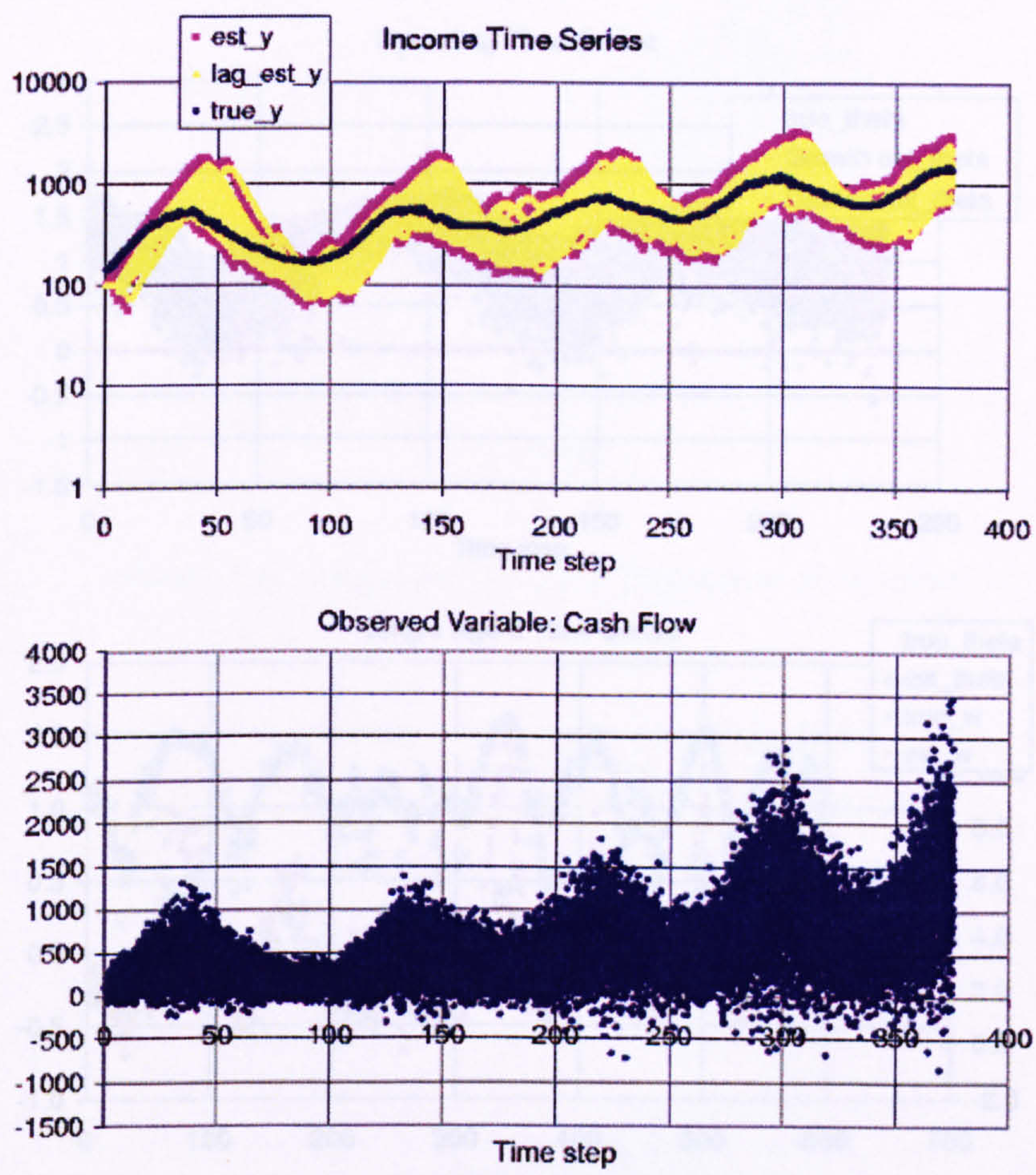
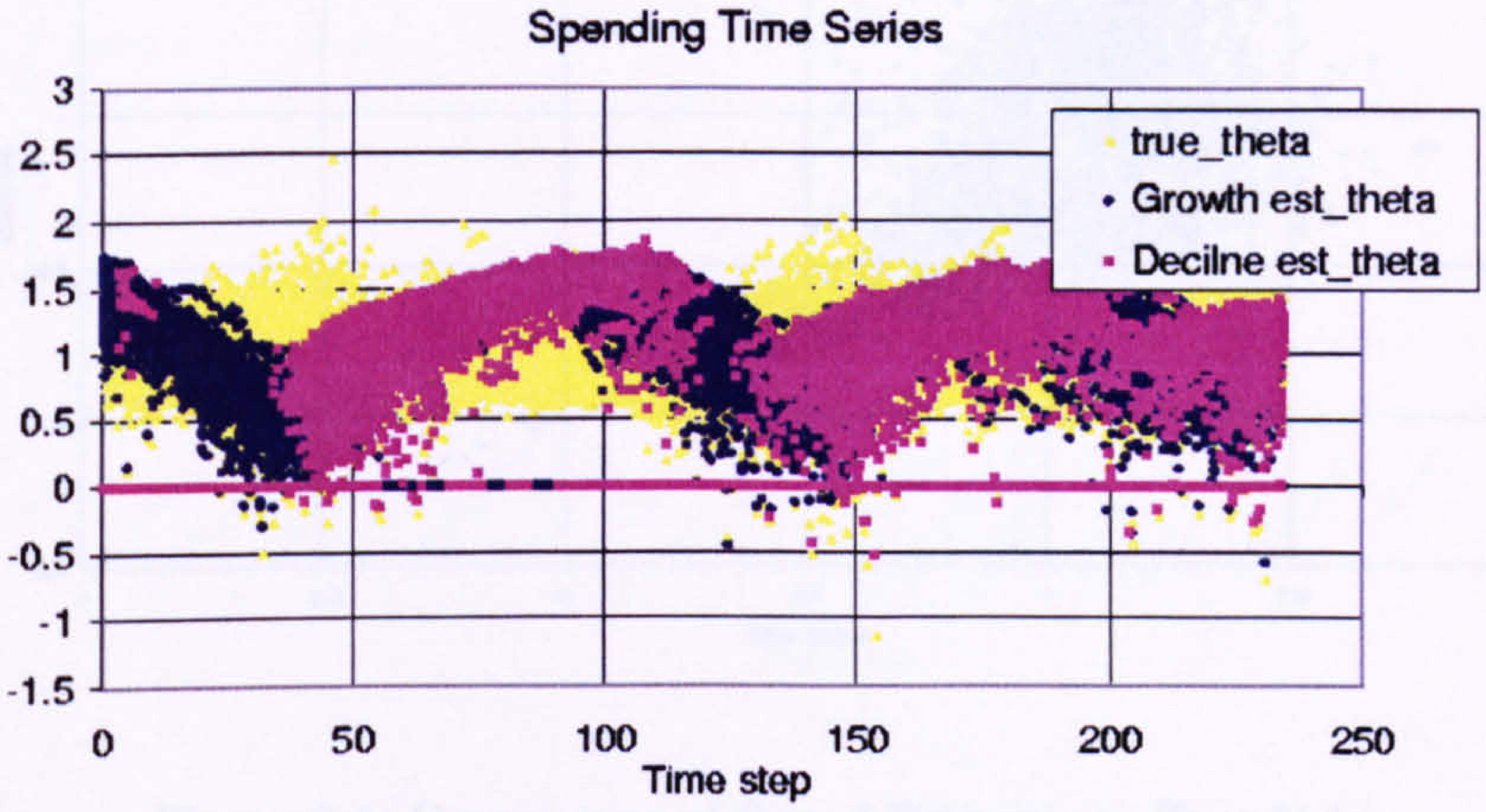


Figure 2.2: Time series of cash flow and income



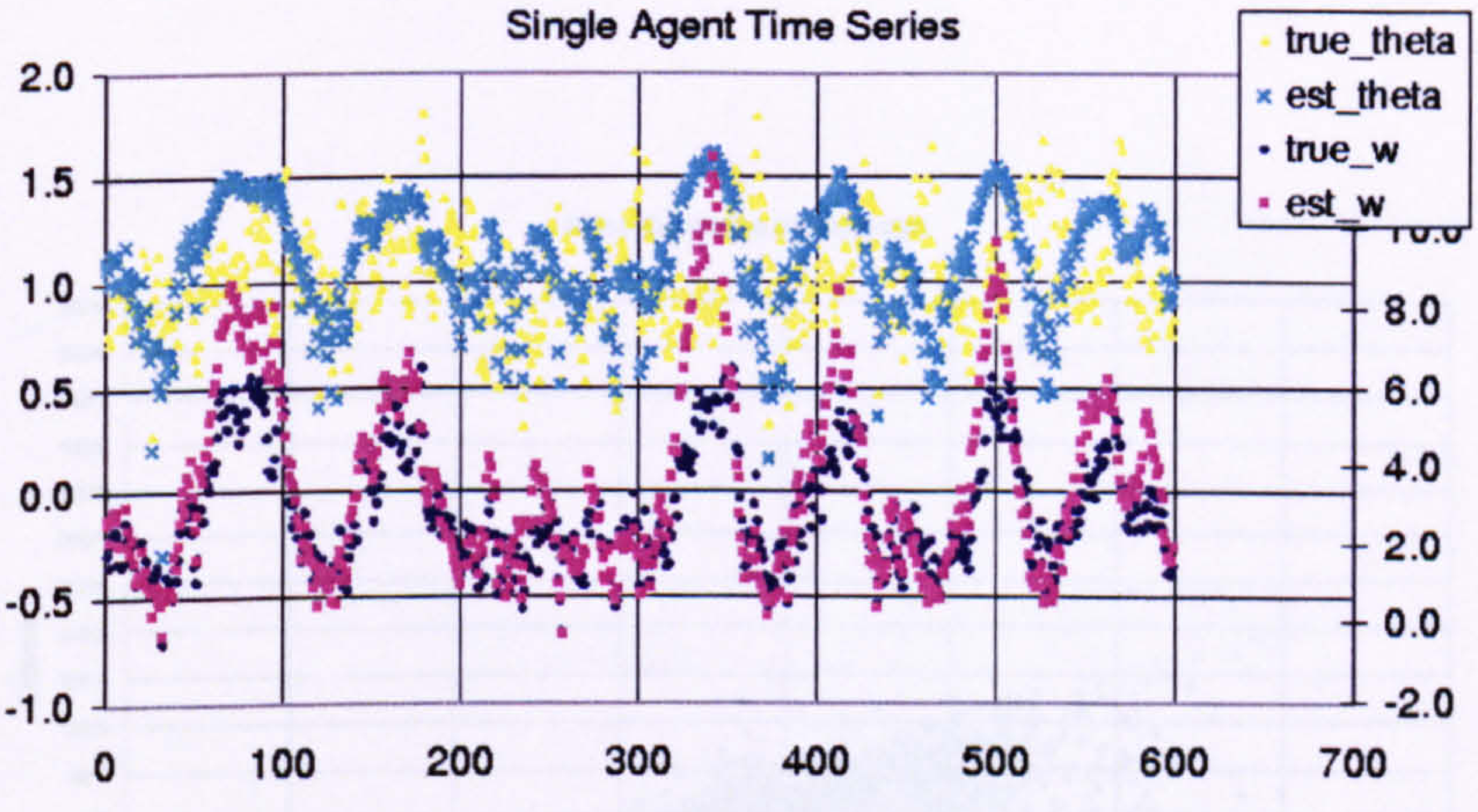


Figure 2.3: Time series of spend and wealth ratios

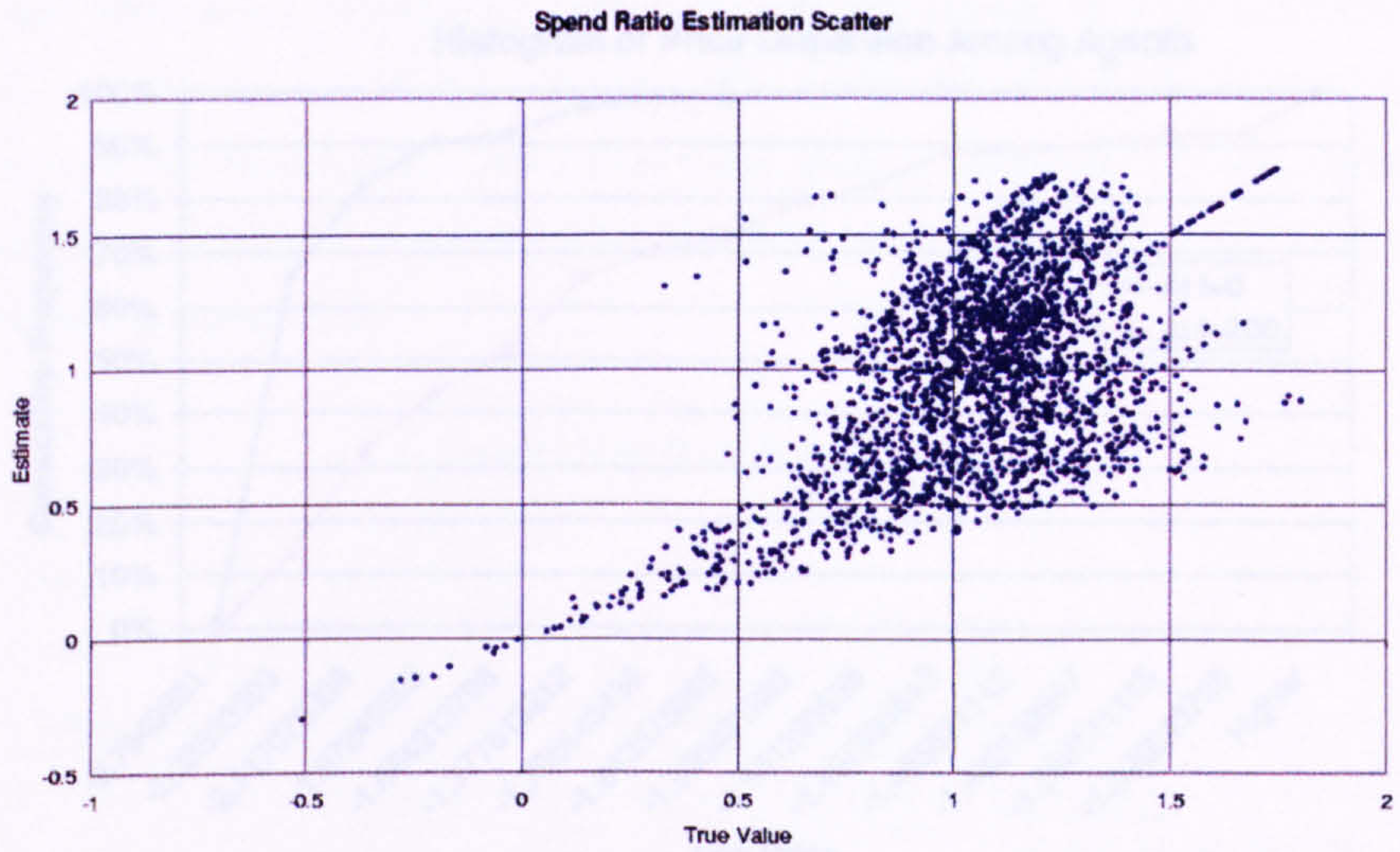


Figure 2.4: Comparison of Spend Estimate to True Value

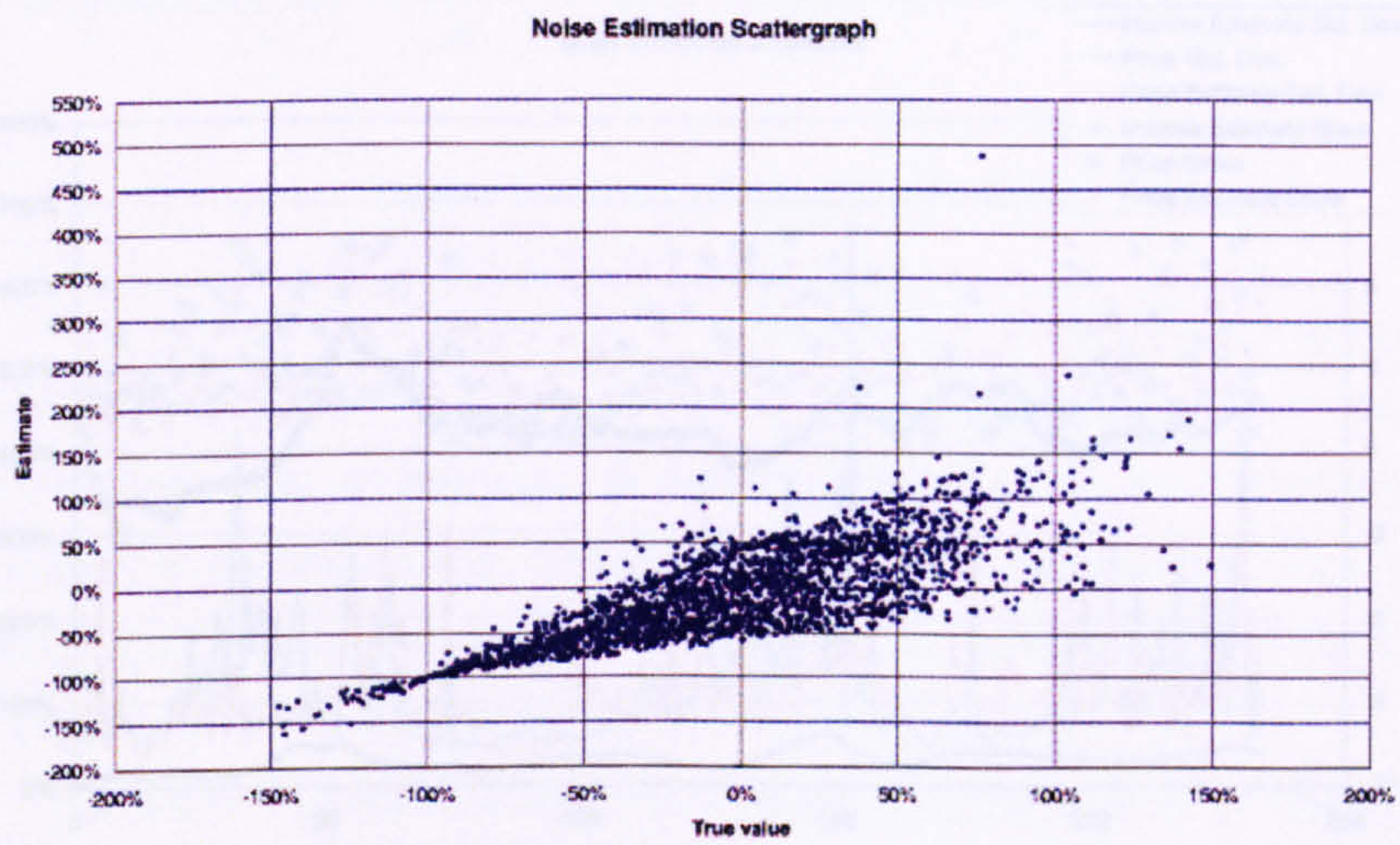


Figure 2.5: Comparison of Noise Estimate to True Value

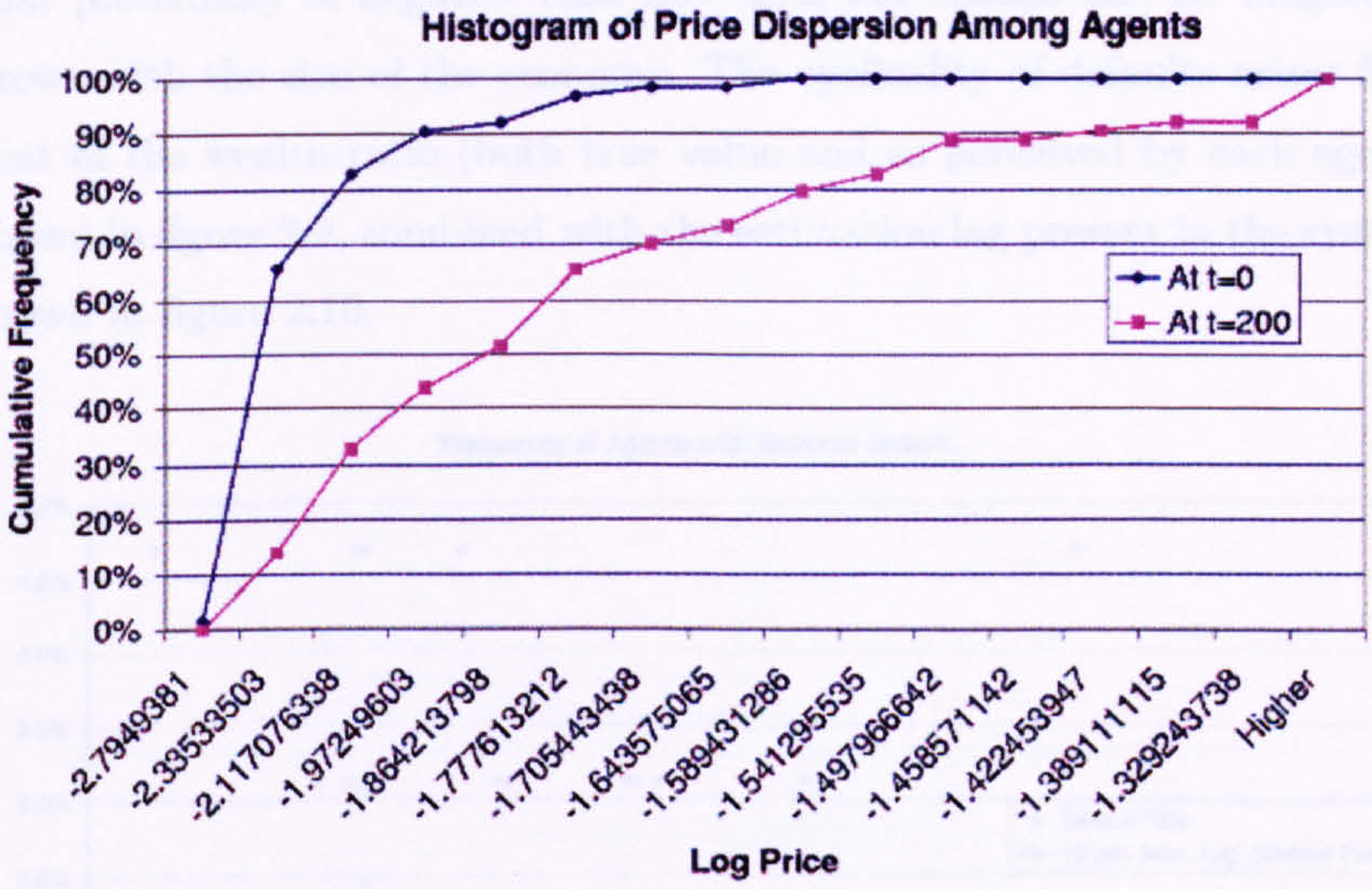


Figure 2.6: Marginal prices across agents at two points in time

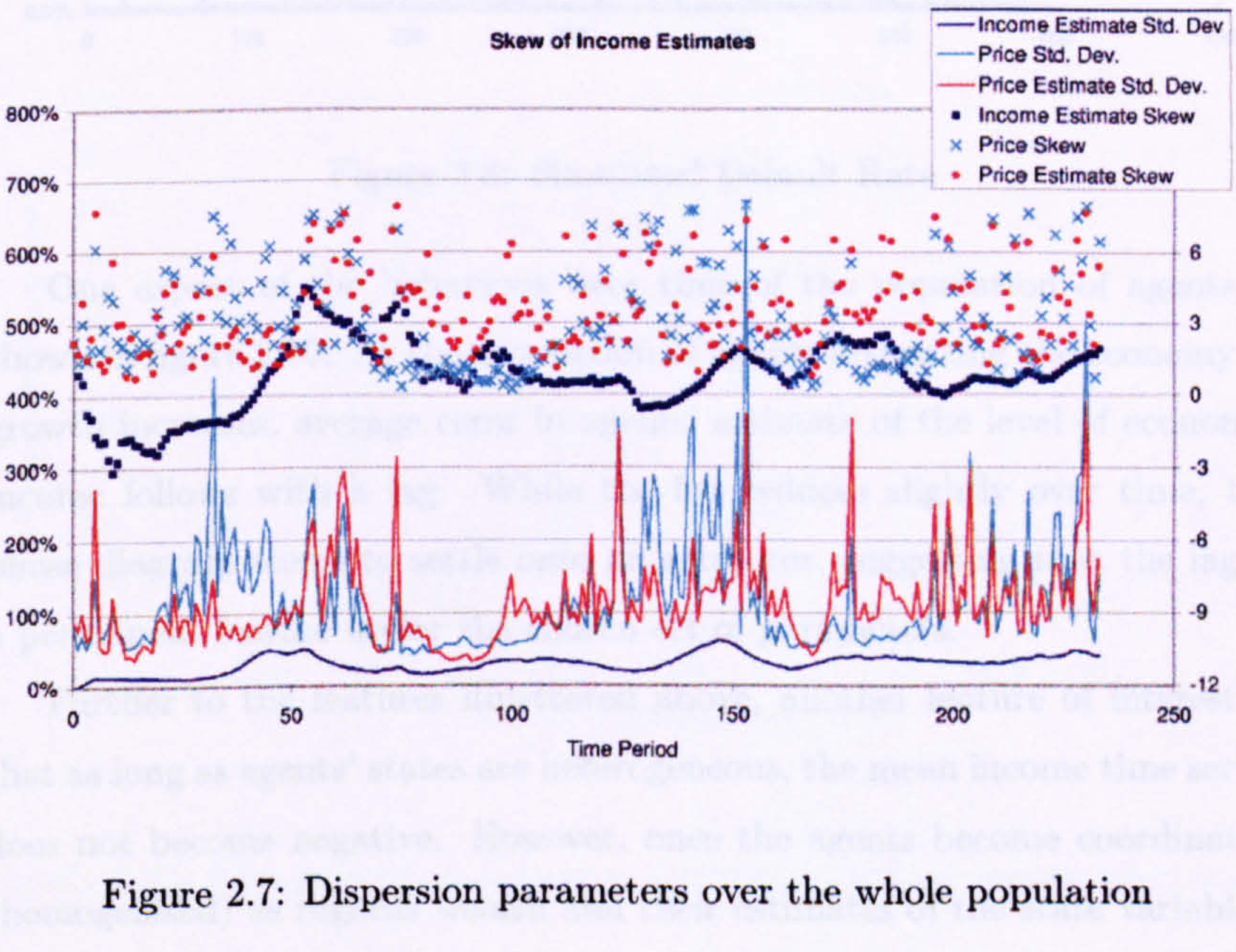


Figure 2.7: Dispersion parameters over the whole population

the outset, since the cash flow noise term for each agent is multiplicative so that probability of negative cash flow does not change and its magnitude grows with the size of the economy. The cyclicality of defaults arises from that of the wealth ratio (both true value and as perceived by each agent), shown in figure 2.9, combined with the estimation lag present in the system, shown in figure 2.10.

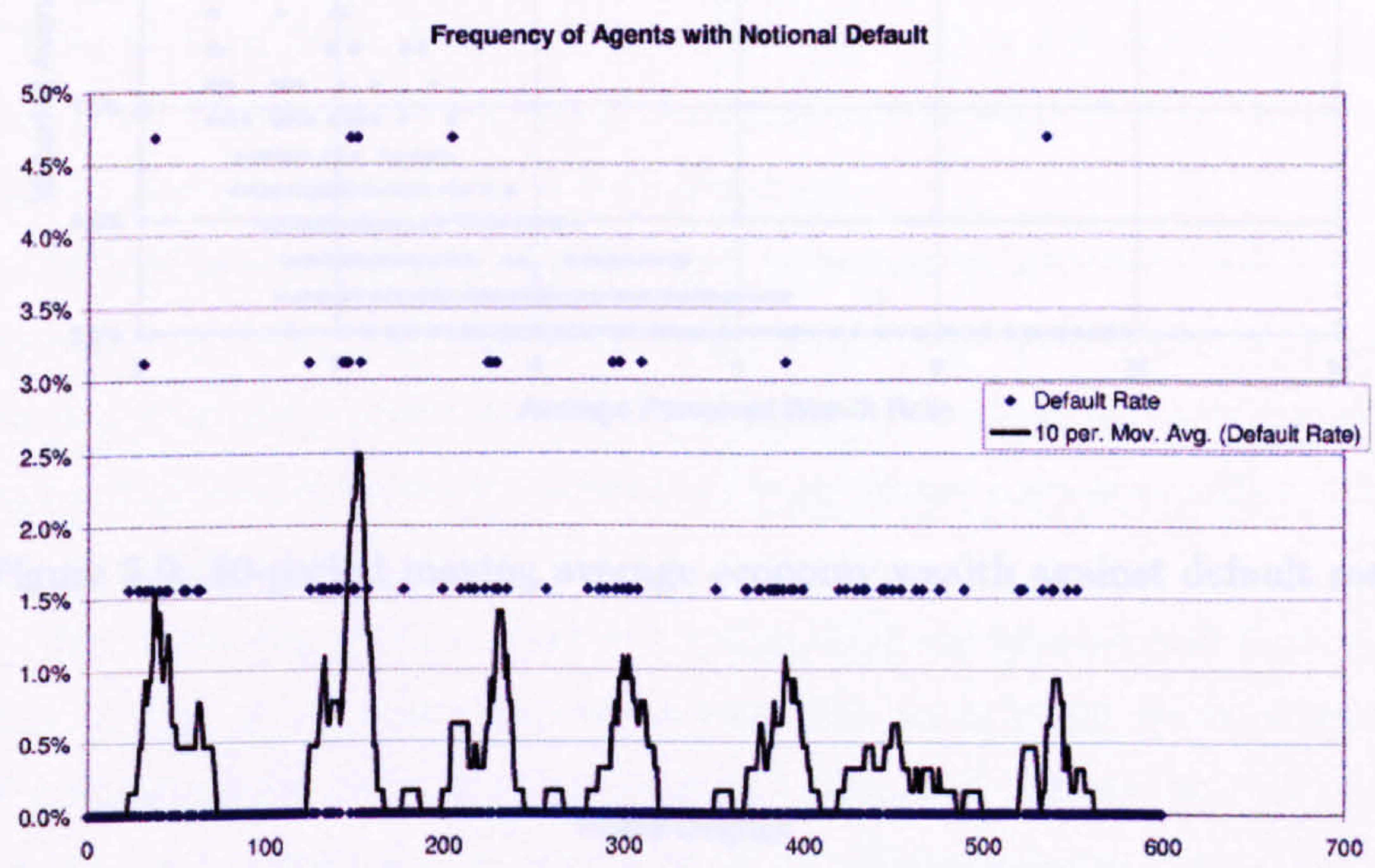


Figure 2.8: Simulated Default Rate

One aspect of the behaviour over time of the population of agents is shown in figure 2.10. As the proportion of agents estimating the economy in growth increases, average error in agents' estimate of the level of economic income follows with a lag. While the lag reduces slightly over time, the phase diagram seems to settle onto an attractor, suggesting that the lag is a permanent feature under the chosen set of parameters.

Further to the features illustrated above, another feature of interest is that as long as agents' states are heterogeneous, the mean income time series does not become negative. However, once the agents become coordinated (homogenised) as regards wealth and their estimates of the state variables,

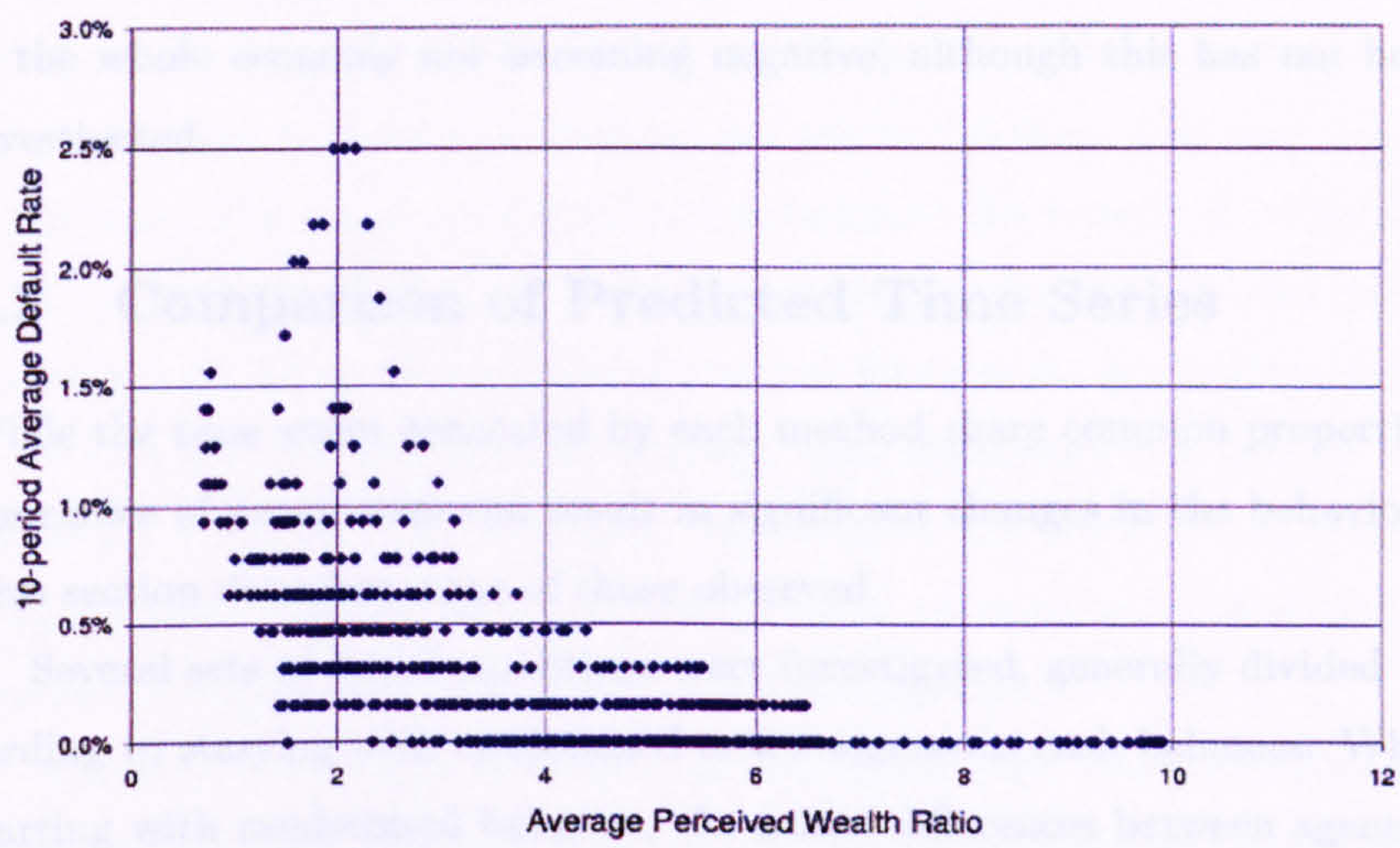


Figure 2.9: 10-period moving average economy wealth against default rate

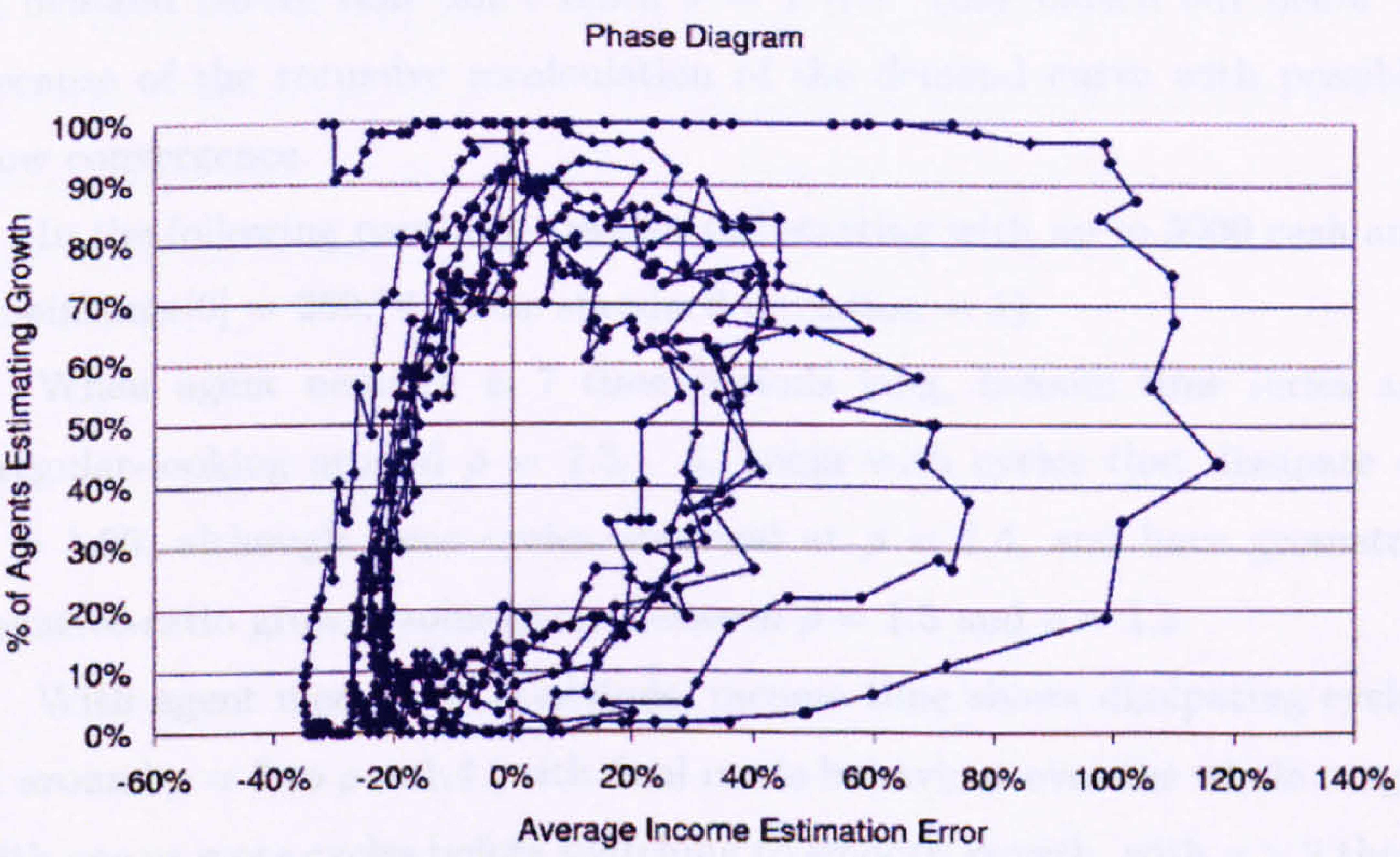


Figure 2.10: Phase diagram of income estimation against regime estimation

then a sign-change becomes possible. It seems that an exogenous random factor is necessary to maintain a degree of certainty of the mean income of the whole economy not becoming negative, although this has not been investigated.

2.5 Comparison of Predicted Time Series

While the time series generated by each method share common properties, the choice of parameters can result in significant changes in the behaviour. This section describes some of those observed.

Several sets of initial conditions were investigated, generally divided according to starting with randomised or homogeneous cash balances. When starting with randomised balances, the initial differences between agents – both actual states and their estimates of those states – are gradually evened out over time.

When starting with homogeneous rather than randomised cash balances, three modes of behaviour appear, noting that some might be an artefact of demand curves that don't reach $\theta = 1$ (i.e. they flatten out below 1) because of the recursive recalculation of the demand curve with possibly slow convergence.

In the following parameter ranges (all starting with up to 3000 cash and trueincome[0] = 269.74, noise standard deviation = 1):

When agent memory is 7 time periods long, income time series are irregular-looking around $\rho = 2.5 - 6$, begin with cycles that dissipate at $\rho = 1.95$, although some cycles observed at $\rho = 1.4$, and have geometric negative-ratio growth somewhere between $\rho = 1.5$ and $\rho = 1.2$.

With agent memory of 6 periods, income time shows dissipating cycles at around $\rho = 2$ to $\rho = 1.4$ (with dual mode behaviour over the whole range, with one or more cycles before switching to smooth growth, with $\rho > 2$ there are cycles for at least 600 steps, but later reversion is not ruled out) and geometric negative-ratio growth at $\rho = 1.3$.

Shortening agent memory to 5 steps results in dissipating cycles at $\rho =$

1.99885 (with a sharp transition at this value), although there is dual mode behaviour at $\rho = 2.1$ (and not at 2.0), flat to geometric negative-ratio growth below $\rho = 1.5$. In an economy where agents have shorter memory, the ranges of ρ with different behaviour are not well-defined and may not be continuous. The large number of parameters and the nature of their effect on the behaviour of the model indicates that it is at the limit of tractability. However, the ability of the model to recreate features similar to those found in various parts of the economy suggests that framework is a fruitful area for future research. The suggested extension to the work of this chapter is to compare the time series to market data, in order to dismiss certain parameter ranges as well as to refine the assumptions.

Although the models developed in this thesis bear closer semblance to imperfect information models than staggered price adjustment, the following is worth noting. The implications of models with staggered price adjustment depend on their assumptions. The models developed in this thesis implicitly assume that contracts set prices for a fixed period and that the price is constant, rather than varying in a predetermined manner during the period of the contract. Additionally, although prices derived respond to changes in the state of the economy, prices only change at fixed periods and will respond to any magnitude of change of state. Altering either of these features can have important consequences.

Chapter 3

Synthesis of Time Series Using Wavelets

Agents in the economy are not homogeneous. The way in which they differ to each other can be in terms of the time scale of planning horizon, complexity of analysis of the available information, memory length, etc. The variety of time scales over which agents store information and plan behaviour suggests that different sectors of the economy behave in different ways, corresponding to the scale appropriate to the sector. Wavelet analysis has several features, not present in other methods, that help it to capture these features if present.

This section describes the salient aspects of wavelets and analyses example outputs of the model to show the capabilities of this line of research.

3.1 Wavelet-based construction

Wavelets can be thought of as a generalisation of Fourier analysis and associated techniques. The immediate difference between the two is that sines and cosines have global support — that is, they are non-zero over infinite time — whereas wavelets are designed to have compact support.[1] To represent a local function, vanishing outside a short interval of space or time, a

[1] "... otherwise, Fourier is virtually unbeatable", as noted in G. Strang (1993) [199].

global basis needs to ensure cancellation over the entire range. Reasonable accuracy needs many terms of the Fourier series. Therefore wavelets, as a local basis, are more appropriate for approximating or analysing time series that have changing properties over time.

A wavelet is not just one function (or pair of functions) like the Fourier basis, but a large family of functions that share common relationships. It is often a significant advantage to be able to choose from a dictionary or design specific wavelets in order to satisfy certain properties.

Wavelets are a broad class of basis functions not dissimilar to the Fourier basis functions ($e^{i\omega t+\phi}$). A wavelet is a function $\psi \in \mathcal{L}^2(\mathbb{R})$ with a zero average:

$$\int_{-\infty}^{+\infty} \psi(t)dt = 0.$$

It is normalised $\|\psi\| = 1$ (where $\|\cdot\|$ is the 2-norm) and centered in the neighbourhood of $t = 0$, but not necessarily symmetric. A family of time-frequency atoms is obtained by *dilation* of ψ by s and *translation* by u:

$$\psi_{u,s}(t) = \frac{1}{\sqrt{s}}\psi\left(\frac{t-u}{s}\right).$$

These atoms remain normalised $\|\psi_{u,s}\| = 1$. The wavelet transform of a function $f \in \mathcal{L}^2(\mathbb{R})$ at time u and scale s is

$$\mathcal{W}f(u,s) = \int_{-\infty}^{+\infty} f(t)\frac{1}{\sqrt{s}}\psi^*\left(\frac{t-u}{s}\right)dt = \langle f, \psi_{u,s}\rangle$$

where the * represents the complex conjugate. In this paper we only deal with real-valued wavelets. The wavelet transform can be rewritten as a convolution product:

$$\mathcal{W}f(t_0,s) = f \star \overline{\psi}_s(u)$$

with

$$\overline{\psi}_s(t) = \frac{1}{\sqrt{s}}\psi^*\left(\frac{-t}{s}\right).$$

The noted advantage of wavelets over sinusoids is that wavelets can have compact support, meaning that each wavelet is non-zero only over a finite range. This makes wavelets better suited than the Fourier basis to localising features in time: the Fourier transform does not immediately tell us where a feature is located until we include phase information, whereas the wavelet transform is immediately linked to the position of the feature. If a signal $f(t)$, defined on the interval $0 \leq t \leq 1$, is zero after $t = \frac{1}{4}$, only a quarter of the later basis functions are involved. The wavelet expansion directly reflects the properties of f in physical space, while the Fourier expansion is perfect in frequency space. The commonly used 'windowed Fourier transform' is an ad hoc approach whereas wavelets are a systematic construction of a local basis.

The wavelet transform of a time-series or signal is just the correlation function of the signal to the wavelet. Repeating the correlation with time-shifted wavelets at different scales, we end up with a series of correlation functions. Putting these next to each other, we have a two dimensional array that can be converted into an image called a scalogram. Scalograms are similar to spectrograms, in that they show the 'frequency' content of a signal, with the addition of showing how the content changes over time. Figure 3.1 shows an example of a time series and its corresponding scalogram.[2]

A scalogram provides a convenient way of showing the characteristics of a signal and how they change as we zoom in or out. This multiresolution property of wavelets makes them suitable for separating different levels of detail, as well as providing a link to fractals (of which Brownian motion is an example).

The following condition ensures that the wavelet transform provides a complete, stable and redundant representation of the signal:

$$C_{\psi} = \int_{0}^{+\infty} \frac{\left|\hat{\psi}(\omega)\right|^{2}}{\omega} d\omega < +\infty$$

where $\hat{\psi}$ is the Fourier transform of the wavelet ψ. The wavelet transform is then left invertible and the redundance implies the existence of a reproducing kernel.)

[2]Figure is from Mallat (1999) [142], made publicly available to aid oral presentations.

A Wavelet Tour of Signal Processing
Stéphane Mallat, Academic Press 1999 (2nd edition)

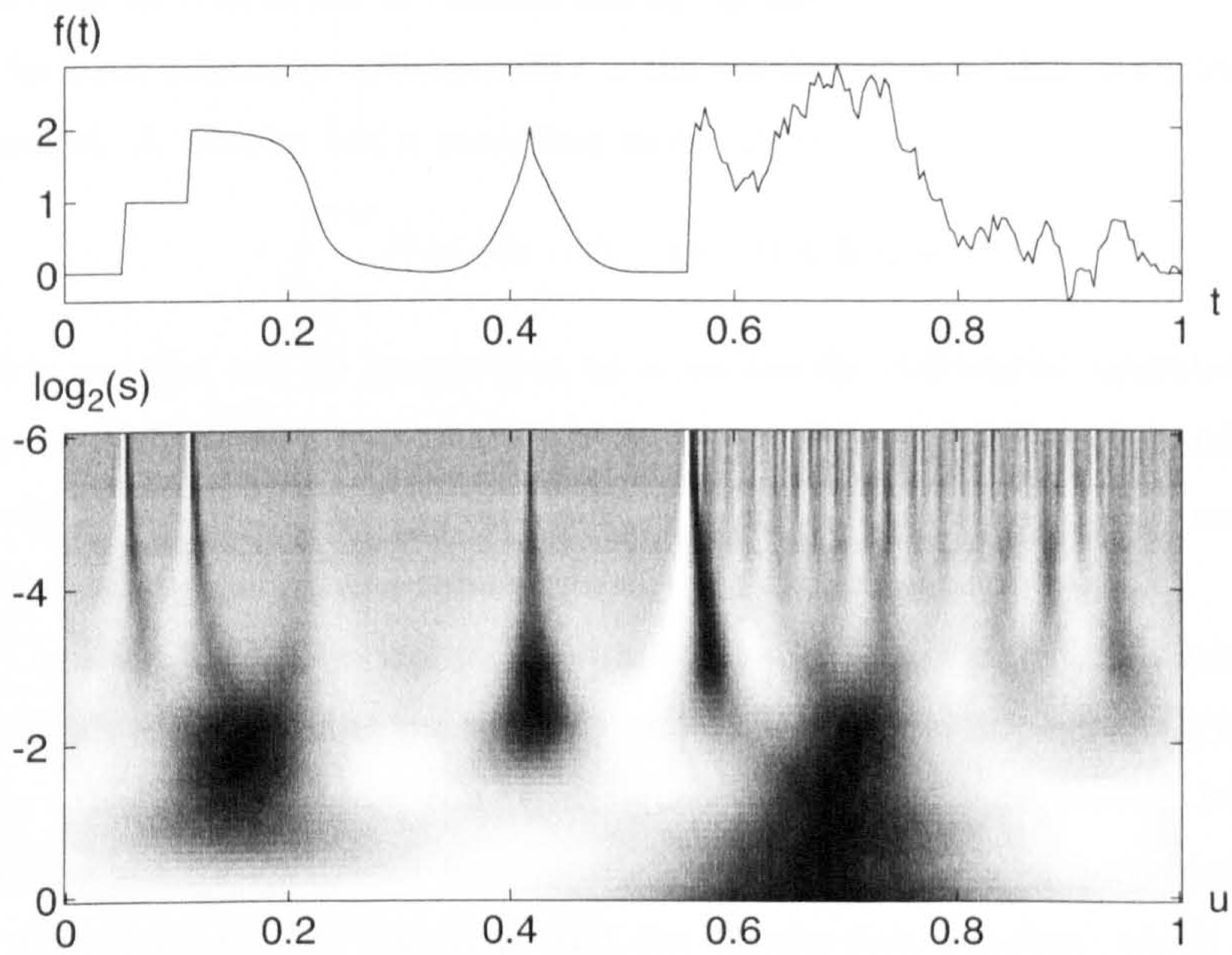


Figure 4.7: Real wavelet transform $Wf(u,s)$ computed with a Mexican hat wavelet. The vertical axis represents $\log_2 s$. Black, grey and white points correspond respectively to positive, zero and negative wavelet coefficients.

Figure 3.1: A time series and its scalogram

Wavelets can be made orthogonal and it is a very common feature due to the computational efficiencies that derive from the property as well as the fact that they are used as a more complicated form of Fourier analysis. Orthogonality is defined as

$$\langle \psi_i, \psi_j \rangle = 0, \qquad i \neq j.$$

With carefully constructed wavelets, we can efficiently approximate any signal using dyadic - non-overlapping - scales, $s = 2^j$, where $j \in \mathcal{Z}$ (the set of integers). This is most useful for signal processing, where minimising data size is important, and allows the formulation of a fast wavelet transform similar to the fast fourier transform. It may also be useful in this investigation by allowing us to calibrate models using a finite set of data points. Wavelets can be discretised and various filters can be used to represent their behaviour as well as aid in reconstructing signals.

An issue related to orthogonality is the number of vanishing moments of a wavelet. A wavelet has n vanishing moments if:

$$\int_{-\infty}^{+\infty} t^k \psi(t)dt = 0, \quad \text{for} \quad 0 \leq k < n.$$

Such a wavelet can be interpreted as a multiscale differential operator of order n. This yields a first relation between the differentiability of f and its wavelet transform decay at fine scales. Vanishing moments are thus useful for analysing regularity of time-series such as Brownian motion.

The Haar basis gives rise to the simplest wavelets, with the fundamental ('mother') wavelet taking the value +1 on $[0, 1/2)$ and the value -1 on $[1/2, 1]$. Many other wavelet bases can be designed with specific properties. Some examples are the 'Mexican hat' wavelet, which is the second derivative of the normal probability function, and the Daubechies wavelets, which are optimised to have the most compact support while having a specified number of vanishing moments.

3.2 Proposed Applications

Ramsey & Zhang (1997) [171] analyse time series under highly redundant representations using 'waveform dictionaries' (see Mallat & Zhang (1993) [143]) covering wavelets and Fourier analysis. Matching pursuit algorithms are used to choose dynamically the best set of waveforms (called 'atoms') that yield a parsimonious representation of the function. They report the following results from analysing financial time series:

> "...despite the relatively low number of atoms needed to provide a very good approximation to the data, about 100 is sufficient, there is little opportunity for improved forecasting. This is because, while relatively few structures are needed to represent the

data, the bulk of the power is in chirps and there does not seem to be any way of predicting the occurrence of the chirps."[3]

Capobianco (2003) [21] is among several authors to have used the same technique to analyse volatility. It may also be appropriate to analyse financial time series using wavelet packets. This technique is a further generalisation of the main recursive analysis and synthesis algorithms used in wavelet and allows the use of a wider range of wavelet shapes that remain orthogonal in some manner.

As a result of investigations to date, there are two proposed uses of a wavelet-based model of prices. First, to provide a volatility model that can be used to price derivatives in the manner suggested by liquidity options. Second, to provide a closer link to micro-structure and (macro-)economic models of price movements through the design of appropriate waveform dictionaries.

The ability of wavelet techniques to deal efficiently with multi-resolution models of time series (of one or more dimensions), makes them suitable tools for analysis and synthesis of price processes similar to Brownian motion. This is because Brownian motion itself is a fractal and can be constructed in a very elegant manner as an "integrated Gaussian noise", as shown by Lévy and Ciesielski (1969) [36]. Recent work by Dobric, Gundy and others (see for example Gundy (2002) [80]) aims to formalise the mathematics of stochastic processes using wavelets. Many papers in the past two decades have shown how to synthesise other processes, such as fractional Brownian motion (see e.g. Flandrin (1992) [68]).

The construction of Brownian motion allows proof of almost all properties, except continuity of the sample path of the process. This includes the ability to show the quadratic variation of the sample path in the theoretical case where the process is defined using an infinite number of wavelet scales. We propose to create wavelet-based stochastic processes in the same manner and show the quadratic variation property in each case. It may

[3] from Ramsey (1999) [169].

then be possible to extend the model to cover a dictionary of wavelets and have a conditional volatility model which will be a subset of the stochastic volatility models of Britten-Jones and Neuberger (2000) [20] and similar to regime-switching models discussed in the paper.

The second application of wavelet based models can be seen as deriving a dictionary based on economic models of rational bubbles. Following the literature from the initial papers starting with Blanchard (1979) [14], there has been much discussion on their usefulness as a model, both in terms of being testable and empirically accurate. Several papers by Sornette and various co-authors develop rational bubble models that involve super-exponential growth of the market price with stochastic critical time at which the price would reach a singularity. At such a point there would be a crash with certainty. However, they show that the probability of a crash increases in a characteristic manner before the critical time, corresponding to a hazard rate model.[4] Such a model could be used to develop a dictionary of wavelets to apply to financial time series modelling in order to pick out the existence of any such behaviour in the market (analysis and calibration) and to simulate price processes (synthesis) for the purpose of pricing derivatives.

3.3 Discrete time synthesis and validation

The first step taken in this investigation was to test the range of validity of numerical simulations, via the Haar wavelet construction, of Brownian motion. This involved writing Matlab code and generating various realisations of Brownian motion with different simulation parameters, such as the number of wavelet scales used and the number of data points in the time-series. To test the success of these simulations, two main properties of Brownian motion were tested: (a) the dependence of the conditional mean and variance of the process only on the time interval and (b) the lack of correlation between increments.

[4] See Sornette and Malevergne (2001) [195] for a description of the model.

(a) The first property is framed as follows

$$\mathbb{P}[x_{t+u} - x_t] \sim N\left(\mu_B, \sigma_B^2(u, s, L)\right) \tag{3.1}$$

where u, s and L are time-shift, scale and signal length respectively. The aim is to test whether or not

$$\mathbb{P}[x_{t+u} - x_t] \rightarrow\sim N(0, u) \quad \text{as} \quad s, L \rightarrow \infty \tag{3.2}$$

and the range over which the approximation is acceptable. This was investigated using the non-parametric Kolmogorov-Smirnov test, which measures the maximal vertical difference between the empirical (cumulative) distribution function and the distribution under test. The most accurate Brownian motion (i.e. including the smallest wavelet possible) fit very closely (to within 99% confidence levels) a normal distribution with mean zero and variance of 2. Figure 3.2 shows the empirical distribution function and a standard normal distribution. The empirical variance is not unity because of the normalisation of the Haar functions. This test was carried out on series with 16,384 entries.

(b) The autocorrelation function of the Brownian motion increments should be zero. This test was straightforward to apply. From this test we can develop the Ljung-Box statistic which, for any given time increment (or 'lag') provides a measure of the correlation. The statistic for a lag of k, $Q(k)$, is distributed as Chi-squared with k degrees of freedom. If r_k is the value of the autocorrelation function for a lag of k then

$$Q(k) = T(T+2)\sum_{n=1}^{k} \frac{r_n^2}{T-n}.$$

Figure 3.3 below shows the $Q(k)$ statistic against k for simulated Brownian motion returns, plotted with lines of 90% and 99% one-tailed confidence for the Chi-squared distribution. For values of k (degrees of freedom) greater than 30, the following expression provides approximate values for χ^2:

$$k\left[1 - \frac{2}{9k} \pm \frac{x}{\sigma}\sqrt{\frac{2}{9k}}\right]^3,$$

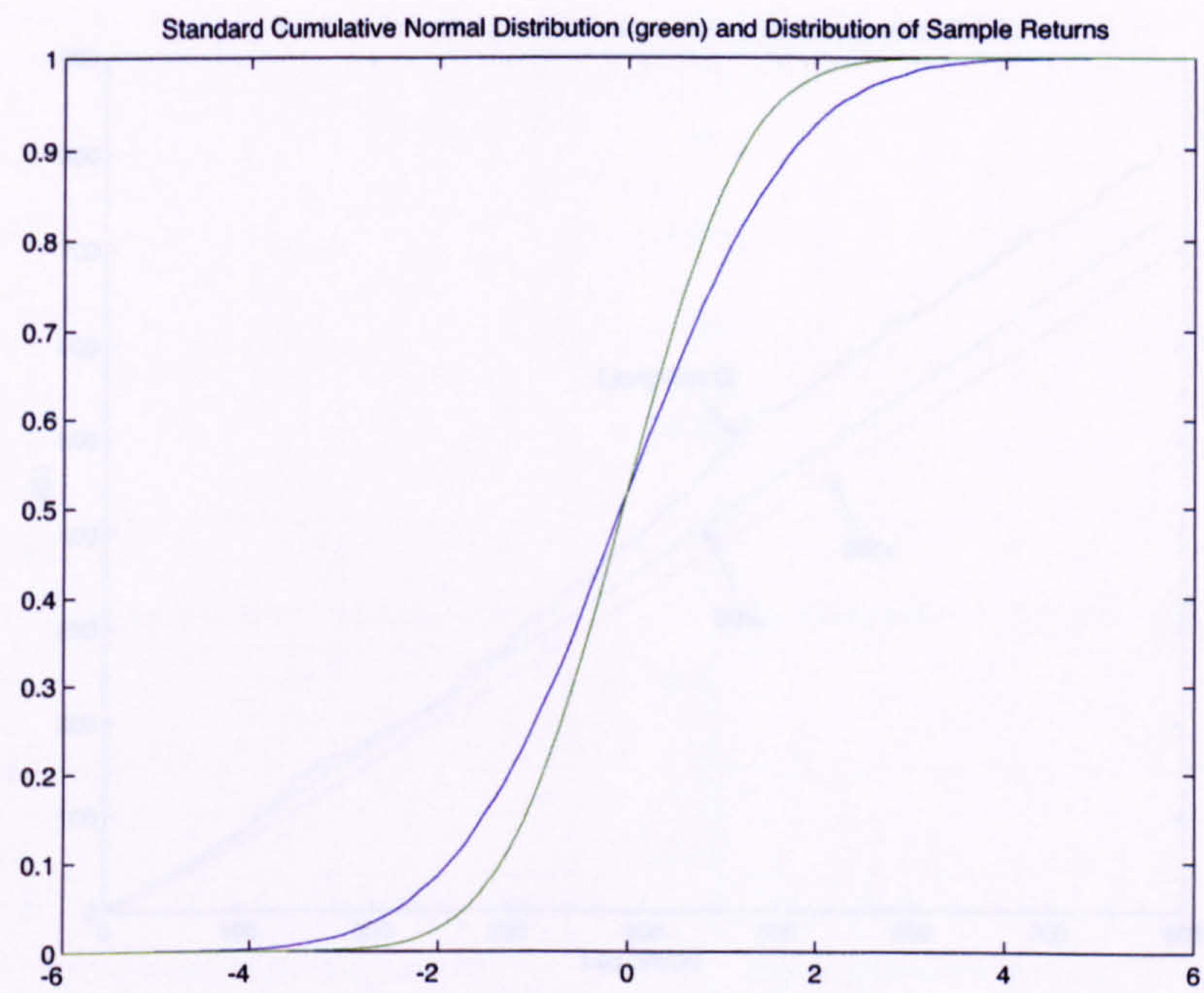


Figure 3.2: Comparison of simulated series and ideal distribution

where $\frac{x}{\sigma}$ is the normal deviate cutting off the corresponding tails of a normal distribution. If $\frac{x}{\sigma}$ is taken at the 0.02 level, so that 0.01 of the normal distribution is in each tail, the expression yields χ^2 at the 0.99 and 0.01 points. For very large values of k, it is sufficiently accurate to compute $\sqrt{2\chi^2}$, the distribution of which is approximately normal around a mean of $\sqrt{2k-1}$ and with a standard deviation of 1.

Figure 3.3 shows that the simulated process falls within the bounds. Points near the boundaries might be due to imperfections in the random number generator used.

The proposed model was described at the end of section 2.2.5 and introduces two regions of scale. Fine scale movements are assumed to be governed by Brownian motion while coarse scale movements are governed by a model chosen from section 2.3. Both of these can be represented under a unified approach using wavelets and the method of choosing these is discussed below.

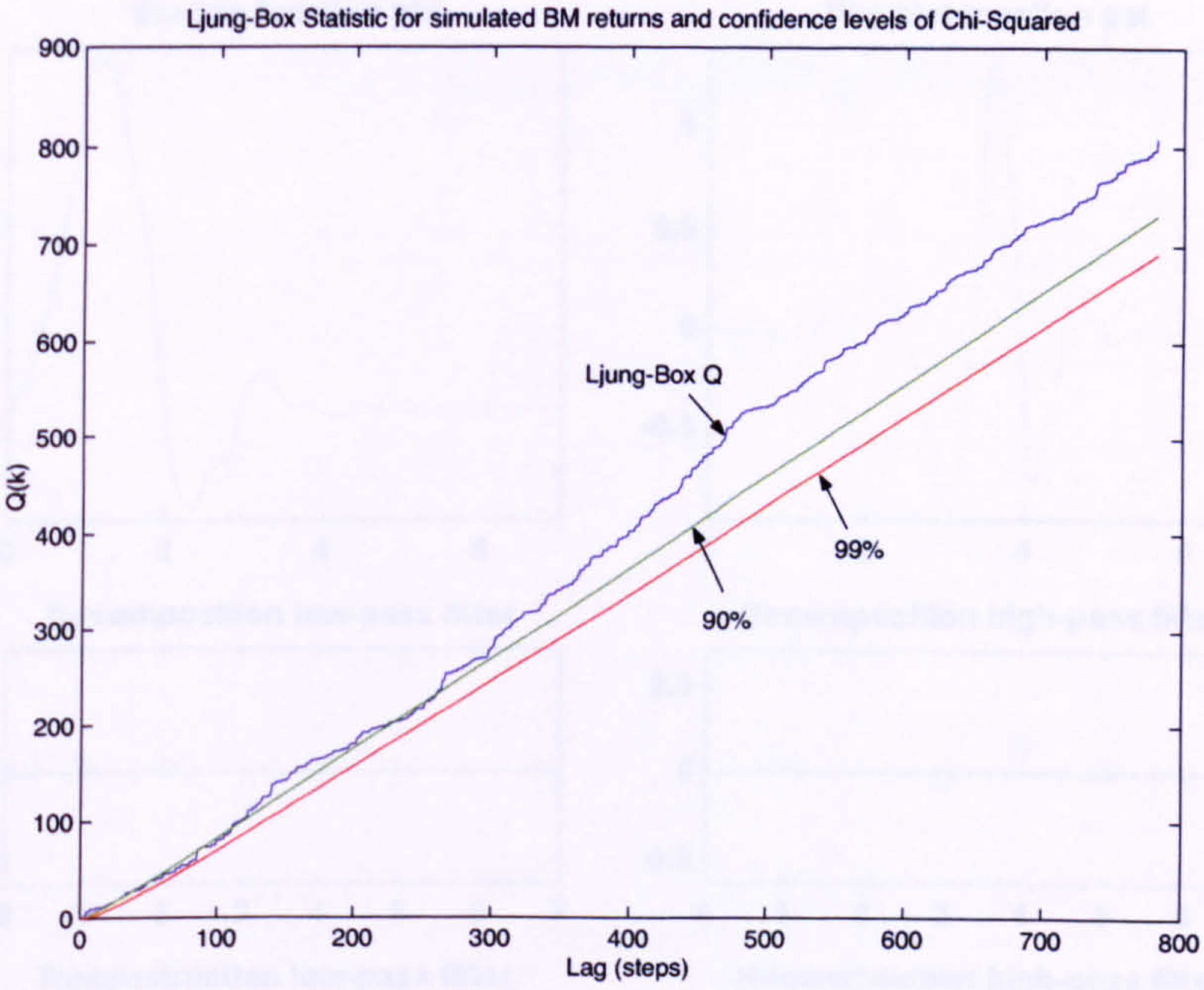


Figure 3.3: Ljung Box analysis of the simulated process

3.4 Multiresolution analysis of financial time series

The time series of income estimates generated in section 2.3 are analysed here using the Daubechies wavelet in order to illustrate a way in which the tool can be used. The Daubechies wavelet, shown in figure 3.4, is chosen because it has the most compact support for the given number of vanishing moments (in this case 4). The wavelet transform at each scale is used to remove short-term fluctuations from the time series, denoted by the letter 'd' in figure 3.5 and a subscript for the scale to which it corresponds, leaving the approximation time series remaining. Histograms of the wavelet coefficients are shown in figure 3.6.

By comparing the s and a_{10} series in figure 3.5, as well as the histograms of s and a_{10} in figure 3.6, the wavelet-based approximation technique has separated short-term fluctuations from the medium-term cycles of income.

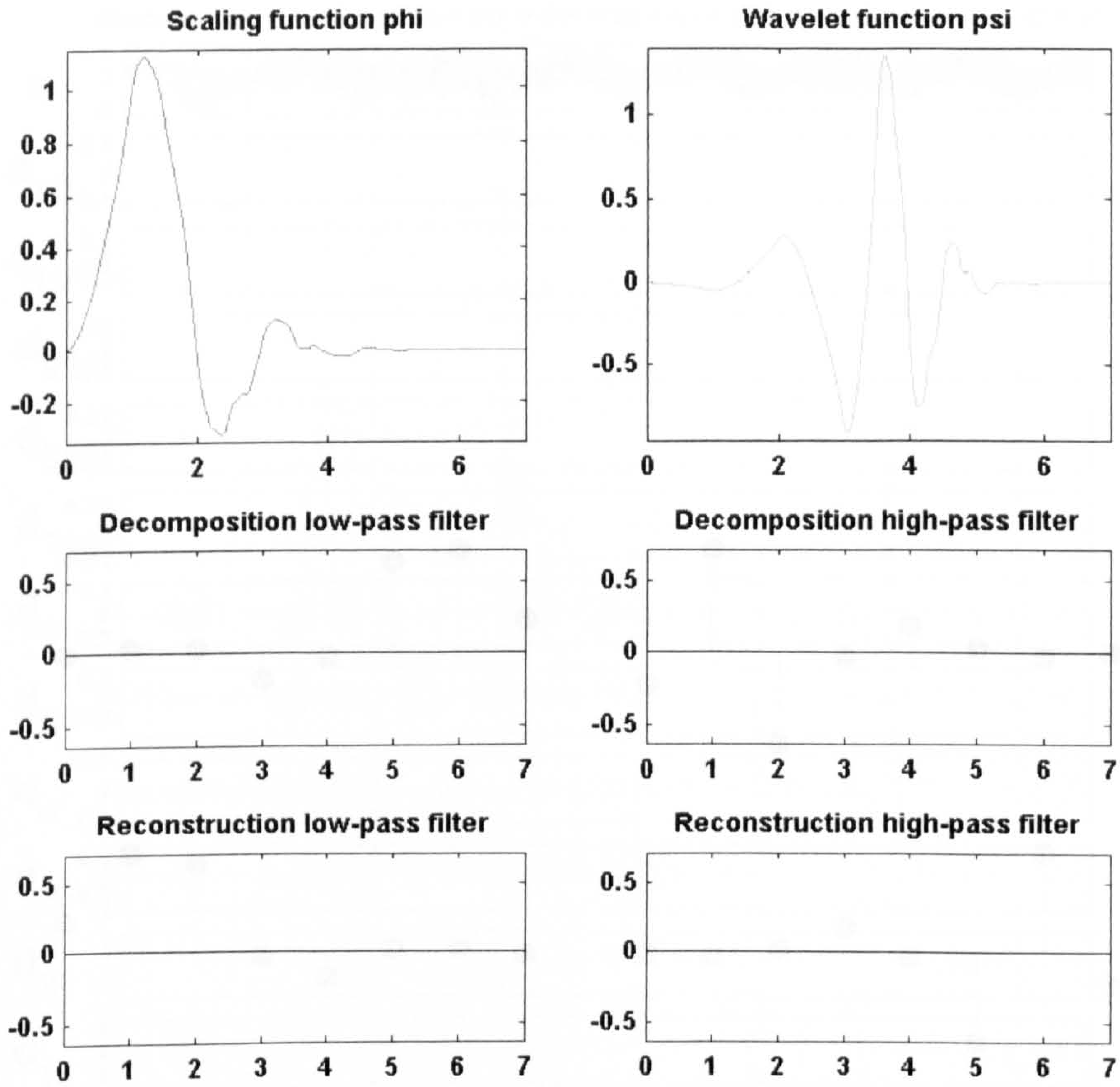


Figure 3.4: Daubechies level 4 (DB4) scaling function and wavelet

This could be quantified in terms of the power of each of the detail time series (d_1 to d_{10}), which would be approximately constant across scales and over time. The lack of apparent structure in any of the detail series or their histograms suggests that there is no further structure present at those scales and/or the chosen wavelet method was not the appropriate one to find any structure that may exist.

The choice of wavelet was made on account of its compact support. However, in order to check for predicted features in empirical series, it is possible to choose wavelets according to other criteria. Instead of using a matching pursuit algorithm, it is possible to design a wavelet basis from the

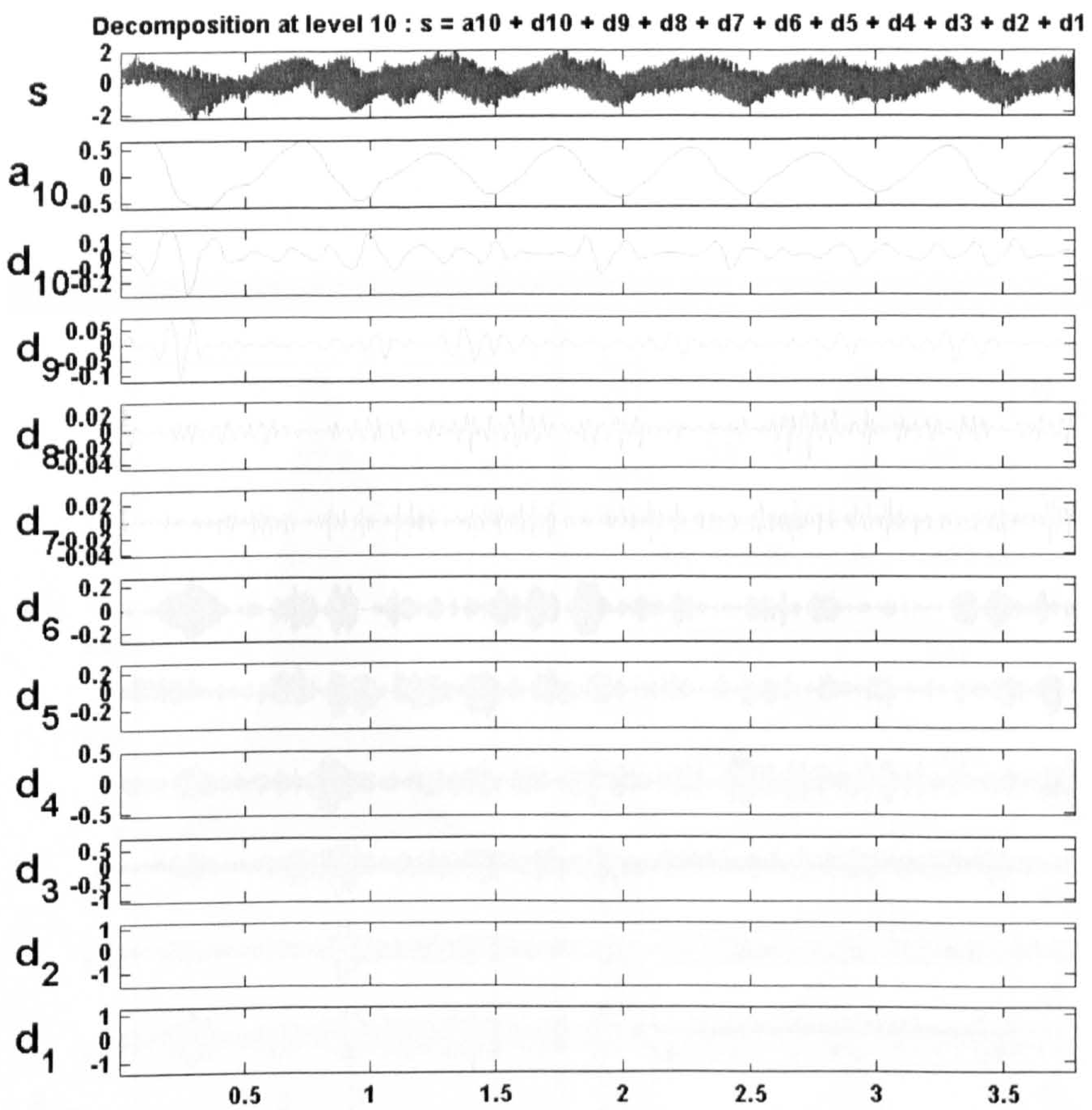


Figure 3.5: Analysis of log-differenced multi-agent estimated income series

features being sought. One method is described in Chapa & Rao (2000) [34] which uses a time series approach, while other papers tend to use use frequency-based features for wavelet design. Development of this method is left for future research.

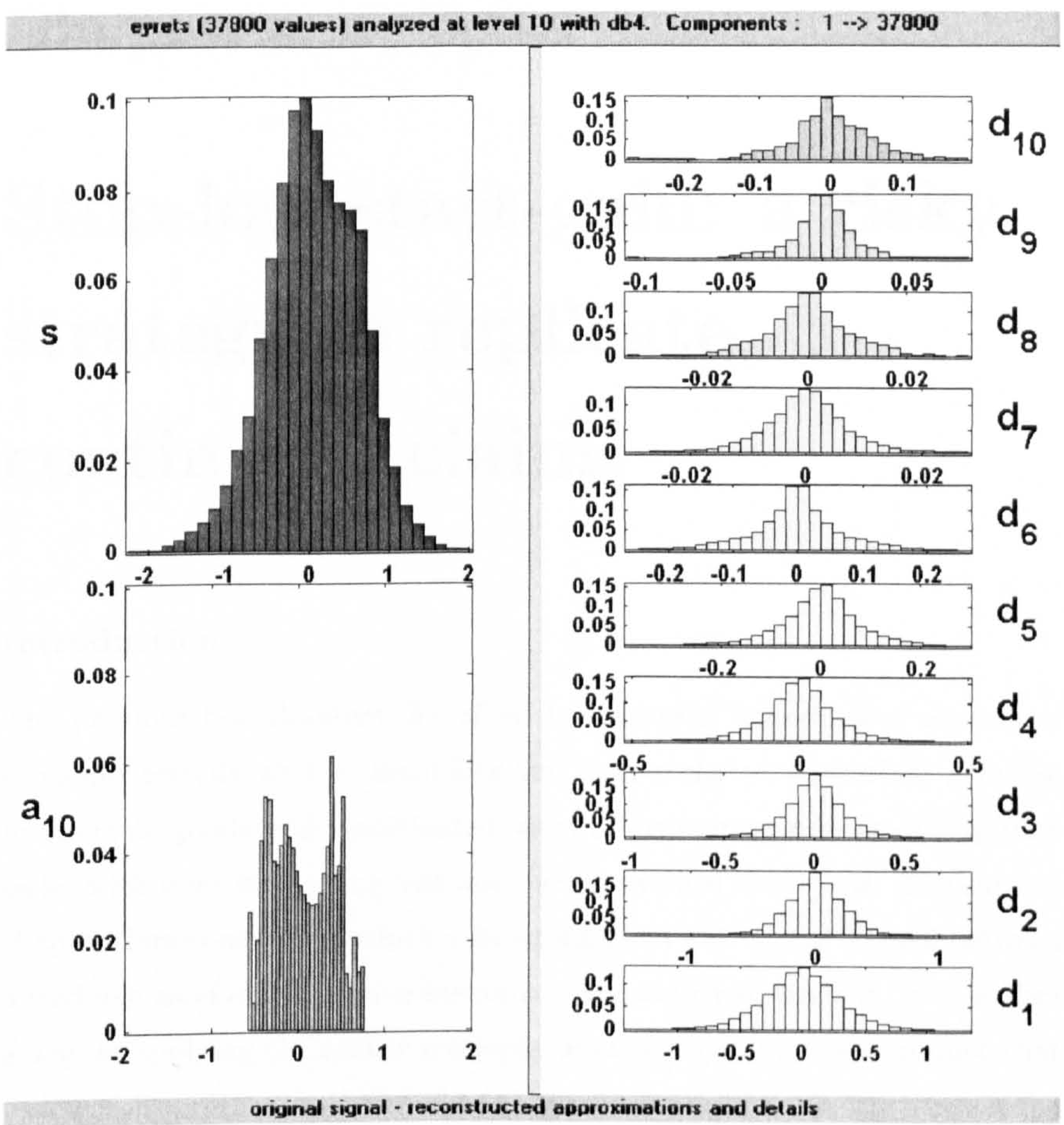


Figure 3.6: Histograms of log-differenced multi-agent estimated income series

Chapter 4

Stop-loss start-gain: a risky strategy to replicate contingent claims

Introduction

The previous two chapters described how certain fundamental aspects of economic activity, such as flexibility and human choice, potentially give rise to tradable goods and investigated ways of generating and analysing time series with some interesting features such as cyclical behaviour. The current chapter shows one way in which a direct financial link can be created between a tradable good and the value inherent in the ability to trade it. It describes a way of applying the earlier concepts to creating a financial product that yields cash returns – and therefore increases financial flexibility – when the availability of cash from other sources is itself falling.

The chapter investigates a trading strategy – mentioned earlier in the thesis – that creates a stochastic quantity of returns depending upon the outcome of the time series of the underlying asset's price. The distribution of the returns from the strategy can, however, be known if the underlying time series falls into a fairly broad category of mathematical forms. Accordingly, the risk-adjusted expected returns from the strategy can be priced based on these assumptions. It must be noted that in addition to the assumptions that are stated explicitly, other potentially crucial assumptions are not investigated. For example, the assumption of being able to buy and sell assets with equal ease (the ability to search for and find a counterparty to the trade, negligible spread between bid and offer prices, etc.) will be maintained, despite evidence in many markets that such liquidity is not present in equal measure across different states of the world.

Idealisations and approximations

Following the discussion in section 1.3, it seems at first possible to replicate without cost the payoff of a European-exercise option by trading the underlying asset according to the stop-loss start-gain strategy ('SLSG'). The strategy aims to hold the entire committed quantity of the underlying asset when its spot price is above the strike price of the option and to hold none when it is below.[1] If the spot price happens to cross the strike price during the hedge, the trader performs the necessary trade as the price crosses the boundary. Since all trades are supposed to take place at the strike price, the strategy appears to be self-financing.

This idealisation is not possible in practice due to time-lags between observations of the spot price and other market frictions. However, the principal observation and source of ambiguity is the result that this apparently self-financing strategy has a zero set-up cost if the option is initially out-of-the-money (and a cost equal to the spot price when it is initially in-the-money) and not the values suggested by the Black-Scholes theory – despite being based on the same assumptions. This paradox was resolved in a series of papers, see Seidenverg (1988) [187], Omberg (1989) [157] and Carr & Jarrow (1990) [25].[2]

[1] The converse is true if replicating a put option rather than a call: below the strike price, the trader shorts the committed quantity, and holds no position if the price is above the strike.

[2] The result has been extended to fractional (arithmetic and geometric) Brownian motions, see [95] for an accessible derivation for Hurst parameter $H > 0.5$ (positive auto-

There are two approximations used to describe the intuition behind the trading strategies and resolving the paradox.

Carr & Jarrow (1990) [25] and Seidenverg (1988) [187] assume continuous monitoring of the price process. There is a self-financing boundary, equal to the strike price of the option, where the portfolio that the trader holds is arbitrary. Therefore the portfolio is decided (arbitrarily in advance and in a non-anticipating manner). The crucial observation is that there must be a stop-loss boundary which is not self-financing and located arbitrarily close to the self-financing boundary. This is because the price may move in the wrong direction from the self-financing boundary given whichever arbitrary portfolio decision had been made and this will occur about one-half the time. Trades at this stop-loss boundary require positive financing. By letting the stop-loss boundary approach the self-financing boundary, the external financing required for each trade becomes infinitessimally small. The papers show that, due to the infinite variation of the process, there is a corresponding increase in the number of trades that require external financing and that the product of the two tends to the local time of the process at the self-financing boundary.

The framework developed by Omberg [157] can be interpreted as a rule dictating trading at observed crossings of the self-financing boundary where the process is monitored at discrete time intervals chosen in a non-anticipating manner. Even though there is infinite variation of the process between two trading dates, the infinite number of crossings in one trading interval may not generate a trade at all, if the process is on the same side of the boundary at the two monitoring times. The paper shows that the cumulative cost of these trades, that almost always occur off the self-financing boundary, is equal to the cumulative cost of one-period (i.e. near-expiration) and near-the-money options (call or put depending on the portfolio held at the start of each period). This cost is equal to the quadratic variation of the process at the strike price.

correlation) and [60] for the complete range $H \in [0, 1]$.

Despite the very different approximations, the limiting results are the same in both cases. Trading at deterministic times is consistent with standard constructions of stochastic integrals and is closer to standard pricing methods that use multinomial trees with deterministic trading intervals. Both the price-trigger and time-trigger models can be close approximations to real market trading and which one is a closer model will depend on the application.

4.1 Local Time and Variation of Diffusions

The results for local time have been extended to general diffusions in [93]. Corresponding concepts exist for Lévy processes as local time and p-variation and these have been discussed by Eisenbaum (2001b) [59]. The analysis in [93] examines the half-line local time of a diffusion along a line: that is, the relative amount of time spent by a diffusion along a line of the form $y = at + b$ (for $a, b \in \mathbb{R}$) over the period $t \in [0, \infty)$. This differs slightly from the local time referred to above, which is a continuous stochastic process defined for every $t \in [0, \infty)$ and is equivalent to the density of its occupation time. The implications will be discussed briefly below. A useful didactic result from the paper is the identification of the probability density of this relative local time. It is shown that the distribution is characterised by a weighted mixture of an exponential function and a Dirac delta function (point mass) at zero.[3]

We now derive the Tanaka formulas for a non-standardised geometric Brownian motion following

$$df = \mu f dt + \sigma f dz,$$

using the exposition in [157]. The derivation for a standardised arithmetic Brownian motion follows a similar method. Given a continuous twice-

[3] See Karatzas and Shreve (1988) for similar results for Brownian half-line local time.

differentiable function $F = F(f,t)$, Itô's theorem states that

$$\begin{aligned}\int_0^T dF &= F(f_T,T) - F(f_0,0)\\ &= \int_0^T \frac{\partial F}{\partial f_t} df_t + \int_0^T \left[\frac{\partial F}{\partial t} + \frac{\sigma^2}{2} f_t^2 \frac{\partial^2 F}{\partial f_t^2}\right] dt.\end{aligned}$$

Consider the function

$$F(f_t) = \max\left[f_t - X, 0\right]$$

with derivatives

$$\begin{aligned}\frac{\partial F}{\partial f_t} &= 1 \quad \text{for } f_t > X\\ \frac{\partial F}{\partial f_t} &= 0 \quad \text{for} f_t < X\\ \frac{\partial^2 F}{\partial f_t^2} &= 0 \quad \text{for } f_t \neq X\\ \frac{\partial^2 F}{\partial f_t^2} &= +\infty \quad \text{for } f_t = X\\ \frac{\partial F}{\partial t} &= 0 \quad \text{everywhere.}\end{aligned}$$

The obvious barrier to applying Itô's theorem is that the function is not twice-differentiable at the point $f = X$ where the function is kinked. The approach is to smooth the function and make it twice-differentiable by inserting a small quadratic curve at the kink, then apply Itô's theorem and take the limit as the curve vanishes. Consider the smoothed function

$$\begin{aligned}F(f_t) &= f_t - X \quad \text{for} f_t \geq X + \epsilon,\\ F(f_t) &= 0 \quad \text{for} f_t \leq X - \epsilon,\\ F(f_t) &= \frac{[f_t-(X-\epsilon)]^2}{4\epsilon} \quad \text{for} X - \epsilon < f_t < X + \epsilon,\end{aligned}$$

with derivatives

$$\begin{aligned}\frac{dF}{df_t} &= 1 \quad \text{for } f_t > X + \epsilon\\ &= 0 \quad \text{for } f < X - \epsilon\\ &= \frac{[f-(X-\epsilon)]}{2\epsilon} \quad \text{for } X - \epsilon < f < X + \epsilon\end{aligned}$$

$$\frac{d^2F}{df^2} = 0 \quad \text{for } f > X + \epsilon, f < X - \epsilon$$

$$= \frac{1}{2\epsilon} \quad \text{for } X - \epsilon < f < X + \epsilon$$

$$\frac{dF}{dt} = 0 \quad \text{everywhere.}$$

Substitute the smoothed function and its derivatives into Itô's theorem and take the limit as $\epsilon \to 0$. From the definition of local time,

$$L(T, X, f_t) = \lim_{\epsilon \to 0} \frac{1}{2\epsilon} \int_0^T 1_{[X-\epsilon, X+\epsilon]}(f_t) dt$$

the Tanaka formula for a non-standardised geometric Brownian motion

$$\max [f_T - X, 0] = \max [f_0 - X, 0] + \int_0^T 1_{[X,+\infty]}(f_t) df_t + \frac{\sigma^2}{2} X^2 L(T, X)$$

follows immediately. A second formula

$$\max [X - f_T, 0] = \max [X - f_0, 0] - \int_0^T 1_{[-\infty,X]}(f_t) df_t + \frac{\sigma^2}{2} X^2 L(T, X)$$

can be established by a parallel proof and a third

$$|f_T - X| = |f_0 - X| + \int_0^T \text{sign}[f_t - X] df_t + \sigma^2 X^2 L(T, X)$$

by combining the first two.

4.1.1 Quadratic Variation

Quadratic variation is a deterministic property of Brownian motion. It is defined as $Q_n(t) = \sum_{i=1}^n (W_{t_i} - W_{t_{i-1}})^2$ in the limit as $n \to \infty$ of the partition $0 = t_0 < t_1 < \ldots < t_n = t$ of $[0, t]$.

It is possible to show convergence with a uniformly spaced partition of $[0, t]$. Define

$$Z_{n,i} := \frac{W_{\frac{it}{n}} - W_{\frac{(i-1)t}{n}}}{\sqrt{t/n}}$$

and rewrite $Q_n(t) = t \sum_{i=1}^n \frac{Z_{n,i}^2}{n}$. The weak law of large numbers shows that the distribution of the sum tends to a point mass at the expectation of each $Z_{n,i}^2$. Since $Z_{n,i}$ are I.I.D. standard normally distributed random variables, $\mathbb{E}[Z_{n,i}^2] = 1$ and so $Q_n(t) \to t$ as $n \to \infty$.

The relationship between quadratic variation and local time is apparent when the Tanaka formula

$$|f_T - X| = |f_0 - X| + \int_0^T \frac{\partial |f_t - X|}{\partial f_t} df_t + \sigma^2 X^2 L(T, X)$$

is compared to the Itô formula

$$F(f_T, T) = F(f_0, 0) + \int_0^T \frac{\partial F}{\partial f_t} df_t + \int_0^T \left[\frac{\partial F}{\partial t} + \frac{\sigma^2}{2} f_t^2 \frac{\partial^2 F}{\partial f_t^2} \right] dt$$

Other things being equal, an option written on a process with higher quadratic variation will have a higher time-value, so it is important to model the quadratic variation accurately.

4.2 Applying the Models to a Financial Product

A few of the largest corporations can borrow directly through the market mechanism, thereby avoiding banks in their role as financial intermediary. The remainder of businesses need to use bank debt to borrow money. In order to signal to third parties that a company can continue trading in the event of a sudden problem until other arrangements are made, some loan facilities are arranged that are usually never drawn. The remote chance of a drawdown means that the bank can allocate much less capital to the loan and can charge a smaller commitment fee to the obligor.

The loan facility acts as a form of insurance against short-term liquidity problems. Insurance companies tend to offer insurance against more specific events, in part because of the difficulty in identifying and quantifying the severity of a liquidity shortfall. Similarly, derivative contracts such as forwards and options provide protection against adverse changes, but are only useful if there is a single market price that can be identified. More recently, derivatives have been developed that have payoffs depending on the values of several market prices, especially in credit derivatives where default risk of several obligors can be correlated.

The advantage of derivatives over bank facilities and insurance contracts is that the price of the risk is determined to a large part by market participants, making the nature of the instrument being sold more transparent and reducing transaction costs. This section develops a derivative that aims to replicate the role of a standby loan facility. In doing so, the price of the risk as estimated by the market can be compared to that of the bank's risk management.

4.2.1 Volatility and Liquidity with RESLSG Trading Strategy

The Tanaka formulae shown above suggest a way of isolating the volatility of a price process through trading in an appropriate portfolio. In this section, we show how this is done and the properties of the dependence on volatility.

The central result was derived by Carr and Madan (1998) [27], where they show how to price and hedge a variance swap under the assumption of a continuous semi-martingale price process, but no other assumption about the behaviour of volatility. Their derivation uses two results: first, a contract that pays the log of the spot price can be easily delta-hedged, so that the hedged payoff is equal to the realised variance (Neuberger (1994) [155]); and second, a log price payoff can be replicated from European options (Breeden and Litzenberger (1978) [18]). When trading is done using the SLSG, it is equivalent to assuming zero (and therefore deterministic) volatility and equivalent to hedging a variance swap with a zero-volatility fixed leg.

The opposite end of the spectrum to a variance swap is a contract that pays the future variance of the price process along a line. This is replicated in the manner suggested by the Tanaka formulae. We will show the case using one call option, although versions with a put option or with a put and call are also possible. Rearranging the formula gives

$$\max\left[f_T - X, 0\right] - \max\left[f_0 - X, 0\right] + \int_0^T -1_{[X,+\infty]}(f)df = \frac{\sigma^2}{2}X^2 L(T,X).$$

The first term on the left can be replicated by purchasing a call option of maturity T and strike X. The second term corresponds to selling a futures contract and borrowing X if the futures price is greater than X and trading nothing otherwise, i.e. starting the reversed SLSG portfolio and the third term describes the cumulative gains from trade according to the reversed SLSG. The term on the right is the quadratic variation of the futures price process at X, or the time value of an option of maturity T with strike X. The actual payoff may be greater or less than the time value, but are equal in (risk-neutral) expectation.

Between the two extremes is the corridor variance swap, where the contract is only active within a price band. This is done by restricting the log-price terminal payoff to a price band, with the payoff outside that band following a constant gradient equal to the gradient of the payoff at the nearest edge of the band. Since these payoffs are replicated with options and terminal payoffs of options end with constant gradients, the corridor variance swap involves buying options with strike prices covering a band of prices. The corresponding trades of futures follows the reversed SLSG strategy for each option bought. Note that, in order to isolate sensitivity to variance, rather than the absolute quadratic variation, we need to scale each contract by the inverse of the square strike price, since the payoff from each contract is proportional to $\sigma^2 K^2 L(T, X)$.

Due to the fact that a price corridor will only be covered by a finite number of strikes, there will be a limited number of reversed SLSG strategies and, therefore, limited trading in the portfolio. The portfolio can equivalently be thought of as being approximated by a finite number of contracts for the variance along a line. Following this idea, a *liquidity option* is created by weighting these contracts to form a desired density of *local liquidity*, or variance-sensitivity, across different time periods and strike prices. The term local liquidity arises from the intuition in Carr *et. al.* (2000) [28] that a European option provides the effective payoff arising from the ability to run a SLSG strategy and trade always at the strike price even when the market price is not at strike. The cumulative gains due to this property can be considered as total liquidity, with the interim arrival rate of gains from trade as local liquidity provided by the option.

In principle, the properties of the overall derivative can vary drastically from one price and time to another. In practice, though, these will be limited by the number of strike prices and maturities of commonly available options as well as the fact, noted above, that the required weighting for low price-levels is high. For example, in order to maintain constant variance-sensitivity for all prices, the position at a price 25% of the current level will require 16 times as many options and will result in trades of 4 times the market value. If the liquidity option is to cover price levels significantly below their current level, it will often not be possible due to lack of liquid options of the required strike price. Even if such options are found, it may face trading difficulties in the event of the price reaching that level.

A point to note is that the strategy dictates reverse trades to the delta-hedge: when the price of the underlying is low, the hedger holds more futures (is less short) than when the price is high. This reduces the likelihood of problems arising from self-reinforcing price fluctuations, which are especially prominent if the market for the underlying is small and/or dominated by agents following a similar strategy to the hedger. It also reduces the problem of an otherwise liquid market drying up in the most crucial instances when liquidity becomes most valuable, since the hedger will effectively be supporting the market.

Note that if the strategy is performed using only put options rather than calls, there is no possible requirement to short the underlying asset. This fact can be used to accommodate short-selling constraints, although the purchase of large numbers of put options will send a negative signal to the market. Increased use of put options instead of calls will, however, reduce the contrarian dynamics of the strategy described in the previous paragraph.

The flexibility in choosing the specific implementation of the strategy, as noted above, suggests that it should be possible to trade any reasonably liquid security, even the company's own stock and bonds. However, the conclusion is treated with caution since the market dynamics caused by use of this strategy and the initial large trades involved in setting up the portfolio are not investigated further.

A further favourable point is that the cost of the product is related to the time value of the options that need to be bought. Since most market traders (here meaning banks) have short options positions and therefore want to buy volatility, the operation of a few large liquidity option contracts will satisfy this requirement as well as result in economies of scale due to the cancellation of a significant portion of dynamic trading requirements.

Also related to the time-value of the options, note that we can use calls, puts or a combination of the two types of options in creating a liquidity option. Since, at high price levels, the option premium is greater for (in-the-money) puts than (out-of-the-money) calls, and vice-versa for low price levels, the value of the initial trades can be minimised by buying only OTM options. This also reduces the initial position in futures (this can be seen from put-call parity). However, it does not ensure that future positions and future trade sizes will be small.

Finally, the risk arising from the individual profit/loss on the contracts with different clients can be partly hedged through diversification — given that the hedger forms contracts with a wide variety of clients. It is anticipated that, in a world with predominantly risk-averse corporations, the swap contracts will almost always be set with the *fixed* leg volatility at zero. This was assumed above, since the SLSG strategy is effectively delta-hedging with the assumption of zero volatility. This is in order to induce in liquidity options the behaviour of a financial hedge since corporations will use liquidity options to reduce exposure to price level changes first and foremost. Reducing exposure to changes in the variance rate will be of secondary importance.

4.2.2 Modelling Behaviour and Probability of Exercise from Commodity Model

Definition of Liquidity Option Payouts

The guarantee pays out if the firm's optimal behaviour would require negative investment, in which case physical voluntary investment formally becomes zero. The model can be expressed algebraically as follows:

$$
\begin{aligned}
a_t &= y_t(1+\epsilon_t) && (4.1)\\
y_t &= y_{t-1}(1+g_t) && (4.2)\\
x_t &= q_t + a_t && (4.3)\\
q_{t+1} &= (1+r)(x_t - c_t + l_t) && (4.4)\\
l_t/y_t &= \max(-\theta_t, 0) && (4.5)\\
c_t/y_t &= \max(\theta_t, 0), && (4.6)
\end{aligned}
$$

where θ_t is a function of the state variables x_t, g_t and ϵ_t. Combining equations 4.4, 4.5 and 4.6 and noting that $\theta_t \leq w_t$ due to the constraint in equation 2.25 yields

$$
\begin{aligned}
\frac{q_{t+1}}{1+r} &= x_t + l_t - c_t,\\
&= x_t + y_t\left(\max\left(-\theta_t, 0\right) - \max\left(\theta_t, 0\right)\right)\\
&= x_t - y_t\theta_t\\
&\geq x_t - y_t w_t \equiv 0
\end{aligned}
$$

showing that the non-negative cash constraint is satisfied.

As discussed in the introduction to section 2.2.1, the price of liquidity must be chosen in such a way as to avoid going to infinity and creating a borrowing constraint again. If the marginal return function λ is set as the required return on payouts from the guarantee, returns during negative investment periods have create no added value for the owners of the company (who would, according to standard theory, cede control of the company to creditors). The choice has the added convenience of avoiding a discontinuous slope in the firm's return-on-investment function.

In order to minimise moral hazard, the payout of cash from the guarantee must not confer a net gain to the decision-makers in the firm. But the cash effect of the cost (excess) must be delayed relative to the cash payout in order to ease short-term liquidity constraints, since the firm will be lacking the liquidity needed to cover any excess. It is possible to limit the number of projects or subsidiaries of a company that are insured, but the cash flows between subsidiaries as well as effects of the seniority structure of claims and control rights in the event of financial distress could result in the need for complicated arrangements for the excess to be effective.

A risk-neutral lender would set the required excess based on estimates of default probability and severity, but the method (coupled with asymmetric information problems) is the main cause of borrowing constraints. Using the fact that the guarantor receives an initial premium to cover expected losses, it seems possible to choose a price for cash payouts that is less than the risk-neutral cost of borrowing yet exceeds the benefit to the firm's decision-makers, thereby allowing the firm to increase its financial flexibility.

Setting the price of cash payouts equal to λ suggests a simple way of charging the firm for the cash payout. Since the firm's value mirrors the value of its assets, transfer of ownership of a group of assets in exchange for the cash payout will align the firm's incentives with the guarantor. It is a close parallel to asset sales, which often take place when a company is in financial distress.[4]

The transfer of ownership is best arranged through the sale of ordinary or preferred shares or asset-backed securities. Equity will have to be from a stock of authorised share capital reserves and the size of the approved reserves in effect sets the exposure limit for the guarantor. The use of asset-backed securities provides better security for the guarantor, but the existence of negative pledges to current creditors may be an obstacle as may the problem of finding ring-fenced assets that are suitable for piecemeal

[4]Except that there is not necessarily one single large transaction and there are fewer problems associated with fire sales and the lack of appropriate buyers because the transaction size (and, in theory, the price) are deterministic functions of the state.

securitisation.

The value of investments transferred to the guarantor are

$$\begin{aligned} S(\theta_t) &= (v(0) - v(\theta_t))\mathbf{1}_{\{\theta_t<0\}} \\ &= \left(\frac{by_t}{\rho}(k-1) - \frac{by_t}{\rho}(k - e^{-\rho\theta_t})\right)\mathbf{1}_{\{\theta_t<0\}} \\ &= -\frac{by_t}{\rho}(1 - e^{-\rho\theta_t})\mathbf{1}_{\{\theta_t<0\}}, \end{aligned}$$

which is a positive number when $\theta_t < 0$ and is also independent of the particular choice of k. The cash payout in this situation is $l_t = -\theta_t y_t \mathbf{1}_{\{\theta_t<0\}}$, so the return when $\theta < 0$ is

$$\begin{aligned} R(\theta_t) &= \frac{S(\theta_t)}{l_t} \\ &= \frac{b}{\rho\theta_t}(1 - e^{-\rho\theta_t}). \end{aligned}$$

$R(\theta_t) \to +\infty$ as $\theta_t \to -\infty$ so that the largest payouts would be very profitable for the guarantor.[5] For $\theta_t \nearrow 0$, l'Hôpital's rule shows that $R(\theta_t) \to b$ and is valid for all $\rho \neq 0$. If $b > 0$ (the NPV of the first marginal unit of discretionary investment is positive), the guarantor makes a positive return on all payouts. Figure 4.1 compares the return resulting from guarantee payments to the marginal return on investment λ.

The expected sum of losses from cash payouts discounted at the guarantor's cost of funds gives the total premium for the guarantee:

$$P = \mathrm{E}_0 \left[\sum_{t=0}^{T} \frac{-S(\theta_t)}{(1+r)^t} \right]. \tag{4.7}$$

Simulation

This section shows how the payoffs from liquidity options can be simulated. For simplicity, we consider here only the case of a RESLSG using call options. The extensions to using put options or both follow the same method. The net position, equal to the accumulated liquidity that will result at the end

[5]In practice, because $\theta_t \geq 1 + \epsilon_t$ and ϵ_t is discretised, $\rho\theta_t$ will be bounded. In most cases, the bound will ensure that all payouts are unprofitable for the guarantor.

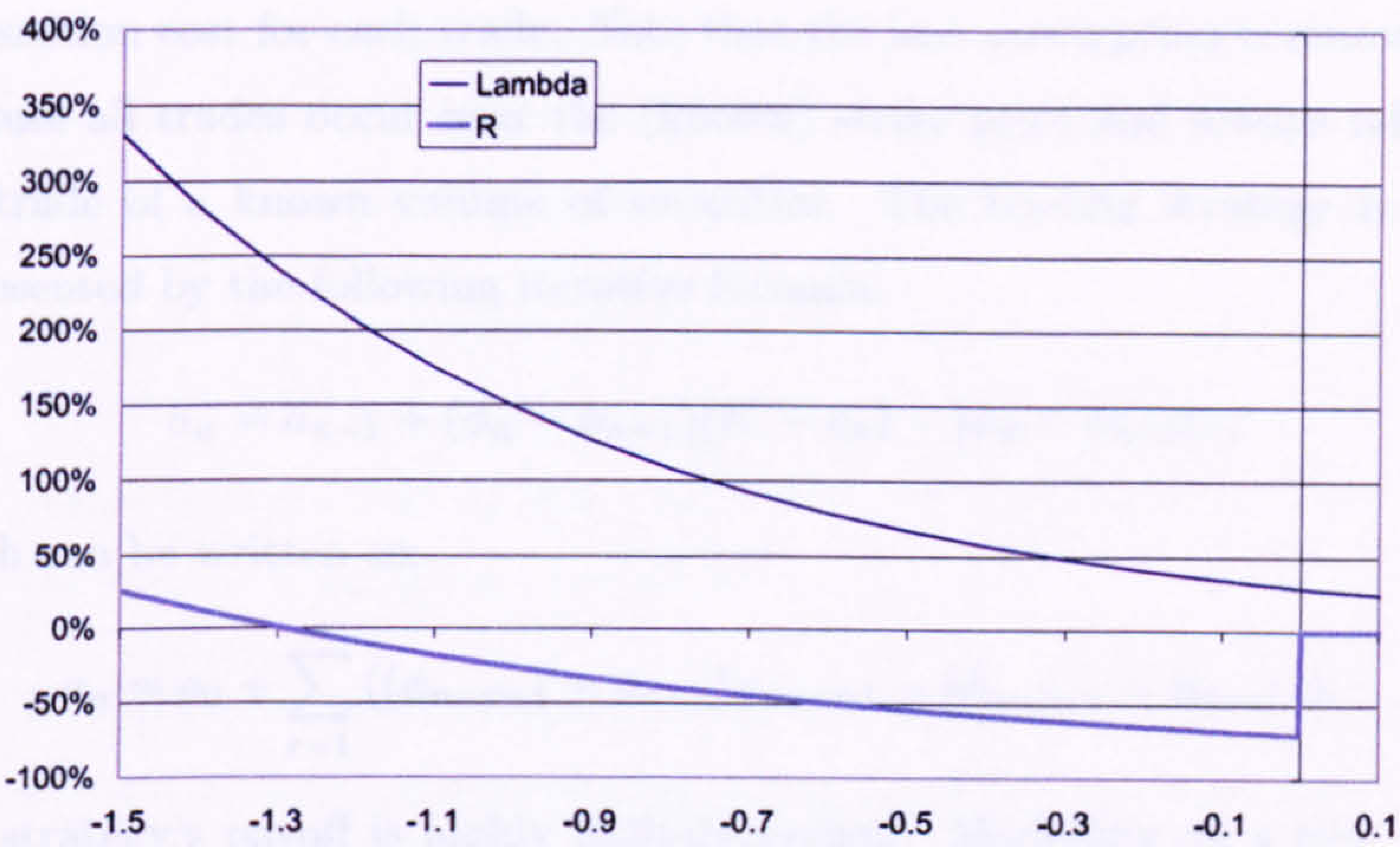


Figure 4.1: The return on topups (R) and on investment (λ) for $b = 30\%$ and $\rho = 1.6$

of the strategy, is given by calculating net assets (in an accounting sense). Assets are cash and the intrinsic value of the option; liabilities are from the short position in the underlying security. When $x \geq K$,

assets = intrinsic value of call option + cash = $(x - K)$ + cash

liabilities = (short) share = x

When $x < K$,

assets = intrinsic value of call option + cash = 0 + cash

liabilities = 0

Overall, net assets

$$a = \text{cash } -1_{\{x>K\}} \cdot K \equiv \text{cash} + \text{number of shares} \cdot K.$$

Introducing an index n to denote the trading step,

$$a_n = c_n + \phi_n \cdot K. \tag{4.8}$$

The amount of cash is given by $c_n = c_{n-1} - (\phi_n - \phi_{n-1})x_n - |\phi_n - \phi_{n-1}|\,\epsilon$, where ϵ is the (assumed constant, as opposed to price- or volume-proportional) transaction cost for each trade. Note that the last assumption is reasonable because all trades occur near the (known) strike price and always refer to the trade of a known volume of securities. The trading strategy is then represented by the following iterative formula:

$$a_n = a_{n-1} + (\phi_n - \phi_{n-1})(K - x_n) - |\phi_n - \phi_{n-1}|\,\epsilon, \tag{4.9}$$

which can be written as:

$$a_n = a_0 + \sum_{r=1}^{n} ((\phi_{n-r+1} - \phi_{n-r})x_{n-r+1} - |\phi_{n-r+1} - \phi_{n-r}|\,\epsilon).$$

The strategy's payoff is highly path-dependent. Modelling on a tree, as is done first in this paper, would require a non-recombining tree. However, it is preferable to use a (recombining) lattice considering the easier computation, storage and visualisation of a lattice on a computer. This requires mapping many nodes in a tree onto one node in a lattice and could make calculations very complex. The fact that the payoffs are path-dependant, but not the strategy, allows us to make this mapping. Note that this approximation also relies on the transaction cost formulation given above. Surveys of the effects of transaction costs and discrete trading, as well as the parameters of the Black-Scholes formula, have been presented in various papers such as Figlewski (1989) [66] and Toft (1996) [200] for delta-hedging. Note that [66] includes the (non-reversed) SLSG as a special case of portfolio indivisibility, although the paper does not analyse it beyond pointing out that the replication cash flow variance is very large. I performed corresponding simulations for the RESLSG.

For reasons outlined later, the tree-based and the other explicit discrete trading-times models are not very convenient. As a result, the proposed method for valuation is to use the analytical, continuous trading-times, arbitrage-free formula for option time value derived in [25] as a benchmark. Then, using Monte Carlo simulations along the lines of [66], the analytical results are adjusted for the effects of key market imperfections, namely, trading only at finite intervals and non-zero transactions costs. The next best pricing method is to use a trinomial tree and try to account for inaccuracies of the model.

The effect of the interest rate is not presently included for two reasons.

1. If the trading strategy is adjusted to be active at the present value of the strike price and we look at the present value of cash flows, then the result matches that of this paper with the stock price re-based with the riskless bond price as numeraire.

2. In implementation, the risky asset traded is likely to be futures on the underlying asset. Due to the low margin requirements, the opportunity cost of holding risky assets rather than an interest-bearing money-market account is considerably smaller than the risk-free interest rate.

It should be noted that when trading bonds, the natural trading criterion is the price. The price includes both the (credit) spread and interest rate effects, with the result that the payoff from the liquidity option is also affected. Since the price falls with rising credit risk and rising interest rates, the product effectively hedges the effect of interest rate risk as well as credit risk on refinancing costs. This will be investigated later.

Each option with its associated dynamic trading portfolio provides liquidity, in the form generated cash, whenever the price crosses the strike before the maturity of the option. By setting up many such strategies at different strikes and with different maturities, we can arrange for a particular rate of cash generation contingent upon the underlying price and time. Under zero transactions costs, the value of such a compound product is simply the sum of the prices of the constituent strategies. However, with realistic transactions costs, it is possible under trading strategies other than the SLSG, that some portfolios will have offsetting positions. Due to the fact that the offsetting positions are determined by the controllable and entirely known portfolio weights, and also that transactions costs are assumed proportional to traded volume, it is possible to include them for pricing purposes.

The state-contingent cash flows can take on a more complicated form if we trade in several underlying securities. For example, if using a multi-dimensional tree to calculate incremental mean-variance formulae, the formula for each step will contain many more terms and increase rapidly with additional dimensions. Note that we still only need to store two numbers at each node: the expectation and variance of net assets. However there will be more parameters, such as correlation of transition probabilities, that need to be defined, modelled and estimated. It may then be appropriate to revert to using a trinomial lattice as the basic building block of the multi-dimensional tree in order to reduce computational requirements.

Corresponding continuous time analytic formulae have been developed. Notably, Föllmer and Protter (2000) [70] provide a d-dimensional generalisation of Itô's formula (further extended to a class of Lévy processes by Eisenbaum (2001a) [58]) where the quadratic covariation of the process replaces the quadratic variation terms in the Itô formulas.

4.3 Valuation Methods

4.3.1 Discretisation and Risk-Minimisation Issues

The convergence of actual cash flows to the continuous-time results described earlier is an important issue when using discrete-time trading to approximate the stop-loss start-gain strategy of a Black-Scholes market. We now consider two methods for quantifying imperfections in trading using analytic results.

There is a large literature investigating the optimal strategies for replicating option payoffs in imperfect markets. The approach taken in Bouchaud and Sornette '94 [16] and several subsequent papers including [188] emphasises that there is a continuous range of strategies available to the hedger, each of which is optimal in some sense. They show that in the case of a market satisfying the Black-Scholes assumptions, the delta-hedge eliminates *all* risk and that, if the assumptions are not satisfied (for example discrete-time trading or correlated or non-Gaussian distribution of returns), then the delta-hedge is not necessarily the optimal strategy. The analysis is set in an implicit utility-maximisation framework with mean-variance preferences, which has been investigated more rigorously elsewhere (see, e.g., [186]). However, the main idea drawn from the paper is the adjustment of the delta-hedge function in order to minimise risk.

First, we note that the local time of a continuous stochastic process is itself a stochastic process. A comparison can be drawn to approximate the variability of payoffs from the SLSG. The trading strategy can be seen as a delta-hedge performed under the assumption of zero volatility. The price of an option under zero volatility is given by its intrinsic value. The cost of an option under non-zero volatility is given by the sum of its intrinsic value and time value. As a result (and shown in [25]), the expected payoff to the SLSG hedge is the time value of the option. However, the variation of the payoff around the expected value will be significant. It follows that strategies with equivalent volatility assumptions between zero and the market-price option implied volatility will result in non-zero riskiness of the payoffs, even in a Black-Scholes world. (This issue is described in [16].)

Combining this idea with those of Arrow-Debreu pricing theory, it becomes possible to replicate all contingent claim payoffs using combinations of forwards and options. The example shown in figure 4.2 shows how we can approximate the Black-Scholes delta hedge using a finite number of stop-loss orders (four in the figure).

Using this framework allows us to analyse the variability of cash flows from trading based on deterministic boundaries for price changes. This is similar to the approximation used by Carr and Jarrow (1990) [25] where trades take place at a price-boundary. Note, however, that their model has very different properties to the discrete trading-time model followed by [157]. The latter approach will be used to arrive at prices and might also be more robust to the fact that there are deviations from even this model.

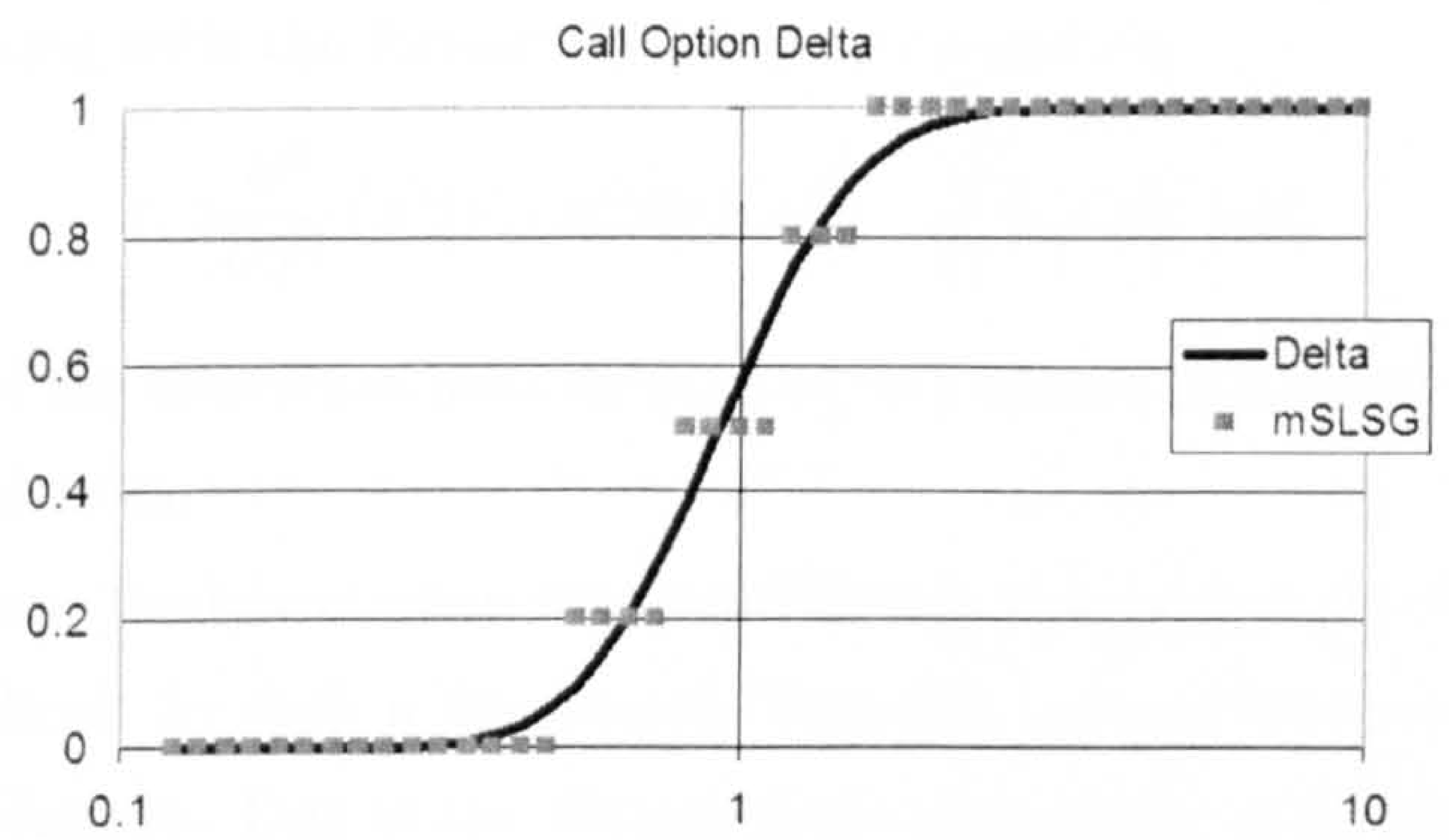


Figure 4.2: Approximation of delta-hedge using liquidity option construct

4.3.2 Models for Calibration to Market Prices

In order to model price processes for pricing of liquidity options, the volatility characteristics must be modelled accurately – principally because the instrument is a volatility derivative based on variance swaps. We investigate two broad classes of models: deterministic (or local) volatility and stochastic volatility. They are based on using market prices of options of various strikes and maturities to calculate implied price processes.

The first result comes from Breeden and Litzenberger (1978) [18], which derives the risk-neutral probability density of the stock price at time t and price S as:

$$\Phi(S,t) = \frac{\partial^2}{\partial K^2} C(K,t),$$

assuming that the price of a European call option $C(K,t)$ is twice differentiable. The result corresponds to twice differentiating the Black-Scholes formula although valid for a wider range of option price formulae.

The next result, by Dupire (1994) [55], is commonly known as local volatility. It assumes that volatility is a deterministic function of time and asset price, with the resulting identity:

$$\sigma(S,t) = \frac{2\frac{\partial C(K,t)}{\partial t}}{K^2 \frac{\partial^2 C(K,t)}{\partial K^2}}.$$

The identity can be proved using the Arrow-Debreu pricing reasoning of [18] and combining with the forward Kolmogorov equation

$$1/2\frac{\partial^2}{\partial S^2}\left(\sigma^2(S,t)S^2\Phi(S,t)\right)-\frac{\partial}{\partial t}\Phi(S,t)=0.$$

A version of the above has been developed by Derman and Kani (1994) [50] for binomial trees.

The above models assume that volatility is a deterministic function of price and time. In such a framework, liquidity options are used to hedge price-level changes. Due to the deterministic changes in volatility, the trading portfolios can be designed exactly to produce a certain contingent cash-flow rate. While this is a neat result, it highlights the fact that such a model assumption is not realistic and ignores volatility risk.

Britten-Jones and Neuberger (2000) [20] describe a general stochastic volatility model and show that there are many price processes for the underlying that are consistent with market prices of options. The paper gives two conditions that must be met by such models (i.e. two forecasts that can be made from options prices), given in discrete and continuous time forms. First, the expected variance rate at some time t in the future

$$\mathbb{E}[\sigma_t^2]=2\int_0^\infty\frac{1}{K^2}\frac{\partial C(K,t)}{\partial t}dK$$

and second, the expected average variance rate between two times t and T

$$\mathbb{E}\left[\int_t^T\left(\frac{dS_t}{S_t}\right)^2\right]=2\int_0^\infty\frac{C(K,T)-C(K,t)}{K^2}dK.$$

We will use this framework as it provides calibration results that are valid for non-Markovian volatility processes as well as more standard time- and price-dependent models.

4.3.3 Tree-based pricing

The reversed, enhanced (with an option position) stop-loss start-gain strategy (RESLSG) has a payoff that increases linearly with time spent at the strike price and with the variance of the underlying price process. This reflects the fact that the time value of an option is greatest when the underlying price is near the strike price and increases generally as volatility of the underlying increases. Therefore, a first approximation to a price using a binomial tree price model would be to calculate the expected product of the two. In the simplest case with constant volatility:

$$\mathcal{P}(x_t = x_{t,s}|p_0) = {}^t C_s \cdot \pi^s \cdot (1-\pi)^{(t-s)}$$

$$\text{Price} = E_0(\text{cashflow}) \simeq \frac{\sigma^2}{2} \cdot \sum_{t=1}^{T} [P(x_t = K|x_0)]$$

where the node corresponding (most closely) to the strike price, K, is (t, s_K) where

$$s_K = \frac{\left[\log \frac{K}{x_{0,0}} - t \cdot \log d\right]}{\log u/d},$$

rounded to the nearest integer. Selecting

$$u = e^{\mu\tau + \sigma\sqrt{\tau}}, d = e^{\mu\tau - \sigma\sqrt{\tau}}$$

results in $\pi = 0.5$. This provides an approximation to the limiting case of continuous trading and a continuous price process. However, it is difficult to incorporate transaction costs, does not explicitly model monitoring frequency and does not give an indication of the spread of possible payoffs (of which the price is the mean).

On a lattice, the calculated value of local liquidity, $a_{n,s}$, at each node is the expectation of $a_{n,s}$ from nodes that can lead to it and, as such, each $a_{n,s}$ is a weighted sum of several of formula 4.9, with the weights being the corresponding transition probabilities, so:

$$a_{n,s} = \sum_i [\pi_i(a_{n-1,s-L+i} + (\phi_{n,s} - \phi_{n-1,s-L+i})(K - x_{n,s}) - |\phi_{n,s} - \phi_{n-1,s-L+i}|\epsilon)], \tag{4.10}$$

and the sum will consist of between 1 and L terms, where L is the number of branches from each node and s is the node height.

By assuming a shape for the distribution of the local liquidities at any lattice node, it is possible to derive a formula for the variance of the local liquidity at the next lattice node in time. In particular, a normal distribution assumption allows one to specify fully the distribution of local liquidity at each node by mean and variance, so the formula for mean and variance at the next step are relatively compact.

Figure 4.3 illustrates the process and formulae are given below. The solid lines represent distributions at various nodes at time n and the dashed lines are the distributions as changed by any trading between time n and n+1. The heights of the distributions correspond to the transition probability from

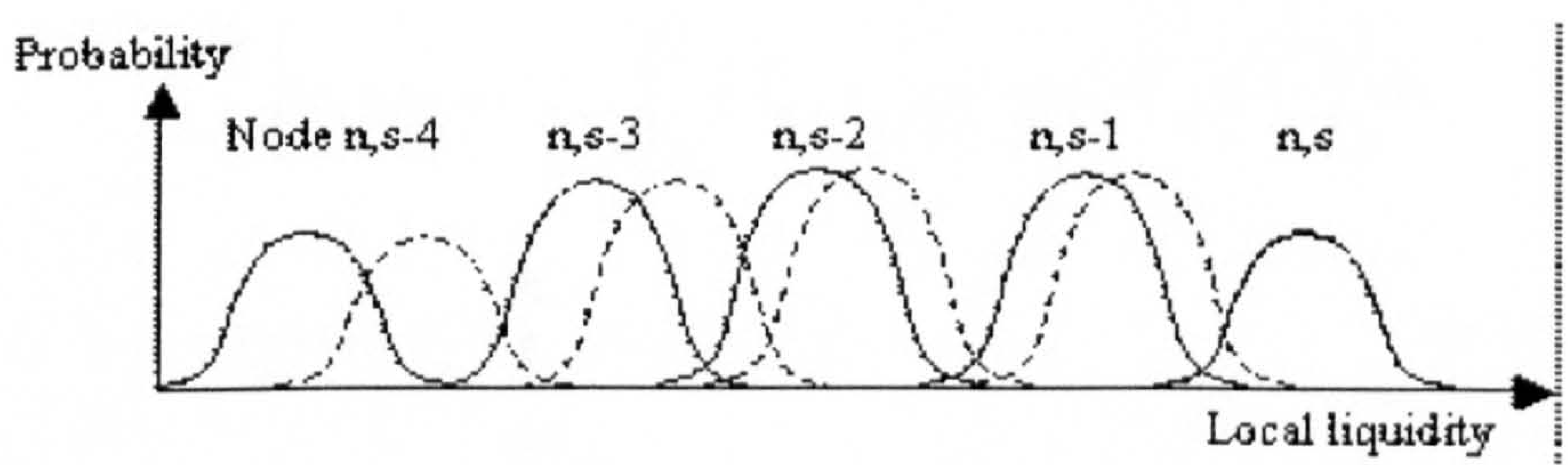


Figure 4.3: Lattice Liquidity Distributions

the indicated node to node (n+1,s), so the area under the whole graph is unity. The mean and variance of the distribution at node (n+1,s) is then the mean and variance of the resulting mixture-of-normals probability density function. In practice, these distributions will overlap much more than shown, so summarising the overall distribution as a single bell shape is acceptable for the current purposes. The particular figure shown corresponds to a case where the prices at the top two nodes (n,s) and (n,s-1) are above and below strike respectively and the price at node (n+1,s) is above strike. No trade occurs between (n,s) and (n+1,s) so the dashed line and the solid line coincide, whereas the prices at the other nodes are below strike so a trade takes place in moving across to (n+1,s) and alters the liquidity position. Note that the distributions will not generally have the shown order. For example, one of the middle nodes could have the highest liquidity by nature of having 'spent more time' near the strike price in previous steps.

Also note that trading does not affect the spreads (or heights) of the constituent distributions, as it shifts all the points within each distribution by an identical amount (see equation 4.13). Mean-shifting and the non-negative payoffs of the strategy can induce skew in the overall distribution, so inclusion of skew (and higher moments) in the calculations would increase accuracy, although at the expense of increased complexity of the algorithm.

The mean payoff at each node is the expectation of the trade-adjusted means of the previous nodes (4.11). The variance of the payoffs is given by applying an incremental formula (4.12), which is the solution of

$$\int_{-\infty}^{+\infty}\left[(a-\mu_{n+1},s)^2\cdot\sum_{i=low}^{high}\left(\frac{\pi_i}{\bar{\pi}}\cdot\frac{1}{\sigma_i\sqrt{2\pi}}\cdot\exp\frac{-(a-\mu_i')^2}{2\sigma_i^2}\right)\right]\mathrm{d}a$$

$$\begin{aligned}
\mu_{n+1,s} &= \sum_{i=low}^{high}\mu'_{n,s-L+i}\cdot\frac{\pi_i}{\bar{\pi}}, && (4.11)\\
\sigma^2_{n+1,s} &= \sum_{i=low}^{high}\frac{\pi_i}{\bar{\pi}}\cdot\left(\sigma^2_{n,s-L+i}+(\mu'_{n,s-L+i}-\mu_{n+1,s})^2\right), && (4.12)\\
\mu'_{n,s-L+i} &= \mu_{n,s-L+i}+(x_{n+1,s}-x_{n,s-L+i})\,(K-p_{n+1,s}) && (4.13)\\
&\quad -|x_{n+1,s}-x_{n,s-L+i}|\,\epsilon && (4.14)\\
\bar{\pi} &= \sum_{i=low}^{high}\pi_i && (4.15)
\end{aligned}$$

L is the branch-order, as before, and 'low' and 'high' account for the edges of the lattice. They are given by: $low = max(1, L-s)$, $high = min(L, (L-1)*n+L-s)$. The mean payoff at any point in time (i.e. across all nodes at that time) is equal to

$$\mu_n = \sum_{s=0}^{(L-1)n}\left[\mu_{n,s}\cdot\mathbb{P}(p_n = p_{n,s}|p_0)\right], \tag{4.16}$$

and the variance is given by

$$\sigma_n^2 = \sum_{s=0}^{(L-1)n}\left[\left(\sigma^2_{n,s}+(\mu_{n,s}-\mu_n)^2\right)\cdot\mathbb{P}(p_n = p_{n,s}|p_0)\right]. \tag{4.17}$$

The most important numbers for pricing the strategy are the terminal mean and variance, μ_T and σ_T^2, which give the expected payoff of the strategy (equal to the option's time value) and its riskiness.

All of the aforementioned calculations depend on the number of branches, L, in the lattice. The payoff of a liquidity option depends on the price-variance of the underlying security as well as having a path-dependent value. Therefore, the lattice used for valuation must reflect non-constant volatility. Due to the fact that trading occurs at discrete time steps, we should also use a price model that reflects the market's inter-trade price changes accurately. As a result, the optimal price model may be different for different instances of the same product, in particular for different markets and different maturities.

Simple binomial lattices are unsuitable, as they cannot model time-varying volatility. The next step up, to a trinomial lattice, provides for time-varying volatility if we accept having different transition probabilities along the time-line of the lattice.

The formula for the probability of getting to any particular node is a little more complicated than in the binomial case. Furthermore, it is only valid in that form if we do away with the ability to change local volatility. This issue remains with higher-branched lattices if we wish to explicitly control local volatility, but it is possible to circumvent that problem as shown earlier. Another weakness is that the trinomial tree has only two significantly different sizes of price step: apart from the middle branch that entails a, typically very small, drift (ud), the other branches 'up' (u^2) and 'down' (d^2) are roughly equal in size. This limits the extent to which it can model the possibility of 'jumping' across the strike, which has a bearing on the nature of the cash flows generated by trading.

For these reasons a pentanomial lattice is investigated, which permits incorporating skew and kurtosis of price returns to reflect a non-constant volatility and the possibility of 'jumps'. Note that, in the selected pentanomial framework, it is not possible to model stylised facts like persistence of volatility and the relationship between volatility and price level.

To create the pentanomial lattice, a set of equations as described in Yamada and Primbs [212] and shown in Appendix B are solved. Assuming valid probabilities are found, it is possible to use a shadow lattice as mentioned in the appendix to store the probability of being at any node, which we only need to calculate once. Also, given a node number and time, the value of σ is given by the user-specified function so π_i can be calculated explicitly if required. By rearranging the order of calculations, this process can be made only slightly more time-consuming than the constant-volatility case.

In the program, I use $\sigma_{n,s}^2 = \hat{\sigma}^2 \left(0.5 + \frac{1}{(2+n/s)}\right)^2$ which varies between $\frac{4}{9}\hat{\sigma}^2$ and $\hat{\sigma}^2$, being equal to $\hat{\sigma}^2$ at the lowest node in each time period and equal to $\frac{9}{16}\hat{\sigma}^2$ along the mean growth path. Changing the formula to increase $\sigma_{n,s}^2$ beyond $\hat{\sigma}^2$ results in negative transition probabilities with the formula for α being used. It is not possible to use a stochastic or autoregressive volatility model without more significant changes to the valuation procedure.

4.3.4 Explicit discrete trading under continuous prices

The expected profit for a general independent-increments process is given by:

$$E[a_T] = \pi_0 + \sum_{n=0}^{N-1} \{X_{Up} + X_{Down}\} \tag{4.18}$$

where

$$\begin{aligned} X_{Up} &= E[K - x_{n+1} | x_n \geq K, x_0] \\ &= \int_{-\infty}^{K} \int_{K}^{\infty} (K - \lambda)\mathcal{P}(x_{n+1} = \lambda | x_n = \mu) d\lambda \mathcal{P}(x_n = \mu | x_0) d\mu \end{aligned}$$

and

$$\begin{aligned} X_{Down} &= E[K - x_{n+1} | x_n < K, x_0] \\ &= \int_{K}^{\infty} \int_{-\infty}^{K} (K - \lambda)\mathcal{P}(x_{n+1} = \lambda | x_n = \mu) d\lambda \mathcal{P}(x_n = \mu | x_0) d\mu. \end{aligned}$$

For a Gaussian absolute increments process the integral can be simplified (and assuming zero drift for brevity) to give:

$$X_{Up} = \int_{-\infty}^{K} \left\{ (K-\mu)\mathrm{N}\left(\frac{-|K-\mu|}{\sigma\sqrt{\tau}}\right) - \sigma^2\tau\mathrm{N}'\left(\frac{K-\mu}{\sigma\sqrt{\tau}}\right) \right\} \mathrm{N}'\left(\frac{\mu-x_0}{\sigma\sqrt{n\tau}}\right) d\mu$$

and

$$X_{Down} = \int_{K}^{\infty} \left\{ (K-\mu)\mathrm{N}\left(\frac{-|K-\mu|}{\sigma\sqrt{\tau}}\right) + \sigma^2\tau\mathrm{N}'\left(\frac{K-\mu}{\sigma\sqrt{\tau}}\right) \right\} \mathrm{N}'\left(\frac{\mu-x_0}{\sigma\sqrt{n\tau}}\right) d\mu$$

where $\mathrm{N}'(\cdot)$ denotes the standard Normal probability density function and $\mathrm{N}(\cdot)$ the cumulative. Beyond this stage, numerical approximations can be used to compute prices.

The formulae for Gaussian relative increments are not analytically solvable. It is possible to evaluate the double integral numerically and can be used to price the overall product if it has a simple structure. However, if the product is extended to cover multiple underlying assets, the effect of inter-dependencies between the asset prices will make the computation much more involved.

There may be a way to avoid the cumbersome integrals in the case when the overall liquidity option required has a smooth, put-option like payout profile. In this case, it may be possible to follow work by various authors such as [200] and [172] who have investigated analytical formulae for the payoffs from delta-type hedging of options. This is because the trading that takes place over the whole portfolio of a liquidity option may be closer to simulation using a continuous function rather than the special case of the SLSG.

4.3.5 Continuous model with discrete-trading adjustment

Due to the difficulties in pricing liquidity options using multinomial trees and the above discrete-time trading approaches, a third way is investigated whereby the analytical formula for the expected time-value is used to arrive at a benchmark price for the liquidity option, which is then adjusted to take into account the limited number of trades and the existence of transactions costs. This would allow a simple and fast formula to be used to price the components of a liquidity option.

In [25] and [95], the prices for European call and put options are decomposed in terms of intrinsic and time value. The former paper analyses the classical Black-Scholes market with a geometric Brownian motion as the underlying risky asset price, whereas the latter paper extends this to geometric fractional Brownian motions with Hurst parameter H greater than $1/2$ (where $H = 1/2$ corresponds to the Black-Scholes case), i.e. where there are positive long-term correlations between price increments. The method used is based on the Meyer-Tanaka formula. Both papers give usable results, but the fractional price process has not been simulated in this thesis. The fractional price process is a good example where application of a wavelet based construction technique can be convenient and make it possible to get tractable analytical results.

Monte Carlo simulations were made with various parameter values, that followed the exact trading strategy in the underlying asset, including discrete trading and transaction costs. The simulated cash flows were then stored with the corresponding parameter values that are drawn uniformly from the following ranges:

Tenor, 0.5 to 10 years
Drift, 0 to 10% p.a. continuously compounded
Variance, 0 to 60% standard deviation (p.a.)
Strike price, (relative to x_0)
Number of trades, 2 to 200
Transaction cost, 0 to 10 basis points

500 independent realisations of the price process from each set of parameters were used to get an average cash flow. This was repeated for 1000 different parameter sets. The resulting data set is regressed against the risk-neutral expected local time of the price process with the given parameters as well as the number of trades and transaction costs. For comparison, regressions were also done with the basic parameters alone, without transformation into

local time. Both versions of regressions were also repeated on the same data sets as polynomial regressions up to order 3. The resulting R^2 values are as follows:

Plain multiple regression $R^2 = 0.41$
Polynomial regression $R^2 = 0.45$
Plain local time $R^2 = 0.28$
Polynomial local time $R^2 = 0.36$

Note that it is also possible to calculate the variance, or any other measure of spread, of the cash flows for any parameter set. In this respect, the model is flexible and does not fall behind the tree-based approach. The regression results show that the local time transformation does not improve the basic regression model and all models struggle to explain more than 60% of the variation of the *average* cash flow. As such, the approach does not present an adequate general pricing formula and so the pricing of each component of a liquidity option would need to be based on simulations with the specific parameter values.

4.3.6 Model comparison

The three models investigated — discrete trades and prices modelled on multinomial trees, discrete trades with continuous prices modelled with integrals, and adjusted analytical formulae modelled with regressions — all have weaknesses. Tree based approaches are faster but susceptible to errors due to the small number of allowable prices. Adjusting analytical formulae using regression has shown to be limited in its scope. Explicit modelling of the trades, with continuous prices, leaves one with complicated numerical integration, but is the least susceptible to errors and seems to be the best of the relatively fast methods investigated so far. Monte Carlo simulation of the payoffs of each liquidity option structure is the last resort.

There are two directions that can be followed to arrive at a more elegant pricing solution. One is to apply the more involved analytical model of the distribution of cash flows (local time) under ideal conditions, following Hjort and Khasminskii (1993) [93]. The paper shows that the density function is a mixture of a delta-function at zero at an exponential distribution. By matching histograms of simulated cash flows, such as shown in figure 4.4, to the analytical solution for the equivalent case, a calibration method can be used to fit the model's predictions to market prices of other instruments.

The other direction is to skip the pricing of individual components of the liquidity option and, subject to constraints on the shape of the local liquidity surface, apply the delta-hedging related results ([200], [172]) mentioned earlier in this chapter.

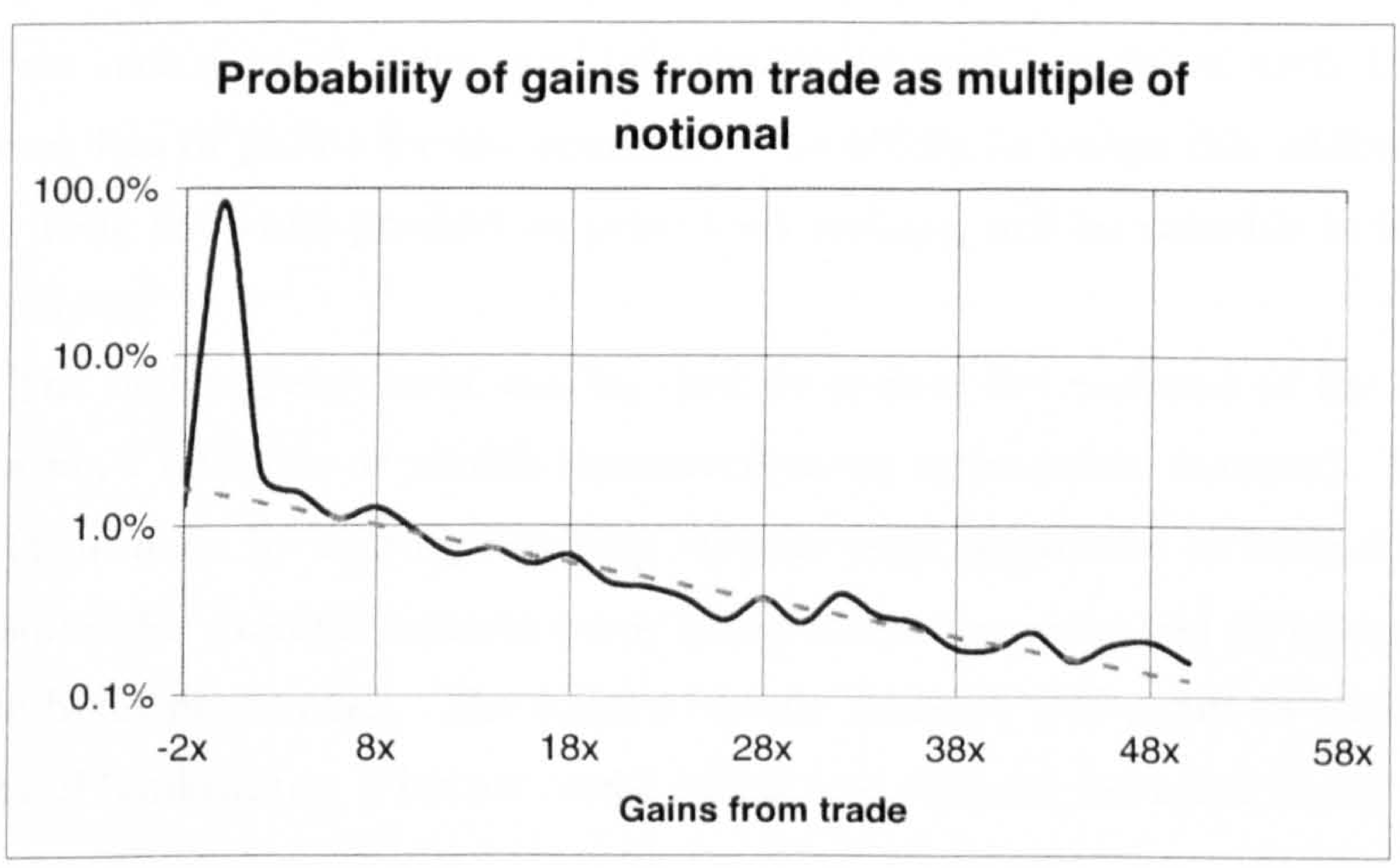


Figure 4.4: Distribution of cash flows from an example liquidity option. The best-fit line for the right tail of the distribution has formula $0.015e^{-0.05x}$ and an R^2 of 0.918.

4.3.7 Construction of Liquidity Options

Liquidity options can be used to approximate a straightforward price level hedge. By increasing the local liquidity far away from some optimal price level (which can change over time), the buyer of the hedge receives more cash in the event of the price leaving its current level. This straightforward application is more versatile and, in some cases, more appropriate than more common hedges such as forwards, options and swaps. There are two distinguishing features.

1. The payoff depends (to varying degrees) on the prices during the whole period of the contract, in contrast to forwards and options.

2. The payoff depends on the volatility of the prices during the period of the contract, which is a feature not directly present in any of the other hedges.

The presence of positive volatility sensitivity is a useful feature in that increased uncertainty in market prices (for commodities, currencies or even indices such as stock prices and inflation) is at least associated with, if not causes, loss of profits for the company. The ability to hedge this additional risk using the same product as price level hedging will be valuable in most situations.

The hedging argument can be used to reduce the variance of the corporation's revenues or profits (measured in an appropriate manner). This could be done by writing liquidity options with sensitivity to several key variables, for example interest rates, foreign exchange rates and an industry-wide bond price index. The reduced profit variance will result in reduced costs of bankruptcy, a better credit rating and reduced financing costs.

4.3.8 Next steps

It would be useful to immunise the payoff of liquidity options from the volatility of volatility in order to create a derivative that is based purely on the local *time* of the underlying process(es).

Early termination options are frequent in commercial contracts, either for one or both parties. The choice of termination could feasibly depend on any one of several factors including price, realised gains and so on. The optimal decision rule can be determined using techniques similar to those for American options or credit default swaps. Davis *et. al.* (2001) [43] have developed an elegant static hedging methodology for this type of problem when applied to European options.

4.3.9 Calibration

During calculation of the model, the theoretically continuous state variables need to be discretised. This allows the equilibrium functions θ_t and p_t to be calculated and stored for a few combinations of the state variables. Two methods are used here, linear interpolation and finite-state approximation. The amount at hand ratio w_t is approximated by a finite grid of points with linear interpolation used for intermediate values. It is possible to prove that, under some conditions, the amount at hand ratio will only occupy a finite range in all sample paths (notwithstanding the initial ratio, which may theoretically be outside that range).[6] Finding the range requires trial and error but the process can, in practice, be avoided by calculating the solution over a very wide range and using linear extrapolation in cases where the range is insufficient.

The other two state variables, average income growth g_t and cash flow noise ϵ_t, are approximated by a finite number of states. Transitions between states are defined with low-order Markov chains designed to approximate the autocorrelation and variance of the underlying time series.

ϵ_t follows an autocorrelated (usually negatively) time series

$$\epsilon_{t+1} \sim N(\phi_\epsilon \epsilon_t, \sigma_\epsilon) \quad \text{where} \quad \phi_\epsilon \leq 0.$$

It is necessary to ensure that at least one state of a_t is negative in order to ensure the possibility of top-ups, so the state values are chosen so that at least one state satisfies $1 + \epsilon < 0$. The parameter K dictates the states values, and needs to be at least 2.5 to provide one negative state. If $K > 5$, there are two or more negative states. Given the state values, the transition

[6] Proofs of this for closely related models can be found in [175] and [45]. Specific proofs for this model are not included.

probabilities are to approximate the autocorrelation and variance.

$$\begin{aligned}
\mathbb{P}[\epsilon_{t+1} = \epsilon_j | \epsilon_t = \epsilon_i] &= \frac{\tau_{ij}}{\sum_j \tau_{ij}} && (4.19)\\
\tau_{ij} &= \frac{1}{\sqrt{2\pi}} \exp\left(\frac{(\epsilon_j - \phi\epsilon_i)^2}{2\sigma_\epsilon^2}\right) && (4.20)\\
-1 < \phi &< 0 && (4.21)\\
\epsilon_i &= \left(\frac{(i+0.5)}{5} - 0.5\right) \cdot K && (4.22)\\
i, j &= 0, 1, ..., 4. && (4.23)
\end{aligned}$$

Note that noise states have zero mean and that normalisation is required to convert the density function to a valid discrete probability mass function.

Average income growth is approximated with two states that can be thought of as boom and slump. This becomes similar to a regime-switching model that can be calibrated using business cycle data. However, the model also includes the expected average growth rate, μ_t, forecast for each period in the business plan and needs to compensate for any bias in the forecast.

$$\mathbb{P}[g_{t+1} - \mu_{t+1} = \hat{g}_j | g_t - \mu_t = \hat{g}_i] = \mathbb{P}_{ij}^T \quad \text{where} \quad i, j = 1, 2$$

The probabilities and values are estimated directly, rather than being designed to approximate an AR(1) time series. The transition probabilities for g are taken from Hamilton (1989) [82], which estimates business cycle data using a regime switching model similar to the one used here.

The particular values of parameters are calibrated to market prices where available. In this vein, the two relevant parameters (b and ρ) of the profit function $v = \frac{b}{\rho}(k - e^{-\rho\theta})$ are estimated using equity market data.

It may be noticed that, although cash income in a period can be negative due to cash flow noise, it is assumed that average income is strictly positive. However, it is possible that a business plan includes periods where it expects negative cash income on average, leading to a breakdown of the solution method in this paper. Such situations indicate that the business plan is showing cash flows at too fine a level of detail, where influences of cash flow noise are showing. When this happens, the model can be changed to take longer time periods as the basic increment or, equivalently, the cash flows can be smoothed (in a mean-preserving manner). If average income of the business does indeed become negative, it points to inherent unprofitability and neither the liquidity guarantee nor other forms of financing are available to businesses subject to this kind of risk.

4.4 Results

The model was applied to a telecommunications company that had recently emerged from a financial restructuring. As part of the restructuring, the company was obliged to provide management forecasts covering five years and a detailed history of cash flows covering a period of six months. The forecasts contain projected figures for revenues, cost of goods sold, operating expenses and capital expenditure. These were combined with interest payments, maturing of bonds and the effect of a rights issue, assuming that the principal of some bonds would be paid while others refinanced. Figures for debtor and creditor days were used to approximate the degree of financing used in payments and receipts. Capital expenditure was split into voluntary investment and maintenance components.

The cash proportions of all forecast figures were estimated from creditor and debtor days provided in historical performance figures, using a steady-state approximation. Once credit is used to finance investment, voluntary or not, subsequent cash flows are fixed charges and less controllable by the firm. As a result, only the cash portion of voluntary investment was termed voluntary, while the remainder was added to fixed charges. All other receipts and payments are assumed to be uncontrollable in the time scale required to deal with short-term liquidity constraints.

The shape of investment function θ shows that noise is more important than growth when liquidity is low. Figure 4.5 plots the investment functions for all ten different cash flow states (two growth rates and five noise states). The cash flow noise dominates behaviour near the origin so that functions for both growth states overlap. The converse is true when liquidity is high:

the growth state is more important than cash flow noise.

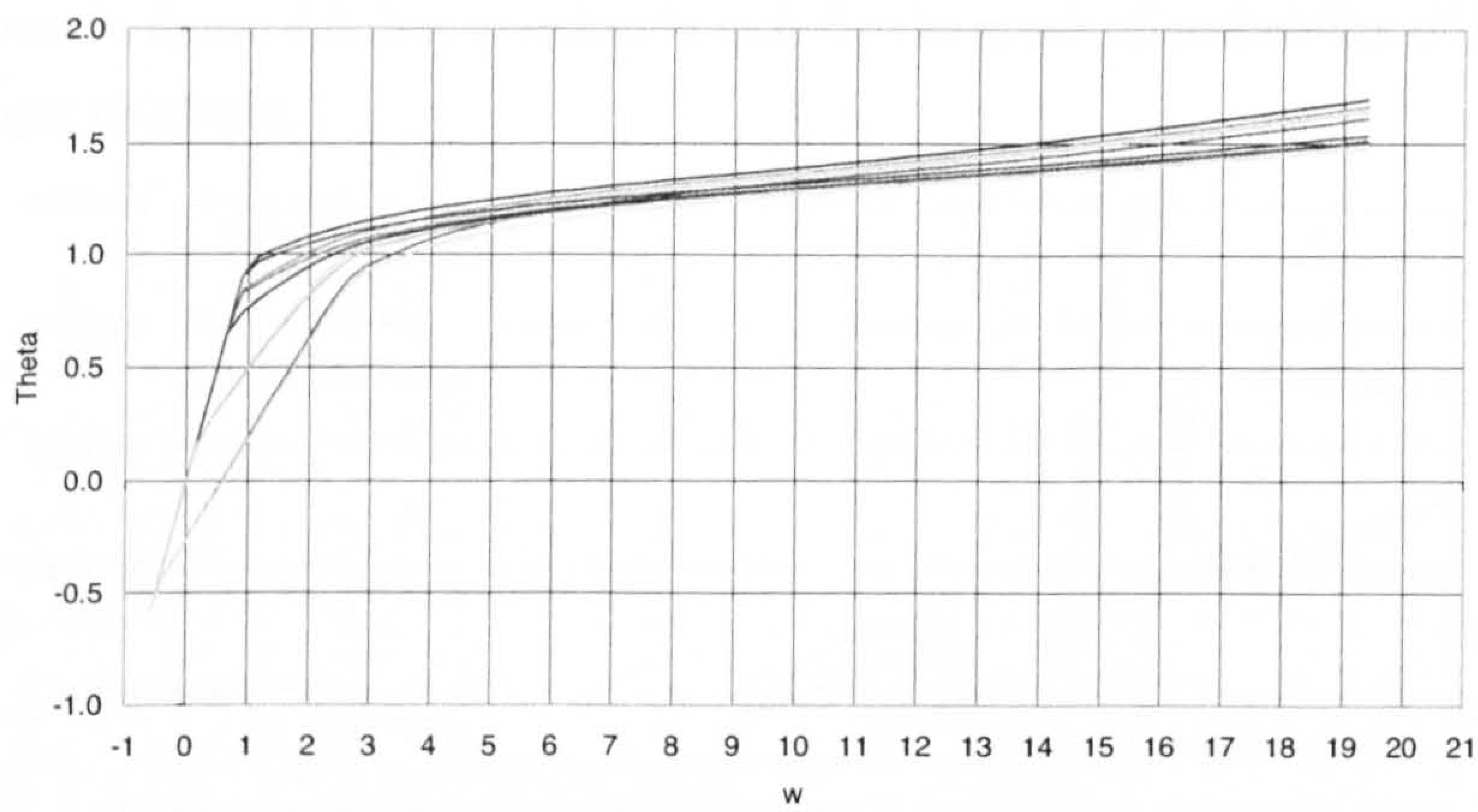


Figure 4.5: Investment functions

Figures 4.6, 4.7 and 4.8 show a typical sample where the guarantee is exercised. The periods are one quarter year in length. Figure 4.6 shows the cash flows in absolute amounts according to one possible sample realisation. These are the variables observed by the agent and its responses to those observations. After a negative cash flow in period 0.5, the agent almost depletes its liquidity and eventually needs to receive external funds as indicated by the top-up line.

Figure 4.7 shows unobserved variables: the Amount on Hand ratios and spend ratio (θ) which are related to the agent's spending decisions, as well as an indicator of the growth state g_t (state 1 meaning higher growth than the zero state). Although the average profitability grows at an above-expected rate throughout the 5-year period, the noise component of cash flow results in the Amount on Hand ratio being consistently below the equilibrium, or target, ratio for 3 years.

The rise in the target Amount on Hand ratio during periods 1.00 to 2.00 in figure 4.7 reflects the expected temporary dip in average profit level during that time. It shows that the agent acts with a degree of prudence by aiming to store more cash as a ratio of profits during those periods in anticipation of larger scale uncertainty in cash flows thereafter. This is particularly clear since expected cash flow is still positive and grows after 2.00, as clearly shown in figure 4.8, but aversion to risk (in the sense of probability of loss) causes prudence.

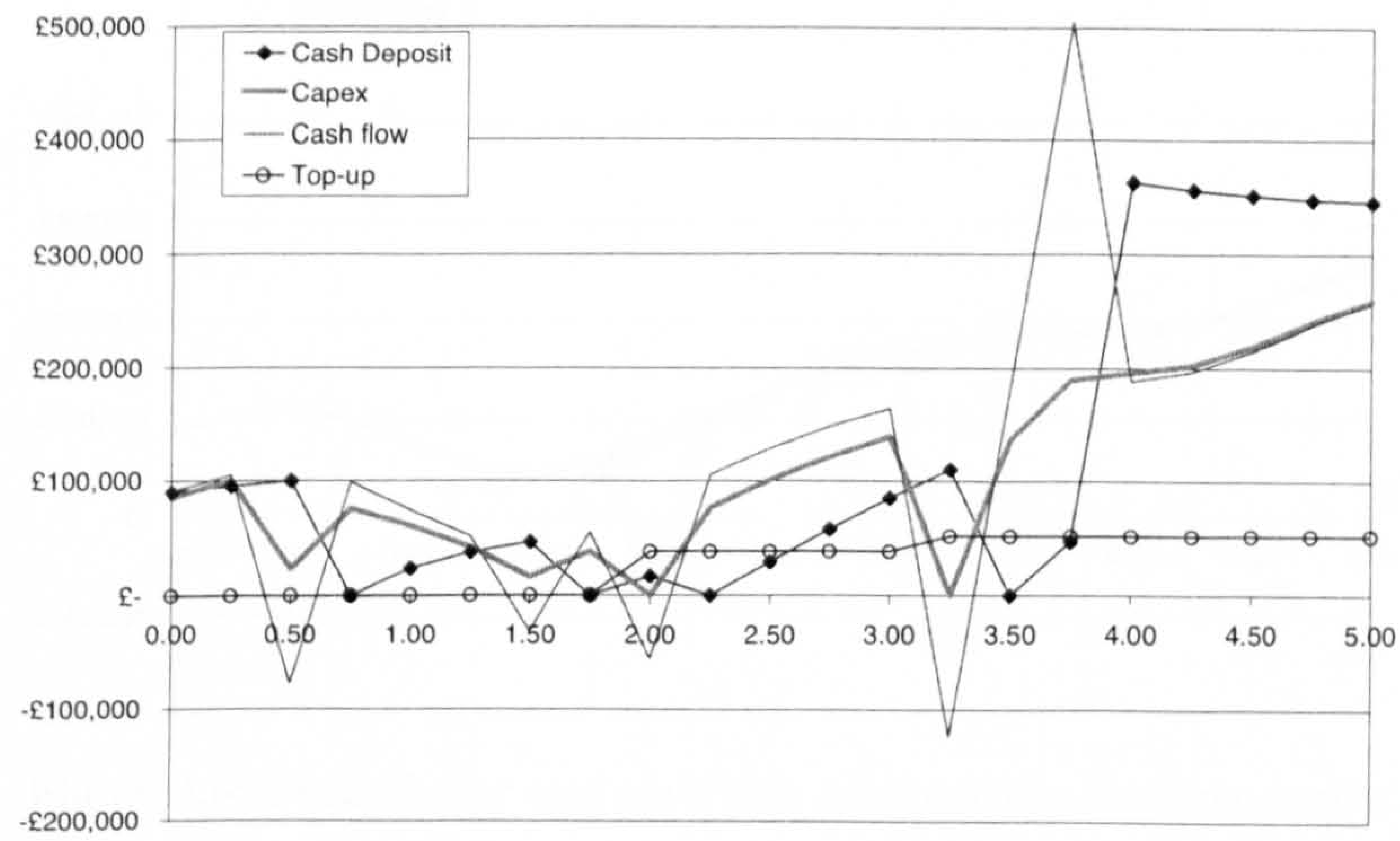


Figure 4.6: Absolute values

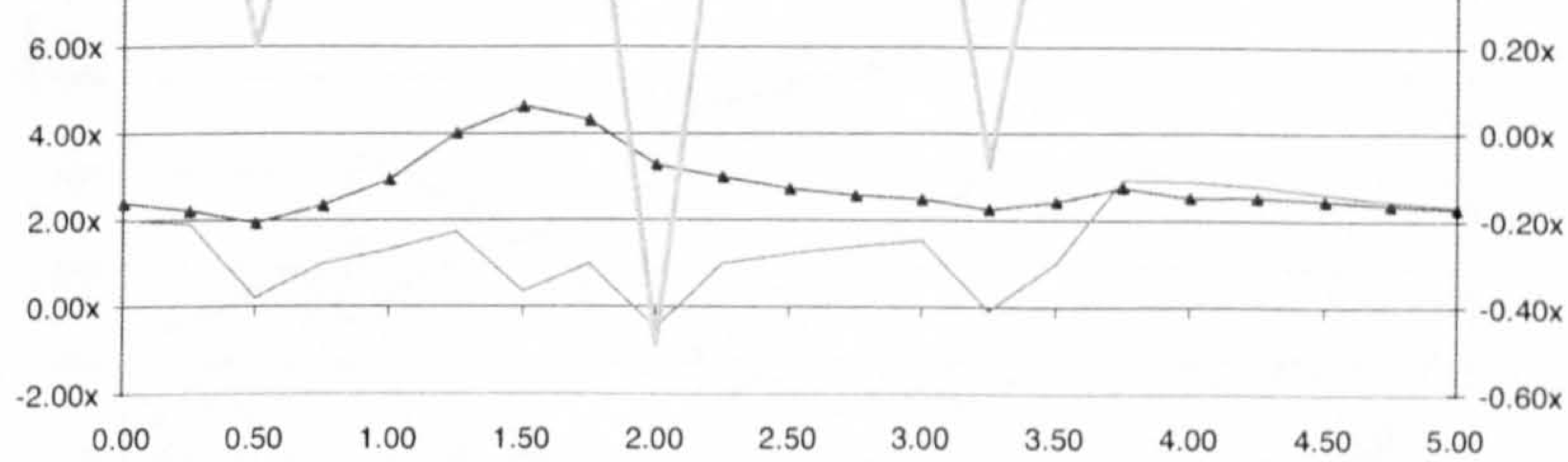


Figure 4.7: Ratio variables

Figure 4.9 shows the base case forecast of the business, illustrating how the discretionary spending plans of the company are related to accounting figures. Free Cash Flow (FCF) after fixed charges are mapped to the ex-

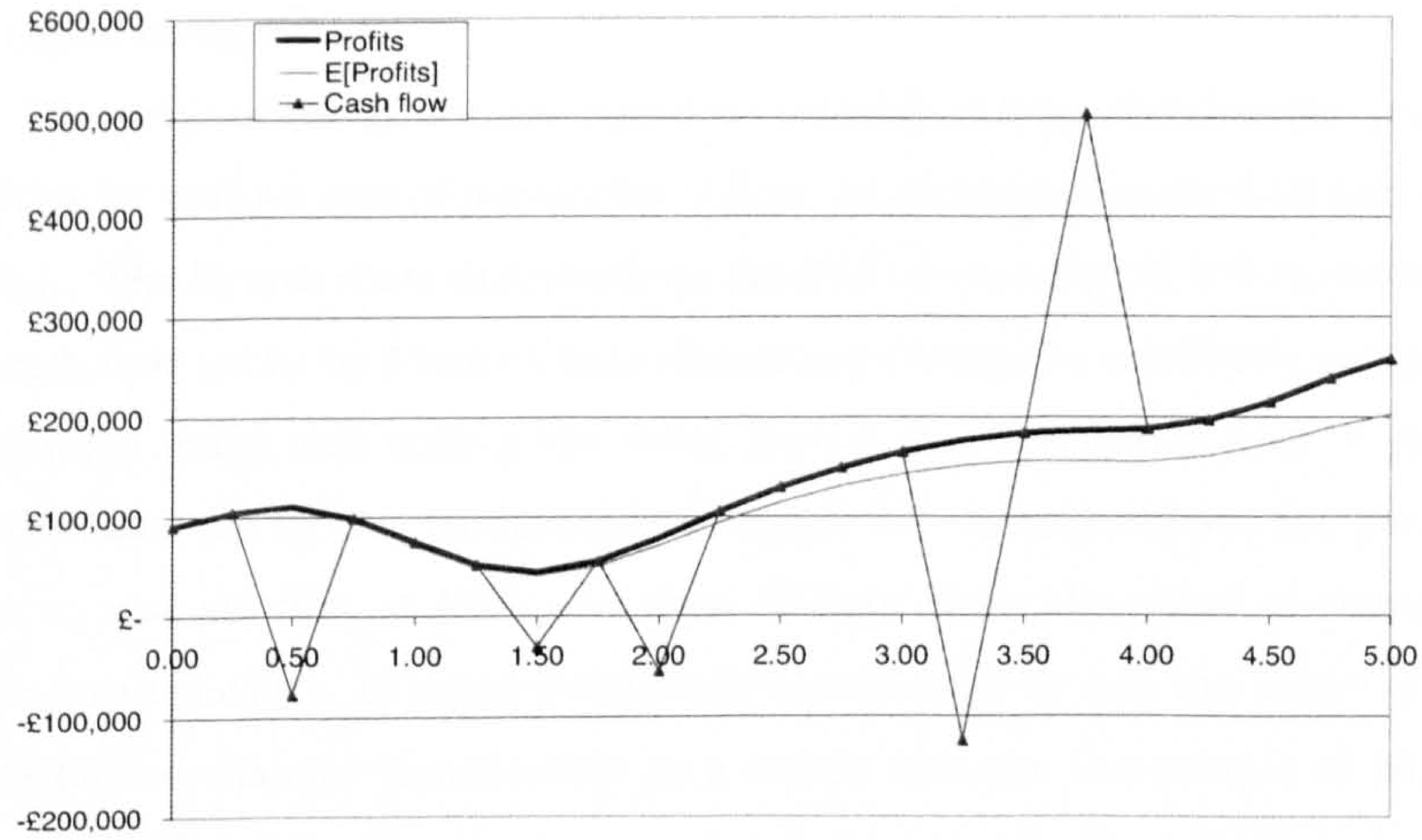


Figure 4.8: Profitability and cash flow compared to forecast profits

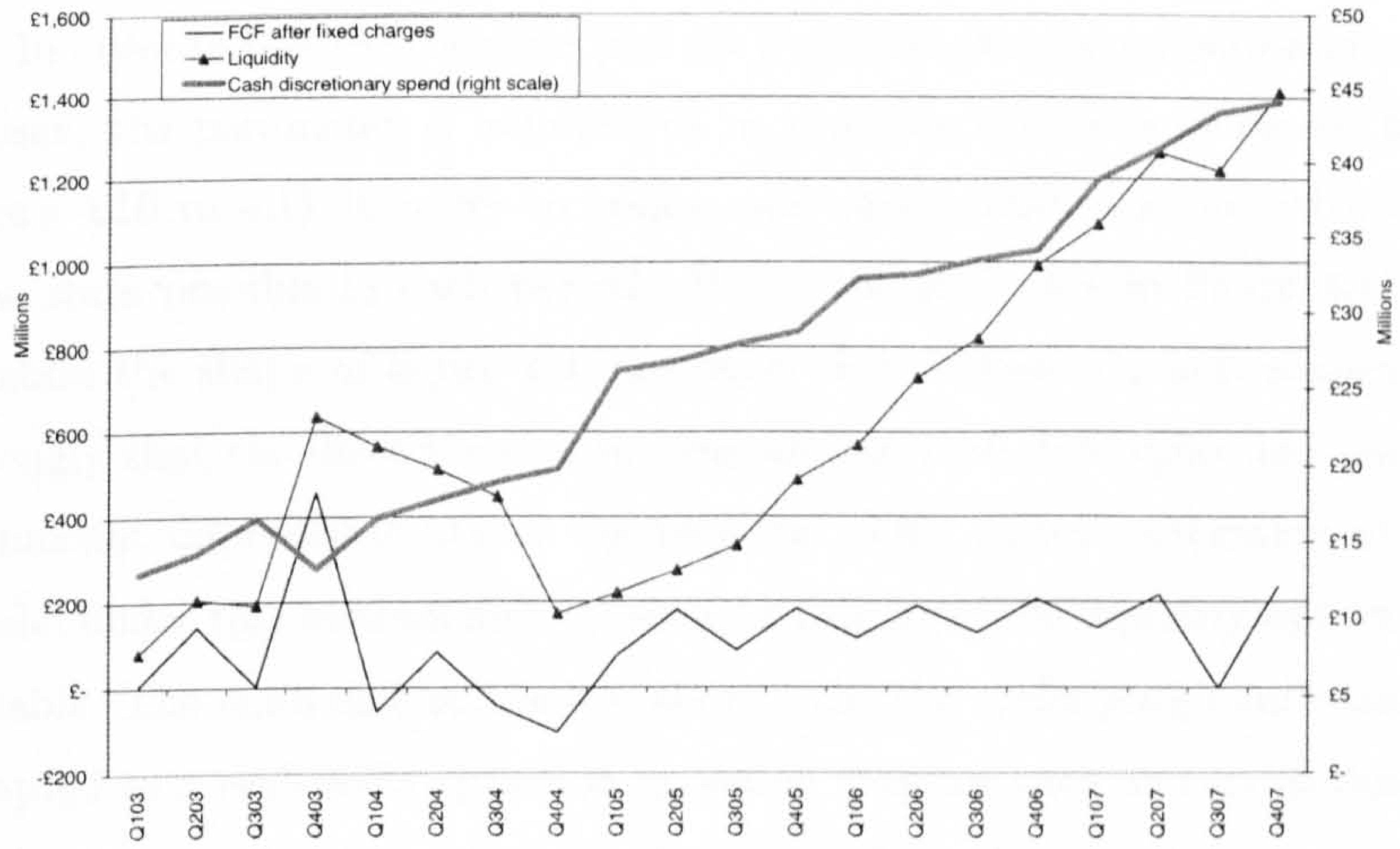


Figure 4.9: Cash flows according to forecast

pected profit in figure 4.8, liquidity is mapped to cash deposit in figure 4.6 (as long as the top-up is not used) and cash discretionary spend is mapped to capex in figure 4.6.

The price of the guarantee could be calculated from distribution probabilities for various sets of parameter values, as shown in figures 4.10 and 4.11 below. The figures show distributions derived by running 50,000 realisations of cash flow paths by Monte Carlo simulation (using the antithetic sampling technique only) and noting the total top-up amount drawn over 5 years. This makes the figures comparable to figure 4.4. In each figure, the parameter σ_ϵ was set first at 60% and then 80% to show the effect of changing cash flow volatility. It is not possible to say whether or not the shape of the distribution changes significantly as a result, because the sample of 50,000 seems insufficient to approach convergence to a stable shape and no formal hypothesis test was performed to assess confidence levels for the shape of the distribution. It can be noted that in both figures, the 60% volatility case has fewer cases of very large payouts and lower frequency overall of any payouts at all. These reflect an intuitive assessment of extensions to the analytical theoretical frameworks described earlier in the report.

In order to test the sensitivity of the results to the discretisation method chosen, the parameter K referred to in equation 4.19 was increased from figure 4.10 to 4.11 in order to create two, rather than one, negative cash flow state possible in each period. While the 60% case in figure 4.11 resembles the shape of figure 4.4, the other distributions do not, suggesting strongly that the discretisation in time and/or cash flow space has caused significant unpredictability of the payouts. This makes calibration of the model under this method and consequent pricing of the liquidity option unreliable. The analysis has however shown that the underlying model can be adapted to a real-world situation including features such as non-stationary profitability and still produce results that reflect observed behaviour such as forward-looking spending and saving decisions that take into account the non-stationarities.

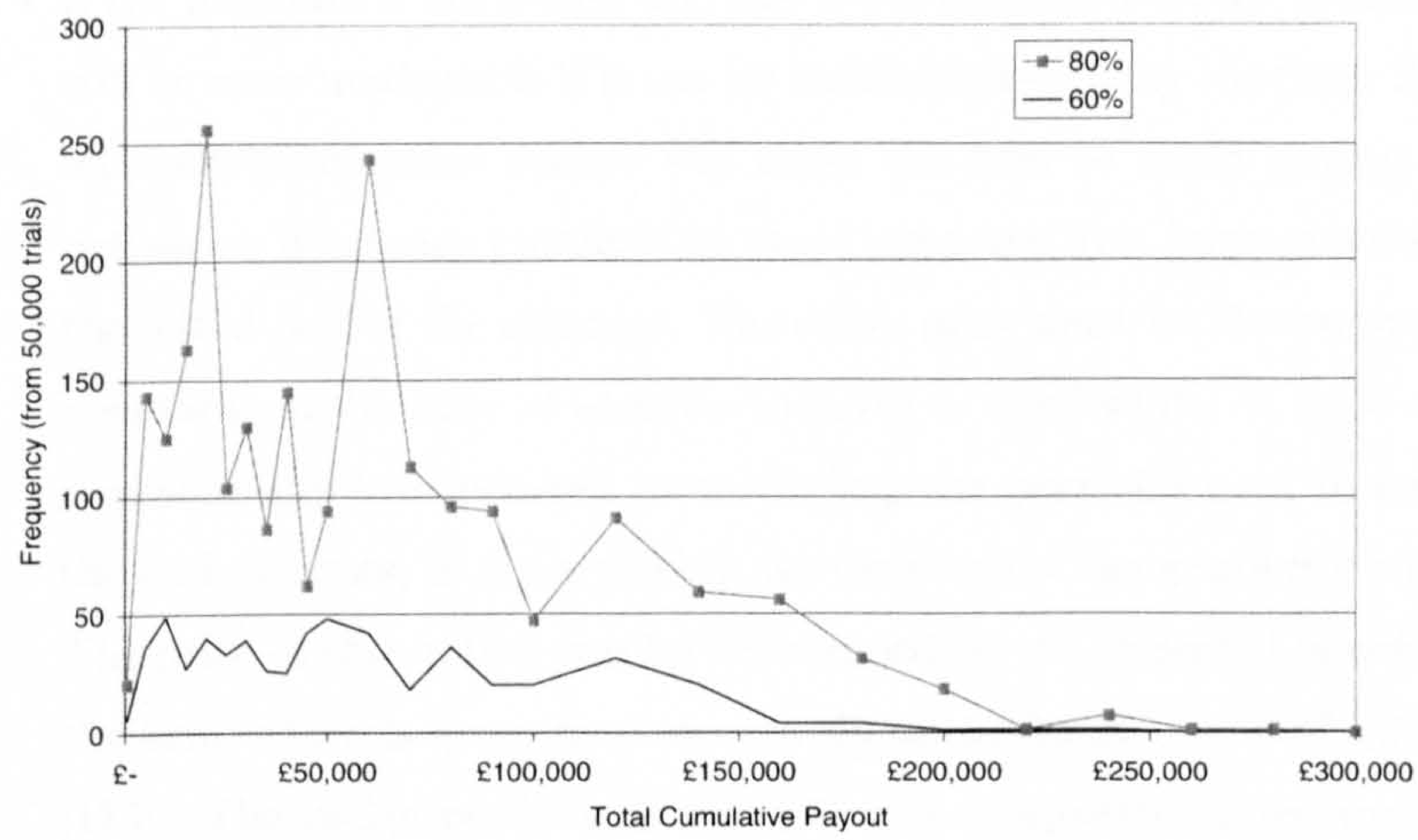


Figure 4.10: Distribution of payouts

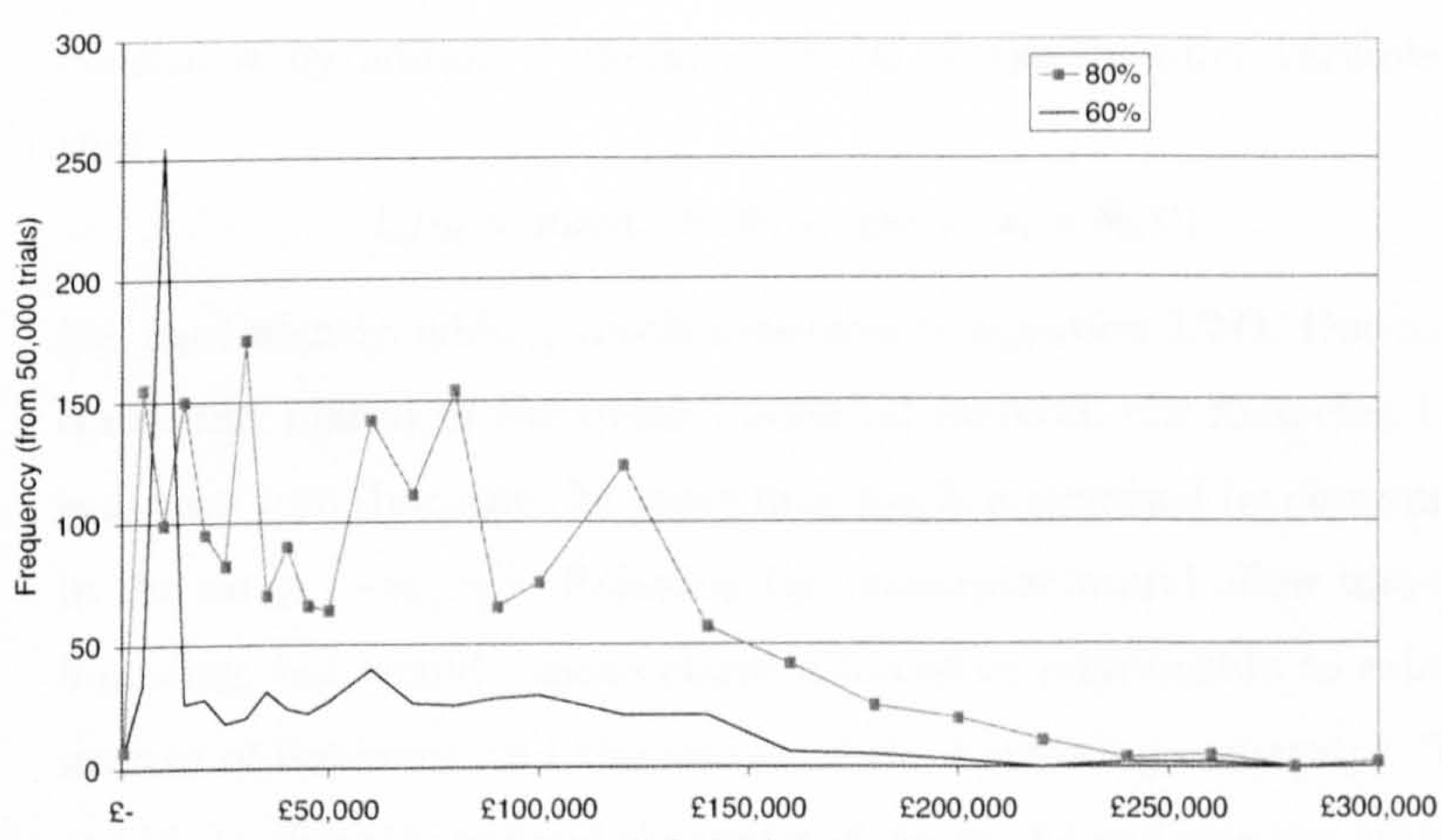


Figure 4.11: Distribution under a wider range of noise states

4.5 Extensions

- If the premium is amortised over the life of the contract, the guarantee will be more marketable if it can be terminated early by the firm. Such an early-termination option will allow the firm to avoid paying the remaining insurance premia if its state improves by a large amount in the initial part of the contract. The strike price must be chosen to give a suitable probability of exercise to make it worthwhile. It must also minimise any loss incurred in unwinding the portfolio used to hedge the cash payouts, in order to keep the exposure of the guarantor small. The price of this option can be determined by the expected benefit to the firm, which is given by a process shown in Routledge *et.al.* (2000) [175]. The hedge portfolio is described in a separate paper and the combination of the two methods is expected to result in a range of suitable options.

- It is tempting to include the possibility of $\theta_t > w_t$, where it would be optimal for the firm to invest more than is currently available. Such a situation could be implemented while satisfying the non-negative cash constraint by adding a 'financing' term to the liquidity variable, so that

$$l_t/y_t = \max(-\theta_t, 0) + \max(-x_t + \theta_t, 0)$$

  (or, equivalently, adding another variable to equation 2.24). Due to the constraint placed in the present solution method, the financing term is always zero (because the value function is maximised by choosing θ_t in the range $[-\infty, x_t]$). Relaxing the constraint would allow non-zero financing, but would require clarification of its relationship to existing sources of financing and the nature of the borrowing constraint. This would significantly expand the scope of the model and was not followed in order to keep the problem tractable.

- The details of the model differ from a model of autocorrelated growth in [44], since it has been modified to allow for losses. Many models in the referenced literature result in stock-outs: when consumption is equal to income and there are no savings/inventory. However they occur only during high growth states that have persisted several periods, which is contrary to the evidence. They do provide two useful characteristics: (a) voluntary expenditure after the end of a slump is high, reflecting the importance of having sufficient funds to pay for inventory-building at the start of a boom; (b) voluntary spending falls rapidly following the onset of a slump and cash balances rise initially.

Both of these effects are present, although less pronounced, in economic data. One attractive explanation rests on relaxing an important assumption is that the decision-makers in the firm are immediately aware of the current state (as well as the history of previous states). While it can be reasonably assumed to apply to the variables ϵ_t and x_t, the firm's estimate of g_t may be subject to more uncertainty. If the firm must estimate the state in order to make expenditure decisions, the delay and residual uncertainty may improve the realism of the firm's behaviour. Optimal estimation of a regime switching model shown in Hamilton (1989) [82] may yield a suitable method, but has not been applied in this paper.

Chapter 5

Conclusions

The thesis has investigated concepts surrounding finite financial or operational flexibility and behavioural features suggested by real options literature.

Chapter 2 postulated a rational economic agent with certain restrictions and analysed several versions of it. Its choice criterion was kept parsimonious throughout and focus was placed on restrictions in its choice criterion arising from two broad categories: first physical constraints and second unobservable variables and related information constraints. An extension was made to add effects of coordination problems and prior beliefs, following further observation restrictions and information constraints. The review of concepts in chapter 1 provided a background to the role of information according to existing literature in price discovery and formation in the various parts of an economy. It also described scarcity of resources leading to constraints on behaviour in various guises and tried to highlight the fact that both scarcity of information and physical resources can have similar results. The models developed in chapter 2 showed that simple representations of either kind of scarcity can yield behaviour resembling that observed in many organised and un-organised markets as well as in situations where an agent interacts with physical nature alone.

Chapter 3 linked time series created using one of the models of the previous chapter to the wavelet method of constructing and deconstructing time series. One avenue investigated in chapter 2 was the possibility of generating part-localised income series in an economy of agents who interact only locally with immediate geographic neighbours. While the multi-agent economy model did yield some structure under certain parameter sets, there were no discernible local patterns that could be exploited by multi-scale separation techniques of wavelets. Newer wavelet techniques were pointed out that can potentially expand the area of research and its applications. Even without use of wavelet technique, chapter 4 investigated implications of a result that it is possible to theoretically hedge a derivative that yields cash flows that would protect an idealised entity against the impact of idealised liquidity constraints. It does this by exploiting a dynamic trading strategy based on a mathematical result related to variation of continuous-time stochastic processes (whether they be continuous or allow for discontinuities in state space). By attempting to map the stochastic process to prices of relevant tradable goods, as well to map the characteristics of the idealised company and idealised measure of liquidity to a real example, some progress was made in showing how such a hedge might be constructed and priced in practice. However, due to complications arising from discretisations (including those required to implement the trading strategy, those needed to perform computer calculations of probabilities and those desirable to increase speed to calculations) the task of calibration and assessment of risks became unreliable without further investigation.

Among the various more advanced discretisation techniques available, the wavelet method is proposed as a promising technique because of its close ties to the characteristics of stochastic processes that have been most prominent in this work: quadratic variation (or other measures of fluctuation, as discussed in relation to information uncertainty and risk aversion arising from liquidity constraints) and differences in behaviour at different scales (as illustrated in the model of an economy of similar agents that develops large-scale patterns in activity). It is hoped that future work can exploit the fundamental links to generate more versatile and realistic time series and at the same time to analyse risks and rewards of derivative contracts that depend on these characteristics.

Appendix A

Exchange Model Extensions

Correlations

The variables c_t and λ_t were assumed to be mutually independent as well as independent of other state variables. However, such a strong assumption is not necessary to arrive at results and introducing dependence between the two variables may be fruitful. In particular, a wavelet-based model of the evolution of c_t and λ_t allows the incorporation of several alternative regimes of explicit correlations, where determining which regime is dominating any given data set becomes the principal task.

Continuous time limit

By adjusting the definition of c_t, we can include the time-step length as a parameter. Remembering that c_t is a (possible) growth rate over time, we define

$$\begin{aligned} y_{t+\delta t} &= y_t(1 + a_t c_t \delta t) \\ \Rightarrow \frac{(y_{t+\delta t} - y_t)}{y_t} &= a_t c_t \delta t \\ x_{t+\delta t} &= x_t - a_t \lambda_t y_t c_t \delta t \end{aligned}$$

and let δt tend to zero. In this case, we can relax the distributional assumption on c_t to a (non-log) Gaussian while maintaining the non-negativity of y_t. By iteratively substituting the formulæ, we arrive at equations for x_T

and y_τ in terms of the exogenous random variables over times $0 \le t < \tau$ in sum or product form

$$\begin{aligned}
\ln(y_\tau) &= \ln\left(y_0 \lim_{\delta t \downarrow 0} \prod_{n=0}^{\tau/\delta t} (1 + a_{n\delta t} c_{n\delta t} \delta t)\right) \\
&= \ln(y_0) + \lim_{\delta t \downarrow 0} \sum_{n=0}^{\tau/\delta t} a_{n\delta t} c_{n\delta t} \delta t \\
&= \ln(y_0) + \mu_{(a_t c_t, \tau)}, \\
x_\tau &= x_0 - \lim_{\delta t \downarrow 0} \sum_{n=0}^{\tau/\delta t} a_{n\delta t} \lambda_{n\delta t} y_{n\delta t} c_{n\delta t} \delta t \\
&= x_0 - \mu_{(a_{n\delta t} \lambda_{n\delta t} y_{n\delta t} c_{n\delta t}, \tau)},
\end{aligned}$$

where $\mu_{(x_t,\tau)}$ is the mean value of x_t over $0 \le t < \tau$. The result should still hold if c_t and λ_t are described by any finite variance distributions. The joint distribution of $a_t c_t$ and λ_t becomes very important.

Quadratic Variation

Let $Q_n(t) = \sum_{i=1}^{n} (W_{t_i} - W_{t_{i-1}})^2$ where $0 = t_0 < t_1 < \ldots < t_n = t$ is a partition of $[0, t]$. So,

$$\begin{aligned}
\mathbb{E}(Q_n(t)) &= \mathbb{E}(\sum_{i=1}^{n} \Delta W_{t_i}^2) \\
&= \sum_{i=1}^{n} \mathbb{E}(\Delta W_{t_i}^2) \\
&= \sum_{i=1}^{n} \mathrm{Var}(W_{t_i} - W_{t_{i-1}}) \\
&= \sum_{i=1}^{n} t_i - t_{i-1}
\end{aligned}$$

so $Q_n(t) = t$ in expectation. Now consider

$$\mathrm{Var}(Q_n(t)) = \mathbb{E}((Q_n(t) - t)^2).$$

Since $W_{t_i} - W_{t_{i-1}}$ are independent of one another, it follows that $(W_{t_1} - W_{t_0})^2, (W_{t_2} - W_{t_1})^2, \ldots (W_{t_n} - W_{t_{n-1}})^2$ are also independent. By a slight extension of the lemma,

$$\begin{aligned}
\mathrm{Var}(Q_n(t)) &= \mathrm{Var}(\sum_{i=1}^{n} \mathbb{E}(\Delta W_{t_i}^2)) \\
&= \sum_{i=1}^{n} \mathrm{Var}(\Delta W_{t_i}^2).
\end{aligned}$$

Now

$$\begin{aligned}
\mathrm{Var}(\Delta W_{t_i}^2) &= \mathbb{E}(\Delta W_{t_i}^4) - \mathbb{E}(\Delta W_{t_i}^2)^2 \\
&= \mathbb{E}(\Delta W_{t_i}^4) - \left[\mathrm{Var}(W_{t_i} - W_{t_{i-1}})^2\right] \\
&= \mathbb{E}(\Delta W_{t_i}^4) - (t_i - t_{i-1})^2 \\
&= 3(t_i - t_{i-1})^2 - (t_i - t_{i-1})^2 = 2(t_i - t_{i-1})^2.
\end{aligned}$$

Hence,

$$\begin{aligned}
\mathrm{Var}(Q_n(t)) &= 2\sum_{i=1}^{n} (t_i - t_{i-1})^2 \\
&\leq 2(\sum_{\pi} (t_i - t_{i-1}))\, \mathrm{mesh}\pi \\
&= 2t\, \mathrm{mesh}\pi \\
&\to 0 \text{ as } \mathrm{mesh}\pi \to 0.
\end{aligned}$$

In other words,

$$\mathbb{E}((Q_n(t) - t)^2) \to 0 \text{ as } n \to 0.$$

Appendix B

Pentanomial Lattice Construction Method

There follow the formulae used to construct the pentanomial tree used in section 4.3.3.

$$\begin{aligned}
\phi_1 + \phi_2 + \phi_3 + \phi_4 + \phi_5 &= 1, \\
\left(\frac{\tau}{\alpha}\right)^{0.5} \sum_{i=1}^{5} [\phi_i (6 - 2i)] &= 0, \qquad \text{(drift)} \\
\left(\frac{\tau}{\alpha}\right)^{1} \sum_{i=1}^{5} [\phi_i (6 - 2i)^2] &= \sigma^2 \tau, \qquad \text{(variance)} \\
\left(\frac{\tau}{\alpha}\right)^{1.5} \sum_{i=1}^{5} [\phi_i (6 - 2i)^3] &= (\sigma\sqrt{\tau})^3 s\tau, \qquad \text{(skewness)} \\
\left(\frac{\tau}{\alpha}\right)^{2} \sum_{i=1}^{5} [\phi_i (6 - 2i)^4] &= (\sigma\sqrt{\tau})^4 (3 + k\tau), \qquad \text{(Fisher kurtosis)}
\end{aligned}$$

and

$$u = \exp\left(\frac{\mu\tau}{4}\right) + \sqrt{\frac{\tau}{\alpha}}, \qquad d = \exp\left(\frac{\mu\tau}{4} - \sqrt{\frac{\tau}{\alpha}}\right).$$

describing the log-returns process (X and i being random variables)

$$X = \ln\left(\frac{p_{n+1}}{p_n}\right) = \mu\tau + (6 - 2i)\sqrt{\frac{\tau}{\alpha}}$$

so that $p_{n+1,s} = p_n u^s d^{(4-s)}$ with $0 \leq s \leq 4n$. The node number corresponding to a price K is (n, s_K) where

$$s_K = \left[\log\left(\frac{K}{p_0}\right) - 4n\log(d)\right] / \log\left(\frac{u}{d}\right).$$

The solution of the equations gives formulae for ϕ_i:

$$\phi_1 = (-1 + s\tau\sqrt{(\alpha\sigma^2)} + \alpha\sigma^2/4(3 + k\tau))\alpha\sigma^2/96,$$
$$\phi_2 = (16 - 2s\tau\sqrt{(\alpha\sigma^2)} - \alpha\sigma^2(3 + k\tau))\alpha\sigma^2/96,$$
$$\phi_3 = (64 + \alpha\sigma^2(-20 + \alpha\sigma^2(2 + k\tau)))/64,$$
$$\phi_4 = (16 + 2s\tau\sqrt{(\alpha\sigma^2)} - \alpha\sigma^2(3 + k\tau))\alpha\sigma^2/96,$$
$$\phi_5 = (-1 - s\tau\sqrt{(\alpha\sigma^2)} + \alpha\sigma^2(3 + k\tau))\alpha\sigma^2/96.$$

The choice of α affects the probabilities but not the moments of the implied distribution. α must be chosen in the range that gives $\phi_i \in [0, 1]$, since that has not yet been specified in the constraints. The expressions for the valid ranges are complicated and depend on the moments that are being simulated. Instead,

$$\alpha = 12/[\sigma^2(3 + k\tau]$$

is used which gives valid probabilities for moderate values of skewness and kurtosis (i.e. absolute skewness below about 8 and kurtosis greater than -10).

In order to use an explicit deterministic volatility model, we define $\sigma = f(\sigma, n, p)$ where σ is a constant. To keep the tree recombining, we set $\alpha = 12/[\sigma^2(3 + k\tau]$ so that u and d are constant across the whole lattice. Only the transition probabilities are variable in this method, so calculating probabilities associated with any given node is slightly more complicated. The formula for α becomes less successful at finding valid transition probabilities, in particular, it gives negative probabilities when trying to model skew less than about -2.

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