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Leonhard Euler's early lunar theories 1725–1752

Part 1: first approaches, 1725–1730

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Abstract Leonhard Euler (1707–1783) published two lunar theories in 1753 and 1772. He also published lunar tables in 1745, 1746, and—anonymously—in 1750. There are notebook records, unpublished manuscripts, and manuscript fragments by Euler reflecting the development of his lunar theories between about 1725 until about 1752. These documents might be used to reconstruct Euler's theory on which he based his calculations of those lunar tables and to analyze the development of his lunar theories within this time span. The results of this analysis will be published here in three parts representing three stages of Euler's research on this topic: First approaches (about 1725–1730), developing the methods (about 1730–1744), and the breakthrough (about 1744–1752). In this part, I analyze Euler's manuscripts and, predominantly, Euler's records of his first two notebooks written between 1725 and 1730. I found that his early theoretical approach is coined by his development of analytical (rational) mechanics of punctiform bodies moved by central forces. He tried to describe the Moon's motion in terms of two simultaneously acting centripetal forces, Huygens' centrifugal theorem, and associated osculating radii.

In memoriam Emil A. Fellmann (1927–2012).

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1 Introduction

1.1 Overview

The motions of Sun and Moon are by far the most influential astronomical phenomena for cultural development. Their apparent daily, monthly, and yearly movements on the celestial sphere are perceptible by everybody. The orbital motions of these celestial bodies are important for the daily life in various social and scientific respects. They dominate the calendars of and through all cultures, and they had a crucial impact on the development of the exact sciences. The determination of time by measuring of lunar distances (defined as angular separations between the Moon and selected stars) or accurate predictions of (solar and lunar) eclipses represent the most prominent examples. Even today, the Earth's variable rotation may only be reconstructed most accurately by using ancient solar eclipse observations (cf. Stephenson 1997).

The determination of the Moon's motion was one of the most difficult tasks in mathematical astronomy before the computer era. While the apparent motion of the Sun (reflecting the orbital motion of the barycenter of the Earth-Moon-System around the Sun) may be treated in good approximation as a so-called "two-body-problem" in terms of Keplerian orbital parameters, the determination of the motion of the Moon has to be regarded as a "three-body-problem" with the Earth and the Sun representing the central and the perturbating body, respectively. Isaac Newton (1643–1727), in his *Principia*, struggled with this problem, and his successors failed to solve the lunar problem exactly as it is derived from the principle of universal gravitation. The solution of the equations of motion for the Moon in an analytically closed form is impossible, as the mathematician Henri Poincaré (1854–1912) proved in his famous paper on the three-body-problem of 1889 (cf. Poincaré 1890; Barrow-Green 1997). With the establishment of the Leibnizian calculus, of the trigonometric functions, and of the equations of motion by Leonhard Euler (1707–1783) it was possible to solve the coupled differential equation system of second order for the Moon's motion approximately using series expansions by the early 1740s. In that time the first analytical lunar theories based on those mathematical methods and physical principles were developed simultaneously by Euler, Alexis-Claude Clairaut (1713–1765), and Jean le Rond d'Alembert (1717–1783).

In the history of science, the development of lunar theories was addressed frequently during the last decades. The historical studies, however, were focussed mainly on the contributions by Newton (cf. Cohen 1975; Waff 1976b, 1977; Smith 1999a,b; Kollerstrom 2000). In particular, the problem of the apsidal motion of the Moon's orbit and its solution by Clairaut was analyzed with great emphasis (cf. Waff 1976a, 1995). D'Alembert's unpublished manuscripts on lunar theory were published in the context of the edition of his collected works (cf. Chapront-Touzé et al. 2002). The appreciation of the significance and importance of Euler's lunar theories of 1753 and 1772 is, however, still a missing part in the history of exact sciences. The analysis and edition of his unpublished manuscripts and notebooks related to lunar theory and of his anonymously published lunar tables are preparatory works for addressing and tackling this task. Euler's manuscripts will give us insight into his outstanding role in the development and history of the lunar theories as an essential part of the history of eighteenth century celestial mechanics.

1.2 The scientific significance of lunar theories

The description of the Moon's motion was one of the most important problems in the history of positional astronomy. From the antiquity, e.g. Ptolemy, until the middle of the seventeenth century geometric and kinematic theories were developed to predict solar and lunar eclipses and to construct calendars based on the observed motions of the Sun and the Moon (cf. Brack-Bernsen 1997; Toomer 1984). The determination of the Moon's position with an accuracy of less than one arc minute became an official challenge for eighteenth century scientists by the "Longitude Act" of 1714, initiating the search for a method to find the longitude at sea and, thus, to solve the most important problem of navigation (cf. Longitude Act 1714; Andrewes 1996). One of the most important astronomical solutions proposed was the method of lunar distances: from topocentric measurements of angular distances between the limb of the Moon's disk and nearby stars and its elevations at the local time of the observer one obtained geocentric angular distances, which then were compared with predicted and calculated values for the meridian of London (or Paris) tabulated in lunar tables (cf. Howse 1996). The differences between observed and tabulated distances represent the time differences between these locations and thus yield the longitudes of the observer. The quality of this method depends crucially on the correctness and precision of the lunar tables involved in this procedure and consequently on the quality of the lunar theory used to construct them.

Although Isaac Newton established the principle of universal gravitation in his *Principia* (cf. Newton 1687, 1713, 1726) and even published a "Theory of the Moon's Motion" in 1702 (cf. Newton 1702; Cohen 1975; Kollerstrom 2000), the first lunar theories based on the law of gravitation and treated as a so-called "three-body-problem" including the application of equations of motion formulated in three dimensions were only developed in the 1740s by Leonhard Euler, Alexis-Claude Clairaut and Jean le Rond d'Alembert. The reason for this delay was the lack of adequate mathematical methods and physical principles to cope with the analytical treatment of this three-body-problem (cf. Verdun 2010). It was Euler who first formulated general equations of motion of the three-body-problem for the Earth–Sun–Moon system and introduced trigonometric series to integrate them using the method of undetermined coefficients. Using his (unpublished) "embryonic" lunar theory of 1743/1744 (cf. Verdun 2010), Euler constructed and published between 1744 and 1750 numerous lunar tables incorporating perturbational effects of the Sun (cf. Verdun 2011). But only in 1753 his "first" lunar theory was published, one year after Clairaut's (cf. Euler 1753; Clairaut 1752). They solved the problem of the apsidal motion of the Moon's orbit, which was actually first noted already by Newton but which reappeared in 1747 when neither Euler nor Clairaut could derive this motion from the inverse square law of gravitation in accordance with observational data. Their lunar tables based on those theories (cf. Verdun 2011; Clairaut 1754), however, were not as accurate as the tables constructed by Tobias Mayer (1723–1762), which were

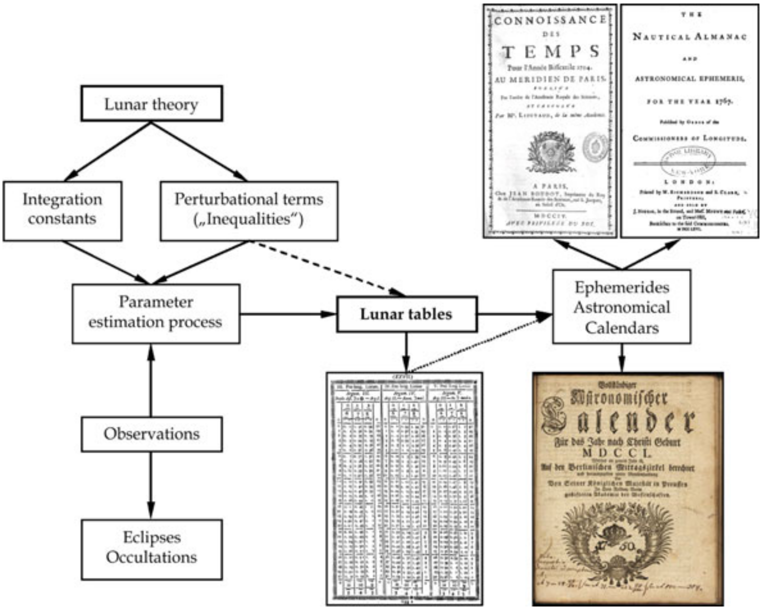

Fig. 1 The construction and function of lunar tables and their relation to ephemerides and astronomical calendars. Lunar tables may also be incorporated in astronomical calendars

published first in 1753 and 1754, then improved in 1770 (cf. Mayer 1753, 1754, 1770), although his lunar theory is similar to that of Euler (cf. Mayer 1767). He developed and applied empirical methods to bring theory in close agreement with observations (cf. Wepster 2010). It was—among others—just such kind of processes, i.e., the determination of integration constants resulting from theory and the "correction" or improvement of the theoretically derived inequalities in the Moon's motion (represented by the coefficients of the integrated series expansions), with which Euler struggled when he tried to bring his "first" lunar theory of 1753 into closer agreement with observation.

This process is displayed graphically in Fig. 1, illustrating the construction and function of lunar tables in general and their relation to ephemerides and astronomical calendars, which are calculated from them. The perturbational terms resulting from theory were not directly tabulated in the lunar tables, because the equations of motion could be integrated only approximately. Therefore, the theoretically determined inequalities had to be corrected using observations, e.g. from solar and lunar eclipses or occultations of stars by the Moon. Such phenomena allowed the observation of the Moon's position with high accuracy, which was used to improve the inequalities in a parameter estimation process. These inequalities were then tabulated as a function of the various linear combinations, e.g., of the relevant geocentric angular distances between Sun and Moon (so-called "elongations") and their (geocentric or heliocentric) ecliptic arc lengths. The precision of such tables, constructed for the meridian of London or Paris, became fundamental for solving the longitude problem. This is why the development of lunar theories and the construction of lunar tables play an important and significant role in the history of science. Numerous prize questions of the Academies of sciences of Paris and St. Petersburg addressed this problem, including the development and improvement of lunar theories. Lunar tables were used for navigational purposes even in the nineteenth century. The mathematical methods and physical principles emerging from the development of lunar theories had an immense impact on the exact sciences of the eighteenth and nineteenth centuries: Trigonometric series and Fourier analysis, physical and variational principles, methods of analytical and numerical integration, parameter estimation methods, to mention but a few methods resulted from these theoretical works (cf. Verdun 2010).

The relevance of lunar theory and its history within the field of celestial mechanics was best expressed, with an emphasis on the methods invented, by two authorities in positional astronomy, the Swiss astronomer Alfred Gautier (1793–1881) and the Canadian-American astronomer Simon Newcomb (1835–1909). In 1817, Gautier wrote in the preface to his *Essai historique* (cf. Gautier 1817, pp. v–vij):

> On a dit déjà que l'histoire de l'Astronomie donne, plus qu'aucune autre, une juste idée des progrès de l'esprit humain. Si cela est vrai en général, cela s'applique spécialement, ce me semble, à l'histoire des travaux que la découverte de l'attraction a fait naître. […] l'examen de la théorie de la Lune de Newton, fait par Clairaut et d'Alembert, peut servir à l'apprécier jusqu'à un certain point; la discussion entre ces deux derniers Géomètres nous éclaire sur les avantages et les inconvéniens de leurs méthodes; la franchise d'Euler nous découvre les imperfections des siennes […].
>
> (It was already said that the history of astronomy gives, more than any other science, an exact idea of the progress of human esprit. If this is true in general, this applies in particular – as it seems to me – to the history of works made for the discovery of the attraction. The examination of Newton's lunar theory by Clairaut and d'Alembert can be used to appreciate it up to a certain point; the discussion between these two Geometers clarify the advantages and disadvantages of their methods; the ingenuity of Euler's let us discover the imperfections of his theory […])

Introducing his article of 1908, Newcomb wrote (cf. Newcomb 1908, p. 1):

> Parmi les problèmes de la mécanique céleste, celui du mouvement de la lune occupe un haut rang, à cause de sa difficulté, et du nombre des questions intéressantes auxquelles il a donné naissance. Il nous offre un bon exemple des méthodes générales de la science par lesquelles nous prédisons les phénomènes de la nature. […] C'est dans la mécanique céleste que ces méthodes de recherche ont leur plus complète manifestation. La loi fondamentale est celle de la gravitation universelle suivant la formule de Newton […] Depuis l'énoncé de la loi de Newton une succession de grands geomètres, tels que d'Alembert, Clairaut, Laplace, Lagrange, Euler, Plana, Damoiseau, Hansen, – je ne nomme pas les vivants – ont développé et perfectionné les méthodes de déduction, tandis que les maîtres de l'astronomie théorique ont corrigé sans cesse les éléments astronomiques au moyen des observations.

> (Among the problems of celestial mechanics the one on the motion of the Moon occupies a high rank due to its difficulty and the number of interesting questions which have emerged from it. It offers us a good example of general methods of science with which we predict the phenomena in nature. [...] It is in celestial mechanics where these research methods are manifested most completely. The fundamental law is the one of universal gravitation according to Newton's formula. [...] Since the announcement of Newton's law a series of great Geometers such as d'Alembert, Clairaut, Laplace, Lagrange, Euler, Plana, Damoiseau, Hansen,—I do not mention those alive—one has developed and perfected the methods of deduction, whereas the masters of theoretical astronomy improve permanently the astronomical elements using observations.)

These statements underline the importance of the methods developed during the history of lunar theory not only for celestial mechanics but for the exact sciences in general. This is confirmed by the number of recent published monographs and papers on this topic (cf., e.g., Wilson 2008, 2010; Wepster 2010; Steele 2012).

1.3 The structure of this article

In Sect. 2, I will first describe the main body of Euler's published (Sect. 2.1) and unpublished (Sect. 2.2) writings related to lunar theory. I will refer to this section in the follow-up parts 2 and 3 of this series of papers. In Sect. 2.3, I will describe in more detail the unpublished documents which may regarded as the beginnings of Euler's studies on lunar theory and which therefore are addressed in this part 1. The content of these documents is given almost in full length in the appendix of this article. In the main Sect. 3, I will focus on the reconstruction of Euler's first approaches to lunar theory using summaries of these documents. On one hand these summaries should give us insight in Euler's very first thoughts on the topic, on the other hand they will raise important questions, e.g., about the sources probably used by Euler to get into lunar theory, about the theoretical equipment available to him to tackle the problems, and about the mathematical and physical deficiencies which prevented him to solve the lunar problem successfully from the beginning. This will be the subject of the last Sect. 4, where I will venture a final assessment of Euler's first approaches to lunar theory.

2 The corpus of Euler's documents on lunar theory

2.1 Euler's published contributions to lunar theory

Euler was concerned with lunar theory almost during his entire scientific career. His earliest manuscripts and notes on lunar theory were written between 1725 and 1727, his latest records are dated in the 1770s. He left us an extensive work on that topic which consists of published and unpublished treatises, notebook entries, and a huge correspondence related to some extent to the motion of the Moon. Nearly all of his original published works on lunar theory and lunar tables and a large part of that correspondence are included in the second and fourth series of his collected works, *Leonhardi Euleri Opera Omnia*, edited and published since 1910 by the Euler Commission of the Swiss Academy of Sciences in four series: Series prima (mathematics), Series secunda (physics and astronomy), Series tertia (Miscellanea), Series quarta A (Correspondence), and (the planned and postponed) Series quarta B (manuscripts and notebooks). Until now, 76 volumes in quarto have been published. The following list summarizes Euler's published works of the second Series sorted by their relation to his two lunar theories. In parentheses, I added to each item the years when its manuscript approximately (Estimation according to Verdun 2010) was finished and when it was published. They are numbered according to the bibliography of Euler's printed works by Gustaf Eneström (cf. Eneström 1910). The numbers are accompanied by the letters "E" and "A" referring to works by Leonhard Euler or by his son Johann Albrecht, respectively, which most probably were written under the supervision of his father. The chronological development (which is by far not identical with the corresponding dates of publication) and brief summaries of these treatises as well as all of Euler's treatises on celestial mechanics are discussed in Verdun (2010). It is worth noting that Euler's first lunar theory (E 187) actually remained uncommented in the *Opera omnia* due to the death of the responsible editor. The editorial board used as a substitute for the editor's introduction to this work a text written by Félix Tisserand (1845–1896), which fails, however, to meet modern historiographical standards (cf. Tisserand 1894).

Lunar theory treated as a two-body-problem:

– E 15: Mechanica sive motus scientia analyticae exposita, Chapter 5 (1730–1734, 1736)

Lunar theory treated as a three-body-problem as preparatory works for Euler's first lunar theory (E 187):

– E 138: De motu nodorum lunae ejusque inclinationis ad eclipticam variatione (1744, 1750)
– E 112: Recherches sur le mouvement des corps célestes en général (1747, 1749)

Treatises related to Euler's first lunar theory (E 187):

– E 187: Theoria motus lunae exhibens omnes eius inaequalitates (1751–1753, 1753)
– E 193: De perturbatione motus planetarum ab eorum figura non sphaerica oriunda (1749, 1753)
– E 204: Extract of a letter from professor Euler, of Berlin, to the rev. Mr. Caspar Wettstein (1751, 1753)
– E 304: Considerationes de motu corporum coelestium (1762, 1766)
– E 371: Considerationes de theoria motus Lunae perficienda et imprimis de eius variatione (1763, 1769)
– E 399: Réflexions sur les diverses manières dont on peut représenter le mouvement de la Lune (1763, 1770)
– E 401: Nouvelle manière de comparer les observations de la Lune avec la théorie (1766, 1770)
– A 22: Réflexions sur la variation de la lune (1766, 1768)

Treatises related to Euler's second lunar theory (E 418):

- E 418: Theoria motuum lunae, nova methodo pertractata una cum tabulis astronomicis, unde ad quovis tempus loca lunae expedite computari possunt incredibili studio atque indefesso labore trium academicorum: Johannis Alberti Euleri, Wolfgangi Krafft, Johannis Andreae Lexell (1768, 1772)
- E 485: Réponse à la Question proposée par l'Académie Royale des Sciences de Paris, pour l'année 1770. Perfectionner les méthodes sur lesquelles est fondée la théorie de la Lune, de fixer par ce moyen celles des équations de ce Satellite, qui sont encore incertaines, et d'examiner en particulier si l'on peut rendre raison, par cette théorie de l'équation séculaire du mouvement de la Lune (1769, 1777)
- E 486: Réponse à la Question proposée par l'Académie Royale des Sciences de Paris, pour l'année 1772. Perfectionner les méthodes sur lesquelles est fondée la théorie de la Lune, de fixer par ce moyen celles des équations de ce Satellite, qui sont encore incertaines, et d'examiner en particulier si l'on peut rendre raison, par cette théorie de l'équation séculaire du mouvement moyen de la Lune (1772, 1777)
- E 504: De theoria Lunae ad majorem perfectionis gradum evehenda (1775, 1780)
- E 548: De variis motuum generibus, qui in satellitibus planetarum locum habere possunt (1777, 1783)
- E 549: De motibus maxime irregularibus, qui in systematae mundane locum habere possunt, una cum methodo huismodi motus per temporis spatium quantumvis magnum prosequendi (1777, 1783)

Treatises related to theory and observation (verification of the theory and parameter estimation):

- E 837: De emendatione tabularum lunarium per observationes eclipsium Lunae (1746, 1862)
- E 838: Tria capita ex opere quodam majori inedito de theoria Lunae (1747, 1862)
- E 113: Méthode pour trouver les vrais momens tant les nouvelles que les pleines lunes (1747, 1749)
- E 114: Méthode de trouver le vrai lieu géocentrique de la lune par l'observation de l'occultation d'une étoile fixe (1748, 1749)
- E 115: Méthode de determiner la longitude des lieux par l'observation d'occulations des étoiles fixes par la lune (1748, 1749)

Treatises related to the determination of the lunar parallax:

- E 838: Tria capita ex opere quodam majori inedito de theoria Lunae (1747, 1862)
- E 113: Méthode pour trouver les vrais momens tant les nouvelles que les pleines lunes (1747, 1749)
- E 172: De la parallaxe de la lune tant par rapport à sa hauteur qu'à son azimuth, dans l'hypothèse de la terre sphéroïdique (1747, 1751)
- E 117: Réflexions sur la dernière éclipse de Soleil du 25 juillet a. 1748 (1748, 1749)
- E 141: Sur l'accord des deux dernières éclipses de soleil et de la lune avec mes tables pour trouver les vrais momens des pléni-lunes et novilunes (1748, 1750)
- A 19: Beantwortung über die Preisfrage: In was für einer Verhältniss sowohl die mittlere Bewegung des Monds, als auch seine mittlere Entfernung von der Erde mit den Kräften stehen, welche auf den Mond wirken? (1762, 1767)

– E 529: Theoria parallaxeos, ad figuram terrae sphaeroidicam accomodata (1780, 1782)

Lunar tables:

– E 76: Novæ et correctæ tabulae ad loca lunae computanda (1745, 1745)
– E 87: Tabulae astronomicae solis et lunae (1746, 1746)
– E 418A: Leonhardi Euleri novae tabulae lunares singulari methodo constructae, quarum ope loca lunae ad quodvis tempus expedite computare licet (1768, 1772)

Euler published lunar tables not only in 1746 and 1772. I was able to establish Euler's authorship of many anonymously published lunar tables (cf. Verdun 2011). These "unknown" Eulerian lunar tables appeared in different versions of the Astronomical Calendars of the Berlin Academy of Science. These periodicals are nowadays extremely rare and thus hardly known to the scientific community. In addition, I showed that the lunar tables (E 87) incorporated in Euler's *Opuscula varii argumenti* (E 80, cf. Euler 1746) of 1746 are not identical with those (E 76) published in 1745 (cf. Euler 1745). Up to now, it had been assumed, that E 76 is a preprint of E 87, which is why E 76 was not included in the *Opera omnia*. Until now, I succeeded to establish Euler's authorship for the following tables (cf. Verdun 2011):

– Calendarium ad annum Christi MDCCXLIX. Pro meridiano Berolinensi. Cum approbatione academiae regiae scientiarum et elegantiorum literarum Borussicae. 1749. pp. [L1r]–[M4v]. (14 p), containing
  – Tabula Noviluniorum & Pleniluniorum mediorum quæ post initium cujusvis Anni Epocharum sequentium contingent Ad Meridianum Parisinum accommodata
  – Tabula Noviluniorum & Pleniluniorum pro Annis Expansis
  – Tabula Successionum Noviluniorum & Pleniluniorum intervallo unius Anni
  – Tabula exhibens Anomaliam Lun excentricam
  – Tabula æquationum I. Pro momentis Noviluniorum
  – Tabula æquationum I. Pro momentis Pleniluniorum
  – Tabula æquationum II. Pro momentis Noviluniorum & Pleniluniorum
  – Tabula æquationum III, IV, V, VI. Pro momentis Novi- & Pleniluniorum
  – Reductio momentorum Novi & pleniluniorum in orbita ad momenta eorum in Ecliptica
  – De Tabulis. Vera momenta noviluniorum ac pleniluniorum exhibentibus
– Vollständiger Astronomischer Calender Für das Jahr nach Christi Geburt MDCCL. Welches ein gemein Jahr ist, Auf den Berlinischen Mittagszirkel berechnet und herausgegeben unter Genehmhaltung Der Von Seiner Königlichen Majestät in Preussen In Dero Residenz Berlin gestifteten Akademie der Wissenschaften. 1750. pp. [G4v]–[H4r]. (8 p), containing
  – Tafeln, die stündliche Bewegung des Monds zu berechnen
– Calendarium ad annum Christi MDCCL. Pro meridiano Berolinensi. Cum approbatione academiae regiae scientiarum et elegantiorum literarum Borussicae. 1750. pp. [G4v]–[H4r]. (8 p), containing
  – Tabulæ Ad computandum Lunæ motum horarium
– Almanac Astronomique pour l'an de Grace MDCCL. au meridien de Berlin, publié par l'ordre et avec privilege de l'academie royale des sciences et belles lettres de

Prusse. 1750. A Berlin, imprimé chez Chretien Louis Kunst. pp. [G8v]–[M2r]. (69 p), containing

- Tables de la Lune pour le Meridien de Paris
- Tables pour corriger le lieu moyen de la Lune
- Tables pour calculer la distance de la Lune à la Terre
- Tables pour corriger le lieu moyen du Noeud & pour trouver l'Inclinaison de l'Orbite de la Lune à l'Ecliptique

The tables published in 1750 in the *Almanac Astronomique* are the most important ones of this list, because the Moon's position and motion can be determined for any epoch and any place on Earth from them. The other tables were calculated for special purposes using the former ones. It is remarkable to note that Euler constructed so many lunar tables until 1750 although his "first" lunar theory was published only in 1753. In a letter to Johann Caspar Wettstein (1695–1760) dated March 29, 1746, Euler wrote that he had no chance yet to print his lunar tables [of 1746] but he intends to publish them together with the theory (cf. original letter registered as R 2749 in Juškevič et al. 1975, to be published in Volume O.IVA 7 of the *Opera omnia*, as well as Juškevič et al. 1976, p. 258, where the citation differs somewhat from the original text):

> Je n'ai pas encore trouvé occasion de les [E 87] faire imprimer mais je conte de les publier avec la theorie.

This is why historians of Euler's work speculated time and again that there must have exist an "early" (i.e. pre-1753) lunar theory on which Euler based the calculations of his lunar tables published between 1745 and 1750. There is, in fact, strong evidence that Euler's alleged missing "embryonic" lunar theory is preserved, although fragmentarily, in his unpublished manuscript Ms 281 (cf. Sect. 2.2 and Table 1 below), which will be discussed in part 3 of this series of papers. Euler probably used this theory for the construction and calculation of his lunar tables published in 1745 (E 76) and 1746 (E 87) as well as of his anonymously published lunar tables of 1749 and 1750 (cf. Verdun 2010, 2011). I intend to analyze this matter in a follow-up study, addressing in particular the reconstruction of Euler's "embryonic" lunar theory from Ms 281 and from his lunar tables.

2.2 Euler's unpublished contributions to lunar theory

Most of Euler's manuscripts are now preserved in the St. Petersburg Archive of the Russian Academy of Sciences. They were registered, tentatively dated, and numbered in 1962 by J. C. Kopelevič, M. V. Krutikova, G. K. Mikhailov, and N. M. Raskin (cf. Kopelevič et al. 1962). Table 1 lists the manuscripts and manuscript fragments related to lunar theory and gives the number of original manuscript pages. Some of them—indicated in this table by an Eneström number—were already published, but some remain unpublished until now. The published version must not necessarily be identical with the manuscript. The most prominent example, however, is Euler's draft version Ms 167 for his "Mechanica" (E 15), which was published in 1736 (cf. Euler 1736). This manuscript differs considerably from the printed version, which is why it was published in 1965 by Gleb K. Mikhailov (cf. Mikhailov 1965, pp. 93–224).

Table 1 List of lunar theory related manuscripts by Euler (cf. Kopelevič et al. 1962)

Ms	E	Title	Pages
167	–	Mechanica seu scientia motus	197
180	–	De Motu corporum a pluribus viribus centralibus sollicitatis	5
251	–	De trium corporum mutua attractione	2
271	–	De Motu Lunæ in Ellipsin	6
272	–	Dissertatio de Motibus Lunæ	4
273	–	[Sex propositiones de perturbatione motus Lunae a Sole]	22
274	138	De motu nodorum Lunae ejusque inclinationis ad eclipticam variatione	8
275	139	Quantum motus Terrae a Luna perturbatur accuratius inquiritur	4
276	–	De Motu Lunæ	8
277	187	Theoria motus et anomaliae Lunae	162
278	837	De emendatione tabularum lunarium per observationes eclipsium Lunae	20
279	485	Réponse à la question proposée par l'Academie Royale des sciences de Paris pour l'année 1770	160
280	–	[Deux fragments d'un ouvrage sur la théorie du mouvement de la Lune]	12
281	–	[Fragmenta ex opere quodam de motu Solis ac Lunae]	32
282	838	Quinque capita ex opere quodam majori inedited de theoria Lunae	48
283	–	[Applicatio theoriae motus Lunae ad observations eclipsium lunarium]	12
284	–	[Fragmenta ex opere quodam de motu Lunae]	36
Total number of original manuscript pages			736

There are further examples: Ms 282 contains two chapters (named "d" and "e" by Kopelevič et al. 1962 on p. 88) entitled "Constitutio elementorum motus Lunæ" (The foundation of the elements of the Moon's motion, Sects. 1–6) and "Applicatio theoriæ ad observationes eclipsum lunarium" (The application of the theory to lunar eclipse observations, Sects. 1–17), which were not included in the posthumously published version E 838, and also Ms 277 contains a chapter entitled "Alia methodus easdem inæqualitates eruendi" (A further method with which the inequalities may be found), which was not included in the printed "first" lunar theory E 187. On the other hand, the Chapters XVI–XVIII as well as the "Additamentum" of E 187 are missing in the original manuscript Ms 277. Consequently, all "published" manuscripts need to be checked for the purpose of an adequate reconstruction of Euler's lunar theories, as well. Some of the manuscripts mentioned in Table 1, for example Ms 281, turned out to be very important for this reconstruction, as we shall see in part 3 (cf. Verdun 2010).

Of similar importance are Euler's notebooks. There are 12 of them, listed in Kopelevič et al. (1962) with No. 397–No. 408, which I will call Ms 397–Ms 408, respectively, and which comprise about 4,200 pages, whereof about 100 pages concern lunar theory.

All these documents—manuscripts and notebook records—have never been analyzed before in the context of the development of Euler's lunar theory. A first approximate analysis was done in Verdun (2010). That study showed the potential inherent in these documents. Although some of the manuscripts are unfinished or only fragmentarily preserved, and although the records contained in the notebooks often are extremely fragmentary, they may be used to put together the pieces for the development of Euler's early lunar theories. This is a difficult task solved to some extent in Verdun (2010). This reconstruction, although incomplete, will disclose the basic approach, the scientific context, and the principle methods available to Euler for dealing with the lunar problem.

For this first of three essays on the subject, only two manuscripts and the first two notebooks are relevant. It is impossible to date them exactly to within, e.g., 1 or 2 years. Consequently, the order I present them here is mainly for didactic reasons. Not every derivation of mathematical formulae and not every explanation and comment by Euler can be reproduced here. This will be the goal of a critical edition of these works, which is also being prepared. Figures (sketches) are reproduced as exact copies of Euler's original drawings, including all inconsistencies and inaccuracies (e.g., faultily drawn tangential or perpendicular lines). When reconstructing intermediate results and formulae which Euler did not derive explicitly, I often will refer to Volume I, Chapter V, of his "Mechanica" (cf. Euler 1736), which Euler finished in 1734 and which contains much of the results developed by him already several years earlier.

2.3 Description and dating of the unpublished documents

Four documents preserved in the Archives of the Russian Academy of Science in St. Petersburg may be considered as the earliest manuscripts by Euler that contain his first but still tentative approaches to lunar theory: two of Euler's unpublished manuscripts, here referred to as Ms 272 and Ms 180, as well as his first two notebooks, also called "Adversaria mathematica" I and II, or the "Basel notebook" and the "travel diary" respectively, but here referred to as Ms 397 and Ms 398. Ms 397 was written during the period when Euler lived in his hometown Basel until 1727, and Ms 398 contains records of a "Diarium" describing his journey from Basel to St. Petersburg in 1727 (cf. Fellmann 2007). Due to the uncertainty in the exact dating of these documents, I will present them in a hypothetical, didactically motivated order thus underlining the evolutionary character of the development of Euler's early lunar theory.

2.3.1 Euler's manuscript Ms 272

This document is listed as number 272 in the catalogue of Euler's unpublished manuscripts (see Kopelevič et al. 1962, p. 85). It consists of two folios written on both sides in a carefully executed clean copy style handwriting. Kopelevič et al. (1962) dates this manuscript to the years 1726–1728, perhaps because of its characteristic style (ductus litterarum) of Euler's early handwriting. It is entitled "Dissertatio de Motibus Lunæ" (Treatise on the Moon's motion) and contains eight paragraphs consisting solely of text and no formulae. There are only very few corrections and emendations by Euler. One may conclude that Euler probably intended to use it as an introductory passage to a larger treatise on the Moon's motion stating its principle problems and observable inequalities. It may therefore be considered an unfinished work by Euler.

2.3.2 Euler's manuscript Ms 180

This document is registered as number 180 in the catalogue of Euler's unpublished manuscripts (see Kopelevič et al. 1962, p. 64). It consists of three folios written on both sides in a carefully executed clean copy style handwriting. Kopelevič et al. (1962) dates this manuscript to the years 1725–1726, perhaps for the same reason as for Ms 272. It is entitled "De Motu corporum a pluribus viribus centralibus sollicitatis" (On the motion of bodies driven by multiple central forces) and contains two "lemmata" and four "problemata", illustrated by four marginal figures. There are only very few corrections and emendations by Euler. He probably intended to use it as part (chapter) of a larger treatise on mechanics. Euler left this manuscript unfinished, ending abruptly while solving the forth problem.

2.3.3 Euler's notebook Ms 397 (Adversaria mathematica I)

The composition of the "Basel notebook" Ms 397 was dated by Kopelevič et al. (1962), p. 114, to the years 1725–1727. It consists of 213 double-side written folio pages. The strongest evidence for this dating is the fact that it contains in the middle of the notebook (cf. Ms 397, fol. 82r–84v) a fragment of a draft letter by Euler to Daniel Bernoulli (1700–1782) written in November 1726 (Juškevič et al. 1975, No. 91, p. 17). In this letter, Euler mentions his dissertation ("Habilitationsschrift") on the theory of sound (E 2), with which he competed in spring 1727 for the physics professorship in Basel, as well as the public prize competition announced in 1726 by the Paris Academy on the optimal way of setting up the masts on ships (cf. Fellmann 2007). Euler wrote (cf. Ms 397, fol. 84r):

> Constat sine dubio de Problemate Nautico quod Academia Regia Scientiarum Gallica hoc anno proposuit, cujusque præmium pro[xi]mo paschate distri adjudicetur [sic!]. Illius quoque solutionum misi, ita autem illud sonat, Quelle est la Meilleure maniere de master les Vaisseaux, tant par rapport à la situation qu'au nombre et hauteur des Masts.
>
> This statement may be translated as:
>
> You know without doubt about the nautical problem that the Royal French Academy of Sciences proposed this year, and for which the prize will be awarded next Easter. I have sent a solution of this, too; [the question is] as follows: "What is the best way of masting ships, both with regard to the position and to the number and height of the masts?"

The notebook was almost in its entirely used by Euler as a clean copy of a textbook on mechanics, in which he transformed the relevant content of Newton's *Principia* into his own analytic language using the Leibnizian calculus as developed further by his teacher in Basel, Johann I Bernoulli (1667–1748). Evidence for this are numerous marginal references to the corresponding pages in the *Principia*. These references pertain to the second edition of the *Principia* (cf. Newton 1713; Mikhailov 1965, footnote on p. 42), which confirms the starting date of composition of this notebook before the third edition of the *Principia* which was published in 1726 and to became available to Euler in 1727 (cf. Newton 1726; Ms 398, Sect. 2.3.4 below).

2.3.4 Euler's notebook Ms 398 (Adversaria mathematica II)

The "travel diary" was dated by Euler himself on folio 2r with "A[nno] 1727". His "Diarium" of the journey from Basel to St. Petersburg started on April 1, 1727. It was published by Mikhailov (1959), pp. 275–278. This notebook consist of 85 double-side written folio pages. It shows the characteristics of a waste book and contains not only scientific but practical notes concerning, e.g., Russian words listed in dictionary manner, names of persons Euler met during his journey, etc. Further details about this notebook are given in Mikhailov (1959). The notebook contains values of parameters and orbital elements which Euler copied from the third edition of Newton's *Principia* (cf. Ms 398, fol. 50v–51r), a fact that confirms Euler's own dating of this notebook. A peculiarity of the travel diary concerns the curiosity that Euler wrote into this notebook starting first from the front side until folio 40r, then turned it around by 180°, and then continued to use it from the back side. Another "anomaly" concerns the interim use of a pencil instead of an ink feather. This is the more pitiable because the constant rubbing-off of the lead particles from the paper (caused by the mutual friction between the folio pages over the centuries) resulted in the loss of legibility of most of the folios of Ms 398. Nowadays, modern technology (e.g., ultraviolet, infrared, or X-ray spectroscopy or photometry) might allow us to uncover and analyze the information hidden in these folios. Unfortunately, there are—as far as I know—no plans for such a project in this direction, which would help to reconstruct an essential part of Euler's intellectual biography.

3 Reconstruction of the development of Euler's first approaches to lunar theory

I analyzed these unpublished manuscripts and notebooks written by Euler between 1725 and 1730 containing his earliest tentative approaches to lunar theory. There is evidence that Euler learned the basic empirical facts on the Moon's motion from Ismael Boulliau's *Astronomia philolaica* of 1645 and from the books by David Gregory and Charles Leadbetter. He was most probably motivated to engage with lunar theory by Newton's statements on the motion of the lunar apsides in the *Principia*. Euler formulated his very first thoughts on the motion of the Moon in terms of central forces, osculating or curvature radius, Huygens' centrifugal theorem, and multiple force centers (represented by the Earth and the Sun) acting simultaneously on the Moon. His first approaches were dominated by the problem of central force motion, which at that time (about 1710) took an important role in the proof of the inverse problem of central forces by Johann I Bernoulli and Jacob Hermann. Pierre Varignon published a series of papers on that topic using the Leibnizian calculus between 1703 and 1712. When Euler was transforming Newton's *Principia* into analytical language resulting in what we now call rational mechanics, he was following a tradition of mechanical and mathematical methods prepared by Bernoulli, Hermann, and Varignon (cf. Guicciardini 1999, Chap. 8). We may conclude that Euler's first steps emerged from this context and thus may be judged as quite unspectacular with respect to innovative ideas. Euler's grappling with multiple force centers and associated osculating radii led him to the concept of the osculating ellipse. This conflict actually concealed a fundamental problem, namely the choice of an appropriate (origin of) reference frame, that Euler disentangled some time later in his "Mechanica" by the discovery of what he called "genuina methodus" (genuine method), but what I call the principle of the transference of forces. This principle turned out to be the most important step towards a powerful lunar theory which Euler took between 1730 and 1744. I will explain these statements now in more detail using summaries of the relevant documents.

The content of each document addressed in this part is presented and analyzed in detail in the Appendix. Disregarding the mathematical aspects, here I will summarize and discuss the document's contents according to their main topics, stressing their outlines, the principal goals and ideas focussed by Euler as well as the strategies, methods, and arguments used by him. In addition, I will explain the astronomical and physical concepts and principles that play a role in these documents.

3.1 Summary of Ms 272

In Ms 272 (cf. Appendix A) Euler tried to get an overview of the principal *inequalities* which are observed in the Moon's motion. In celestial mechanics the term "inequality" means something which has to be added to (or subtracted from) the mathematical or geometrical representation describing the circular motion of a celestial body under consideration in order to make it accord with the observation of this motion. Thus, the concept "equality" refers to a uniform circular motion, so that everything that disturbs it (which is called a perturbation) transforms this "equality" (in the mathematical sense) into an "inequality". In general, the inequalities or deviations of the real motion from the simple circular model occur "along track" (i.e., along the body's trajectory) or "across track" (i.e., out of the body's orbital plane). The former inequalities are called *longitudinal*, the latter *latitudinal* with respect to a reference frame defined by the body's mean orbit. Due to this definition, longitudinal inequalities may even be caused by non-uniform circular motion or by non-circular (e.g., elliptical) orbits varying distance (or "altitude" as Euler calls it) continually from the central body and thus changing the velocity in the orbit due to Kepler's laws. This is why elliptic (or Keplerian) orbits cause the largest longitudinal inequalities and are therefore called *first inequalities*. Euler discusses other such inequalities in longitude, which today are called *evection* and *variation*. He describes also the inequalities in latitude defined as (periodic) deviations of the body's trajectory from a great circle defined by the mean orbit considered as reference plane. The excentricity of the body's orbit caused by its ellipticity or by any other non-circular form of the orbit induces a variation of the body's distance or altitude from the central body and thus (due to the law of gravitation) a variation of the orbital velocity. Euler describes the effect of this kind of inequality as well. He then turns to the contributions by Newton who solved the direct and inverse problem thus connecting Kepler's laws with the inverse square law of gravitation. Euler points out that there remained, however, a serious problem unsolved by Newton. It concerns the fact that, in general, orbits are not fixed in space but rotate around the central body situated in one focus, which Euler—following Newton—called “mobile” orbits. In the case of the Moon’s elliptic orbit this kind of mobility results in a continuous motion of its apsidal line as well as of its nodes (defined by the intersection between the Moon’s and the Earth’s orbital planes) with respect to inertial space or (in Euler’s words) with respect to the zodiac defined by the sequence of its signs. There was a considerable discrepancy (up to a factor 2) in the lunar apsidal motion between theory and observation, which had already been recognized by Newton, who left this problem unsolved. On one hand, this fact was probably Euler’s motive to get into lunar theory in the first place, on the other hand he ascribed this difficulty to the unsolved problem of treating mathematically multiple force centers (e.g., Sun and Earth) acting simultaneously on another body (e.g., the Moon), one of these centers being in motion as well. To solve this problem by using the differential calculus was Euler’s motive and starting point for his early approaches.

3.2 Summary of Ms 180

The manuscript Ms 180 (cf. Appendix B) may be considered a first trial to cope with the problem of multiple central forces. The most important parameter which Euler tries to determine is the velocity (or speed) of the body at any place of its orbit around the central body (or bodies). In the eighteenth century, it was not possible to measure the velocity of, e.g., a body falling onto the Earth. This is why the final speed of a freely falling body on the Earth’s surface was substituted by its height of fall. Therefore, the speed or absolute value of velocity of any body on Earth or even of a celestial object was expressed by its corresponding altitude of free fall on Earth (neglecting air resistance and fictitious or pseudo forces). Thus, the concepts “speed” and its corresponding “altitude” were used synonymously by Euler. The general approach to determine this speed is—at first glance—quite simple. Equating the gravitational (or centripetal) force acting on the body at any point of its orbit by the central body with the centrifugal force defined by the osculating radius and the velocity at this point as given by Huygens’ centrifugal theorem. This equation may easily be solved for the body’s velocity, which may be used to determine the tangential and normal force components acting on the body at the given point. The centrifugal force is then defined by the normal component, which depends on the osculating radius at this given point. Given these parameters, Euler is able to derive de Moivre’s theorem in a straightforward manner. This theorem is a generalization of Huygens’ theorem (which is applicable only for circular motions) and allows the determination of the centripetal force for non-circular motions in terms of the body’s velocity at a given point of its trajectory, of the osculating radius at that point, of the distance between that point and the force center, and of the perpendicular from the force center to the tangent passing through that given point (i.e., the straight line SA in Fig. 2). However, there remains the painful difficulty that the osculating radius has to be determined for every point of the trajectory. If the orbit is defined by a closed curve, e.g. by an ellipsis, this problem might be solvable. In the central force problem, where just one force center is acting on the body, there is no problem. But what happens if there are two force centers acting simultaneously on the body, one of them even moving with respect to the other, and therefore the resulting orbit of the body will not be a closed curve? Euler postponed this problem for further investigation (see Ms 397 and Ms 398) and simplified the static problem by considering both force centers as fixed in inertial space. In this case, the tangential and normal force components produced by each of the force centers can be superimposed, i.e., added together, to one resulting force (consisting of two terms) of each kind. The resulting velocity at a given point of the body's curve due to the double component normal force is then also composed of two terms. The integration of the corresponding first order integro-differential equation is beyond Euler's capabilities at that time. This is probably why he did not continue his manuscript and left it unfinished.

3.3 Summary of the records in Ms 397 and Ms 398

The analysis of Euler's notebooks Ms 397 and Ms 398 is quite more difficult than of his manuscripts, because (1) the individual entries are distributed over almost the whole books and therefore, in a first step, have to be recognized and identified as relevant for lunar theory, (2) they often are only sparsely commented by Euler, and the meaning of the mathematical symbols used by him is not always explained, (3) the entries have to be related to a special topic of lunar theory and put together according to some common relevant topics. The last task is not manageable without an a priori hypothesis about the contents of each entry. The result is thus inevitably affected by a certain degree of subjectivity.

There are four topics transpiring through the tangle of entries in Ms 397 (cf. Appendix C), which are related to lunar theory. In a first one (cf. Appendix C1), Euler determines the Moon's motion in a spatially fixed orbit which he assumes to be coplanar with the ecliptic, i.e. the Earth's orbital plane. "Motion" of the Moon means for Euler its velocity defined at any given point in its orbit. Therefore, the main goal is the determination of this velocity in a most general way. For this purpose, he derives equations for the resulting normal and tangential forces acting on the Moon by the Earth and the Sun. In this sense he actually treats what is called today a restricted three-body-problem. The overall strategy is nevertheless the same as already described above: having found the resulting normal component acting on the Moon, the centrifugal force is determined using Huygens' theorem, which then is equated to the resulting centripetal force defined by the gravitational action of Sun and Earth. Most striking is the fact that Euler did not apply any form of what we now call equations of motion. Instead, Euler's derivations are dominated by his attempt to find useful geometric relations to express in a simple way the different distance and force ratios which occur in his calculations. Even more striking is the fact that he did not introduce time as an independent parameter. The final differential equations for the resulting normal and tangential force components become functions of the masses of Earth and Sun, the distances between them and the Moon, and its first and second derivatives, i.e. the line elements. Euler is able to solve them only for the special case when the lunar apogee coincides with its opposition to the Sun. This solution is still far from what we would call a successful description of the Moon's motion. As we will see in the third part of this series of papers, dealing with the epoch from 1744 to 1752, the use of general equations of motion and its parametrization for the time argument, which would led to such success, will be Euler's fundamental breakthrough in that period.

Some other entries may be identified as related to a second topic (cf. Appendix C2), where Euler tries to find differential equations to describe the motion of the lunar apsides. Here, we find him dealing with mobile orbits, i.e. elliptic orbits rotating around one of their foci, and thus taking up the famous problem Newton left unresolved. He is searching for relations between two forces, one of them acting in the immobile orbit, causing the Moon to move in its orbit, and the other acting in the mobile orbit, causing the Moon's elliptic orbit to move around the center of force being in one focus. Having found this ratio, he supposes these forces as being given and the motion of the orbit as to be determined in terms of the body's velocity in its orbit and the angular velocity of the rotating orbit. Using this result he derives a second order differential equation for the line element of the body's trajectory and its components, but leaves this equation unsolved.

Quite a lot of Euler's notebook entries concern the problem of general central force motion, which may be applied not only to the Moon's motion but to the motions of the planets as well. This third topic (cf. Appendix C3) is related to some chapters of his draft version of the treatise on mechanics, which he began to compose in this notebook and which were published in 1965 (cf. Mikhailov 1965, Chap. I, pp. 38–41, Chap. II, pp. 41–47, Chap. XII, pp. 61–62). While these chapters do not specifically concern the central force motion of celestial bodies, the records contained in the remaining folios do so. There Euler derives the velocity of the Keplerian motion, i.e., of the two-body-problem, together with the central force causing this motion. Using an equation defining the perpendicular distance between the force center and the tangent passing through the body's current place in its orbit (i.e., the straight line SA in Fig. 2), he determines the "orbital parameters" algebraically and provides formulae for them which are adapted for applying them to the special case of the Moon's motion around the Earth and which are simplified by the insertion of numerical values thus making them usable for the easy computation of, e.g., lunar tables. The theorems that Euler derives in connection with this result depend on algebraic relations which follow from the geometry of the ellipsis, which he found "earlier" in this notebook. In one of these theorems he claims that if the centripetal force is increasing, then the orbit's major axis is decreasing, and vice versa. The proof is based both on the equation for the above-mentioned perpendicular distance (SA) and on the equation defining the "latus rectum", i.e. the parameter of the conic section. In Ms 398 (cf. Appendix D), Euler develops his considerations on the general central force motion further and amends his notes in Ms 397 on that topic. Again (cf. Appendix D1), he focuses on the determination of the "orbital parameters" and on the derivation of differential equations which relate them to each other. As fundamental orbital parameters Euler considers the distance between the force center and the body, whose motion has to be determined, the instantaneous radius osculating the curve at a given point where this body is located, the body's velocity at that point, and the perpendicular line SA as defined in Fig. 2, disregarding any inclination of the body's orbit thus using it as a reference plane. He first derived Keill's theorem, which actually represents de Moivre's theorem in differential form. He substitutes (at this point for the first time) the line element of the body's orbit by the time element and obtains, using de Moivre's theorem, a differential equation for this time element, which depends on the above-defined orbital parameters and on the arc length covered by the body in its orbit during this time element. Euler immediately re-substitutes it by the corresponding line element of the curve to obtain equations for the parameters SA and the osculating radius as a function only of given distances and their first and second derivatives. At that point we catch Euler keeping the time parameter out of his equations, not yet recognizing that this would be an important step forward (as we will see in part 3). At that time, he seemed still be biased by the idea that the body's trajectory must also be determined only by distance elements, i.e., by the central force induced geometry of the orbit. Finally, he derives this result again with a slightly different kinematic approach.

There are also "standing alone" entries in both of Euler's notebooks dealing with the relative motion of two bodies, with observational data of the Moon's motion, and with the general three-body problem. The first of these topics (cf. Appendix D2) is very interesting with respect to its innovative momentum. It pertains to one of the crucial ideas—I speculate—that will lead Euler to the principle of the transference of forces (see part 2). This is suggested by the question Euler formulated to describe the relative motion of two bodies *as seen from one of them considered to be at rest*. Some time later, Euler will transform this idea into a rule which defines how to apply the forces acting mutually between three bodies in such a way that the motion of one of them can be described *as seen from another of them considered to be at rest*. The records specifying orbital data of the Moon's motion (cf. Appendix D3) turned out to be important for the identification of one of Euler's sources from where he learned the basics of lunar theory.

Another miscellaneous topic in Ms 397 is Euler's very first approach to the general three-body-problem representing the theoretical background and basis of the lunar problem (cf. Appendix C4). The result is sobering or even disappointing. His "success" does not go beyond some geometric relations between the positions of the three bodies. The way of solving this problem reflects Euler's misinterpretation of its complexity and difficulty, of which he will become fully aware only in the 1740s. Euler's unpublished manuscript Ms 251, which was published only in 1992 (cf. Knobloch 1992), gives evidence that Euler completely undervalued this problem at that time (around 1730).

The last one of the topics treated by him in his notebooks brings the corresponding entries of both notebooks together. These entries may be interpreted as Euler's first approach to cope with the problem of two force centers, one of them is moving around the other one, and the resulting trajectory of a third body being subject of the central forces acted by the others has to be determined (cf. Appendix E). His approach was a failure. Nevertheless, it led him probably to the concept of the so-called *osculating ellipse*. This is an ellipse which instantaneously fits best the body's trajectory at a given point. So the ellipse is changing its parameters, which characterize its form and position, continuously with time. Unfortunately, Euler left us only fragmentary notes on his ideas to solve that problem. From the four sketches he drew to illustrate what is going on we may, however, assess the difficulty with which Euler was confronted. Two of these figures, representing the Sun–Earth–Moon system, are closely related to each other. One figure shows the situation where the centers of the osculating radii coincide with the two force centers of the Sun (S) and Earth (T), implying that Euler assumes here the Moon (L) describing a circular orbit around the Earth. In the other figure he drew the situation where the osculating radius associated with the centripetal force acting on the Moon by the Earth is larger than the instantaneous distance between these two bodies, implying that Euler assumes here the Moon moving in a non-circular orbit. The difficulty arising from this latter situation concerns the problem that it seems that it was not clear for Euler how to determine the resulting osculating radius assumed as a superposition of the two radii associated with each force centers. Euler tries to find a relation between these two osculating radii resulting in an equation that defines the position and orientation of the osculation center associated with the Earth with respect to the Sun being considered as fixed in inertial space. The result is quite complicated and did not allow Euler to deduce any conclusions, which could have been useful for further investigations in that direction.

4 Final assessment of Euler's first approaches to lunar theory

Let me infer from the manuscripts and notebook records discussed here for the period from about 1725 until about 1730 the crucial points which were important in the development of Euler's first approaches to lunar theory during this period of time. These documents reveal three issues which have to be addressed: 1. The sources, from which Euler learned about lunar theory, 2. the theoretical framework which was available to Euler at that time, and 3. the main deficiencies in Euler's first approaches to tackle the lunar problem successfully.

4.1 Euler's sources to get into lunar theory

From Euler's studies of Newton's *Principia* Newton (1687, 1726) as reflected in his entries on mechanics written in his first "Basel notebook" (Ms 397) and his "travel diary" (Ms 398) respectively, we may assume that he was acquainted with Newton's lunar theory. In addition, from Euler's "Catalogus Librorum meorum" (Euler's own catalogue of his library, written between 1747 and 1749) of his sixth notebook (cf. Ms 402, fol. 192r–201 v) we know that he possessed Gregory's *Astronomiæ* (cf. Ms 402, entry No. 453, fol. 200v; Gregory 1702, 1726, which contains Newton's lunar theory), from which he adopted at least numerical values of orbital parameters and from which he most probably learned the state-of-the-art on lunar theory, as well as Leadbetter's *Astronomy* and *Astronomy of the Satellites* (cf. Ms 402, entries no. 95–97, fol. 194r; Leadbetter 1728, 1729). But there is strong evidence, that Euler studied the basic phenomena related to lunar theory from a treatise of the seventeenth century which became a standard textbook at the time, namely Book III (De lunæ motibus) of Boulliau's *Astronomia philolaica* (cf. Boulliau 1645). I compared astronomical concepts, statements, and numerical values from Ms 272 and fol. 38v of Ms 398 with Boulliau (1645), Lib. III, and found them in very good coincidence with each other, as the following examples illustrate. In Ms 272, fol. 1r (Sect. 2), Euler wrote:

> Dum linea synodi a 270 g. anomaliæ in 90. properat, tantisper secundæ inæqualitates, a conjunctione ad oppositionem, ablativæ sunt, ab oppositione ad conjunctionem adjectivæ.

On page 102 of Boulliau (1645), Lib. III, Boulliau wrote:

> Dum autem percurrit linea Synodi à 170.g. anomaliæ in 90. tantisper secundæ inæqualitates in primo semicirculo Synodicæ revolutionis, hoc est à coniunctione ad oppositionem, ablativæ sunt; in secundo adiectivæ.

The digit "1" of the number "170" is a misprint and should be "2". Euler's statement is distinctively an abbreviated version of Boulliau's. The concept "linea synodi" means the line conjoining the syzygies. Newton or Gregory only used the term "syzygia". In Ms 272, fol. 1r (Sect. 2), Euler wrote:

> Cum enim linea synodi cum linea absidum congruit, maxima tum observatur hæc secunda inæqualitas per totam Lunationem. Si autem cadit in lineam mediarum longitudinum, nulla fere est inæqualitas secunda.

On page 101 of Boulliau (1645), Lib. III, Boulliau wrote:

> Cum etenim linea Synodi Luminarium convenit cum linea absidum, maxima tunc contingit inæqualitas synodica per totam lunationem. Si verò ista Synodi linea cadat in medias longitudines, nulla fere tunc est per totam Lunationem inæqualitas secunda.

Aside from the rearrangement of the words the similarity between Euler's summarized and Boulliau's full statement is striking. In Ms 398, fol. 38v, Euler noted three numbers:

> Luna Zodiacum absolvit tempore
> 27 d: 7 H. 43′.
> Rursus vero ad Aphelium fertur
> 27. d. 13. H. 12′[.]
> Mensis synodicus. 29. d. 12H 44′[.]

Exactly the same numbers appear even in the same order on page 101 of Boulliau's book. Moreover, on the same folio Euler continued with:

> Linea axem majorem in centros
> normaliter secans dicetur diacentros[,]
> in foco - - - - diagæos[.]
> Luna ab Apogæo singulis diebus movetur
> circa terram angulo 13° 3′ 54″ .

The meaning of the terms "diacentros" and "diagæos" are explained only by Boulliau on page 103 of his book using a figure. And on page 157, Boulliau wrote:

> Luna verò ab Apogæo *E* movetur circa terram singulis diebus g.13.′3.″54.

The agreement between Boulliau's and Euler's statements prove not only the latter's dependence on the former but also the close relationship between Ms 272 and the notes on folio 38v in Ms 398. This implies that Ms 272 most likely was composed in 1727, which confirms the perfect agreement with the estimation made by Kopelevič et al. (1962).

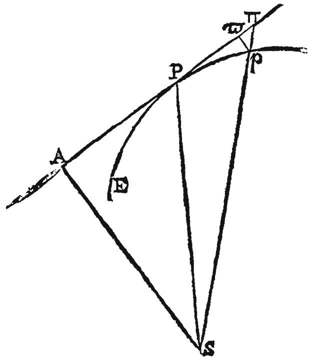

Fig. 2 The figure in Bernoulli's paper of 1712 (cf. Bernoulli 1712) used to determine the centripetal force acting on the body P by the force center S, thus driving P into the motion along the trajectory EPp around S, the straight line $AP\pi$ being the tangent passing through P

Concerning the theory of central forces, we know from Ms 180, fol. 2r, and Prop. 74, Coroll. 4, of the "Mechanica" (cf. Euler 1736, Sect. 592) that Euler must have been acquainted with the paper of Abraham de Moivre (1667–1754) published 1717 in the Philosophical Transactions (cf. Moivre de 1717; Guicciardini 1995), wherein (p. 624) it is referred to the corresponding publications of Johann I Bernoulli (cf. Bernoulli 1712; Guicciardini 1995), John Keill (1671–1721) (cf. Keill 1708; Guicciardini 1995), and Jacob Hermann (1678–1733) (cf. Hermann 1716). When Euler started his position in the St. Petersburg Academy in June 1727 (cf. Fellmann 2007, p. 29), the most important Academy journals must have been available to him. Therefore, I assume, that he most probably knew the papers of Pierre Varignon (1654–1722) and others (cf. Bomie 1708) on central forces and on the determination of osculating radii as well (cf. Varignon 1703a,b,c, 1704, 1705, 1707a,b, 1712). All these publications led Euler to the frontier of scientific research of his time and provided him the then up to date state-of-the-art tools useful for treating the lunar problem.

Finally, let me speculate even further about Euler's motive to put the focus of his research on the use of central forces to describe the Moon's motion. Eric J. Aiton showed (cf. Aiton 1989) that Johann I Bernoulli and Jacob Hermann successfully proved the inverse problem of central forces in 1710, which was published in the Paris Memoires of 1712 (cf. Hermann 1712; Bernoulli 1712; Guicciardini 1995). This was unquestionably an awesome demonstration of the power inherent in this method. The researches of Varignon were straightened to use multiple force centers for the determination of planetary orbits (cf. Varignon 1705; Aiton 1989, p. 52). Why could this approach not be successfully also for solving the lunar problem? If we inspect the Bernoulli paper of 1712, we find some analogies in Euler's first approaches in Ms 180 and Ms 397, in particular concerning the method used to determine the centripetal force and the derivation of de Moivre's theorem (cf. Bernoulli 1712, pp. 529–532, and Fig. 2). Bernoulli and Varignon were without doubt the leading scientists with respect to the development of the theory of central force motions in the years between 1700 and 1720 (cf. Costabel et al. 1992).

4.2 Euler's theoretical equipment to tackle the lunar problem

The sequence of Euler's records in Ms 397 reveals, that already in Basel he was deeply involved in transforming Newton's *Principia* into the analytic notation of rational mechanics. In this process, he must have been confronted with Section IX "De Motu Corporum in Orbibus mobilibus, deque motu Apsidum" (The motion of bodies in mobile orbits, and the motion of the apsides, cf. Cohen 1999, pp. 534–545), of book I in Newton's *Principia*. Therein he would have read Newton's famous statement, that "the [advance of the] apsis of the moon is about twice as swift" (cf. Cohen 1999, p. 545) thus admitting the discrepancy by a factor of 2 between theory and observation. According to Euler's very first record in Ms 397 concerning the Moon's motion, this fact—I surmise—might probably have been Euler's motivation to begin a serious engagement with lunar theory, which lasted all his life. His first approach, however, was considerably directed by the theory of central force motion as reflected in Chapter V of his "Mechanica". At that time, the central key for Euler needed to describe the Moon's motion was the determination of the normal (centripetal) and tangential forces moving both the Moon in its orbit and the orbit around the center of force resulting in the rotating motion of the apsidal line. This is why he also studied extensively how to determine the osculating radius (cf., e.g., Ms 397, fol. 0 [i.e., inside of front cover], fol. 55v, 98v, 120v, and 206v).

Euler formulated his first approaches to celestial mechanics in terms of central force motion combined with Huygens' theorem, applied to multiple acting force centers, which cause mobile orbits changing their form and position at any time due to varying radii of osculation. This strategy is explicable in his statements in the "Mechanica" (cf. Euler 1736, Prop. 83, 84, and 89):

> § 694. With regard to the figures that bodies describe under the action of given forces, it is not worth the effort to add more here, as in Physics and Astronomy, the hypotheses of centripetal forces other than those in proportion to the inverse square of the distances have no use. Yet in Astronomy, when a body must be considered to be acted on by several forces of this kind, of which one exercises a maximum influence on the body over the others, these extra forces, as the problem demands, do not have to be introduced into the calculation, as they only augment or diminish a little, that by which even the approximate motion of the body is known. Therefore in these cases the curve the body describes does not disagree much with an ellipse. For this reason, Astronomers usually consider the curve to be in the form of an ellipse, however one which is not fixed, but mobile, so that thus they consider the body moving in an ellipse which in turn is rotating around the focus. Hence the mobility of the orbits of the planets arise, where the lines of the apses continually move to another place. We will, when proceeding closer to the truth, besides the mobility of the axis, also consider the form of the ellipse as a variable. Therefore we will proceed in such a way, that – with respect to any element of the curve the body describes – we can determine the ellipse (having the focus at the center of force) from which this element is part of. Hence the position and the nature of the ellipse can become known. Moreover all these ellipses in turn have one of their foci in the center of forces, because the body is continually attracted towards it.

> § 701. The theory of osculating ellipses is not to be confused with the motion of bodies in moving orbits, concerning which Newton and others after him have worked on. For here, however, we determine the very ellipse each element of the

> curve described by a body is part of. But, when the talk is about moving orbits, the very centripetal force is being investigated causing the body to move in a given curve which is rotating around the center of forces.
>
> § 740. The curves described by bodies acted on by centripetal forces of this kind are hardly to be recognized in a different way, and it is not at all possible to determine their form without applying the considerations made above. Therefore investigations of centripetal forces of this kind have the maximum use for curves generated by some given conditions, and from which in turn, from the given centripetal forces, the curves themselves together with their properties can be derived. For the motions of heavenly bodies there occur so complex expressions of the forces acting on them, that none of their orbits could be determined at all, if these forces were – by chance – not in such a state, for which the centripetal force could has been found a posteriori.

This may probably be the reason for the remarkable delay of the use and application of the equations of motion as given, e.g. by Jacob Hermann (cf. Hermann 1716, p. 57), a textbook Euler studied intensively as well. Euler seemed to be caught by the method of central forces as investigated by, e.g., his teacher Johann I Bernoulli, Jacob Hermann, and Pierre Varignon, which thus prevented him from recognizing the power of the equations of motion. This will be the essential step which Euler only took in the 1740s.

4.3 Euler's deficiencies to cope with the lunar motion

Euler's own words as cited above make clear that his first approaches to lunar theory were dominated by the problem of central force motion: equating the centrifugal forces (given by the osculating radius at any point of the body's trajectory and Huygens' theorem) with the gravity acting at this point by the central body yields the motion of the body. But what happens if there are two force centers (e.g., the Sun and the Earth) acting simultaneously on a third one (e.g., the Moon), one of them (e.g., the Earth) even in motion as well (e.g., moving around the Sun)? How then is the resulting central force (or its radial and tangential components) to be determined, or, which is the relevant osculating radius in this case? If this radius is changing for every element of the curve which the body describes, how can its resulting trajectory be found? These problems led Euler to the conflict between a mobile ellipse rotating around the common force center and a resulting ellipse changing its parameters (form and position) continuously. The presented documents reveal that Euler obviously struggled with this kind of problems.

Actually, this problem is—as we know today—closely intertwined with the choice of an appropriate reference frame. Euler did not know yet that this could be an important issue. Only a few years later will he stumble on the crucial discovery which I call the *principle of the transference of forces*. If the Moon's motion is to be described in an earth-fixed reference frame, all the forces acting on it by the Sun and Moon have to be transferred in reverse direction to these bodies thus leaving the Earth at rest. He formulated this principle (Euler called it *genuine method*, "genuina methodus", cf. Euler 1736, Prop. 96, Schol. 2) in his "Mechanica" Euler 1736, Prop. 97), which was completed by 1734. I will address this important development in part 2 of this series of papers, covering the years about 1730–1744.

5 Conclusions

Documents written by Euler between 1725 and 1730, presented and analyzed in this paper, reveal that he struggled considerably with different approaches to find a method or strategy by means of which a powerful lunar theory could be constructed. Most of his early approaches were not successful. Neither did he already use time parameterized equations of motion nor did he realize the importance of the choice of an appropriate reference frame in that early phase of the development of his researches. The theory of central forces, as formulated by Euler in terms of Leibnizian calculus, the use of Huygens' centrifugal principle, of de Moivre's and Keill's theorems, as well as the determination of the osculating radius, dominated his ideas due to his simultaneous transformation of Newton's *Principia* into the analytic language of rational mechanics. Euler was embedded in a tradition of mechanical and mathematical methods, which were prepared by his teacher Bernoulli, Hermann, and Varignon in such a way that his first steps emerging from this context may be judged as quite unspectacular with respect to innovative ideas. These approaches may be judged as inadequate to solve the problem of multiple central forces, on which Euler based his first theoretical approach to cope with the lunar problem. Nevertheless, Euler's first trials and ideas contained the nuclei which some time later led him to the concept of the osculating ellipse and to the principle of the transference of forces. This will be the subject of my next paper (part 2).

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Appendix A: The content of Ms 272

In the short "Treatise on the Moon's motion", Euler describes the principal inequalities that may be observed in the motion of the Moon and its possible causes. The treatise reflects his studies of the relevant textbooks and his understanding of the problem.[1]

In the first paragraph, Euler mentions three inequalities in the Moon's motion, namely in longitude, in latitude, and in altitude (i.e., distance from the Earth). Euler

[1] From a systematic point of view these inequalities in the Moon's motion are best explained, theoretically and quantitatively, in Moulton (1914), Chapter IX, in Beutler (2005a), Chapter 6, and in Beutler (2005b), Chapter 2.2.1, respectively.

describes the first of them (concerning, as it does, the Moon's sidereal and synodic motions) which is confirmed by observations. The Moon moves neither uniformly along the Zodiac nor does it cover equal arcs in the ecliptic in equal time intervals. There are three such longitudinal inequalities, of which the first and second ones were known since Hipparchus (c. 190 BC–c. 120 BC), while the third one was noted first by Tycho de Brahe (1546–1601). The first of these longitudinal inequalities affects the Moon's periodic motion around the Earth and causes its non-uniform passing through the Zodiac. It depends on the Moon's distance from the Earth: the Moon's motion is growing faster when it becomes closer to the Earth, and it is slowing down when it goes further away. This inequality is periodically reset in the time interval starting from one apogee to the next, when the Moon is returning back to the same place. We now know that this so-called *first inequality* is caused by the equation of the center, (i.e., by the ellipticity of the Moon's orbit) having a period of 27.5581 days and an amplitude of 6.2815° (cf. Beutler 2005b, Table 2.3, p. 31).

In the second paragraph, Euler explains the second longitudinal inequality, which was named *evection* by Ismael Boulliau (1605–1694) (cf. Boulliau 1645, Lib. III; Lynn 1908). It refers to the Moon's synodic motion and is periodically reset within a synodic month, namely from one conjunction of Moon and Sun to the next one. Characteristically, it sometimes vanishes and sometimes it is perceptible. If the synodic line coincides with the apsidal line the maximum value of this second inequality is observed during the whole lunation. But if it is placed in medium longitudes the second inequality nearly vanishes. When the synodic line is changing its anomaly from 270° to 90° the second inequalities are decreasing from conjunction to opposition, but increasing from opposition to conjunction. The contrary holds if the synodic line proceeds from 90° to 270°. We can complete Euler's explanation by adding the evection's period of 31.8230 days and amplitude of 1.2759° or $1°16'33''$ (cf. Beutler 2005b, Table 2.3, p. 31).

In the third paragraph, Euler describes the third inequality in longitude which was called *variation* by Tycho. It depends only on the Moon's synodic motion and affects the motion of the Moon in such a way, that from conjunction to the first quadrature it is accelerated and from the quadrature to the opposition it is retarded. Then again from opposition to the second quadrature it is accelerated and from the second quadrature to the conjunction it is retarded. This is why this inequality is added to the Moon's motion when proceeding from the syzygies to the quadratures, but subtracted when the Moon moves from the quadratures to the syzygies. Again, we can add to Euler's statements that the variation has a period of 14.7670 days and an amplitude of 0.6638° or $39'50''$ (cf. Beutler 2005b, Table 2.3, p. 31).

Euler devoted the forth paragraph to the inequality in latitude, caused by the Moon's orbital plane which deviates from the ecliptic plane (by about 5°). Considering them as great circles, astronomers often put them into one and the same plane. However, Tycho Brahe demonstrated by most accurate observations that the Moon's orbital plane does not take a constant angle with the ecliptic but it is inclined sometimes more and sometimes less with respect to this plane. In fact, the inclination of the Moon's orbit with respect to the ecliptic varies between about 5° and 5.3° (cf. Beutler 2005b, Figure 2.10, p. 27). According to Tycho's observations, Euler continues, it is certain that this inclination angle increases when the Moon is moving from the syzygies to the quadratures, but diminishes when proceeding from the quadratures to the syzygies. Furthermore, the intersections between the Moon's orbit with the ecliptic, the so-called nodes, are not always located at the same place in the sky, but they regress against the sequence of the signs. We may add, that the period of one revolution of the line of node is about 18.6 years.

In the next paragraph, Euler mentions the inequality which depends on the variable distance of the Moon from the Earth. The Moon's position is sometimes closer and sometimes further away from the Earth. And even the point of maximum distance, the apogee, is not fixed, but—as may be concluded from observations—moves continuously along the sequence of the signs (i.e., "in consequentia"). Even the excentricity of the Moon's orbit, conceived as difference between the maximum and minimum distance of the Moon from the Earth, is found to be subject to variation. It is largest when the apsidal line is coinciding with the syzygies, and smallest when the apsides are aligned with the quadratures. Indeed, the numeric excentricity of the lunar orbit might vary between 0.02 and 0.08 even on relatively short time scales (cf. Beutler 2005b, Figure 2.8, p. 26).

All these phenomena of the lunar motion, together with the motions of the other planets and comets, were—according to Euler—most elegantly explained by the most sagacious Newton, who derived them from laws of motion when most of the astronomers had tried without success to find its cause. He succeeded in this explanation by assuming this unique hypothesis that every body in the world is endowed with a force to attract the surrounding bodies in the inverse square ratio of their mutual distances. Actually this is a mere hypothesis, but because all phenomena might be derived most accurately from it with correct conclusions, there is no doubt that it is really true. This is why the attraction of the bodies in the world is—considered as phenomenon—generally accepted; however, its physical cause has not yet been discovered.

Newton has proved, Euler continues, that the planets describe ellipses around the Sun which is located in one of its foci. It was Kepler who observed this for the first time, and then also Boulliau who derived it from observations but did not determine the motions of the planets correctly. Those astronomers attributed an elliptic orbit to the Moon similarly as to the primary planets. But due to the Moon's inequalities, as described above, its orbit is by no means an ellipsis. Therefore, they claimed that the Moon actually moves in an ellipsis, but one which is mobile, perhaps even changing its shape because the transverse axis and the excentricity sometimes increase and sometimes decrease. This is why Newton determined with capacious labor the motion of the apsides, the motion of the nodes, and the variation in the lunar orbit. Although it would be possible to find out the correct lunar orbit, hardly any use would result from this due to the excessively great complexity and unhandiness of the equations. Neither the position of the apsides nor the position of the nodes nor the motion of the Moon itself might be determined from it, Euler points out. Therefore, Euler admitted, he too will use that simplifying supposition of an elliptic lunar orbit.

Finally, Euler closes his introduction with some statements relating to his own work. According to Euler, Kepler already attributed the cause for the inequalities in the Moon's motion to the Sun. But Newton first derived these inequalities with sufficient accuracy from the Sun's and Earth's double source of attraction. But because nobody knowledgable about the theory of centripetal forces was aware how difficult it is to determine the motion of a body driven by two force centers, one of them even being mobile, it is reasonable why Newton too determined the Moon's motion only approximatively. But because in many kinds of such propositions one justly might doubt whether the neglect of the least circumstance not will produce a big discrepancy in the solution, one should not rely too much on that matter. When Euler first dealt with that topic, he wrote, he found by calculation some things which seem to be useful for astronomy. Because he also found the magnitude of the Sun's attractive force to be much bigger than Newton did, major perturbances in the Moon's motion must consequently occur. This is why he will present his findings concerning this question in this treatise and will, as good as he can, determine the absolute perturbance of the Moon's motion from the known force of the Sun in order to be able to compare subsequently the theory with observations.

From these last statements by Euler, one might conclude that he must have been concerned with the problem of lunar theory already for some time and that he must have made some progress in his investigation. Unfortunately, neither this unfinished manuscript nor any other manuscript of that time is preserved containing calculations which would confirm his claims. However, two things become clear from this manuscript: 1. Newton's failure to describe the motion of the lunar apses became a serious motive for Euler to get into lunar theory, and 2. The determination of "the motion of a body driven by two force centers, one of them even being mobile" seemed to be *the* key for Euler to solve the lunar problem as described in the following documents.

Appendix B: The content of Ms 180

This small unfinished treatise represents Euler's first theoretical approach to cope with the Moon's motion. As its title "On the motion of bodies driven by multiple central forces" reveals, Euler intended to apply the general solution of this kind of problem to the special case of the description of the Moon's orbit, identifying the two centers of force by the Earth and the Sun. There are entries in Euler's notebooks Ms 397 and Ms 398 showing that he was developing the problem of multiple central forces further (cf. Sect. E). In a first Lemma, Euler states Huygens' centrifugal theorem (cf. Huygens 1673, 1703; Bomie 1708): Given a body (cf. Fig. 3) moving along an arbitrary curve AM and having in M the velocity corresponding to the altitude v, then the ratio between the normal force in M and the gravitational force [in M] corresponds to the ratio between the altitude v and the half of the osculating radius MO. In a corollary, he

Fig. 3 Euler's sketch explaining the first Lemma (Huygens' centrifugal theorem) in Ms 180

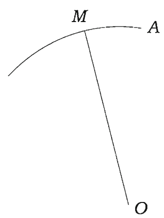
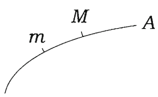


Fig. 4 Euler's sketch illustrating the second lemma in Ms 180



restates it analytically: Let $r = MO$ be the osculating radius, A and N the gravitational and normal forces, respectively, then

$$N = \frac{2Av}{r} \, . \tag{1}$$

Some years later (by 1734), Euler derived this theorem in Volume 1 of his "Mechanica" (cf. Euler 1736, Prop. 70). In the second Lemma, Euler claims: If a body (cf. Fig. 4) is moving along the curve AM then the ratio between the tangential force in M and the gravitational force (in M) is as the element or increment of the velocity's corresponding altitude, across the element Mm, to just this line element Mm itself. The demonstration is straightforward. Again, Euler restates this lemma mathematically in a separate corollary: Let $Mm = \mathrm{d}s$ and T be the tangential force, then

$$T = \frac{A \, \mathrm{d}v}{\mathrm{d}s} \, . \tag{2}$$

It may also be found in Euler's "Mechanica" (cf. Euler 1736, Prop. 70, Coroll. 3). Euler starts with the first problem: Determine the velocity of a body located anywhere in M [of its trajectory] being attracted to the center of force C. This problem and its solution together with the corollaries are treated by Euler in a similar way in his "Mechanica" (cf. Euler 1736, Prop. 74). Let AM be the body's trajectory, the body being in M, MT the tangent through M, CT the perpendicular from C to MT, $MC = y$, $CT = p$, and the osculating radius $MO = r$. Let the ratio between the central and the gravitational force in M be as an arbitrary function P to 1. If the central force $= AP$, Euler finds the normal force (being central) as $CT(p)$ to $CM(y)$. Using Huygens' centrifugal theorem, the normal force becomes

$$N = \frac{APp}{y} = \frac{2Av}{r} \tag{3}$$

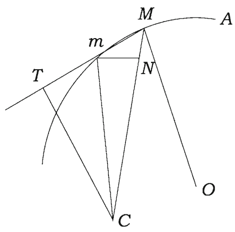
and thus

$$v = \frac{Ppr}{2y} \, . \tag{4}$$

Euler gives an alternative solution as well. Let $Mm = \mathrm{d}s$ be the line element of the body's trajectory. The tangential force may then be found by the ratio between $CM(y)$ and $MT(q)$ in such a way that the central force AP to the tangential force is equal to $\frac{APq}{y}$. Due to lemma II this is $= \frac{A\,\mathrm{d}v}{\mathrm{d}s}$, and therefore

$$\mathrm{d}v = \frac{Pq \, \mathrm{d}s}{y} \qquad \text{thus} \qquad v = \int \frac{Pq \, \mathrm{d}s}{y} \, . \tag{5}$$

Fig. 5 Euler's sketch to the first problem in Ms 180



Euler derives four corollaries from these findings. First, he substitutes $P = \frac{2yv}{pr}$ from Eq. (4) into Eq. (5) and obtains

$$\frac{\mathrm{d}v}{v} = \frac{2q\,\mathrm{d}s}{pr}\ . \tag{6}$$

From Figure 5 it follows that $CM = y$, $MN = -\mathrm{d}y$ and $r = \frac{y\,\mathrm{d}y}{\mathrm{d}p}$, thus

$$\frac{\mathrm{d}v}{v} = \frac{2q\,\mathrm{d}s\,\mathrm{d}p}{py\,\mathrm{d}y}\ . \tag{7}$$

The equivalence of the triangles MNm and MTC yields $MN(-\mathrm{d}y) : Mm(\mathrm{d}s) = MT(q) : MC(y)$. Therefore, $-y\,\mathrm{d}y = q\,\mathrm{d}s$ and thus

$$\frac{\mathrm{d}v}{v} = \frac{-2\,\mathrm{d}p}{p}\ . \tag{8}$$

Consequently, $\ell v = -2\ell p + \ell C$ or

$$v = \frac{C}{p^2}\ , \tag{9}$$

where ℓ denotes the logarithm to base 10. In a second corollary, Euler equates Eqs. (4) and (9), yielding de Moivre's theorem (cf. Moivre de 1717)

$$P = \frac{2Cy}{p^3 r}\ . \tag{10}$$

As a third corollary Euler substitutes $-y\,\mathrm{d}y = q\,\mathrm{d}s$ into the integral of Eq. (5), obtaining $v = -\int P\,\mathrm{d}y$. If the central force is proportional to any power of y, Euler concludes in a fourth corollary, then $P = y^n$ and thus

$$v = C - \frac{1}{n+1} y^{n+1}\ . \tag{11}$$

And this formulae allows easily to determine everything concerning the velocity [of the body] whatever hypothesis [of the force law] may be ("Et hinc facile eruuntur

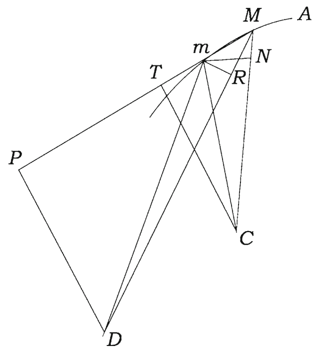


Fig. 6 Euler's sketch to the third problem in Ms 180

omnia quæ velocitates concernunt pro quacunque hypothesi"), Euler comments on this result in an euphoric style.

Then Euler formulates the second problem: Determine the curve [i.e., the trajectory] which a body describes around the center of force being attracted in a given way. His solution is straightforward: Due to the fact that P is a function given by $CM(y)$, de Moivre's theorem defines the nature of the curve, or, since $r = \frac{y\,\mathrm{d}y}{\mathrm{d}p}$, the trajectory is given by $P = \frac{2C\,\mathrm{d}p}{p^3\,\mathrm{d}y}$. Euler closes this topic with two corollaries of minor importance. He then states the third problem: Determine the velocity in each place [of the trajectory] of a body attracted in a given way by two centers of force C and D. To solve it, Euler draws (cf. Fig. 6) the straight lines MC, MD, and the perpendiculars CT, DP to the tangent MP through M. He sets $CM = y$, $DM = z$, $CT = p$, $DP = t$, $MT = q$, $MP = x$, and the osculation radius in $M = r$. He defines the ratio between the central force, with which M is attracted by C, and the gravitational force as P to 1, and the ratio between the [central] force, with which M is attracted by D, and the gravitational force as Q to 1. Therefore the central force by C is $= AP$ and by D is $= AQ$. Then the normal and tangential forces produced by C become $= \frac{APp}{y}$ and $= \frac{APq}{y}$, the ones produced by D become $= \frac{AQt}{z}$ and $= \frac{AQx}{z}$, respectively. Thus the resulting (total) normal and tangential forces become $= \frac{APp}{y} + \frac{AQt}{z}$ and $= \frac{APq}{y} + \frac{AQx}{z}$, respectively. Let v be the altitude that corresponds to the body's velocity in M, the normal force becomes (according to Lemma I) $= \frac{2Av}{r}$, resulting in $\frac{Pp}{y} + \frac{Qt}{z} = \frac{2v}{r}$. Consequently,

$$v = \frac{Ppr}{2y} + \frac{Qtr}{2z} \; . \tag{12}$$

But $r = \frac{y\,\mathrm{d}y}{\mathrm{d}p} = \frac{z\,\mathrm{d}z}{\mathrm{d}t}$, which gives

$$v = \frac{Pp\,\mathrm{d}y}{2\,\mathrm{d}p} + \frac{Qt\,\mathrm{d}z}{2\,\mathrm{d}t} \; . \tag{13}$$

There is, however, an alternative solution: Assuming M situated approximately in m, one has $Mm = \mathrm{d}s$, $MN = -\mathrm{d}y$, and $MR = -\mathrm{d}z$. The tangential force is $\frac{APq}{y} + \frac{AQx}{z}$.

Due to Lemma II this is $=\frac{A\,\mathrm{d}v}{\mathrm{d}s}$, resulting in $\mathrm{d}v=\frac{Pq\,\mathrm{d}s}{y}+\frac{Qx\,\mathrm{d}s}{z}$. But $q\,\mathrm{d}s=-y\,\mathrm{d}y$ and $x\,\mathrm{d}s=-z\,\mathrm{d}z$, thus $\mathrm{d}v=-P\,\mathrm{d}y-Q\,\mathrm{d}z$. Integrating this latter equation yields

$$v=C-\int P\,\mathrm{d}y-\int Q\,\mathrm{d}z\,, \tag{14}$$

where C denotes an integration constant. If $P=ay^m$ and $Q=bz^n$, as Euler supposes in a first of two corollaries, then $v=C-\int ay^m\,\mathrm{d}y-\int bz^n\,\mathrm{d}z$, thus

$$v=C-\frac{ay^{m+1}}{m+1}-\frac{bz^{n+1}}{n+1}\,. \tag{15}$$

Substituting $P=ay^m$ and $Q=bz^n$ into Eq. (13) yields as second corollary the result

$$v=\frac{ay^m p\,\mathrm{d}y}{2\,\mathrm{d}p}+\frac{bz^n t\,\mathrm{d}z}{2\,\mathrm{d}t}\,. \tag{16}$$

Finally, Euler formulates a fourth problem: Determine the curve [i.e., the trajectory] which the body describes due to the attraction of two centers [of force]. To solve this problem, Euler refers to the Eqs. (13) and (14) and equates them:

$$\frac{Pp\,\mathrm{d}y}{2\,\mathrm{d}p}+\frac{Qt\,\mathrm{d}z}{2\,\mathrm{d}t}=C-\int P\,\mathrm{d}y-\int Q\,\mathrm{d}z\,. \tag{17}$$

This is what we now call a first order integro-differential equation. There was no chance for Euler to solve it at that time. This may probably be the reason for the abrupt end of this manuscript.

Appendix C: The relevant records in Ms 397

Actually, Euler's first notebook Ms 397 is a draft version of a larger treatise on analytical mechanics, showing Euler deeply involved with the re-formulation and translation of the content of Newton's *Principia* into the analytical language based on Leibnizian calculus. The text is structured into Propositions, Scholia, Corollaries, etc. It was published in Mikhailov (1965), pp. 38–62. At the beginning only rarely, then more and more frequently, it is interrupted by other topics concerning astronomy, mathematics, and even music theory. These notes are, however, rather fragmentary, sometimes continuing and referring to each other, sometimes standing alone. The records concerning celestial mechanics and, in particular, lunar theory, clearly reflect Euler's approach to tackle the problem: there are trials, errors, and dead ends. Often Euler takes up one and the same problem for several times, restating it over and over. It is hard to recognize a central theme. Sometimes Euler is concerned with planetary motion and solar theory, sometimes with lunar theory or with the general three body problem. I will try to put together the relevant fragments essential for the formation and development of his early lunar theories. The records may be grouped together according to two main topics: the Moon's motion in its orbit (cf. Ms 397, fol. 124r–126r) and the motion of this orbit around the center of force resulting in the motion of the apsidal line (cf. Ms 397, fol. 25r, 25v, 34v, 125r–126v). Euler treated these two problems separately at that time. In addition, he developed the motion of the Earth around the Sun (so-called Solar theory), and even more generally, he already studied the motion of a body around the central body when its central force is moving, thus changing the resulting orbit (cf. Ms 397, fol. 121v–122v, 125v–126v). From the theoretical point of view, this topic is intimately connected with lunar theory because we learn from it important procedures and results Euler applied to lunar theory. This is why this issue is treated here as well. Finally, there are records in Ms 397 which are difficult to assign one of the topics mentioned above. A most prominent example is Euler's first trial with the general three-body-problem. I will present this topic separately at the end of this section.

C1: The relevant records of Ms 397 concerning the Moon's motion in its orbit

The notes on lunar theory start on folio 124r with Euler's statement of the central "Problema" (cf. Fig. 7):

> Sit T terra[,] L luna, S sol[;] requiritur motus lunæ.

"Let T be the Earth, L the Moon, and S the Sun. Find the motion of the Moon." This question is a special case of the general three-body-problem: given three celestial bodies, determine their motions due to the mutual gravitational forces. Applied to the Sun–Earth–Moon system this is the most general (and, we may add, by far the shortest) way to formulate this famous problem. The principle solution is similar to the one Euler used in the manuscript Ms 271, which I will present in part 2: Determination of the relevant force components acting on the Moon and equating the centripetal with the centrifugal force according to Huygens' formula. First, Euler considers the three bodies having masses T, L, and S, respectively, and assumed to be situated in one and the same plane. He sets $ST = a$, $TQ = x$, $TL = y$, $LS = z$, $TP = p$, and $LP = q$. The forces acting on the Earth and on the Moon by the Sun are $\frac{S}{a^2}$ and $\frac{S}{z^2}$, respectively. The components of the latter force along LT and LV are $\frac{Sy}{z^3}$ and $= \frac{Sa}{z^3}$, respectively. The difference between the forces acting on Earth and Moon by the Sun along the line LM (which is parallel to TS and LV) is $LM = \frac{Sa}{a^3} - \frac{Sa}{z^3}$. Now Euler gives without any derivation two formulae for the ratios $LN : LR$ and $LN : NR$, which he later will

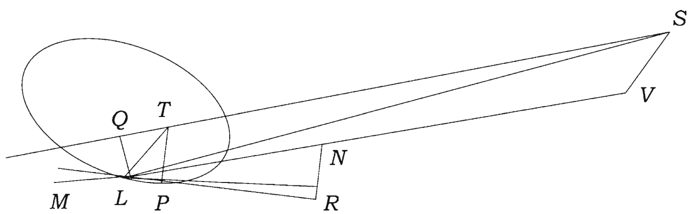


Fig. 7 Euler's sketch explaining the geometry in the System Earth (T), Moon (L), and Sun (S), showing the elliptic orbit of L with T in one focus and S lying in the same plane

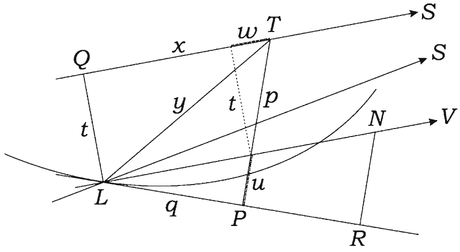


Fig. 8 Redrawn detail of Fig. 7

use for the determination of the tangential and normal force components. I reconstruct this missing derivation by using a redrawn detail of Fig. 7.

In addition to Euler's notations I introduce ad interim the straight lines u and w (cf. Fig. 8) and abbreviate $t = \sqrt{y^2 - x^2}$ as Euler will do so below. Then the following relations hold:

$$\frac{LN}{LR} = \frac{x - w}{q} \,, \qquad \frac{LN}{NR} = \frac{x - w}{u} \,. \tag{18}$$

Fig. 7 reveals two relations

$$u^2 + q^2 = (x - w)^2 \,, \qquad \frac{u}{q} = \frac{w}{t} \,, \tag{19}$$

which may easily be solved for u and w. This includes the solution of the quadratic equation

$$\left(t^2 - q^2\right) w^2 - 2t^2 x\, w + t^2\left(x^2 - q^2\right) = 0 \,, \tag{20}$$

yielding

$$w = \frac{t^2 x \pm q t \sqrt{t^2 + x^2 - q^2}}{t^2 - q^2} \,. \tag{21}$$

Regarding $t^2 - q^2 = p^2 - x^2$, $t^2 + x^2 = y^2$, $q^2 = y^2 - p^2$, and using only the solution with the positive sign, one finds Euler's results

$$\frac{LN}{LR} = \frac{qx - p\sqrt{yy - xx}}{xx - pp} \,, \qquad \frac{LN}{NR} = \frac{qx - p\sqrt{yy - xx}}{pq - x\sqrt{yy - xx}} \,. \tag{22}$$

Euler is now able to determine the force component which increases the tangential force or which reduces the normal force acting in L, defined by $LM \cdot \frac{LR}{LN}$ and $LM \cdot \frac{NR}{LN}$, yielding, respectively,

$$\frac{\overline{xx - pp} \cdot S \cdot \overline{z^3 - a^3}}{a^2 z^3 \cdot \left(qx - p\sqrt{yy - xx}\right)} \quad \text{and} \quad \frac{\overline{pq - x\sqrt{yy - xx}} \cdot S\overline{z^3 - a^3}}{a^2 z^3 \left(qx - p\sqrt{yy - xx}\right)} \,. \tag{23}$$

The tangential and normal forces are reduced or increased by the Sun's force $\frac{Sy}{z^3}$ acting on the Moon by it's components $\frac{Sq}{z^3}$ and $\frac{Sp}{z^3}$, respectively. Summing up, the resulting normal force becomes

$$= \frac{Sp}{z^3} + \frac{Sa\left(pq - x\sqrt{yy-xx}\right)}{z^3\left(qx - p\sqrt{yy-xx}\right)} - \frac{S\left(pq - x\sqrt{yy-xx}\right)}{a^2\left(qx - p\sqrt{yy-xx}\right)}\,, \tag{24}$$

and the resulting tangential force becomes

$$= \frac{Sq}{z^3} + \frac{Sa(xx-pp)}{z^3\left(qx - p\sqrt{yy-xx}\right)} - \frac{S\cdot\overline{xx-pp}}{a^2\left(qx - p\sqrt{yy-xx}\right)}\,. \tag{25}$$

Having derived the normal and tangential components acting on the Moon by the Sun, Euler now determines these components acting on the Moon by the Earth. The Moon is attracted by the Earth with the force $\frac{T}{y^2}$, which increases the normal component by $\frac{Tp}{y^3}$ and reduces the tangential component by $\frac{Tq}{y^3}$. Calling the Moon's velocity v, the curve element $\mathrm{d}s$, and the osculating or curvature radius r, the tangential and normal forces become $\frac{v\,\mathrm{d}v}{\mathrm{d}s}$ and $\frac{vv}{r}$, respectively. At this point, Euler applies the equation of motion (cf. Euler 1736, Prop. 20). Using the relations $\mathrm{d}s = \frac{y\,\mathrm{d}y}{q}$ and $r = \frac{y\,\mathrm{d}y}{\mathrm{d}p}$, the equations of motion for the resulting tangential and normal components thus become

$$\begin{aligned}
\frac{v\,\mathrm{d}v}{\mathrm{d}s} &= \frac{qv\,\mathrm{d}v}{y\,\mathrm{d}y}\\
&= \frac{S\cdot\overline{xx-pp}}{a^2\left(qx - p\sqrt{yy-xx}\right)} - \frac{Sa\cdot\overline{xx-qp}}{z^3\left(qx - p\sqrt{yy-xx}\right)} - \frac{Sq}{z^3} - \frac{Tq}{y^3} \qquad (26)\\
\frac{vv}{r} &= \frac{vv\,\mathrm{d}p}{y\,\mathrm{d}y}\\
&= \frac{Tp}{y^3} + \frac{Sp}{z^3} + \frac{Sa\left(pq - x\sqrt{yy-xx}\right)}{z^3\left(qx - p\sqrt{yy-xx}\right)} - \frac{S\cdot\left(pq - x\sqrt{yy-xx}\right)}{aa\left(qx - p\sqrt{yy-xx}\right)}\,. \qquad (27)
\end{aligned}$$

Again without derivation (which may be found in Euler's "Mechanica", Euler 1736, Sect. 601), Euler uses the relations

$$p = \frac{\overline{y\,\mathrm{d}x - x\,\mathrm{d}y}\cdot\sqrt{y}}{\sqrt{y\,\mathrm{d}y^2 + y\,\mathrm{d}x^2 - 2x\,\mathrm{d}x\,\mathrm{d}y}}\,, \qquad q = \frac{\mathrm{d}y\sqrt{y^3 - yxx}}{\sqrt{y\,\mathrm{d}y^2 + y\,\mathrm{d}x^2 - 2x\,\mathrm{d}x\,\mathrm{d}y}} \tag{28}$$

to rewrite the factors in brackets of the above equations (cf. Eqs. 26 and 27):

$$\begin{aligned}
qx - p\sqrt{yy-xx} &= \frac{\overline{2x\,\mathrm{d}y - y\,\mathrm{d}x}\sqrt{y^3 - yxx}}{\sqrt{y\,\mathrm{d}y^2 + y\,\mathrm{d}x^2 - 2x\,\mathrm{d}x\,\mathrm{d}y}}\\
pq - x\sqrt{yy-xx} &= \frac{\overline{y^2\,\mathrm{d}x\,\mathrm{d}y - 2xy\,\mathrm{d}y^2 - xy\,\mathrm{d}x^2 + 2x^2\,\mathrm{d}x\,\mathrm{d}y}\sqrt{yy-xx}}{y\,\mathrm{d}y^2 + y\,\mathrm{d}x^2 - 2x\,\mathrm{d}x\,\mathrm{d}y} \qquad (29)
\end{aligned}$$

Euler then substitutes these equations and the relations for p and q into Eq. 27, which yields the equation of motion for the normal component depending only on the distances and masses of the bodies acting on the Moon:

$$\begin{aligned}\frac{vv}{r} &= \frac{Tp}{y^3} + \frac{Sp}{z^3} + \left(\frac{x\,\mathrm{d}x - y\,\mathrm{d}y}{\sqrt{yy\,\mathrm{d}y^2 + yy\,\mathrm{d}x^2 - 2xy\,\mathrm{d}x\,\mathrm{d}y}}\right)\left(\frac{Sa}{z^3} - \frac{S}{aa}\right) \\ &= \frac{1}{\sqrt{yy\,\mathrm{d}y^2 + yy\,\mathrm{d}x^2 - 2xy\,\mathrm{d}x\,\mathrm{d}y}}\left(\frac{Tyy\,\mathrm{d}x - Tyx\,\mathrm{d}y}{y^3}\right. \\ &\quad \left. + \frac{Syy\,\mathrm{d}x - Syx\,\mathrm{d}y}{z^3} + \frac{Sax\,\mathrm{d}x - Say\,\mathrm{d}y}{z^3} \frac{-Sx\,\mathrm{d}x + Sy\,\mathrm{d}y}{aa}\right) \end{aligned} \tag{30}$$

Setting $LQ = t$ and regarding $yy = tt + xx$, Euler transforms the square root denominator

$$\sqrt{yy\,\mathrm{d}y^2 + yy\,\mathrm{d}x^2 - 2xy\,\mathrm{d}x\,\mathrm{d}y} = t\sqrt{\mathrm{d}t^2 + \mathrm{d}x^2} = t\,\mathrm{d}s\ . \tag{31}$$

Thus, the normal component becomes

$$\frac{vv}{r} = \frac{1}{t\,\mathrm{d}s}\left(\frac{Tt^2\,\mathrm{d}x - Txt\,\mathrm{d}t}{\overline{tt + xx}\sqrt{tt + xx}} + \frac{Stt\,\mathrm{d}x - Sxt\,\mathrm{d}t - Sat\,\mathrm{d}t}{z^3} + \frac{S\,\mathrm{d}t}{aa}\right)\ . \tag{32}$$

He then substitutes the osculating radius given by the relation $r = \frac{\mathrm{d}s\,\mathrm{d}t}{\mathrm{dd}x}$. Although used by Euler from the beginning of his notes in Ms 397, he presented the derivation of this formula in a paper entitled "Methodus facilis investigandi radium osculi ex principio maximorum et minimorum petita" (Easy method to find the osculating radius deduced by the principle of maximum and minimum, E654) only in September 11, 1776. It was published posthumously only in 1793, cf. Euler 1793). The normal component (cf. Eq. 27) then becomes

$$\frac{vv\,\mathrm{dd}x}{\mathrm{d}t} = \frac{Tt\,\mathrm{d}x - Tx\,\mathrm{d}t}{(tt + xx)^{\frac{3}{2}}} + \frac{St\,\mathrm{d}x - Sx\,\mathrm{d}t - Sa\,\mathrm{d}t}{z^3} + \frac{St\,\mathrm{d}t}{aa}\ . \tag{33}$$

Using the relations (29), Euler integrates the equation of motion (26) for the tangential component,

$$\begin{aligned}\frac{qv\,\mathrm{d}v}{y\,\mathrm{d}y} &= \frac{S\,\mathrm{d}x}{aa\,\mathrm{d}s} - \frac{Sa\,\mathrm{d}x - Sx\,\mathrm{d}x - St\,\mathrm{d}t}{z^3\,\mathrm{d}s} - \frac{Tx\,\mathrm{d}x - Tt\,\mathrm{d}t}{\mathrm{d}s\cdot(xx + tt)^{\frac{3}{2}}} = \frac{v\,\mathrm{d}v}{\mathrm{d}s} \\ v\,\mathrm{d}v &= \frac{S\,\mathrm{d}x}{aa} - \frac{Sa\,\mathrm{d}x - Sx\,\mathrm{d}x - St\,\mathrm{d}t}{z^3} - \frac{Tx\,\mathrm{d}x - Tt\,\mathrm{d}t}{(xx + tt)^{\frac{3}{2}}}\ ,\end{aligned} \tag{34}$$

in a straightforward manner, yielding

$$\frac{vv + 2A}{2} = \frac{Sx}{aa} + \frac{S}{\sqrt{\overline{a + x}^2 + tt}} + \frac{T}{\sqrt{xx + tt}}\ . \tag{35}$$

Euler remarks that the latter equation gives the velocity generating altitude by which the Moon is moved in its orbit. He rewrites $\frac{vv}{2}$ as

$$\frac{S \cdot QT}{ST^2} + \frac{S}{LS} + \frac{T}{TL} - A\ . \tag{36}$$

Multiplying Eq. (33) by $\mathrm{d}t$ and Eq. (35) by $2\,\mathrm{dd}x$ and equating the results for $vv\,\mathrm{dd}x$, produces

$$\begin{aligned} &\frac{2Sx\,\mathrm{dd}x}{aa} + \frac{2S\,\mathrm{dd}x}{\sqrt{\overline{a+x}^2+tt}} + \frac{2T\,\mathrm{dd}x}{\sqrt{xx+tt}} - 2A\,\mathrm{dd}x \\ &\quad = \frac{Tt\,\mathrm{d}t\,\mathrm{d}x - Tx\,\mathrm{d}t^2}{(tt+xx)^{\frac{3}{2}}} + \frac{St\,\mathrm{d}x\,\mathrm{d}t - Sx\,\mathrm{d}t^2 - Sa\,\mathrm{d}t^2}{(\overline{a+x}^2+tt)^{\frac{3}{2}}} + \frac{S\,\mathrm{d}t^2}{aa}\ . \end{aligned} \tag{37}$$

If the apogee (Euler erroneously wrote "aphelium") occurs in opposition to the Sun, let the velocity corresponding altitude there be K and the distance of the Moon from the Earth be h. Then $t = 0$ and $x \equiv h$, and Eq. (35) becomes

$$K = B\left(\frac{S \cdot h}{aa} + \frac{S}{a+h} + \frac{T}{h} - A\right)\ . \tag{38}$$

If furthermore c denotes the Earth's radius, then the inverse value of the gravity on the Earth's surface (as "normalizing factor") is $B = \frac{cc}{T}$ and thus one gets

$$A = \frac{Sh}{aa} + \frac{S}{a+h} + \frac{T}{h} - \frac{TK}{cc}\ . \tag{39}$$

But if the velocity corresponding altitude anywhere (in the orbit) is V, then

$$V = \frac{Sccx}{Taa} + \frac{Scc}{Tz} + \frac{cc}{y} - \frac{Scch}{Taa} - \frac{Scc}{T \cdot \overline{a+h}} - \frac{cc}{h} + K\ . \tag{40}$$

Setting $c = 1$ and using the geometric relations of Fig. 9, Euler obtains

$$V = \frac{S \cdot an}{T \cdot as \cdot sn} + \frac{am}{at \cdot tm} - \frac{S \cdot aq}{T \cdot ts^2} + K \tag{41}$$

The remaining records that appear to be related with the topic are quite hard to reconstruct and link together. These records concern five lines on folio 125v and six lines

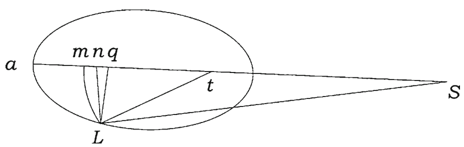


Fig. 9 Euler's sketch showing geometric relations and designations

on folio 126r. They seem to express ideas and trials, and thus do not further contribute to Euler's theory.

C2: The relevant records of Ms 397 concerning the motion of the lunar apsides

Euler's records associated with this topic are much more fragmentary than the concise notes of the previous one. The various entries on different folios often do not match together. Many of the intermediate formulae are not derived but are results gained from anywhere. However, I try to trace the main steps as given by the numbering of the folio in Ms 397 (which is not necessary identical with their chronological order) thus showing Euler's progress in dealing with the problem of the Moon's apsidal motion. We must leave some results provisional and unexplained due to the fact that there is not enough information contained in this notebook to be able to reconstruct all these results and formulae without any remaining gaps. Nevertheless, where it was possible, I have done so and have referred in such cases again to Volume I, Chapter V, of Euler's "Mechanica", Euler (1736).

Euler opens this topic by stating the following problem:

Invenire vim quæ facit ut corpus in orbe mobili moveatur.

"Find the force causing a body to move in a mobile orbit." The sequence of records in Ms 397 associated with this topic reveals that the term "orbe mobili" (mobile orbit) means the lunar elliptic orbit rotating around the center of force being in one focus, i.e., the Earth. We witness here Euler taking up Section IX, on the motion of bodies in mobile orbits, and the motion of the apsides, of Newton's *Principia*. To solve it, Euler sets the ratio between the orbital velocity ("velocitas orbis") and the velocity of the radius vector ("velocitas radii vectoris"), i.e. the angular velocity of the rotating orbit, as $m : n$. Furthermore, he sets $m + n = l$, and uses $CL = y$, $Lr = \mathrm{d}x$, and $lr = \mathrm{d}y$ (cf. Fig. 10). Then the ratio between the force acting in the immobile orbit ("vis in orbe immobili"), causing the Moon to move in its orbit, and the force acting in the mobile orbit ("vis in orbe mobili"), causing the Moon's elliptic orbit to move around the center of force being in one focus, is given by

$$\mathrm{d}x^2 + \mathrm{d}y^2 + \frac{y\,\mathrm{d}y\,\mathrm{dd}x}{\mathrm{d}x} : \mathrm{d}y^2 + \frac{ll\,\mathrm{d}x^2}{nn} + \frac{y\,\mathrm{d}y\,\mathrm{dd}x}{\mathrm{d}x}\,. \tag{42}$$

Setting $CP = p$ (and $\mathrm{d}s = \sqrt{\mathrm{d}x^2 + \mathrm{d}y^2}$), Fig. 10 yields the ratio

$$y : p = \sqrt{\mathrm{d}x^2 + \mathrm{d}y^2} : \mathrm{d}x \left(= \frac{\mathrm{d}s}{\mathrm{d}x} \right). \tag{43}$$

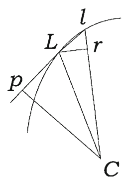


Fig. 10 Euler's sketch of the problem concerning the motion of the apsides

Now Euler introduces the radius osculating the curve in point L by

$$r = \frac{y\,\mathrm{d}s^3}{\mathrm{d}s^2\,\mathrm{d}x + y\,\mathrm{d}y\,\mathrm{dd}x}\,, \tag{44}$$

At that time Euler seemed to be already acquainted with this formula because he wrote it without any derivation. He solves the relation (Eq. 44) for

$$y\,\mathrm{d}y\,\mathrm{dd}x = \frac{y\,\mathrm{d}s^3}{r} - \mathrm{d}s^2\,\mathrm{d}x \tag{45}$$

and puts this into Eq. (42), gaining

$$\frac{y\,\mathrm{d}s^3}{r} : \frac{\overline{2mn+mm}}{nn}\,\mathrm{d}x^3 + \frac{y\,\mathrm{d}s^3}{r} = \frac{y^4}{r} : \frac{\overline{2mn+mm}\,p^3}{nn} + \frac{y^4}{r}\,. \tag{46}$$

Now Euler supposes the centripetal force being equal to the inverse square of the distance y and refers to de Moivre's theorem (cf. Euler's "Mechanica", Euler 1736, Sect. 592), stating that if $r = \frac{y\,\mathrm{d}y}{\mathrm{d}p}$ then

$$\frac{y}{p^3 r} = \frac{A}{yy}\,, \tag{47}$$

where A is the centripetal force (in Sect. 592 it corresponds to the "normalized" centripetal force $\frac{P}{2ch^2}$). Thus, $r = \frac{y^3}{Ap^3}$. In addition, Euler refers to Keill's theorem (cf. ibidem), which states that

$$\frac{\mathrm{d}p}{p^3\,\mathrm{d}y} = \frac{A}{yy}\,, \tag{48}$$

which he integrates to obtain $\frac{1}{2pp} = \frac{A}{y} + B$ with B being the integration constant. Regarding $\frac{y}{p} = \frac{\mathrm{d}s}{\mathrm{d}x}$ (cf. Eq. 43), the theorems substituted in Eq. (46) lead to

$$Ayp^3 : \frac{\overline{2mn+mm}}{nn}\,p^3 + Ayp^3 = Ay : \frac{2mn+mm}{nn} + Ay\,. \tag{49}$$

Euler supposes now the forces as being given and the motion of the orbit as to be determined. Assuming the attracting force to be proportional to the inverse square of the distance. Let the force moving the body in the immobile orbit be V, and let in addition the force T be given. If these forces are given and the motion of the apsides has to be found, then

$$V : T + V = Ay : \frac{2mn+mm}{nn} + Ay \tag{50}$$

and therefore

$$AyT = \frac{\overline{2mn + mm}}{nn} Vn\sqrt{AyT + V} = \overline{m+n}\sqrt{V} \ . \tag{51}$$

Thus,

$$\overline{m+n} : n = \sqrt{AyT + V} : \sqrt{V} \ . \tag{52}$$

Because the ratio between the body's velocity and his velocity in the immobile orbit is $\frac{1}{p}$, it follows

$$y : p = \frac{1}{p} : \frac{1}{y} \ . \tag{53}$$

If the angular velocity of the radius vector ("velocitas angularis radii vectoris", cf. Moivre de 1717 for the concept of angular velocity) $\frac{1}{yy}$ is substituted by n, then

$$m = \frac{1}{yy}\sqrt{\frac{AyT + V}{V}} - \frac{1}{yy} = \frac{1}{yy}\left(\sqrt{\frac{ATy}{V} + 1} - 1\right) . \tag{54}$$

But if $\frac{1}{A} = \frac{b}{2}$, where b denotes the latus rectum of the conic section, then it follows from de Moivre's theorem (Eq. 47)

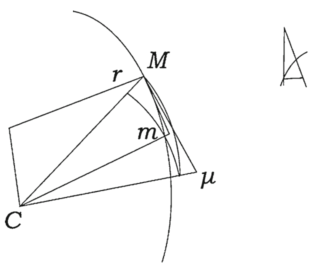
$$r = \frac{by^3}{2p^3} \ . \tag{55}$$

Denoting, respectively, the lengths of the major and minor axes by a and b, Euler gives without any comment the formulae for the osculating radius as

$$r = \frac{4(ay - yy)^{\frac{3}{2}}}{ac} \ . \tag{56}$$

On folio 34v, we find the derivations of the Eqs. (42) and (44). However, several lines containing formulae for the derivation of the first one are crossed out by Euler. We skip them here as well. The right hand side of the folio contains as a marginal note the derivation of r. Restating the relation $\mathrm{d}s : \mathrm{d}x = y : p$ and setting (see Fig. 11)

Fig. 11 Euler's sketch for the derivation of Eqs. (42) and (44)



$Mr = \mathrm{d}y$ and $rm = \mathrm{d}x$, Euler calculates the differential $\mathrm{d}p$ as the first derivative of $p = \frac{y\,\mathrm{d}x}{\mathrm{d}s}$, yielding

$$\begin{aligned}\mathrm{d}p &= \frac{\mathrm{d}s\,\mathrm{d}y\,\mathrm{d}x + y\,\mathrm{d}s\,\mathrm{dd}x - y\,\mathrm{d}s\,\mathrm{dd}s}{\mathrm{d}s^2} \\ &= \frac{\mathrm{d}s^2\,\mathrm{d}y\,\mathrm{d}x + y(\mathrm{d}x^2 + \mathrm{d}y^2)(\mathrm{dd}x - \mathrm{dd}s)}{\mathrm{d}s^3} \\ &= \frac{\mathrm{d}s^2\,\mathrm{d}y\,\mathrm{d}x + y\,\mathrm{d}y^2\,\mathrm{dd}x}{\mathrm{d}s^3}\,, \end{aligned} \tag{57}$$

supposing $\mathrm{d}x\,\mathrm{dd}x \approx \mathrm{d}s\,\mathrm{dd}s$. This result, substituted in $r = \frac{y\,\mathrm{d}y}{\mathrm{d}p}$, yields Eq. (44). For the derivation of Eq. (42) Euler substitutes r and p into the relation $\frac{Mm^2}{pr}$, where $Mm = \mathrm{d}s$, obtaining

$$\frac{Mm^2}{pr} = \frac{\mathrm{d}s^3 \cdot \overline{\mathrm{d}s^2\,\mathrm{d}x + y\,\mathrm{d}y\,\mathrm{dd}x}}{yy\,\mathrm{d}s^3\,\mathrm{d}x}\,. \tag{58}$$

Then, without considering the factor y^2 in the denominator, Euler goes ahead with

$$\frac{(\mathrm{d}x^2 + \mathrm{d}y^2)\,\mathrm{d}x + y\,\mathrm{d}y\,\mathrm{dd}x}{\mathrm{d}x} : \frac{nn\,\mathrm{d}y^2 + ll\,\mathrm{d}x^2}{nn} + \frac{y\,\mathrm{d}y\,\mathrm{dd}x}{\mathrm{d}x}\,, \tag{59}$$

which yields Eq. (42).

On folio 125v, we see Euler restating the problem again (see Fig. 12): Find the forces retaining both the body in any mobile orbit AB and around the center O of the osculating circle. Let v be the velocity of the body L in the immobile orbit and u the orbit's [angular] velocity. The tangential force in the immobile orbit is then given by $\frac{v\,\mathrm{d}v}{Ll}$, the tangential force in the mobile orbit by $\frac{v\,\mathrm{d}v + v\,\mathrm{d}u + u\,\mathrm{d}v + u\,\mathrm{d}u}{Ll}$. The normal force in the immobile orbit is given by $r = \frac{vv}{LO}$, and the normal force in the mobile orbit by $\frac{vv+2vu+uu}{LO}$. Thus, the ratio between the tangential components in the immobile and the mobile orbits becomes $v\,\mathrm{d}v : v\,\mathrm{d}v + v\,\mathrm{d}u + u\,\mathrm{d}v + uu$. If $u = nv$, then this ratio becomes $= v : v + 2nv + nnv = 1 : \overline{1+n}^2$. Therefore, the ratio between the normal components in the immobile and the mobile orbits is $vv : (v+u)^2$, and with $u = nv$, it becomes $1 : \overline{1+n}^2$.

In Ms 397, fol. 126r and 126v, there are two fragmentary records which also most probably concern the motion in a mobile orbit. However, it is quite hard to identify

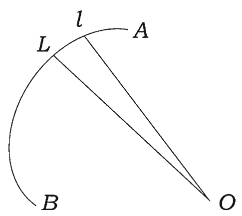


Fig. 12 Euler's sketch used to define the forces moving the body L in the orbit and the orbit around the center of force O

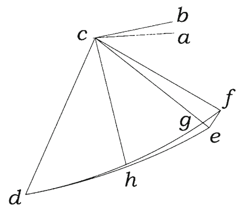


Fig. 13 Euler's sketch illustrating a rotating (mobile) orbit

their meaning and in what respect they are connected with this topic. For the sake of completeness, I will present them as they are without any interpretation.

Euler denotes the decrement of the orbit's axis by $\mathrm{d}p$ and uses it in a decreasing sense. Then he sets $dc = y$, $ch = q$, the axis $= b$, $cf = b - y$, $fg = \mathrm{d}x$, $de = t$, and $he = v$ (cf. Fig. 13). Then he claims that the ratio $\angle acb : \angle fde$ (in which the axis moves according to the signs, i.e. "in consequentia") is as $t\,\mathrm{d}x : v\,\mathrm{d}x - q\,\mathrm{d}p$. And the sinus of the angle d is given by

$$\sin d = \frac{v\,\mathrm{d}x - q\,\mathrm{d}p}{t \cdot \overline{b - y}}\ , \tag{60}$$

thus

$$v = \frac{bb - 2by + tt}{2t} \tag{61}$$

if $\mathrm{d}x = 0$.

On folio 126v, both the figure and the formulae are even more puzzling. Euler did not define any symbols nor does he give any other hint as to what is going on. There are only some "stand-alone" relations. I speculate that they have something to do with the motion in a mobile orbit MA rotating around the center of force S (cf. Fig. 14). Euler defines $\angle SPp =$ rect and $\angle qPp = sST$, then he notes the following relations:

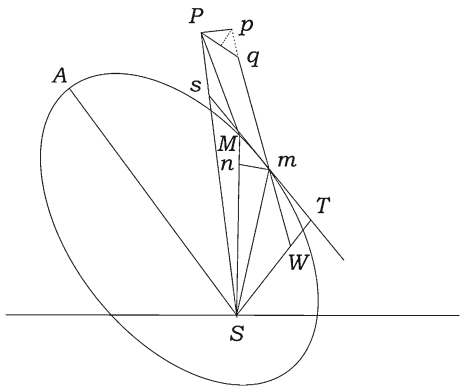


Fig. 14 Euler's sketch illustrating geometric relations in a mobile orbit rotating around the center of force

$$MP : mp = A^2 : \left(A + \frac{MS \cdot mn}{2}\right)^2 = A : A + MS \cdot mn \tag{62}$$

$$MP : qp = A : MS \cdot mn = A : Pq \cdot ST \tag{63}$$

and $\triangle Prp \backsim STs$ as well as $\triangle Prp \backsim STs$, where $\backsim$ designates the proportionality symbol. Finally, he states that $Ss : sT = Pp : pr$ and $pq : pr = Wm : WT$.

C3: The relevant records of Ms 397 concerning the general central force motion

During the years 1725 until 1730 the problem of central force motion was one of Euler's main research topics, not only in the context of general mechanics, but primarily in the context of celestial mechanics. There are at least five unpublished manuscripts by Euler written at that time addressing that issue (cf. Kopelevič et al. 1962, pp. 64–65, No. 178, 180, 181, 185, and 186). In addition, Ms 167 contains chapters that are also devoted to this matter. It is important to understand Euler's first approaches to lunar theory principally as a problem of central force motion. His main task in that period was to transform the corresponding Propositions in Newton's *Principia* into the analytic language of rational mechanics and thus to develop mathematically the theory of central force motion "from scratch". I will therefore comment here only on some "chapters" from Euler's first notebook, that are related to that topic and that represent his theoretical equipment, i.e. analytical mechanics, to tackling the lunar problem.

There are twelve chapters in Ms 397 on free and constrained motion of bodies in vacuo or in a resistant medium, but only four of them are related to the free central force motion problem. Unfortunately, the figures to which Euler refers in the margins are missing throughout. He began his notebook with a carefully written chapter entitled "De Motu Corporum à vi quacunque centrali agitatorum" (On the motion of bodies which are agitated by an arbitrary central force) and opens this chapter with three "definitiones":

I. A central force is a force with which a body is attracted towards any fixed point or from which it is repulsed. In the former case, we call it centripetal force, in the latter centrifugal force.
II. A mutative force ("vis mutatrix", Euler's neologism?) is a force which changes the speed of a moving body, i.e., either increases or decreases it. In the former case, we call it an accelerating, in the latter case a retarding force. NB: This force only changes the speed, leaving the direction of the motion unchanged.
III. The central force may be resolved into its components ("laterales"), namely in one whose direction is tangential to the curve describing the motion, and in one whose direction is normal to the other, calling the former tangential force, but the latter normal force.

This is followed by three propositions including demonstrations, nine corollaries, and two scholia. The next chapter concerns the motion of a body when no mutative force is acting ("De Motu corporum vi nulla existente mutatrice"). In subsequent chapters Euler develops a method to find the mutative force when the trajectory described by the body and the central force are given ("Methodus datis curva et vi centrali inve-

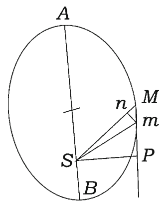


Fig. 15 Euler's sketch explaining the central force motion of the body M around the central body S, A and B being the apo- and pericenter, respectively

niendi vim mutatricem") and he describes the time of revolution of a body rotating around any center and driven by any mutative force ("De Temporis periodicis corporum circa centrum aliquod gyrantium, quacunque existente vi mutatricem"). All these chapters were written in a textbook style and reflect Euler's state-of-the-art knowledge on central force mechanics. They were published by Gleb K. Mikhailov in 1965 (cf. Mikhailov 1965, pp. 38–62), which is why there is no need to present them here again.

His research on central force motion are continued on folio 121v of Ms 397. Let, respectively, M and r be the mass and radius of the Earth moving in an elliptic orbit around the body S with mass S [sic!] (cf. Fig. 15). Let A and B be the apo- and pericenter, respectively, and $SA = a$, $SM = y$, and $SP = p$ perpendicular to the tangent line through M. Let further v and K be the altitudes that correspond to the velocities of M and of a body on the Earth's surface, respectively. Without any derivation (which may be found—with correct reverse signs—in Ms 273, fol. 1r. I postpone this derivation to the next paper when the manuscript Ms 273 will be discussed in full length), Euler states:

$$v = K + \frac{rrSy - rrSa}{May} \, . \tag{64}$$

He proceeded with his calculations and, at the end of the folio, recognized, that the second term of this equation has the wrong sign. Having discovered his error, Euler cancelled all his calculations on that folio and restarted on the next folio 122r anew, where he changed (overstriked) four times the plus sign into a minus sign. If $v = \frac{aaK}{pp}$ then

$$\frac{aaK}{pp} = \frac{MaKy - rrSy + arrS}{May} \, . \tag{65}$$

(Note that this time the sign is correct!) Thus,

$$pp = \frac{Ma^3Ky}{MaKy - rrSy + arrS} \, . \tag{66}$$

To obtain the distance BS between the pericenter and the focus S, Euler sets $p = y$ and divides the resulting equation by $y - a$, gaining

$$y = \frac{+MaaK}{-MaK + rrS} = BS \, . \tag{67}$$

Using this result, he determines the length of the major axis AB:

$$AB = a + \frac{MaaK}{rrS - MaK} = \frac{arrS}{rrS - MaK} \ . \tag{68}$$

Let $Mm = \mathrm{d}s$, then the time element, corresponding to $\mathrm{d}s$ given by $\frac{\mathrm{d}s}{\sqrt{v}}$, is thus $= \frac{p\,\mathrm{d}s}{a\sqrt{K}}$. The area covered during this time element is $p\,\mathrm{d}s = 2MSm$. Denoting this area as A, the total time of revolution then becomes $= \frac{2A}{a\sqrt{K}}$. Using Eqs. (68) and (67) the length of the semi-major is given by

$$\sqrt{AB \cdot SB - SB^2} = \sqrt{\frac{Ma^3K}{rrS - MaK}} = a\sqrt{\frac{MaK}{rrS - MaK}} \ . \tag{69}$$

Let $1 : \pi$ be the ratio between the radius and the perimeter ("Sit $1 : \pi$ ut radius ad peripheriam").[2] The area of the ellipsis is, according to this definition of π, given by

$$A = \frac{\pi aarrS}{4rrS - 4MaK}\sqrt{\frac{MaK}{rrS - MaK}} \ , \tag{70}$$

the time of revolution becomes

$$\frac{2A}{a\sqrt{K}} = \frac{\pi arrS}{2rrS - 2MaK}\sqrt{\frac{Ma}{rrS - MaK}} \ . \tag{71}$$

The last two and—consequently—all the following results are faulty because the area of an ellipsis is given by the product of the semi-major and semi-minor axes, but Euler used instead the major and semi-minor axes, resulting in an error of a factor 2 in the denominator. This might be the consequence of a misplaced correction by Euler: after having found the length of the major axis, he stated:

Inveniatur axis minor est $\frac{AB}{\cancel{2}} = \frac{arrS}{\cancel{2}rrS - \cancel{2}MaK}$

In fact, without cancelling the "2", this formulae would have given the correct semi-major axis as it should have been used afterwards. In addition, Euler called it misleadingly "axis minor", instead of major axis. However, if the time of revolution is required in seconds of time, Euler expresses a and r in Rhinelandian or Prussian feet ("scrupulis pedis Rhenani", corresponding 0.313853497 meters), so in this units it becomes

[2] This is probably the very first definition of the symbol π by Euler used to express what we now call Ludolph's number (Mattmüller 2010, p. 185). He used this kind of definition also in Ms 167 and Ms 273, which will be presented in part 2. In his "Mechanica", which Euler finished in 1734, he defined π in the same way as we use it today, namely as the ratio between the *diameter* of a circle and its perimeter: "Posita ratione diametri ad peripheriam $1 : \pi$", cf. Euler 1736, Prop. 76, Coroll. 2).

$$= \frac{2\pi arrS \cdot \sqrt{Ma}}{1000\left(rrS - MaK\right)^{\frac{3}{2}}} \,. \tag{72}$$

Calling the major axis b, Euler substitutes

$$rrS - MaK = \frac{arrS}{b} \tag{73}$$

in Eq. (72), obtaining for the time of revolution

$$\frac{2\pi b\sqrt{Mb}}{1000r\sqrt{S}} = \frac{1256b\sqrt{bM}}{100000r\sqrt{S}} \tag{74}$$

in seconds. If $S = M$, as Euler concludes in a corollary, then

$$\frac{2\pi b\sqrt{b}}{1000 \cdot r} = \frac{1256 \cdot b\sqrt{b}}{100000r} \tag{75}$$

and

$$v = \frac{Sbrr - Srry}{Mby} \qquad \text{thus} \qquad b = \frac{Srry}{Srr - Myv} \,. \tag{76}$$

Finally, Euler evaluates the result (cf. Eq. 74) numerically assuming the Earth's radius r to be 20302353000 Rhinelandian feet, yielding for the time of revolution t in time seconds

$$t = \frac{b\sqrt{Mb}}{1616429300000\sqrt{S}} \,. \tag{77}$$

Setting $n = 1616429300000$, then

$$b = \frac{n^{\frac{2}{3}} t^{\frac{2}{3}} S^{\frac{1}{3}}}{M^{\frac{1}{3}}} = 137592900 \sqrt[3]{\frac{ttS}{M}} \,. \tag{78}$$

Supposing the Moon's synodic time of revolution to be $= 2360580$ seconds, which is 27days 7 h 43 min and corresponds to the value given in Newton's *Principia* (Lib. III, Prop. XXVI, XXVIII, XXIX), and setting $M = S$, Euler finds for the Moon's major axis $b = 2435396100$ Rhinelandian feet or 119.952 Earth radii.

On folio 125v, Euler formulated the following theorem: A body moves in an ellipse around the focus S, in which the attractive force is located. In such a way the absolute force in S increases, the body L will then immediately revolve in another ellipse of which the major axis becomes shorter than before. Euler re-formulates this theorem using Fig. 16. Let M and r be the Earth's mass and radius, respectively, S the mass of the central body S, $LS = y$, and v the altitude corresponding velocity of the body L. The major axis of the orbit being $= \frac{Srry}{Srr-Myv}$ (cf. Eq. 76). Supposing S is increasing,

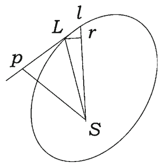


Fig. 16 Euler's sketch illustrating his theorem

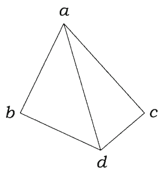


Fig. 17 Euler's sketch of a quadrilateral with right angles at the edges *b* and *c*

then the growing absolute force in S diminishes $\frac{Srry}{Srr-Myv}$, which is why the absolute force is decreasing the orbit's major axis. And, in contrary, a decreasing attractive force increases the orbit's major axis. Euler proves this theorem in the following way on folio 126r: Let b and c be the major and minor axes, respectively, and p the perpendicular SP, which is given by

$$pp = \frac{ccy}{4b - 4y} \ . \tag{79}$$

The following demonstration depends crucially on this equation. Its derivation, however, may be found some folios earlier in the notebook, namely on folio 88v and 120v. On folio 88v, Euler noted, without comment, the formula

$$r = \frac{\sqrt{tt + zz \pm 2tz\sqrt{1 - ff}}}{f} \ , \tag{80}$$

where (see, Fig. 17) $ab = t$, $ac = z$, $ad = r$, $\sin \angle bac = f$, and the angles at b and c are right angles. This equation is a consequence of the law of cosines (cosine rule), applied to the triangle abc and of the fact, that the straight line bc is equal $rf = r \sin \angle bac$, which may be proved by geometric construction. Using this result and identifying a with P, b with T, c with M, and d with A, so that $ad = r$ in Fig. 17 corresponds to $AP = y$ in Fig. 18, Euler derives on folio 120v the parameter p as follows (see Fig. 18): Let $AP = y$, $AB = a$, and $BP = z$ be a certain function of y. The cosine rule of the triangle ABP yields

$$yy - zz + aa = 2aP \ , \tag{81}$$

from where one can conclude that $P = y \cos \angle PAM \equiv MP$.

Setting $PT = t$, Euler claims without proof that

$$tt = \frac{y^3 - yPP}{y - 2\cancel{PP} + y\cancel{PP}} = yy - pp \ , \tag{82}$$

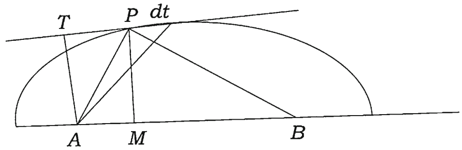


Fig. 18 Euler's sketch illustrating the tangent through the point P of the ellipse with foci A and B, and the perpendicular $p = AT$ to be find

where the meaning of the symbol $\bar{P}$ is also not given by Euler. This equation solved for p yields

$$p = \frac{(y\bar{P} - P)\sqrt{y}}{\sqrt{y - 2P\bar{P} + y\bar{P}\bar{P}}} \,. \tag{83}$$

Setting $yz = bb$ resp. $z = \frac{bb}{y}$, Eq. (81) defining P yields

$$P = \frac{y^4 - b^4 + aayy}{2ayy} \,, \tag{84}$$

and the "unknown" definition of $\bar{P}$ yields

$$\bar{P} = \frac{y^4 + b^4}{ay^3} \,. \tag{85}$$

Thus,

$$p = \frac{(y^4 + 3b^4 - aayy)y}{2\sqrt{2b^3 - aab^4y^2 + 2b^4y^4}} \,. \tag{86}$$

But setting $z = b - y$, the equations defining P and $\bar{P}$ yield

$$P = \frac{2by - bb + aa}{2a} \qquad \text{and} \qquad \bar{P} = \frac{b}{a} \,, \tag{87}$$

and so

$$p = \frac{\frac{bb-aa}{2a}\sqrt{y}}{\sqrt{y - \frac{bby}{aa} + \frac{b^3}{aa} - b}} \,. \tag{88}$$

Finally, by setting $bb - aa = cc$ Euler gained

$$p = \frac{c\sqrt{y}}{2\sqrt{b - y}} \qquad \text{and} \qquad tt = \frac{4byy - 4y^3 - ccy}{4b - 4y} \,. \tag{89}$$

Euler proceeds with the demonstration of his theorem by assuming the angle PLS and the minor axis as given. He supposes in addition that $SP = p = ny$, yielding

$$nny = \frac{cc}{4b - 4y} \qquad \text{resp.} \qquad cc = 4nry(b - y)\ . \tag{90}$$

From Eq. (76) it follows

$$b - y = \frac{Mbyv}{Srr} \qquad \text{and thus} \qquad cc = \frac{4Mn^2by^2v}{Srr}\ . \tag{91}$$

Therefore, the "latus rectum" becomes

$$\frac{cc}{b} = \frac{4Mn^2y^2v}{Srr}\ , \tag{92}$$

from which one may conclude that if the attractive force S is increasing, the latus rectum is decreasing, and vice versa. To find out by which amount the major axis is shortened, Euler calls SV the force with which L is attracted by S and $V = \frac{1}{y^2} : \frac{1}{(y+\mathrm{d}y)^2}$ the force acting in l in such a way that the body describes an ellipsis around S. Neglecting $\mathrm{d}y^2$, this ratio becomes $1 = \frac{yV}{y+2\,\mathrm{d}y}$. But if l is attracted by $V + \mathrm{d}V$, then

$$\frac{yV}{y + 2\,\mathrm{d}y} : V + \mathrm{d}V = S : \frac{SVy + 2SV\,\mathrm{d}y + Sy\,\mathrm{d}V}{yV}\ , \tag{93}$$

where, again, the second order differential $\mathrm{d}y\,\mathrm{d}V$ in the product $(y+2\,\mathrm{d}y)(V+\mathrm{d}V) = Vy + 2V\,\mathrm{d}y + y\,\mathrm{d}V + 2\,\mathrm{d}y\,\mathrm{d}V$ is neglected. Using the results already gained above, Euler concludes that the shortening part becomes

$$\frac{SMrry^2v\,\mathrm{d}V + 2SMrryVv\,\mathrm{d}y}{(SrrV - MyVv)(Srr - Myv)}\ . \tag{94}$$

If $S = M = T$, this part is

$$\frac{rry^2v\,\mathrm{d}V + 2rryVv\,\mathrm{d}y}{V(ST - yv)^2}\ . \tag{95}$$

But if S and T designate the masses of the Sun and the Earth, respectively, and z the distance of the body L from the Sun, then

$$V = \frac{T}{yy} + \frac{Sy}{z^3}\ , \tag{96}$$

and, when y passes on $y + \mathrm{d}y$, the shortening part becomes

$$\begin{aligned}\frac{3Srrvy^4(z\,\mathrm{d}y - y\,\mathrm{d}z)}{z(Tz^3 + Sy^3)(rr - yv)^2} &= \frac{3Sbvy^3(z\,\mathrm{d}y - y\,\mathrm{d}z)}{z(Tz^3 + Sy^3)} \\ &= \frac{3Srryy(b - y)(z\,\mathrm{d}y - y\,\mathrm{d}z)}{z(Tz^3 + Sy^3)}\ .\end{aligned} \tag{97}$$

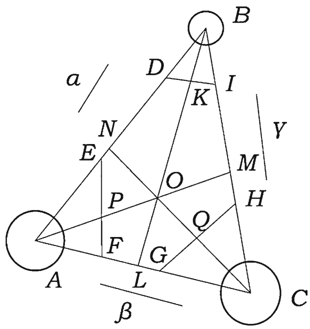


Fig. 19 Euler's sketch used to "solve" the general three-body-problem in Ms 397

C4: Miscellaneous records in Ms 397 related to lunar theory

On folio 123r, just one folio before the very page where Euler formulated the general problem of lunar theory (cf. Sect. C1 and Fig. 7), he treated the general three-body-problem: Given three attracting globes A, B, C (cf. Fig. 19), find the attraction of each one. Let α and β be the absolute forces acting in B by A and in C by A, respectively, and let γ be the mutual force between B and C. Considering AA as $B\alpha$, BD as $A\alpha$, BI as $C\gamma$, CH as $B\gamma$, CG as $A\beta$, and AF as $C\beta$. These lines represent the forces with which adjacent bodies attract each other. Euler bisects the line DI in K and conjoins BK, which intersects AC in L. This is done for EF and GH as well. The following derivations are proved by Euler on folio 29r and 32v–33r as well as in his unpublished manuscript Ms 251, which was published in Knobloch (1992). Euler states:

$$\frac{1}{\sin LBC} : \frac{1}{\sin ABL} = BI : BD \tag{98}$$

or

$$\sin LBC : \sin ABL = BD : BI = CL \cdot AB : AL \cdot BC\ . \tag{99}$$

Therefore

$$CL \cdot AB + AL \cdot BC = BD : BI = A\alpha : C\gamma\ . \tag{100}$$

In addition,

$$AL \cdot BC \cdot A\alpha = CL \cdot AB \cdot C\gamma \tag{101}$$

and thus

$$AL : CL = AB \cdot C\gamma : BC \cdot A\alpha = \frac{AB}{A\alpha} : \frac{BC}{C\gamma}\ . \tag{102}$$

In the same way, the following relations hold:

$$BM : CM = \frac{AB}{B\alpha} : \frac{AC}{C\beta} \quad \text{and} \quad BN : AN = \frac{BC}{B\gamma} : \frac{AC}{A\beta} \,. \tag{103}$$

Due to $CL \cdot AN \cdot BM = AC \cdot BN \cdot CM$, one can see that "ex geometria" the common center must be in O, whereto all three bodies tend to move, namely B with the force $2BK$, A with $2AP$, and C with $2CQ$. But we know from the geometry of the triangle, that

$$2BK = \sqrt{BD^2 + BI^2 - \frac{AB^2 \cdot BD \cdot BI}{AB \cdot BC} - \frac{BC^2 \cdot BD \cdot BI}{AB \cdot BC} + \frac{AC^2 \cdot BD \cdot BI}{AB \cdot BC}} \,. \tag{104}$$

Setting $\alpha = p \cdot AB$, $\beta = p \cdot AC$, $\gamma = p \cdot BC$, and assuming a [massless] body X located in O, the force with which B is attracted to X becomes $= X \cdot pBO$. Therefore

$$X = \frac{2BK}{pBO} = \frac{2\sqrt{\begin{array}{l} A^2 \cdot pAB^2 + C^2 \cdot pBC^2 - \frac{AB^2 \cdot A \cdot C \cdot pAB \cdot pBC}{AB \cdot BC} \\ -\frac{BC^2 \cdot AC \cdot pAB \cdot pBC}{AB \cdot BC} + \frac{AC^2 \cdot A \cdot C \cdot pAB \cdot pBC}{AB \cdot BC} \end{array}}}{pBO} \tag{105}$$

or

$$pBO = \frac{2BK}{X} \,. \tag{106}$$

In 1992, Eberhard Knobloch published a paper on Euler's earliest study of the three-body-problem (cf. Knobloch 1992). It concerns Euler's manuscript fragment Ms 251, which was written in the 1730ies (according to Kopelevič et al. 1962, p. 79) and which was published by Knobloch in the original Latin and in a translated German version. Knobloch claimed, however, that this manuscript was written around 1730 and that he could not verify a hint by Otto Volk (1892–1989). According to Volk, this general three-body-problem was already formulated by Euler in his [first] notebook of 1727. Knobloch was guest professor at the Russian Academy of Sciences in St. Petersburg in 1984 and—thanks to this visit—was able to describe the contents of all of Euler's notebooks in 1989 (cf. Knobloch 1989). Knobloch did not realize that Volk's claim in fact was right, as it was shown above. However, Knobloch must have recognized these notebook entries as well, although he assigned folio 123r of Ms 397 in Knobloch (1989), pp. 300–301, to the field of "mechanics", and not to "astronomy", thus considering its content as a mechanical instead of an astronomical problem as Euler probably did in Ms 251 and as Knobloch himself did as well already in the abstract of his paper:

> Euler was very interested in the *astronomical three body problem* since the beginning of his scientific career up to the end of his life […] (emphasis added)

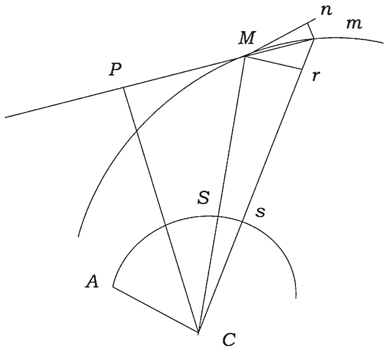


Fig. 20 Euler's first sketch on fol. 26v in Ms 398 concerning the central force problem

Appendix D: The relevant records in Ms 398

As mentioned above, the second notebook Ms 398 has the character of a waste book. The records are thus more fragmented and much less commented than in Ms 397. The most coherent entries concern the problem of central force motion. This is why they are worth to be presented here as well according to three major topics which are most carefully elaborated by Euler.

D1: The relevant records of Ms 398 concerning the general central force motion

Again, Euler aims at obtaining equations for the orbital parameters and its differentials due to a central force in terms of the instantaneous osculating radius. Let G be (in fact, Euler used the symbol A instead of G) the gravitational force and V the centripetal force acting on M by C (cf. Fig. 20). Let $CM = y$, $rm = \mathrm{d}y$, $CP = p$, and assume the velocity v in M equal to the corresponding altitude [of free fall]. Then (cf. Euler 1736, Prop. 71, Coroll. 1, where $P \equiv \frac{V}{A}$) $A : \mathrm{d}y = V : -\mathrm{d}v$, thus

$$-\,\mathrm{d}v = \frac{V\,\mathrm{d}y}{A}\,. \tag{107}$$

If M is moving to m within the time element $\frac{Mm}{\sqrt{v}}$, it covers the distance element $\frac{Mm^2}{4v}$ due to the gravitational force. Therefore,

$$A : \frac{Mm^2}{4v} = \frac{pV}{y} : mn \qquad \text{or} \qquad A : \frac{r}{4v} = V : \frac{y}{p}\,, \tag{108}$$

Hence

$$V = \frac{4Ayv}{pr} = -\frac{A\,\mathrm{d}v}{\mathrm{d}y} \tag{109}$$

and thus

$$-\frac{dv}{v}=\frac{4v\,dy}{pr}\ . \tag{110}$$

Substituting $r=\frac{y\,dy}{dp}$ (cf. Sect. B and Euler 1736, Prop. 74) yields

$$-\frac{dv}{v}=\frac{4y\,dy\cdot dp}{p\cdot y\,dy}=\frac{4\,dp}{p}\ . \tag{111}$$

At this point Euler recognizes an error by a factor of 2 when adding the comment "should be" (debet est)

$$-\frac{dv}{v}=\frac{2\,dp}{p}\ , \tag{112}$$

which in fact is the correct result of the derivation in Prop. 74 of Euler's "Mechanica" (cf. Euler 1736). Now, something strange happens: Euler supposes

$$\frac{c}{v}=pp \tag{113}$$

and sets (without explaining the meaning of the new symbols) $p=f$ and $v=K$, yielding $c=ffK$. Euler re-substitutes this in Eq. (113), gaining $v=\frac{ffK}{pp}$. Anyway, this latter equation together with the one for r are substituted in Eq. (109), which Euler corrected by the factor 2, resulting in Keill's theorem (cf. Keill 1708; Guicciardini 1995)

$$V=\frac{2AffKp\,dp}{p^4\,dy}=\frac{2AffK\,dp}{p^3\,dy}\ . \tag{114}$$

Let the arc lengths $AS=x$ and $Ss=dx$, then the time element dt used to cover the distance $Mm=ds$ will be

$$\frac{ds}{\sqrt{v}}=\frac{p\,ds}{f\sqrt{K}}=\frac{yy\,dx}{f\sqrt{K}}\ , \tag{115}$$

using the equivalence of the two triangles $\triangle CPM$ and $\triangle Mrn$, from which follows the relation

$$\frac{p}{y}=\frac{y\,dx}{ds} \tag{116}$$

if $Mr\approx y\,dx$. Using de Moivre's theorem,

$$V=\frac{2AffKy}{p^3r}\ , \tag{117}$$

Euler solves for $f\sqrt{K}$ and substitutes the result into Eq. (115) for dt, obtaining

$$\mathrm{d}t = \frac{yy\,\mathrm{d}x\sqrt{2Ay}}{\sqrt{Vp^3r}}\ . \tag{118}$$

Due to $\mathrm{d}s = \sqrt{\mathrm{d}y^2 + yy\,\mathrm{d}x^2}$, Eq. (116) yields

$$p = \frac{yy\,\mathrm{d}x}{\sqrt{\mathrm{d}y^2 + yy\,\mathrm{d}x^2}}\ , \tag{119}$$

from which Euler derives the differential

$$\mathrm{d}p = \frac{2y\,\mathrm{d}y^3\,\mathrm{d}x + y^2\,\mathrm{d}y\,\mathrm{d}x^3 - yy\,\mathrm{d}x\,\mathrm{d}y\,ddy}{(\mathrm{d}y^2 + yy\,\mathrm{d}x^2)\sqrt{\mathrm{d}y^2 + yy\,\mathrm{d}x^2}} \tag{120}$$

Substituting this result into $r = \frac{y\,\mathrm{d}y}{\mathrm{d}p}$ yields

$$r = \frac{(\mathrm{d}y^2 + yy\,\mathrm{d}x^2)^{\frac{3}{2}}}{2\,\mathrm{d}y^2\,\mathrm{d}x + yy\,\mathrm{d}x^3 - y\,\mathrm{d}x\,ddy}\ , \tag{121}$$

and thus the denominator in de Moivre's theorem becomes

$$p^3r = \frac{y^6\,\mathrm{d}x^2}{2\,\mathrm{d}y^2 + yy\,\mathrm{d}x^2 - y\,ddy}\ . \tag{122}$$

Finally, Euler derives the formula for $-\frac{\mathrm{d}v}{v}$ as it is given in his "Mechanica" (cf. Euler 1736, Prop. 73, Sect. 578, and Prop. 74, Sect. 587) and where he derived it more elegantly using a dynamic approach with the equation of motion. Let v be the velocity corresponding altitude in M, $CM = y$, $CP = p$, $MN = \mathrm{d}x$, $Nm = \mathrm{d}y$, $Mm = \mathrm{d}s$, and the osculating radius $MO = r = \frac{y\,\mathrm{d}y}{\mathrm{d}p}$. From the geometry of Fig. 21 it may easily be seen that $mn = \frac{\mathrm{d}s^2}{r}$, $m\nu = \frac{\mathrm{d}s^3}{r\,\mathrm{d}x}$, $n\nu = \frac{\mathrm{d}s^2\,\mathrm{d}y}{r\,\mathrm{d}x}$, and $M\nu = \frac{r\,\mathrm{d}s\,\mathrm{d}x + \mathrm{d}s^2\,\mathrm{d}y}{r\,\mathrm{d}x}$. Then Euler compares the time intervals used to cover the distances Mm and $M\nu$, which must be the same, if the associated velocities are appropriately matched, so that this kinematic approach yields

$$\frac{\mathrm{d}s}{\sqrt{v} + \frac{\mathrm{d}v}{2\sqrt{v}}} = \frac{r\,\mathrm{d}s\,\mathrm{d}x + \mathrm{d}s^2\,\mathrm{d}y}{r\,\mathrm{d}x\,\sqrt{v}}\ . \tag{123}$$

Some algebraic reformulations lead to

$$\frac{r\,\mathrm{d}v\,\mathrm{d}s\,\mathrm{d}x}{2\sqrt{v}} + \mathrm{d}s^2\,\mathrm{d}y\,\sqrt{v} = 0\ , \tag{124}$$

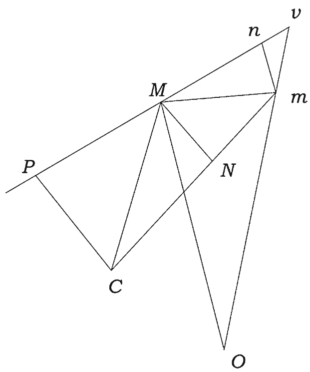


Fig. 21 Euler's second sketch, a detailed enlargement of Fig. 20, on fol. 27r in Ms 398 concerning the central force problem

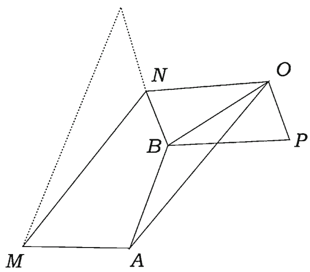


Fig. 22 Euler's sketch the relative motions of two bodies. The dotted lines were cancelled by Euler and thus have no meaning.

which may be solved for

$$-\frac{\mathrm{d}v}{v} = \frac{2\,\mathrm{d}s\,\mathrm{d}y}{r\,\mathrm{d}x} = \frac{2y\,\mathrm{d}s}{rp}\,, \tag{125}$$

where for the latter equation the equivalence of the triangles $\triangle CMP$ and $\triangle MmN$ was used by the relation $\frac{y}{p} = \frac{\mathrm{d}y}{\mathrm{d}x}$.

D2: The relevant records of Ms 398 concerning the relative motion of two bodies

Actually, this topic is connected with the motion in a mobile orbit. It is convenient, however, to present it separately because this issue treated here by Euler for the very first time probably leads to the principle of the transference of forces, which I will analyze in the next paper. Euler starts with the following proposition [I]: Given the uniform motions of two bodies A and B (cf. Fig. 22) along the directions AM and BN, respectively. Find the relative motion of body B, or, the motion as it appears when observed from body A. The solution is straightforward by a geometrical construction: When A has arrived in M, the body B will have reached N. Conjoin the lines AB and MN. Ex A draw the straight line AO parallel to MN which will end in O, which is the relative position of body B corresponding to the true place N. Moreover, the line BO represents the motion of body B as seen from A and its (relative) velocity. In a Corollary [I] Euler addresses the principle of superposition: Due to $NO = AM$ the line BO is the diagonal of the rectangle $NBPO$ being composed by the two laterals

Fig. 23 Euler's sketch of a mobile elliptic orbit

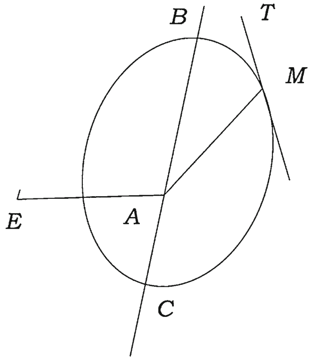
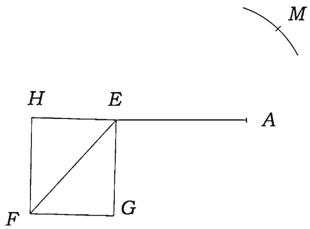


Fig. 24 Euler's sketch of the rectilinear motion from A to E and from E to F of the orbit's central body



BN and BP, which are produced by the forces respectively by the motions of the bodies B and A. And in Corollary II he adds, that the relative motion is performed in a straight line. In proposition II Euler is concerned with the motion of a body in an ellipsis which is subject to a uniform rectilinear motion of its focus. Let A be the attracting central body and M a body moving with such a velocity around A that it describes an ellipsis MBC (cf. Fig. 23). One asks for the ellipsis and the position of its axis described by the body M if the body A is set in motion along the line AE with a given uniform velocity. Euler did not solve this problem, but formulates proposition III instead: Let body A move along the line AE and body M describe an ellipsis around A (cf. Fig. 24). One asks for the ellipsis and the position of its axis the body M will describe if the direction of the central body A is altered into the line EF by a given [uniform] velocity. Euler seemed not to be able to solve this problem successfully at that time. He only gave a short speculative description of what is going on. After the body A has reached E it would proceed further to H. But now a second motion along EG happens in such a way that the former will be destroyed, and one may ask—at least for the case that the body is at rest—whether an ellipsis will result (it will be an ellipsis, Euler claims, if the body has started to move in a straight line).

D3: Miscellaneous records in Ms 398 related to lunar theory

Fol. 38v and fol. 50r of Ms 398 contain records which are strongly correlated with Ms 272 (cf. Sect. A). Euler summarizes the principle data and phenomena of the Moon's motion. On folio 38v, he noted:

> The Moon moves through the zodiac in the time interval of $27^{\mathrm{d}}7^{\mathrm{h}}43'$. On the other hand the Moon needs 27 days 13 h 12 min from apogee to apogee (Euler mistakenly wrote "Aphelium"). The synodic month lasts $29^{\mathrm{d}}12^{\mathrm{h}}44'$. When the synodic line (i.e., the syzygial line) coincides with the apsidal line the synodic inequalities will be maximal. If the syzygies ("synodus") will happen in the quadratures, there are no inequalities of the second kind. From the conjunction to the first quadrature they [the inequalities] have to be added to the Moon's motion. From quadrature to the opposition they have to be subtracted. They have to be added once again from the opposition to the quadrature, and subtracted from the quadrature to the conjunction. This is the third kind of inequality observed by Tycho. The line perpendicular to the major axis is called "diacentros" if it intersects in the center, [otherwise] "diagæos" if it intersects in the focus. From the apogee onwards the Moon covers each day the angle $13°3'54''$ around the Earth.

On folio 50r we find the following entries:

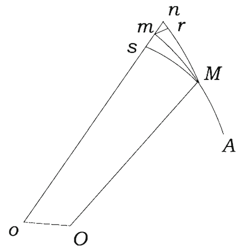
> The motion of the lunar apogee "in consequentia" [i.e., in the order of the signs] amounts to $3°3'$ in each revolution. The major axis ("axis transversus") of the Moon's orbit amounts according to the value corrected by Tycho to 121 Earth radii. The Earth's perimeter is 123249600 Parisian feet according to the French. The monthly revolution period of the Moon is $27^{\mathrm{d}}7^{\mathrm{h}}43'5''$. The daily motion [i.e., rotation period] of the Earth is $23^{\mathrm{h}}56'$, of the Moon $27^{\mathrm{d}}7^{\mathrm{h}}43'5''$ (I skip the data of other solar system bodies). The Earth's radius is 19615800 Parisian feet.

Appendix E: The records in Ms 397 and Ms 398 concerning the motion in a mobile orbit due to two simultaneously acting force centers

There are records in Euler's first two notebooks that document the very early moment when Euler tried to combine the concepts of central force motion, of the mobile orbit, and of the simultaneously acting force centers, concepts, which he treated in previous entries as separate problems. This approach, I may speculate, led Euler to the concept of the so-called *osculating ellipse*, which he defined and refined in his "Mechanica" until 1734 (cf. Euler 1736, Prop. 83, Schol. 2, and Prop. 84, Schol. 1). The notebook entries in Ms 397 and Ms 398 concerning this topic complement one another, thus proving their simultaneous emergence with respect to the dating of these records into the years 1727–1728.

On folios 45r and 44v of Ms 398 Euler tries to describe the motion of a body within a mobile orbit, i.e. an orbit changing its shape, position, and orientation with respect to an inertial reference frame, thus mimicing, e.g., an apsidal motion or a variation of orbital parameters, as we would call it today. Euler states the problem as follows (cf. Fig. 25): A body is moving in the curve AMm around the centers O and o, being either at rest or mobile. (The meaning of this, as it will turn out from the following context, is that O actually is one of two force centers which is moving uniformly on a straight line from O to o.) One asks for the motion of the body in the moving curve

Fig. 25 Euler's sketch in Ms 398 illustrating the central force motion of the body M due to a force center O which is moving to o



AM. Let the ratio between the central force in o and the central force of the Earth, i.e. the gravity, be as V to A. Let the velocity in AM be equal to the corresponding altitude v of free fall, and $v - \mathrm{d}v$ be the corresponding altitude associated with Mm. Draw the line Ms perpendicular to om. Then $A : V = ms : -\mathrm{d}v$, and thus

$$- \mathrm{d}v = \frac{V \cdot ms}{A} \ . \tag{126}$$

Furthermore, the ratio between $Mn(Mm)$ and rn is as the velocity Vv to the element $\frac{-\mathrm{d}v}{2\sqrt{v}}$, namely $Mm : rn = 2v : -\mathrm{d}v$, thus

$$- \mathrm{d}v = \frac{2v \cdot rn}{Mm} \ . \tag{127}$$

Due to $rn : mr = ms : Ms$ it follows

$$- \mathrm{d}v = \frac{2v \cdot mr \cdot ms}{Mm \cdot Ms} \ . \tag{128}$$

Equating this equation with Eq. (126) and setting the osculating radius in $M = r$, then

$$\frac{2v \cdot Mm \cdot ms}{r \cdot Ms} = \frac{V \cdot ms}{A} \tag{129}$$

and therefore

$$\frac{V}{A} = \frac{2v \cdot Mm}{r \cdot Ms} \ . \tag{130}$$

On the next folio 44v Euler tried to gain more insight into the situation by drawing and analyzing it in detail (cf. Fig. 26). In fact, Euler's figure fills out the whole page of

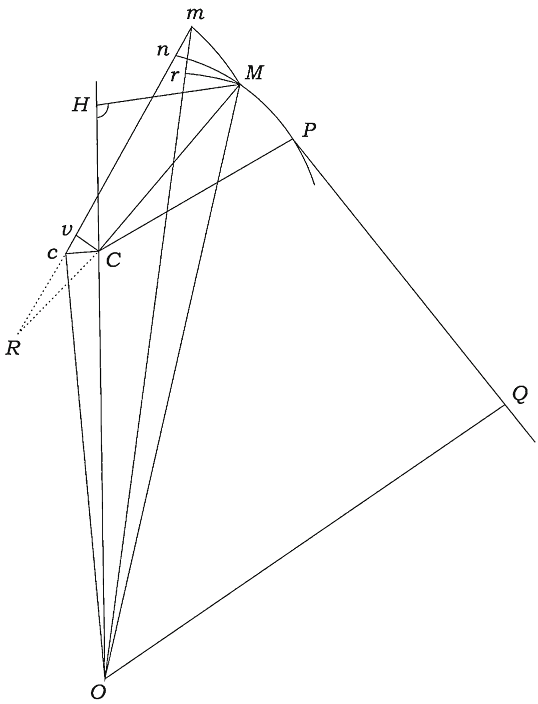


Fig. 26 Euler's sketch in Ms 398 illustrating the central force motion of the body M due to two simultaneous acting force centers C and O

his notebook. He defines the straight lines of the figure: $MH = t$, $OC = a$, $CM = y$, $OM = z$, and $mr = \mathrm{d}z$. Thus, $cv = \frac{t\,\mathrm{d}x}{y}$,

$$Cv = \frac{\mathrm{d}x\sqrt{yy - tt}}{y}\,, \qquad mn = \mathrm{d}y - \frac{t\,\mathrm{d}x}{y}\,, \tag{131}$$

and

$$Mn = \sqrt{\mathrm{d}s^2 - \left(\mathrm{d}y - \frac{t\,\mathrm{d}x}{y}\right)^2}\,, \qquad Mr = \sqrt{\mathrm{d}s^2 - \mathrm{d}z^2}\,. \tag{132}$$

The velocity with which the body M is moving corresponds to the altitude v, and the velocity with which C is moving corresponds to the altitude c (to be distinguished from the point c), so that $\sqrt{c} : \sqrt{v} = Cc : Mm$. But $Cc = \mathrm{d}x$, $\mathrm{d}y = cv + mn$, and $Mm = \frac{\mathrm{d}x\,\sqrt{v}}{\sqrt{c}} = \mathrm{d}s$. Before I proceed with presenting the rest of Euler's records in Ms 398 on that topic, it is useful first to show here his sketches of Ms 397 of the same topic in order to understand the meaning of the symbols introduced by him for this purpose. In Ms 397, fol. 132v, and Ms 398, fol. 43r, Euler defined the straight lines of the Figs. 27 and 28 illustrating the central force motion of L (Luna) due to two simultaneous acting force centers T (Terra) and S (Sol) in the cases where T is the center of the osculating radius in L and where it is not, respectively. For the former case, he wrote the parameters in parentheses.

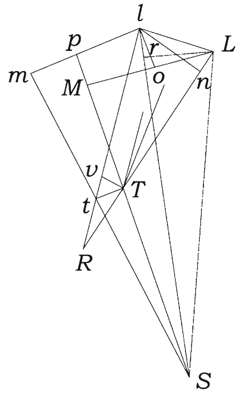


Fig. 27 Euler's sketch in Ms 397 illustrating the central force motion of L (Luna) due to two simultaneous acting force centers T (Terra) and S (Sol) in the case if T is the center of the osculating radius in L

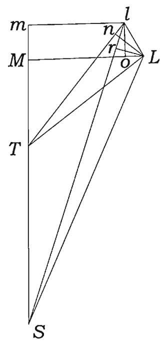


Fig. 28 Euler's sketch in Ms 397 illustrating the central force motion of L (Luna) due to two simultaneous acting force centers T (Terra) and S (Sol) in the case if T is not the center of the osculating radius in L

$$
\begin{array}{rcccl}
\multicolumn{5}{l}{\text{Definitions in Ms 397 :}} \\
TM & = & (TM) & = & x \\
tm & = & (Tm) & = & x + dx \\
LM & = & (LM) & = & y \\
lm & = & (lm) & = & y + dy \\
TL & = & (TL) & = & \sqrt{xx + yy} = z \\
tl & = & (Tl) & = & z + dz \\
ST & = & (ST) & = & a \\
SL & = & (SL) & = & \sqrt{yy + (a + x)^2} = t \\
Sl & = & (Sl) & = & t + dt \\
Tt & = & - & = & do \\
Mp & = & (Mm) & = & dx \\
\multicolumn{5}{l}{\text{Definitions in Ms 398 :}} \\
- & = & (Ll) & = & ds \\
- & = & (Lr) & = & \sqrt{ds^2 - dt^2} \\
LS & = & (LS) & = & t \\
lr & = & (lr) & = & dt \\
ST & = & - & = & a \\
Tt & = & - & = & do
\end{array}
\tag{133}
$$

Let V be the centripetal force in L, A the gravity, and r the osculating radius in L (not to confuse with the point r marked in the figures). Further let c be the altitude corresponding velocity of the body T and v the velocity of the body L. The Eq. 130 found above reads

$$\frac{V}{A} = \frac{2v \cdot Ll}{r \cdot ln}, \tag{134}$$

and Eq. 128 becomes

$$\mathrm{d}v = \frac{2v \cdot Ll \cdot Ln}{r \cdot ln} = \frac{V}{A} Ln \,. \tag{135}$$

Euler then derives in Ms 397 some relations which will be used later on:

$$pm = \frac{\overline{a+x} \cdot do}{a}, \quad tv = \frac{y \,\mathrm{d}o}{z}, \quad Tv = \frac{x \,\mathrm{d}o}{z}. \tag{136}$$

To find the distance TR of the osculating center from T, Euler derives the equations

$$\frac{Tv}{TR} = \frac{\mathrm{d}o}{a} + \frac{(Ln)}{z} = \frac{x \,\mathrm{d}o}{z \cdot TR} \tag{137}$$

and

$$z \cdot TR \cdot \mathrm{d}o + a \cdot (Ln) TR = ax \,\mathrm{d}o, \tag{138}$$

obtaining

$$TR = \frac{ax \,\mathrm{d}o}{z \cdot \mathrm{d}o + a(Ln)} \,. \tag{139}$$

He then formulates the relation between the two situations represented by the two figures:

$$\begin{aligned} LSl - TSt &= (LSl) \\ LRl - TSt &= (LTl) \\ tv - Ln &= (ln) \\ Lo - pm &= (Lo). \end{aligned} \tag{140}$$

Using these “shortcuts” he finds the next series of useful relations:

$$
\begin{aligned}
LR &= \frac{ax\,\mathrm{d}o}{z\,\mathrm{d}o + a(ln)} + z \\
Lo &= \frac{\overline{a+x}\,\mathrm{d}o}{a}\,\mathrm{d}y \\
lo - Mp &= \frac{y\,\mathrm{d}o}{a} \\
lo &= \mathrm{d}x + \frac{y\,\mathrm{d}o}{a}\,.
\end{aligned} \tag{141}
$$

The records concerning this topic reveal that Euler aims at finding a relation between the two osculating radii produced by the two central forces acting in S and in T, resulting in an equation for $\mathrm{d}o = Tt$ (or Cc in Ms 398, cf. Fig. 26). In Ms 398 he obviously found it in an apparently shorter way than in Ms 397. Using

$$
\begin{aligned}
\frac{Lr}{t} &= \frac{\mathrm{d}o}{a} + \frac{\sqrt{\mathrm{d}s^2 - \mathrm{d}t^2}}{t} \\
Lr &= \frac{t\,\mathrm{d}o}{a} + \sqrt{\mathrm{d}s^2 - \mathrm{d}t^2} \\
Ll^2 &= \frac{tt\,\mathrm{d}o^2}{aa} + \mathrm{d}s^2 + \frac{2t\,\mathrm{d}o}{a}\sqrt{\mathrm{d}s^2 - \mathrm{d}t^2}
\end{aligned} \tag{142}
$$

he derives from

$$
Ll^2 = \frac{v\,\mathrm{d}o^2}{c} \tag{143}
$$

the wanted parameter:

$$
\mathrm{d}o = \frac{\frac{t}{a}\sqrt{\mathrm{d}s^2 - \mathrm{d}t^2} \pm \sqrt{\frac{v\,\mathrm{d}s^2}{c} - \frac{tt\,\mathrm{d}t^2}{aa}}}{\frac{v}{c} - \frac{tt}{aa}}\,. \tag{144}
$$

In Ms 397 this process seemed to be quite more involved than in Ms 398: On folio 133r, Euler started with the relation

$$
TR : Tv = LR : ln \tag{145}
$$

and intended to derive $\mathrm{d}o$ from Ln resp. (Ln), for which he finds the equations

$$
(Ln) = \left((Ll)^2 - \mathrm{d}z^2\right) = \sqrt{\mathrm{d}x^2 + \mathrm{d}y^2 - \mathrm{d}z^2} = \frac{y\,\mathrm{d}x - x\,\mathrm{d}y}{\sqrt{xx+yy}} \tag{146}
$$

and

$$
Ln = \frac{y\,\mathrm{d}o}{z} - (ln)\,\mathrm{d}z = \frac{y\,\mathrm{d}o}{z} - \mathrm{d}z + (Ln)^2 + 2(Ln)\frac{ax\,\mathrm{d}o + zr\,\mathrm{d}o}{az}\,. \tag{147}
$$

However, Euler recognized, that the derivation of do seems to be more directly and thus more successfully by using (cf. Eq. 143 from Ms 398)

$$Ll = \frac{\mathrm{d}o\sqrt{v}}{\sqrt{c}} \quad \text{resp.} \quad Ll^2 = \frac{v\,\mathrm{d}o^2}{c}\ . \tag{148}$$

From the geometry of the figures Euler finds

$$Lo = \frac{\overline{a+x}\,\mathrm{d}o}{a}\,\mathrm{d}y \qquad \text{and} \qquad lo = \mathrm{d}x + \frac{y\,\mathrm{d}o}{a} \tag{149}$$

and therefore

$$Ll = \sqrt{\frac{\overline{a+x}^2\,\mathrm{d}o^2}{aa} + \mathrm{d}y^2 - \frac{2\,\mathrm{d}o\,\mathrm{d}y\cdot\overline{a+x}}{a} + \mathrm{d}x^2 + \frac{2y\,\mathrm{d}x\,\mathrm{d}o}{a} + \frac{yy\,\mathrm{d}o^2}{aa}}\ , \tag{150}$$

where $\sqrt{}$ concerns the whole expression on the right hand side of this equation. Equating the square of Ll with Eq. (148) yields the quadratic equation for do:

$$\frac{v\,\mathrm{d}o^2}{c} - \frac{(a+x)^2\,\mathrm{d}o^2}{aa} - \frac{yy\,\mathrm{d}o^2}{aa} = \frac{2y\,\mathrm{d}x\,\mathrm{d}o}{a} - \frac{2(a+x)\,\mathrm{d}y\,\mathrm{d}o}{a} + \mathrm{d}x^2 + \mathrm{d}y^2\ , \tag{151}$$

which Euler solves for do:

$$\mathrm{d}o = \frac{\frac{y\,\mathrm{d}x}{a} - \frac{\overline{a+x}\cdot\mathrm{d}y}{a} \pm \sqrt{-\frac{2y(a+x)\,\mathrm{d}x\,\mathrm{d}y}{yaa} - \frac{(a+x)^2\,\mathrm{dd}x}{aa} - \frac{yy\,\mathrm{d}y^2}{aa} + \frac{v\,\mathrm{d}x^2}{c} + \frac{v\,\mathrm{d}y^2}{c}}}{\frac{v}{c} - \frac{(a+x)^2}{aa} - \frac{yy}{aa}}\ . \tag{152}$$

The Eqs. (135) and (147) yield approximatively

$$\mathrm{d}v = \frac{V}{A}\left(\frac{x\,\mathrm{d}o}{z} - \mathrm{d}z\right)\ , \tag{153}$$

thus

$$\frac{V}{A} = \frac{2\,\mathrm{d}o\,v\sqrt{v}}{r\sqrt{c}\cdot\left(\frac{x\,\mathrm{d}o}{z} + \frac{z\,\mathrm{d}o}{a} + \frac{z\,\mathrm{d}x - x\,\mathrm{d}z}{\sqrt{zz-xx}}\right)}\ . \tag{154}$$

If the Earth's radius is assumed to be 1 and $\frac{V}{A} = \frac{1}{zz}$, Euler finally gains

$$\mathrm{d}o = \frac{\frac{zz\,\mathrm{d}x - az\,\mathrm{d}z - xz\,\mathrm{d}z + ax\,\mathrm{d}x}{a\sqrt{zz-xx}} \pm \sqrt{\frac{v}{c}\,\mathrm{d}s^2 - \left(\frac{z\,\mathrm{d}z}{a} + \mathrm{d}x\right)^2}}{\frac{v}{c} - 1 - \frac{2x}{a} - \frac{zz}{aa}}\ , \tag{155}$$

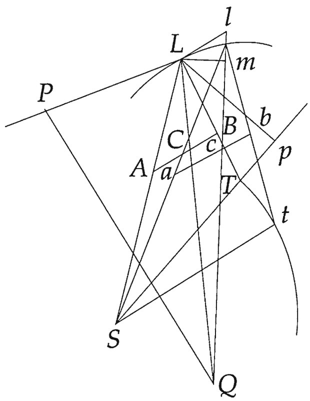


Fig. 29 Euler's sketch in Ms 397 illustrating the geometric constellation and relationship between the Moon L (Luna), the Earth T (Terra) and the Sun S (Sol) in terms of the resulting osculating radius R in L

which is in accordance with Eq. (144) if t and $\mathrm{d}t$ are substituted correspondingly.

Euler continued this line of thought on folio 202v of Ms 397 (cf. Fig. 29). He starts with the following definitions: $AL = a$, $LB = b$, the velocity of $L = v$, the velocity of $T = c$, $LC = x$, $LQ = z$. He then defines the differentials $Tt = \mathrm{d}r$, $Ll = \frac{v\,\mathrm{d}r}{c}$, and $lm = \mathrm{d}z$. Using the equations

$$\frac{\mathrm{d}v}{\mathrm{d}r} = -\frac{x\,\mathrm{d}z}{v\,\mathrm{d}r} \quad \text{and} \quad v\,\mathrm{d}v = -x\,\mathrm{d}z \tag{156}$$

and denoting the osculating radius in L with R, Euler derives the equation

$$v^3\,\mathrm{d}r = Rx\sqrt{vv\,\mathrm{d}r^2 - cc\,\mathrm{d}z^2}\ . \tag{157}$$

Let $QP = p$ be the distance between the center of the osculating radius in L and the perpendicular to the tangent through L. Then the following relations hold:

$$z : p = \frac{v\,\mathrm{d}r}{c} : \frac{\sqrt{vv\,\mathrm{d}r^2 - cc\,\mathrm{d}z^2}}{c} \tag{158}$$

and

$$vvz = Rxp\ , \tag{159}$$

where

$$R = \frac{z\,\mathrm{d}z}{\mathrm{d}p}\ , \tag{160}$$

therefore

$$vvz = \frac{zxp\,\mathrm{d}z}{\mathrm{d}p} \quad \text{or} \quad vv\,\mathrm{d}p = px\,\mathrm{d}z\ . \tag{161}$$

Now Euler wants to determine the Moon's velocity in terms of the parameters introduced above and of these new ones: $Lp = q$, $ST = e$, $SL = y$, $LT = w$, thus $Sp = \sqrt{yy - qq}$, $Tp = \sqrt{ww - qq}$, and $\sqrt{ww - qq} + e = \sqrt{yy - qq}$. For this purpose he compares the distances $LT = w$ and lt between the Moon and the Earth at a given epoch and after elapsing a short time element, respectively. He obtains

$$lt = \frac{w\sqrt{\frac{vv\,\mathrm{d}r^2}{cc} - \mathrm{d}w^2 + \frac{2q\,\mathrm{d}w\,\mathrm{d}r}{w} - \frac{qq\,\mathrm{d}r^2}{ww}}}{\sqrt{\frac{vv\,\mathrm{d}r^2}{cc} - \mathrm{d}w^2 + \frac{2y\,\mathrm{d}w\,\mathrm{d}r}{w} - \frac{yy\,\mathrm{d}r^2}{ww} - \frac{\mathrm{d}r\sqrt{ww-qq}}{w}}}\,. \tag{162}$$

Using Keill's theorem $x\,\mathrm{d}z = \frac{A\,\mathrm{d}p}{p^3}$ and the relative motions m and n introduced earlier in the notebook (cf. Sect. C2) Euler derives a series expansion of v, which he left, however, unfinished:

$$v = \frac{\overline{m-n}\cdot a\,\mathrm{d}y}{n\,\mathrm{d}s} + \frac{ma}{n^2\,\mathrm{d}s^2}\overline{m-n}\,a\,ddy + \frac{m^2a^2}{n^3\,\mathrm{d}s^3}\overline{m-n}\,a\,d^3y \tag{163}$$

which is

$$v = \frac{\overline{m-n}\cdot a\,\mathrm{d}y}{n\,\mathrm{d}s} + \frac{ma}{n\,\mathrm{d}s}\left(\frac{\overline{m-n}\,a\,ddy}{n\,\mathrm{d}s}[\ldots]\right. \tag{164}$$

or

$$\frac{v}{m-n} = \frac{a\,\mathrm{d}y}{n\,\mathrm{d}s} + \frac{ma^2\,ddy}{n^2\,\mathrm{d}s^2} + \frac{m^2a^3\,d^3y}{n^3\,\mathrm{d}s^3} + c + \left([\ldots] + \frac{ma}{n\,\mathrm{d}s}\left([\ldots]\right.\right. \tag{165}$$

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