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Existence and non-existence of Schwarz symmetric ground states for elliptic eigenvalue problems

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Abstract. We determine a class of Carathéodory functions G for which the minimum formulated in the problem (1.1) below is achieved at a Schwarz symmetric function satisfying the constraint. Our hypotheses about G seem natural and, as our examples show, they are optimal from some points of view.

1. Introduction

In this paper we present a broad extension of the use of symmetrization inequalities in treating the following constrained minimization problem:

$$M(c) = \inf\{J(u) : u \in H^1(\mathbb{R}^N) \text{ and } \int_{\mathbb{R}^N} u(x)^2 dx = c^2\}, \tag{1.1}$$

where $c > 0$ is prescribed and

$$J(u) = \int_{\mathbb{R}^N} \frac{1}{2} |\nabla u(x)|^2 - G(|x|, u(x))dx. \tag{1.2}$$

Whereas the use of symmetrization as a means of showing that $M(c)$ is attained by a Schwarz symmetric function is a fairly standard technique, its application has hithertofore been restricted to a rather limited class of functions $G : (0, \infty) \times \mathbb{R} \to \mathbb{R}$. In fact, the method hinges on the inequality

$$J(u^*) \leq J(u) \text{ for all } u \in H^1(\mathbb{R}^N), \tag{1.3}$$

where u^* denotes the Schwarz symmetrization of $|u|$. Now (1.3) follows from

$$\int_{\mathbb{R}^N} G(|x|, u(x))dx \leq \int_{\mathbb{R}^N} G(|x|, u^*(x))dx \text{ for all } u \in H^1(\mathbb{R}^N) \tag{1.4}$$

since it is well known ([14], [15], [24]) that

$$\int_{\mathbb{R}^N} \left|\nabla u^*(x)\right|^2 dx \leq \int_{\mathbb{R}^N} |\nabla u(x)|^2 dx \text{ for all } u \in H^1(\mathbb{R}^N). \tag{1.5}$$

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However, until quite recently, a justification of the inequality (1.4) for a large class of functions G has been lacking and consequently the use of symmetrization in the context of problems like (1.1) has been restricted to cases where $G(r,s) = \widetilde{G}(s)$ with $\widetilde{G} : \mathbb{R} \to [0,\infty)$ continuous, [14], [15], [18] and [20], for example. Then we have that

$$\int_{\mathbb{R}^N} \widetilde{G}(u)dx \le \int_{\mathbb{R}^N} \widetilde{G}(|u|)dx = \int_{\mathbb{R}^N} \widetilde{G}(|u|^*)dx \text{ for all } u \in H^1(\mathbb{R}^N), \qquad (1.6)$$

provided that $\widetilde{G}(s) \le \widetilde{G}(|s|)$ for all $s \in \mathbb{R}$ and $\widetilde{G}(0) = 0$, which justifies (1.3) in this special case. In [10] (see also [11] for some extensions), we established (1.4) for functions G of Carathéodory type having appropriate monotonicity properties and, as we show below, these results now allow us to use symmetrization to treat (1.1) under rather general and natural assumptions about the function G. Here we discuss the minimization problem (1.1) under the following basic assumptions about the function G:

(g1) $G : (0,\infty) \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function in the sense that:
  (i) $G(\cdot, s) : (0,\infty) \to \mathbb{R}$ is measurable on $(0,\infty)$ for all $s \in \mathbb{R}$;
  (ii) $G(r,\cdot) : \mathbb{R} \to \mathbb{R}$ is continuous on $\mathbb{R}$ for all $r \in (0,\infty)\backslash\Gamma$, where Γ has one-dimensional measure zero.

(g2) $G(r,0) = 0$ and $G(r,s) \le G(r,|s|)$ for all $r \in (0,\infty)$ and $s \in \mathbb{R}$.

(g3) $s^{-2}G(r,s)$ is a non-decreasing function of s on $(0,\infty)$ for all $r > 0$.

(g4) For $R \ge r > 0$ and $s \ge t \ge 0$,

$$G(R,s) - G(R,t) - G(r,s) + G(r,t) \le 0.$$

(g5) There are constants $K > 0$ and $\sigma \in [0, \frac{4}{N})$ such that

$$0 \le G(r,s) \le K(s^2 + s^{\sigma+2}) \text{ for all } r, s > 0.$$

Remark 1.1. The positivity of G that is required in (g5) can be relaxed. If G satisfies the hypotheses (g1) to (g4) and

(g5*) There are constants $K > 0$ and $\sigma \in [0, \frac{4}{N})$ such that

$$|G(r,s)| \le K(s^2 + s^{\sigma+2}) \text{ for all } r, s > 0.$$

Then $\lim_{s\to 0+} s^{-2}G(r,s) \ge -K$ for all $r > 0$, and the function $\widetilde{G}$ defined by $\widetilde{G}(r,s) = G(r,s) + Ks^2$ satisfies the conditions (g1) to (g5). The problem (1.1) for $\widetilde{G}$ is clearly equivalent to the problem for G.

Remark 1.2. The assumption (g3) and the positivity of G imply that, for fixed $r > 0$, $G(r,s)$ is a non-decreasing function of s for $s \ge 0$, and the assumptions (g2) and (g4) imply that, for fixed $s \ge 0$, $G(r,s)$ is a non-increasing function of r for $r > 0$. If $G \in C^2((0,\infty)^2)$, the condition (g4) is equivalent to $\partial_1\partial_2 G(r,s) \le 0$ for $r, s > 0$.

Under some regularity assumptions about G, a minimizer for (1.1) satisfies the Euler–Lagrange equation

$$\Delta u(x) + \partial_2 G(|x|, u(x)) + \lambda u(x) = 0 \text{ on } \mathbb{R}^N, \tag{1.7}$$

where $\lambda \in \mathbb{R}$ is a Lagrange multiplier. Let us reformulate our hypotheses in the context of such an elliptic eigenvalue problem:

(f1) $f : (0, \infty) \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function.
(f2) $f(r, 0) \geq 0$ for all $r > 0$.
(f3) $f(r, s)$ is a non-decreasing function of s on $(0, \infty)$ for all $r > 0$.
(f4) $f(r, s)$ is a non-increasing function of r on $(0, \infty)$ for all $s \geq 0$.
(f5) There are constants $K > 0$ and $\sigma \in [0, \frac{4}{N})$ such that

$$f(r, s) \leq K(1 + s^{\sigma}) \text{ for all } r, s > 0.$$

Setting

$$G(r, s) = \int_0^s f(r, |t|)t dt \text{ for } (r, s) \in (0, \infty) \times \mathbb{R}, \tag{1.8}$$

one easily verifies that G satisfies the conditions (g1) to (g5) and that, if $u \geq 0$ is a minimizer for the problem (1.1), then

$$\Delta u(x) + f(|x|, u(x))u(x) + \lambda u(x) = 0 \text{ on } \mathbb{R}^N. \tag{1.9}$$

Our results concerning (1.1) give conditions ensuring that this equation has a Schwarz symmetric ground state. Let us observe that the radial symmetry of G does not by itself guarantee that (1.1) has a minimizer that is radially symmetric, let alone Schwarz symmetric. (See Examples 1 to 4 in Section 3, and also [13], [7], [1], [3] and [6].) Thus it is the appropriate combination of monotonicity and symmetry of G that ensures the existence of a Schwarz symmetric minimizer for (1.1). Note that in the case $N = 1$ our hypotheses about G are well suited to problems concerning symmetric planar waves guides composed of layers of self-focusing media where discontinuities in x may occur at the interfaces between adjacent layers, [21]. The inequality (1.4) has also been discussed in [5] and [22] [23], but under stronger regularity assumptions than ours, [10], [11].

Our main result establishes the existence of a Schwarz symmetric minimizer for (1.1). Recently [9], it has been shown that, under the hypotheses of Theorem 3.1 where G has the form (1.8) but with strict inequality in (f4), all the minimizers of (1.1) are Schwarz symmetric and hence are ground states of (1.9).

Finally let us mention that in [11], we have extended (1.4) to obtain

$$\int_{\mathbb{R}^N} H(|x|, u(x), v(x)dx \leq \int_{\mathbb{R}^N} H(|x|, u^*(x), v^*(x))dx \text{ for all } u, v \in H^1(\mathbb{R}^N)$$

under appropriate assumptions about the function H. This means that our method can be extended, using exactly the same arguments, to vectorial variational problems. Thus one can establish the existence of Schwarz symmetric ground states for the corresponding 2×2 elliptic systems (see [8]).

2. Preliminary results

Recalling that $|u| \in H^1(\mathbb{R}^N)$ whenever $u \in H^1(\mathbb{R}^N)$ with

$$\int_{\mathbb{R}^N} |\nabla u|^2 \, dx = \int_{\mathbb{R}^N} |\nabla \, |u||^2 \, dx,$$

it follows from (g2) that $J(|u|) \leq J(u)$ for all $u \in H^1(\mathbb{R}^N)$ and so henceforth, in dealing with (1.1), we need to consider only $u \in S(c)$, where

$$S(c) = \left\{ u \in H_+ : \int_{\mathbb{R}^N} u(x)^2 dx = c^2 \right\} \text{ and } H_+ = \{u \in H^1(\mathbb{R}^N) : u \geq 0 \text{ on } \mathbb{R}^N\}. \tag{2.1}$$

Let us show first that $M(c) > -\infty$ and that all minimizing sequences are bounded.

Lemma 2.1. *Under the assumptions* (g1) *and* (g5) *we have that there exists a constant $B > 0$ such that*

$$J(u) \geq \left(\frac{1}{2} - \frac{KBN\sigma}{4} \varepsilon^{\frac{4}{N\sigma}} \right) \int_{\mathbb{R}^N} |\nabla u|^2 \, dx - Kc^2 - \frac{KB}{q\varepsilon^q} c^\gamma,$$

for all $\varepsilon > 0$ and all $u \in S(c)$, where $q = \frac{4}{4-N\sigma} > 1$ and $\gamma = 2(2\sigma + 4 - N\sigma)/(4 - N\sigma) > 2$.

In particular, choosing $\varepsilon = (KBN\sigma)^{-\frac{N\sigma}{4}}$ we have that

$$J(u) \geq \frac{1}{4} \int_{\mathbb{R}^N} |\nabla u|^2 \, dx - Kc^2 - \frac{KB}{q\varepsilon^q} c^\gamma \textit{ for all } u \in S(c).$$

This shows that $M(c) \geq -Kc^2 - \frac{KB}{q\varepsilon^q} c^\gamma > -\infty$ for all $c > 0$ and that minimizing sequences for (1.1) *are bounded in* $H^1(\mathbb{R}^N)$.

Proof. By (g5), for any $u \in S(c)$,

$$\int_{\mathbb{R}^N} G(|x|\,, u(x) dx \leq K \left\{ c^2 + \int_{\mathbb{R}^N} u(x)^{\sigma+2} dx \right\},$$

and by the Gagliardo–Nirenberg inequality (see Proposition 1.8.12 in [14], for example),

$$\int_{\mathbb{R}^N} u(x)^{\sigma+2} dx \leq B\{\int_{\mathbb{R}^N} u(x)^2 dx\}^{(1-\alpha)(\sigma+2)/2} \{\int_{\mathbb{R}^N} |\nabla u|^2 \, dx\}^{\alpha(\sigma+2)/2}, \tag{2.2}$$

where $\alpha = \frac{N\sigma}{2(\sigma+2)}$. Then using Young's inequality we have

$$\begin{aligned} &\{\int_{\mathbb{R}^N} u(x)^2 dx\}^{(1-\alpha)(\sigma+2)/2} \{\int_{\mathbb{R}^N} |\nabla u|^2 \, dx\}^{\alpha(\sigma+2)/2} \\ &\leq \frac{1}{p} \varepsilon^p \{\int_{\mathbb{R}^N} |\nabla u|^2 \, dx\}^{p\alpha(\sigma+2)/2} + \frac{1}{q\varepsilon^q} \{\int_{\mathbb{R}^N} u(x)^2 dx\}^{q(1-\alpha)(\sigma+2)/2}, \end{aligned}$$

for any $\varepsilon > 0$ and $p > 1$ where $\frac{1}{p} + \frac{1}{q} = 1$. Choosing $p = \frac{2}{\alpha(\sigma+2)} = \frac{4}{N\sigma}$, we have that

$$\begin{aligned}\int_{\mathbb{R}^N} u(x)^{\sigma+2} dx &\leq \frac{B}{p}\varepsilon^p\{\int_{\mathbb{R}^N} |\nabla u|^2\, dx\} + \frac{B}{q\varepsilon^q}\{\int_{\mathbb{R}^N} u(x)^2 dx\}^{q(1-\alpha)(\sigma+2)/2} \\ &= \frac{B}{p}\varepsilon^p\{\int_{\mathbb{R}^N} |\nabla u|^2\, dx\} + \frac{B}{q\varepsilon^q} c^{q(1-\alpha)(\sigma+2)},\end{aligned}$$

with $q = 4/(4 - N\sigma) > 1$ and $q(1-\alpha)(\sigma+2) = 2(2\sigma + 4 - N\sigma)/(4 - N\sigma) > 2$. Thus

$$\begin{aligned}J(u) &\geq \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2\, dx - Kc^2 - \frac{KB}{p}\varepsilon^p \int_{\mathbb{R}^N} |\nabla u|^2\, dx - \frac{KB}{q\varepsilon^q} c^{q(1-\alpha)(\sigma+2)} \\ &= \left(\frac{1}{2} - \frac{KB}{p}\varepsilon^p\right)\int_{\mathbb{R}^N} |\nabla u|^2\, dx - Kc^2 - \frac{KB}{q\varepsilon^q} c^{q(1-\alpha)(\sigma+2)}.\end{aligned}$$

Remark 2.1. If in (g5) we allow $\sigma = \frac{4}{N}$, the problem (1.1) still makes sense for sufficiently small values of $c > 0$ since in (2.2) we then have $\alpha = 2/(\sigma+2)$ and

$$\int_{\mathbb{R}^N} u(x)^{\sigma+2} dx \leq Bc^{\frac{4}{N}} \int_{\mathbb{R}^N} |\nabla u|^2\, dx \text{ for } u \in S(c).$$

Hence

$$\begin{aligned}J(u) &\geq \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2\, dx - Kc^2 - KBc^{\frac{4}{N}} \int_{\mathbb{R}^N} |\nabla u|^2\, dx \\ &= \left(\frac{1}{2} - KBc^{\frac{4}{N}}\right)\int_{\mathbb{R}^N} |\nabla u|^2\, dx - Kc^2.\end{aligned}$$

Thus $M(c) > -\infty$ and minimizing sequences are bounded in $H^1(\mathbb{R}^N)$ provided that $0 < c < \left(\frac{1}{2KB}\right)^{\frac{N}{4}}$.

Remark 2.2. If $\liminf_{s\to\infty} s^{-l} G(r,s) \geq A > 0$ for some $l > 2 + \frac{4}{N}$, then $M(c) = -\infty$ for all $c > 0$. However see [12] for an interesting treatment of the associated eigenvalue problem (1.9) in such cases.

For any $u \in H_+$, its Schwarz symmetrization u^* is well defined and also belongs to H_+.

Lemma 2.2. *Suppose that G has the properties* (g1) *to* (g4) *and that $G(r,s) \geq 0$ for all $r > 0$ and $s \geq 0$:*

(a) $J(u^*) \leq J(u)$ *for all* $u \in H_+$.
(b) *For any* $c > 0$*, there is a sequence* $\{u_n\} \subset S(c)$ *such that* $u_n = u_n^*$ *and* $\lim_{n\to\infty} J(u_n) = M(c)$.

Furthermore, $M(c)$ is attained if and only if there exists $u = u^* \in S(c)$ such that $J(u) = M(c)$.

Remark 2.3. This lemma is crucial for the use of symmetrization in connection with problem (1.1) and the essential ingredient is the inequality (1.4) which we have justified in [10] under hypotheses which we believe are natural, and, as our examples in Section 3 show, more or less optimal, in the context of (1.1) (see also [11] for some extensions). For example, the proof of (1.4) in [5] [22], [23] requires the assumption that, for some $p \geq 1$,

$$|G(r,s)| \leq A(r) + s^p \text{ for } r > 0 \text{ and } s \geq 0, \tag{2.3}$$

where $\int_0^\infty r^{N-1} A(r) dr < \infty$, but we require no such restriction. Notice that we use (g5) to ensure that $M(c) > -\infty$, and this allows $G(r,s)$ to have different rates of growth for s near 0 and ∞, whereas (2.3) does not.

Proof. (a) Let $u \in H_+$. Considering G restricted to $(0,\infty) \times [0,\infty)$, we see that the hypotheses of Proposition 5.1 of [10] are satisfied and hence the inequality (1.4) is valid since $H_+ \subset F_N$, in the notation of [10]. Using (1.5), it follows that $J(u^*) \leq J(u)$. On the other hand, (1.6) implies that $u^* \in S(c)$ and so $J(u^*) \geq M(c)$. Thus if $u \in S(c)$ and $J(u) = M(c)$, it follows that $u^* \in S(c)$ and $J(u^*) = M(c)$.

(b) Clearly there is a sequence $\{u_n\} \subset S(c)$ such that $\lim_{n\to\infty} J(u_n) = M(c)$ and it follows from part (a) and (1.6) that $\{u_n^*\} \subset S(c)$ is also a minimizing sequence for (1.1).

Lemma 2.3. *Suppose that G has the properties* (g3)*,* (g4) *and* (g5) *and that $G(r,0) = 0$, for all $r > 0$:*

(a) For $r, s > 0$, let $k(r,s) = s^{-2} G(r,s)$. Then $k(r,s)$ is non-increasing in r and non-decreasing in s.
(b) Let $P(r) = \lim_{s\to 0+} k(r,s)$. Then P is non-increasing and we set $P(\infty) = \lim_{r\to\infty} P(r)$. Furthermore, $0 \leq P(r) \leq K$ for all $r > 0$.
(c) $P(\infty) = \inf\{k(r,s) : r > 0$ and $s > 0\} \geq 0$.
(d) Given any $\varepsilon > 0$, there exist $r_0 > 0$ and $s_0 > 0$ such that

$$0 \leq k(r,s) - P(r) \leq \varepsilon \text{ for all } r \geq r_0 \text{ and } s \in (0, s_0].$$

Proof. (a) Putting $t = 0$ in (g4) we find that for $R \geq r > 0$ and $s \geq 0$, $G(R,s) - G(r,s) \leq 0$ and hence $k(R,s) \leq k(r,s)$ for $s > 0$. Thus $k(r,s)$ is non-increasing in r. By (g3) it is also non-decreasing in s.

(b) By (g5), $k(r,s) \geq 0$ for all $r, s > 0$ and hence $P(r) \geq 0$. Since $k(R,s) \leq k(r,s)$ for $s > 0$ for $R \geq r > 0$. It follows that $P(R) \leq P(r)$. Using (g5) we see that $P(r) \leq K$ for all $r > 0$.

(c) For all $r, s > 0$, $k(r,s) \geq P(r) \geq P(\infty) \geq 0$. Given any $\varepsilon > 0$, there exists $r_0 > 0$ such that $P(r_0) \leq P(\infty) + \frac{\varepsilon}{2}$ and there exists $s_0 > 0$ such that $k(r_0, s_0) \leq P(r_0) + \frac{\varepsilon}{2}$. Hence $k(r_0,s_0) \leq P(\infty) + \varepsilon$, showing that $\inf\{k(r,s) : r > 0$ and $s > 0\} \leq P(\infty)$.

(d) Given $\varepsilon > 0$, we have shown in (c) that there exist $r_0, s_0 > 0$ such that $k(r_0,s_0) \leq P(\infty) + \varepsilon$. But, for $r \geq r_0$ and $s \in (0, s_0]$, it follows from (a) and (b) that

$$0 \leq k(r,s) - P(r) \leq k(r_0,s) - P(r) \leq k(r_0,s_0) - P(r) \leq k(r_0,s_0) - P(\infty).$$

Lemma 2.4. *Under the assumptions of Lemma 2.3, we have that*

$$-\infty < M(c) \le -P(\infty)c^2 \text{ for all } c > 0.$$

Furthermore, $c^{-2}M(c)$ is a non-increasing function of c and

$$\begin{aligned}\lim_{c\to\infty} c^{-2}M(c) = Q(\infty) \text{ where } Q(\infty)&\\ = \inf\left\{\int_{\mathbb{R}^N} \frac{1}{2}\,|\nabla u|^2 - L(|x|)u(x)^2dx : u \in S(1)\right\}&\end{aligned}$$

and $L(r) = \lim_{s\to\infty} k(r, s)$. Here $L(|x|)u(x)^2$ is interpreted as zero when $u(x) = 0$, even if $L(|x|) = \infty$.

Remark 2.4. It is easy to see that

$$\begin{aligned}0 &= \inf\left\{\frac{\int_{\mathbb{R}^N} |\nabla u(x)|^2\,dx}{\int_{\mathbb{R}^N} u(x)^2dx} : u \in H^1(\mathbb{R}^N)\right\}\\ &= \inf\left\{\int_{\mathbb{R}^N} |\nabla u(x)|^2\,dx : u \in S(c)\right\} \text{ for all } c > 0, \qquad (2.4)\end{aligned}$$

and this infimum is not attained since the Laplacian has no eigenfunction in $H^1(\mathbb{R}^N)$. This observation will be used several times in the following. Note that (2.4) corresponds to problem (1.1) with $G \equiv 0$ which clearly satisfies all the hypotheses (g1) to (g5) and $M(c) = 0$ is not attained in this case.

Remark 2.5. Clearly $L(r) \ge P(\infty)$ and it follows from (2.4) that $Q(\infty) \le -P(\infty)$.

Proof. For any $u \in H_+$, it follows from Lemma 2.3(c) that

$$\int_{\mathbb{R}^N} G(|x|\,, u(x))dx = \int_{\{x:u(x)>0\}} G(|x|\,, u(x))dx =$$

$$\int_{\{x:u(x)>0\}} k(|x|\,, u(x))u(x)^2dx \ge P(\infty)\int_{\{x:u(x)>0\}} u(x)^2dx = P(\infty)\int_{\mathbb{R}^N} u(x)^2dx$$

and hence that $J(u) \le \int_{\mathbb{R}^N} \frac{1}{2}\,|\nabla u(x)|^2\,dx - P(\infty)c^2$ for all $u \in S(c)$. By (2.4), this implies that $M(c) \le -P(\infty)c^2$.

The monotonicity of $c^{-2}M(c)$ is easily deduced from the fact that $k(r, s)$ is a non-decreasing function of s.

Let $u \in S(1)$ and set

$$w_n(x) = \begin{cases} k(|x|\,, nu(x))u(x)^2 & \text{if } u(x) > 0\\ 0 & \text{if } u(x) = 0\end{cases}.$$

Then $\{w_n\}$ is a non-decreasing sequence of non-negative measurable functions and so by the monotone convergence theorem,

$$\begin{aligned}\lim_{n\to\infty}\int_{\mathbb{R}^N} \frac{G(|x|\,, nu(x))}{n^2}dx = \lim_{n\to\infty}\int_{\mathbb{R}^N} w_n(x)dx &= \int_{\mathbb{R}^N} \lim_{n\to\infty} w_n(x)dx\\ &= \int_{\mathbb{R}^N} L(|x|)u(x)^2dx,\end{aligned}$$

where $L(|x|)u(x)^2 = 0$ if $u(x) = 0$ even when $L(|x|) = \infty$. Hence,

$$\lim_{n\to\infty} \frac{J(nu)}{n^2} = \int_{\mathbb{R}^N} \frac{1}{2} |\nabla u|^2 - L(|x|)u(x)^2 dx$$

and so $\lim_{c\to\infty} c^{-2}M(c) \leq \int_{\mathbb{R}^N} \frac{1}{2} |\nabla u|^2 - L(|x|)u(x)^2 dx$ for all $u \in S(1)$.

But for any $u \in S(c)$, $v = c^{-1}u \in S(1)$,

$$\begin{aligned} J(u) &= \int_{\mathbb{R}^N} \frac{1}{2} |\nabla u|^2 - G(|x|, u(x))dx = c^2 \int_{\mathbb{R}^N} \frac{1}{2} |\nabla v|^2 - c^{-2}G(|x|, cv(x))dx \\ &= c^2 \int_{\mathbb{R}^N} \frac{1}{2} |\nabla v|^2 - k(|x|, cv(x))v(x)^2 dx \geq c^2 \int_{\mathbb{R}^N} \frac{1}{2} |\nabla v|^2 - L(|x|)v(x)^2 dx \\ &\geq c^2 Q(\infty) \text{ since } k(r, s) \leq L(r) \text{ for all } r, s > 0, \end{aligned}$$

showing that $M(c) \geq c^2 Q(\infty)$ for all $c > 0$.

Lemma 2.5. *Suppose that G satisfies the conditions* (g1) *to* (g5). *Let $\{u_n\} \subset H_+$ be a sequence such that $u_n = u_n^*$ for all $n \in \mathbb{N}$ and $u_n \rightharpoonup z$ weakly in $H^1(\mathbb{R}^N)$. Then*

$$\widetilde{J}(z) \leq \lim\inf_{n\to\infty} \widetilde{J}(u_n),$$

where

$$\widetilde{J}(u) = J(u) + P(\infty) \int_{\mathbb{R}^N} u^2 dx.$$

Proof. Setting

$$p(r) = P(r) - P(\infty) \text{ and } F(r, s) = G(r, s) - P(r)s^2 \text{ for } r > 0 \text{ and } s \geq 0,$$

we have that

$$G(r, s) = p(r)s^2 + F(r, s) + P(\infty)s^2$$

and for any $u \in H_+$,

$$\widetilde{J}(u) = \int_{\mathbb{R}^N} \frac{1}{2} |\nabla u|^2 - p(|x|)u(x)^2 - F(|x|, u(x))dx.$$

Certainly,

$$\int_{\mathbb{R}^N} \frac{1}{2} |\nabla z|^2 \, dx \leq \lim\inf_{n\to\infty} \int_{\mathbb{R}^N} \frac{1}{2} |\nabla u_n|^2 \, dx. \tag{2.5}$$

For any $R > 0$, the sequence $\{u_n\}$ converges strongly to z in $L^p(B(R))$, where $B(R) = \{x \in \mathbb{R}^N : |x| \leq R\}$ for any $p \in [1, 2^*)$ where $2^* = \infty$ for $N = 1$ or 2 and $2^* = 2N/(N-2)$ for $N \geq 3$. Now

$$\begin{aligned} &\left| \int_{\mathbb{R}^N} pu_n^2 dx - \int_{\mathbb{R}^N} pz^2 dx \right| \\ &\leq K \int_{B(R)} \left| u_n^2 - z^2 \right| dx + \sup_{r\geq R} p(r) \int_{|x|\geq R} u_n^2 + z^2 dx \end{aligned}$$

and hence

$$\limsup_{n\to\infty} \left| \int_{\mathbb{R}^N} pu_n^2 dx - \int_{\mathbb{R}^N} pz^2 dx \right| \le 2C \sup_{r\ge R} p(r) \text{ for all } R > 0,$$

where $C = \sup_{n\in\mathbb{N}} \int u_n^2 dx < \infty$ since $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$. But $p(r) \to 0$ as $r \to \infty$, so we must have

$$\limsup_{n\to\infty} \left| \int_{\mathbb{R}^N} pu_n^2 dx - \int_{\mathbb{R}^N} pz^2 dx \right| = 0. \tag{2.6}$$

Similarly we observe that

$$\left| \int_{\mathbb{R}^N} F(|x|, u_n(x)) - F(|x|, z(x)) dx \right| \tag{2.7}$$

$$\le \int_{B(R)} |F(|x|, u_n(x)) - F(|x|, z(x))| \, dx$$

$$+ \int_{|x|\ge R} |F(|x|, u_n(x))| + |F(|x|, z(x))| \, dx. \tag{2.8}$$

Now by the basic result on the continuity of Nemytski operators, (see Theorem 2.2 of [2], for example) the condition (g5) implies that $u \mapsto F(|\cdot|, |u(\cdot)|)$ maps $L^{\sigma+2}(B(R))$ continuously into $L^1(B(R))$ and hence

$$\int_{B(R)} |F(|x|, u_n(x)) - F(|x|, z(x))| \, dx \to 0 \text{ for all } R > 0. \tag{2.9}$$

Setting $h(r, s) = k(r, s) - P(r)$ for $r, s > 0$ and $h(r, 0) = 0$ for $r > 0$, we see that

$$\int_{|x|\ge R} |F(|x|, u_n(x))| \, dx = \int_{|x|\ge R} h(|x|, u_n(x)) u_n(x)^2 dx$$

$$\le C \sup_{|x|\ge R} h(|x|, u_n(x)), \tag{2.10}$$

where $C = \sup_{n\in\mathbb{N}} \int u_n^2 dx < \infty$ as before. Given any $\varepsilon > 0$, it follows from Lemma 2.3(d) that there exist $r_0 > 0$ and $s_0 > 0$ such that

$$0 \le h(r, s) \le \varepsilon \text{ for all } r \ge r_0 > 0 \text{ and } s \in [0, s_0].$$

Since u_n is Schwarz symmetric, for any $R > 0$ and any $n \in \mathbb{N}$,

$$C \ge \int_{B(R)} u_n^2 dx \ge u_n^2(R\xi) \, |B(R)| \, ,$$

where $|\xi| = 1$ and $|B(R)| = \int_{B(R)} dx$. Clearly there exists $R_0 > r_0$ such that $C \, |B(R_0)|^{-1} < s_0^2$ and consequently $0 \le u_n(x) \le u_n(R_0\xi) \le s_0$ for all $|x| \ge R_0$. Returning to (2.10) we now have that

$$\int_{|x|\ge R_0} |F(|x|, u_n(x))| \, dx \le C \sup_{|x|\ge R_0} h(|x|, u_n(x)) \le C\varepsilon$$

and a similar argument shows that

$$\int_{|x|\geq R_0} |F(|x|\,, z(x))|\, dx \leq C\varepsilon.$$

Thus, by (2.8) and (2.9),

$$\limsup_{n\to\infty} \left| \int_{\mathbb{R}^N} F(|x|\,, u_n(x)) - F(|x|\,, z(x)) dx \right| \leq 2C\varepsilon,$$

for arbitrary $\varepsilon > 0$, showing that

$$\int_{\mathbb{R}^N} F(|x|\,, u_n(x)) dx \to \int_{\mathbb{R}^N} F(|x|\,, z(x)) dx. \tag{2.11}$$

Combining (2.5), (2.6) and (2.11) we see that the proof is complete.

Remark 2.6. Under the hypotheses of Lemma 2.5, the functional $\widetilde{J}$ may not be weakly sequentially lower semi-continuous on $H^1(\mathbb{R}^N)$.

3. The main result and some examples

Theorem 3.1. *Let G be a function satisfying the hypotheses* (g1) *to* (g5). *If $M(c) < -P(\infty)c^2$, there exists $u = u^* \in S(c)$ such that $J(u) = M(c)$.*

Remark 3.1. If $Q(\infty) < -P(\infty)$, it follows from Lemma 2.4 that there exists $c_0 \geq 0$ such that $M(c) < -P(\infty)c^2$ for all $c > c_0$. Note that if

$$\lim_{s\to\infty} \frac{G(R,s)}{s^2} = \infty \text{ for some } R > 0, \tag{3.1}$$

then $L(r) = \infty$ for all $r \in (0, R]$ and consequently $Q(\infty) = -\infty < -P(\infty)$ in this case.

Remark 3.2. Recalling Remark 2.1, we see (by inspecting the proofs of Lemma 2.5 and Theorem 3.1) that we can allow $\sigma = \frac{4}{N}$ in (g5) provided that we restrict c so that $0 < c < \left(\frac{1}{2KB}\right)^{\frac{N}{4}}$.

Proof. Fix $c > 0$ such that $M(c) < -P(\infty)c^2$. By Lemma 2.2, there is a sequence $\{u_n\} \subset S(c)$ such that $u_n = u_n^*$ and $\lim_{n\to\infty} J(u_n) = M(c)$. Furthermore, by Lemma 2.1 the sequence $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$ and hence, by passing to a subsequence, we may assume that there is an element $z \in H^1(\mathbb{R}^N)$ such that $u_n \rightharpoonup z$ weakly in $H^1(\mathbb{R}^N)$. By Lemma 2.5, $\widetilde{J}(z) \leq \liminf_{n\to\infty} \widetilde{J}(u_n) = M(c) + P(\infty)c^2 < 0 = \widetilde{J}(0)$. Thus $z \neq 0$ and since

$$z \geq 0 \text{ and } \int_{\mathbb{R}^N} z^2 dx \leq \liminf_{n\to\infty} \int_{\mathbb{R}^N} u_n^2 dx = c^2,$$

we have that $tz \in S(c)$ where $t = c\{\int_{\mathbb{R}^N} z^2 dx\}^{-1/2} \geq 1$. But then, since $tz(x) \geq z(x)$, we have that

$$\begin{aligned} M(c) + P(\infty)c^2 &\leq J(tz) + P(\infty)c^2 = \widetilde{J}(tz) \\ &= t^2 \int_{\mathbb{R}^N} \frac{1}{2} |\nabla z|^2 - t^{-2} G(|x|, tz(x)) + P(\infty) z(x)^2 dx \\ &= t^2 \int_{\mathbb{R}^N} \frac{1}{2} |\nabla z|^2 - k(|x|, tz(x)) z(x)^2 + P(\infty) z(x)^2 dx \\ &\leq t^2 \int_{\mathbb{R}^N} \frac{1}{2} |\nabla z|^2 - k(|x|, z(x)) z(x)^2 + P(\infty) z(x)^2 dx \\ &= t^2 \widetilde{J}(z) \leq t^2 \{M(c) + P(\infty)c^2\}. \end{aligned}$$

Hence, $(t^2 - 1)\{M(c) + P(\infty)c^2\} \geq 0$ and consequently, $t^2 \leq 1$ since $M(c) + P(\infty)c^2 < 0$. Recalling that $t \geq 1$, we have shown that $t = 1$. This means that $z \in S(c)$ and that $M(c) + P(\infty)c^2 \leq J(z) + P(\infty)c^2 \leq M(c) + P(\infty)c^2$. Hence $J(z) = M(c)$ as required. By Lemma 2.2, this completes the proof.

To complete our discussion we give some explicit conditions which imply that $M(c) < -P(\infty)c^2$. We also show that, under the hypotheses (g1) to (g5), we can have $M(c) = -P(\infty)c^2$ and that in such cases $M(c)$ may not be attained. First of all we show that without some monotonicity conditions like (g4)/(f4) the radial symmetry of G does not imply that (1.1) has a Schwarz symmetric or even a radially symmetric minimizer. In all of the examples we deal with a function $f : (0, \infty) \times [0, \infty) \to [0, \infty)$ of the form

$$f(r, s) = p(r) + q(r)s^{\sigma}, \tag{3.2}$$

where p and q are non-negative, bounded measurable functions on $(0, \infty)$ and $0 < \sigma < 4/N$. Defining $G : (0, \infty) \times \mathbb{R} \to \mathbb{R}$ by (1.8), it follows easily that G has the properties (g1), (g2), (g3) and (g5) and furthermore, $P(r) = \frac{1}{2} p(r)$. We also have that

$$0 \leq G(r, s) = \frac{1}{2} p(r) s^2 + \frac{1}{\sigma + 2} q(r) |s|^{\sigma+2} \leq \frac{1}{2} f(r, |s|) s^2, \tag{3.3}$$

for all $(r, s) \in (0, \infty) \times \mathbb{R}$. Thus by Lemma 2.1, the problem (1.1) is well-posed and $-\infty < M(c) \leq 0$ for all $c > 0$ by (2.4) and the fact that $G(r, s) \geq 0$ for $r, s > 0$. Furthermore, if $M(c)$ is attained then there exists $u \in S(c)$ satisfying the equation (1.9). Thus

$$\begin{aligned} \lambda \int_{\mathbb{R}^N} u^2 dx &= \int_{\mathbb{R}^N} |\nabla u|^2 - f(|x|, u(x)) u(x)^2 dx \\ &\leq \int_{\mathbb{R}^N} |\nabla u|^2 - 2G(|x|, u(x)) dx \qquad (3.4) \\ &= 2J(u) = 2M(c) \leq 0. \qquad (3.5) \end{aligned}$$

Hence $\lambda \leq 0$. Using the elliptic regularity theory and the maximum principle, we can also assume that

$$u \in C^1(\mathbb{R}^N), \quad \lim_{|x| \to \infty} u(x) = 0 \text{ and that } u(x) > 0 \text{ for all } x \in \mathbb{R}^N. \tag{3.6}$$

However the assumption (3.2) does not ensure that G has the property (g4). Indeed, for $R \geq r > 0$ and $s \geq t \geq 0$,

$$\begin{aligned} &G(R,s) - G(R,t) - G(r,s) + G(r,t) \\ &= \{p(R) - p(r)\}(s^2 - t^2)/2 + \{q(R) - q(r)\}(s^{\sigma+2} - t^{\sigma+2})/(\sigma+2), \end{aligned}$$

from which it follows that (g4) is satisfied if and only if both

$$p \text{ and } q \text{ are non-increasing on } (0,\infty). \tag{3.7}$$

We begin with two examples of radially symmetric functionals where (g4) is not satisfied.

Example 1 (in which $M(c)$ is not attained for any $c > 0$). Consider (3.2) with p and q both non-decreasing and set

$$p(\infty) = \lim_{r\to\infty} p(r) \text{ and } q(\infty) = \lim_{r\to\infty} q(r).$$

The function G does not have the property (g4) and $M(c)$ cannot be attained unless both p and q are constant. To see this we suppose that $u \in S(c)$ with $J(u) = M(c)$ and recall that u has the properties (3.6). The dominated convergence theorem shows that

$$\lim_{\tau\to\infty} J(T_\tau u) = \int_{\mathbb{R}^N} \frac{1}{2}|\nabla u|^2 - \frac{1}{2}p(\infty)u^2 - \frac{1}{\sigma+2}q(\infty)u^{\sigma+2}dx,$$

where $T_\tau u(x) = u(x + \tau\xi)$ and ξ is some fixed unit vector in $\mathbb{R}^N$. Thus

$$\begin{aligned} J(u) - \lim_{\tau\to\infty} J(T_\tau u) = \int_{\mathbb{R}^N} &\frac{1}{2}[p(\infty) - p(r)]u^2 \\ &+ \frac{1}{\sigma+2}[q(\infty) - q(r)]u^{\sigma+2}dx > 0, \end{aligned}$$

unless p and q are both constant. However, $T_\tau u \in S(c)$ for all τ and so in this case $M(c)$ cannot be attained unless p and q are both constant.

Example 2 (in which $M(c)$ is attained but no minimizer is radially symmetric). This example is based on a similar situation studied originally in [13]. Consider (3.2) with $N \geq 2$, $p \equiv 0$ and $q(r) = \chi_{[\tau,\tau+2]}(r) = \begin{cases} 1 & \text{for } \tau \leq r \leq \tau+2 \\ 0 & \text{otherwise} \end{cases}$. We claim there exists $c_0 \geq 0$ such that $M(c)$ is attained for all $c > c_0$ and all $\tau \geq 0$. However, for large values of τ, no minimizer can be radially symmetric. We begin by considering the problem (3.2) with $p \equiv 0$ and $\widetilde{q} = \chi_{[0,1]}$. For this problem all the hypotheses (g1) to (g5) are satisfied and it follows from Theorem 3.1 and the subsequent remark that there exists $c_0 \geq 0$ such that for all $c > c_0$ there exists $z_c = z_c^* \in S(c)$ such that $\widetilde{J}(z_c) = \widetilde{M}(c) < 0$, where

$$\begin{aligned} &\widetilde{J}(u) = \int_{\mathbb{R}^N} \frac{1}{2}|\nabla u|^2 - \widetilde{q}(|x|)\,|u|^{\sigma+2}\,dx \text{ and} \\ &\widetilde{M}(c) = \inf\{\widetilde{J}(u) : u \in S(c)\}. \end{aligned}$$

Now setting

$$J_\tau(u) = \int_{\mathbb{R}^N} \frac{1}{2} |\nabla u|^2 - \chi_{[\tau,\tau+2]}(|x|)\, |u|^{\sigma+2}\, dx,$$

it is easy to see that $J_\tau(T_{\tau+1} z_c) \leq \widetilde{J}(z_c)$ for all $\tau \geq 0$, where $T_\tau u$ is a translation as defined in Example 1, since

$$\int_{\mathbb{R}^N} \chi_{[\tau,\tau+2]}(|x|)\, |T_{\tau+1} u|^{\sigma+2}\, dx = \int_{\mathbb{R}^N} \chi_{[\tau,\tau+2]}(|y-(\tau+1)\xi|)\, |u|^{\sigma+2}\, dy,$$

for all $u \in S(c)$ and $\chi_{[\tau,\tau+2]}(|y-(\tau+1)\xi|) \geq \widetilde{q}(|y|)$ for all $y \in \mathbb{R}^N$. Thus we see that $M_\tau(c) \leq \widetilde{M}(c) < 0$ for all $c > c_0$ and $\tau \geq 0$. Furthermore, the functional

$$\int_{\mathbb{R}^N} \chi_{[\tau,\tau+2]}(|x|)\, |u|^{\sigma+2}\, dx \text{ is weakly sequentially continuous on } H^1(\mathbb{R}^N)$$

since $\chi_{[\tau,\tau+2]}$ has compact support. Using Lemma 2.1 and the fact that $M_\tau(c) < 0$, it follows (by a simple variant of the proof of Theorem 3.1) that for all $c > c_0$ and $\tau \geq 0$, there exists an element $w_c^\tau \in S(c)$ such that $J_\tau(w_c^\tau) = M_\tau(c)$. We now fix c and claim that there exists $\tau(c) > 0$ such that w_c^τ is not radially symmetric for $\tau > \tau(c)$. Indeed, if this were not so, there would be a sequence $\tau_n \to \infty$ such that the functions $v_n = w_c^{\tau_n}$ are radially symmetric for all $n \in \mathbb{N}$. However, using Lemma 2.1, we see that the sequence $\{v_n\}$ is bounded in $H^1(\mathbb{R}^N)$ and so by the radial symmetry there exists a constant $D > 0$ such that

$$|v_n(x)| \leq D\, |x|^{(1-N)/2} \text{ for almost all } x \text{ and all } n \in \mathbb{N}$$

(see [17] or [14]). But then we find that for $\tau_n \geq 1$,

$$\begin{aligned} J_{\tau_n}(v_n) &\geq -\int_{\mathbb{R}^N} \chi_{[\tau_n,\tau_n+2]}(|x|)\, |v_n|^{\sigma+2}\, dx \geq -\omega_N \int_{\tau_n}^{\tau_n+2} r^{N-1} \{D r^{(1-N)/2}\}^{\sigma+2} dr \\ &= -\omega_N D^{\sigma+2} \int_{\tau_n}^{\tau_n+2} r^{-(N-1)\sigma/2} dr. \end{aligned}$$

Since $\int_{\tau_n}^{\tau_n+2} r^{-(N-1)\sigma/2} dr \to 0$ as $\tau_n \to \infty$, this contradicts the fact that $J_{\tau_n}(v_n) = M_{\tau_n}(c) \leq \widetilde{M}(c) < 0$ for all $n \in \mathbb{N}$.

Remark 3.3. This example shows that one should not expect to obtain a radially symmetric minimizer for problem (1.1) if the hypothesis (g4) is weakened.

In the remaining examples all of the hypotheses (g1) to (g5) are satisfied. This still does not ensure that a minimizer for (1.1) exists.

Example 3 (in which $M(c)$ is not attained for any $c > 0$.). Consider (3.2) with p non-increasing and $q \equiv 0$. Suppose in addition that $N \geq 3$. Then there exists a constant $K_N > 0$ such that, if

$$\int_0^\infty r^{N-1} p(r)^{\frac{N}{2}} dr < K_N, \tag{3.8}$$

then there is no value of $c > 0$ for which $M(c)$ is attained. Furthermore, we also have that $M(c) = 0 = -P(\infty)$ since (3.8) implies that $P(\infty) = \frac{1}{2}\lim_{r\to\infty} p(r) = 0$. In this example

$$J(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 - p(|x|)\,|u|^2\,dx$$

so that $M(c) = c^2 M(1) \leq 0$. Thus if $v \in S(c)$ and $J(v) = M(c)$, it follows from (3.5) that

$$\int_{\mathbb{R}^N} |\nabla v|^2\,dx \leq \int_{\mathbb{R}^N} p(|x|)\,|v|^2\,dx \leq \{\int_{\mathbb{R}^N} p(|x|)^{\frac{N}{2}}\,dx\}^{\frac{2}{N}}\{\int_{\mathbb{R}^N} v^{\frac{2N}{N-2}}\,dx\}^{\frac{N-2}{N}}.$$

But since $N \geq 3$, there is a constant $C_N > 0$ such that

$$\{\int_{\mathbb{R}^N} |u|^{\frac{2N}{N-2}}\,dx\}^{\frac{N-2}{2N}} \leq C_N\{\int_{\mathbb{R}^N} |\nabla u|^2\,dx\}^{\frac{1}{2}} \text{ for all } u \in H^1(\mathbb{R}^N)$$

and hence

$$1 \leq C_N^2\{\int_{\mathbb{R}^N} p(|x|)^{\frac{N}{2}}\,dx\}^{\frac{2}{N}} = C_N^2\{\omega_N \int_0^\infty r^{N-1} p(r)^{\frac{N}{2}}\,dr\}^{\frac{2}{N}}.$$

Putting $K_N = (C_N^N \omega_N)^{-1}$, our claim is justified.

Remark 3.4. As is shown in Example 5, there is no result of this kind for $N = 1$ and 2.

Example 4 (in which there exists $c_0 > 0$ such that $M(c)$ is attained by a Schwarz symmetric function for all $c > c_0$ and there exists $c_1 > 0$ such that $M(c)$ is not attained for $0 < c < c_1$). Consider (3.2) with $p \equiv 0$ and $q \not\equiv 0$ a non-increasing function such that

$$\int_0^\infty r^{N-1} q(r)^t\,dr < \infty \text{ for } t = \frac{2N}{4 - N\sigma}.$$

For $N = 1$, we also suppose that $2 < \sigma < 4$, whereas for $N \geq 2$ we simply require $0 < \sigma < \frac{4}{N}$ as usual, so as to ensure that $t > 1$. We begin by showing that there exists $c_1 > 0$ such that $M(c)$ is not attained for $0 < c < c_1$. If $v \in S(c)$ and $J(v) = M(c)$, it follows from (3.5) that

$$\begin{aligned}\frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2\,dx &\leq \frac{1}{\sigma+2}\int_{\mathbb{R}^N} q(|x|)\,|v|^{\sigma+2}\,dx \\ &\leq \frac{1}{\sigma+2}\{\int_{\mathbb{R}^N} q(|x|)^t\,dx\}^{1/t}\{\int_{\mathbb{R}^N} v^{(\sigma+2)s}\,dx\}^{1/s},\end{aligned}$$

where $\frac{1}{t} + \frac{1}{s} = 1$. Furthermore the Gagliardo–Nirenberg inequality (2.2) yields

$$\int_{\mathbb{R}^N} v^{(\sigma+2)s}\,dx \leq B\{\int_{\mathbb{R}^N} v(x)^2\,dx\}^{(1-\alpha)(\sigma+2)s/2}\{\int_{\mathbb{R}^N} |\nabla v|^2\,dx\}^{\alpha(\sigma+2)s/2},$$

where $\alpha = \frac{N}{2}[\frac{(\sigma+2)s-2}{(\sigma+2)s}] = \frac{N}{2}[1 - \frac{2}{(\sigma+2)s}] \in (0, 1)$, and hence

$$\frac{\sigma+2}{2}\int_{\mathbb{R}^N} |\nabla v|^2\,dx \le \{\int_{\mathbb{R}^N} q(|x|)^t dx\}^{1/t} B^{1/s} c^{(1-\alpha)(\sigma+2)}\{\int_{\mathbb{R}^N} |\nabla v|^2\,dx\}^{\alpha(\sigma+2)/2},$$

where $\alpha(\sigma+2)/2 = \frac{N}{2}[1 + \frac{\sigma}{2} - \frac{1}{s}] = \frac{N}{2}[\frac{1}{t} + \frac{\sigma}{2}] = 1$ by the definition of t. Thus

$$\frac{\sigma+2}{2B^{1/s}} \le \{\int_{\mathbb{R}^N} q(|x|)^t dx\}^{1/t} c^{(1-\alpha)(\sigma+2)},$$

and it follows that there exists $c_1 > 0$ such that $c \ge c_1$. On the other hand, since there exists an $R > 0$ for which $q(R) > 0$, it follows from Theorem 3.1 and the subsequent remark that there exists $c_0 > 0$ such that $M(c)$ is attained by a Schwarz symmetric function in $S(c)$ for all $c > c_0$.

Example 5 (in which $M(c)$ is attained by a Schwarz symmetric function for all $c > 0$). Consider (3.2) with the additional assumption (3.7) and suppose also that $p(\infty) = 0$ and that there exists $R > 0$ such that

$$p(R) > 0 \text{ if } N = 1 \text{ or } 2 \text{ and } p(R) > \left(\frac{\alpha_N}{R}\right)^2 \text{ if } N \ge 3,$$

where α_N is the first zero of the Bessel function $J_{\frac{N}{2}-1}$. Since the conditions (g1) to (g5) are all satisfied and $P(\infty) = \frac{1}{2}p(\infty) = 0$, Theorem 3.1 shows that we have to prove only that $M(c) < 0$ for all $c > 0$. Clearly it is enough to show that there exists an element $v \in H_+$ such that

$$\int_{\mathbb{R}^N} |\nabla v|^2 - p(|x|)v^2 dx < 0 \tag{3.9}$$

since then $dv \in S(c)$ and $J(dv) \le \frac{d^2}{2}\int_{\mathbb{R}^N} |\nabla v|^2 - p(|x|)v^2 dx < 0$ where $d = c\{\int_{\mathbb{R}^N} v^2 dx\}^{-1/2}$. To find such a function v we proceed as follows.

Case $N = 1$: Consider $v_t(x) = e^{-t|x|}$ for $t > 0$ and $x \in \mathbb{R}$. It is easily seen that $v_t \in H_+$ for all $t > 0$ and

$$\begin{aligned}\int_{-\infty}^{\infty} (v_t')^2 - p(|x|)v_t^2 dx &= \int_{-\infty}^{\infty} t^2 e^{-2t|x|} - p(|x|)e^{-2t|x|} dx\\ &= \int_{-\infty}^{\infty} te^{-2|y|} - t^{-1} p\left(\frac{|y|}{t}\right) e^{-2|y|} dy\\ &\le t\int_{-\infty}^{\infty} e^{-2|y|} - t^{-1}\int_{-Rt}^{Rt} p\left(\frac{|y|}{t}\right) e^{-2|y|} dy\\ &\le t\int_{-\infty}^{\infty} e^{-2|y|} dy - t^{-1} p(R) e^{-2Rt} 2Rt\\ &\le t\int_{-\infty}^{\infty} e^{-2|y|} dy - 2Rp(R)e^{-2R} \text{ for } 0 < t \le 1.\end{aligned}$$

Thus $v = v_t$ satisfies (3.9) for t small enough.

Case $N = 2$: First we observe that $w \in H_+$ when w is defined by

$$w(x) = \begin{cases} \left[\log \frac{1}{|x|}\right]^{1/3} & \text{for } 0 < |x| < 1 \\ 0 & \text{for } |x| \geq 1. \end{cases}$$

Furthermore,

$$A = \int_{|x|\leq 1} p(|x|)dx > 0 \text{ and there exists } T > 0 \text{ such that}$$
$$w(x)^2 > \frac{1}{A} \int_{\mathbb{R}^N} |\nabla w|^2 \, dx \text{ for } |x| = T.$$

Setting $v(x) = w(Tx)$ for $x \in \mathbb{R}^2 \backslash \{0\}$, we find that

$$\begin{aligned} \int_{\mathbb{R}^2} |\nabla v|^2 - p(|x|)v^2 dx &= \int_{\mathbb{R}^2} T^2 \, |\nabla w(Tx)|^2 - p(|x|)w(Tx)^2 dx \\ &= \int_{\mathbb{R}^2} |\nabla w(y)|^2 \, dy - \int_{\mathbb{R}^2} p(|x|)w(Tx)^2 dx \\ &\leq \int_{\mathbb{R}^2} |\nabla w(y)|^2 - w(T\xi)^2 \int_{|x|\leq 1} p(|x|)dx < 0, \end{aligned}$$

where $|\xi| = 1$, by the definition of T.

Case $N \geq 3$: We define v as follows:

$$v(x) = \begin{cases} \left(\frac{|x|}{R}\right)^{1-\frac{N}{2}} J_{\frac{N}{2}-1}\left(\frac{\alpha_N |x|}{R}\right) & \text{for } 0 < |x| \leq R \\ 0 & \text{for } |x| > R \end{cases}.$$

Then $v \in H_+$ and since

$$-\Delta v(x) = \left(\frac{\alpha_N}{R}\right)^2 v(x) \text{ for } 0 < |x| < R \text{ and } \lim_{r\to 0} \left(\frac{r}{2}\right)^{1-\frac{N}{2}} J_{\frac{N}{2}-1}(r) = \frac{1}{\Gamma\left(\frac{N}{2}\right)},$$

it follows that

$$\int_{\mathbb{R}^2} |\nabla v|^2 \, dx = \left(\frac{\alpha_N}{R}\right)^2 \int_{|x|\leq R} v^2 dx.$$

Hence

$$\begin{aligned} \int_{\mathbb{R}^N} |\nabla v|^2 - p(|x|)v^2 dx &= \left(\frac{\alpha_N}{R}\right)^2 \int_{|x|\leq R} v^2 dx - \int_{|x|\leq R} p(|x|)v^2 dx \\ &\leq \left\{\left(\frac{\alpha_N}{R}\right)^2 - p(R)\right\} \int_{|x|\leq R} v^2 dx < 0. \end{aligned}$$

Note that in Example 5 we can have $q \equiv 0$. We end with a situation in which we can admit $p \equiv 0$.

Example 6 (in which $M(c)$ is attained by a Schwarz symmetric function for all $c > 0$). Consider (3.2) with the additional assumption (3.7) and suppose also that $p(\infty) = 0$ and that there exist $R, A > 0$ and $t \geq 0$ such that

$$0 < \sigma < \frac{2(2-t)}{N} \text{ and } q(r) \geq Ar^{-t} \text{ for all } r \geq R.$$

Given any $c > 0$, we set

$$w_\alpha(x) = \frac{c\alpha^{\frac{N}{4}} e^{-\alpha|x|^2}}{d(N)} \text{ where } d(N) = \{\int_{\mathbb{R}^N} e^{-2|y|^2} dy\}^{1/2}.$$

Then $w_\alpha \in H_+$ for all $\alpha > 0$ and, as is shown in the proof of Theorem 5.4 of [19], there are positive constants $D(N)$ and $I(N)$ such that

$$M(c) \leq \frac{1}{2} c^2 \alpha D(N) - c^{\sigma+2} \alpha^{\frac{N\sigma}{4}+\frac{t}{2}} I(N) \text{ for all } \alpha \in (0, 1].$$

Since $\frac{N\sigma}{4} + \frac{t}{2} < 1$, it follows that $M(c) < 0$.

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