THE SUBCONVEXITY PROBLEM FOR GL$_2$

by Philippe MICHEL[*] *and* Akshay VENKATESH[**]

ABSTRACT

Generalizing and unifying prior results, we solve the subconvexity problem for the L-functions of GL$_1$ and GL$_2$ automorphic representations over a fixed number field, uniformly in all aspects. A novel feature of the present method is the softness of our arguments; this is largely due to a consistent use of canonically normalized period relations, such as those supplied by the work of Waldspurger and Ichino–Ikeda.

CONTENTS

1. Introduction

1.1. *The subconvexity problem*

We refer the reader who is not familiar with L-functions to Section 1.2 for an introduction, in explicit terms, to some of the ideas of this paper.

Throughout this paper, F denotes a fixed number field and $\mathbf{A}$ its ring of adèles. For π an automorphic representation of $\mathrm{GL}_n(\mathbf{A})$ (with unitary central character, not necessarily of finite order), Iwaniec and Sarnak have attached an *analytic conductor* $\mathrm{C}(\pi) \in \mathbf{R}_{\geqslant 1}$; it is the product of the usual (integer) conductor with a parameter measuring how large the archimedean eigenvalues are. More intrinsically, the logarithm of the conductor is proportional to the density of zeros of the corresponding L-function. See Section 4.1.

The *subconvexity problem* is concerned with the *size* of $\mathrm{L}(\pi, s)$ when $\Re s = 1/2$: it consists in improving over the so-called *convexity bound* (see [37, 50] for instance)

$$\mathrm{L}(\pi, 1/2) \ll_{n, \mathrm{F}, \varepsilon} \mathrm{C}(\pi)^{1/4 + \varepsilon}$$

for any[1] $\varepsilon > 0$. The main result of the present paper is the resolution of this problem for GL_1 and GL_2-automorphic representations:

Theorem **1.1.** — *There is an absolute constant $\delta > 0$ such that: for π an automorphic representation of $\mathrm{GL}_1(\mathbf{A})$ or $\mathrm{GL}_2(\mathbf{A})$ (with unitary central character), one has*

$$\mathrm{L}(\pi, 1/2) \ll_{\mathrm{F}} \mathrm{C}(\pi)^{1/4 - \delta}.$$

Remark. — Contrary to appearance, this also includes the question of growth along the critical line, i.e. what is called the *t*-aspect, because

$$\mathrm{L}(\pi, 1/2 + it) = \mathrm{L}(\pi \otimes |\cdot|_{\mathbf{A}}^{it}, 1/2).$$

For example, an interesting corollary is a subconvex bound for the L-function of a Maass form with eigenvalue $1/4 + \nu^2$ at the point $t = 1/2 + i\nu$. Another corollary is a subconvex bound (in the discriminant) for the Dedekind L-function of a cubic extension of F (cf. [20] for an application of the latter).

The above result is a specialization (by taking π_2 to be a suitable Eisenstein series) of the following more general result

Theorem **1.2.** — *There is an absolute[2] constant $\delta > 0$ such that: for π_1, π_2 automorphic representations on $\mathrm{GL}_2(\mathbf{A}_{\mathrm{F}})$ we have[3]:*

$$(\mathbf{1.1}) \qquad \mathrm{L}(\pi_1 \otimes \pi_2, 1/2) \ll_{\mathrm{F}, \pi_2} \mathrm{C}(\pi_1 \otimes \pi_2)^{1/4 - \delta};$$

more precisely, the constant implied depends polynomially on the discriminant of F (for F varying over fields of given degree) and on $\mathrm{C}(\pi_2)$.

---

[1] Recently, Heath-Brown has established a general convexity bound [30], which together with the work of Luo, Rudnick, Sarnak [47] implies the clean convexity bound $\mathrm{L}(\pi, 1/2) \ll_{n, \mathrm{F}} \mathrm{C}(\pi)^{1/4}$.

[2] Independent of the number field F.

[3] More precisely we prove the bound $\mathrm{L}(\pi_1 \otimes \pi_2, 1/2) \ll_{\mathrm{F}, \pi_2} \mathrm{C}(\pi_1)^{1/2 - 2\delta}$. That the latter implies the former is a consequence of the bounds in [10].

The value of δ is easily computable. We have not attempted to optimize any exponent, our goal in this paper being of giving clean proofs in a general context.

Remark. — The bound generalizes (up to the value of δ) a variety of subconvex bounds [6, 8, 9, 12, 16–19, 22, 27, 29, 31, 34, 41, 43, 45, 48, 49, 63, 65, 68]. Its main feature, however, is its uniformity in the various possible parameters (the so-called "conductor", t-aspect, or "spectral" aspects): such bounds are sometimes called "hybrid". The first such hybrid bound is that of Heath-Brown [29] for Dirichlet character L-functions; recent hybrid bounds are to be found in the work of Jutila and Motohashi [41] and in the work of Blomer and Harcos [8]. The present work generalizes these also, again, up to the value of δ.

However, in saying this we have done an injustice to some of those papers; in some cases, the very point was to obtain the best exponent, whereas our emphasis is quite different. For example, let us compare the present result to that of [41]. This paper gives, in particular, the uniform bound for the value $L(\varphi_\lambda, \frac{1}{2} + it) \ll (|t| + |\lambda|)^{1/3}$ where φ_λ is a Maass form. This bound is very strong—the same exponent as the classical Weyl bound; on the other hand, it fails to be subconvex when $t \sim \lambda$, where the conductor drops. The present work fills this lacuna and provides a subconvex bound for that "critical point"; as far as we are aware this subconvex bound is new even over **Q**. On the other hand, while our method presumably leads to a respectable δ, it would not be so strong as the result of [41], one reason being that we are using the *amplification method*. It is also worth observing that the phenomena of the conductor dropping often leads to major difficulties, both in our methods and other treatments.

Remark. — It is reasonable to ask what one hopes by studying hybrid settings—especially given that many applications of subconvexity do not require them.

It is generally believed that the analytic behavior of L-functions are "universal," in that they are controlled by a single scaling parameter, the analytic conductor C [37, 62]. Taking $C \to \infty$ in different ways can correspond to analysis of eigenfunctions of large eigenvalue; analysis of eigenfunctions on a surface of large volume; or sections of a highly ample holomorphic bundle, and the reasons why these should all have similar asymptotic behavior is not clear.

We may hope to achieve some insight into this "universality" by studying hybrid phenomena. Indeed, at many points in the text, the reader will note the close parallel between increasing the level at finite primes, and increasing the eigenvalue at archimedean primes.

Our methods were outlined in our ICM announcement [51]. In particular we do not use trace formulas of any kind. An additional feature (originating in [49]), is that a *special case* of Theorem 1.2 (the case of π_1 a character) enters the proof of the full theorem. That special case is proven in Theorem 5.1, and is based (following [65]) on a study of the equidistribution of cycles on adelic quotients.

We have also tried to make use of the following ideas to simplify the proof (both notationally and conceptually):

– Sobolev norms (cf. [4]; in the adelic context [65]);
– Canonically normalized period formulas (see [33, 67]);
– Regularization of integrals of automorphic forms (we give a self-contained treatment that avoids truncation).

There remain many interesting questions related to the subconvexity story even for GL_2. For instance, the "approximate functional equation" gives a way to numerically compute any given L-function at the central point, in time $C^{1/2+\varepsilon}$, where C is the analytic conductor. An interesting question is whether some of the ideas that enter into the proof of subconvexity can be interpreted to give faster algorithms. There is some suggestion of this in existing fast algorithms for computation of $\zeta(1/2 + it)$.

1.1.1. *An outline of the proof.* — To conclude this section we outline the proof of the main theorem 1.2; a more elementary discussion is in Sections 1.2 and 1.4. Consider two (generic) automorphic representations π_1, π_2; for simplicity we assume that both are cuspidal[4] and that π_2 is fixed. We aim for a subconvex bound for the Rankin/Selberg L-function central value of the form

$$\frac{L(\pi_1 \otimes \pi_2, 1/2)}{C(\pi_1 \otimes \pi_2)^{1/4}} \ll_{\pi_2} C(\pi_1)^{-\delta}, \ \delta > 0.$$

By Rankin/Selberg theory we may realize the left-hand side above as a triple product period $\int_{PGL_2(\mathbf{Q})\backslash PGL_2(\mathbf{A})} \varphi_1 \varphi_2 E(g) dg = \langle \varphi_1, \overline{\varphi_2 E}\rangle$ for suitable $\varphi_i \in \pi_i$ ($i = 1, 2$) and E belonging to the Eisenstein series of type "$1 \boxplus \chi$," where χ is the inverse of the product of the central characters of π_1 and π_2. Now $|\langle \varphi_1, \overline{\varphi_2 E}\rangle|^2$ is bounded by

$$(\mathbf{1.2}) \qquad \langle \varphi_2 E, \varphi_2 E\rangle = \langle \varphi_2 \overline{\varphi_2}, E\overline{E}\rangle,$$

and it suffices to show this is at most $C(\pi_1)^{-\delta}$. Now, $E\overline{E}$ is not square integrable and one needs a regularized version of the above inner product; this is described in Sections 4.3 and 4.4. This being done, we obtain by spectral expansion (see Section 4.3.8)

$$(\mathbf{1.3}) \qquad \langle \varphi_2 \overline{\varphi_2}, E\overline{E}\rangle_{reg} = \langle \varphi_2 \overline{\varphi_2}, \Phi\rangle_{reg} + \int_{\pi} \sum_{\mathcal{B}(\pi)} \langle \varphi_2 \overline{\varphi_2}, \varphi\rangle_{reg} \langle \varphi, E\overline{E}\rangle_{reg} d\mu_P(\pi)$$

where the subscript *reg* denotes regularized inner products (one has of course $\langle \varphi_2 \overline{\varphi_2}, E\overline{E}\rangle = \langle \varphi_2 \overline{\varphi_2}, E\overline{E}\rangle_{reg}$) and Φ is a certain non-unitary Eisenstein series, which is an artifact of the integral regularization; on the other hand, π varies over the automorphic dual of PGL_2, $d\mu_P$ is a "Plancherel measure," and φ varies over an orthonormal basis, $\mathcal{B}(\pi)$, of factorizable vectors in π. Now:

---

[4] However, we devote some time and effort to handling the necessary regularizations in the general case; to our surprise, these modifications are not ugly but rather beautiful.

1. The term $\langle \varphi_2 \overline{\varphi_2}, \Phi \rangle_{reg}$ is handled via the amplification method (Section 5.2.9).
2. The terms $\langle \varphi, E\overline{E} \rangle_{reg}$ are bounded by some negative power of $C(\pi_1)$ (Section 5.2.8). Rankin/Selberg theory implies that $\langle \varphi, E\overline{E} \rangle_{reg}$ factor into a product of local integrals to which bounds for matrix coefficient can be applied (see below) *times* the central value $L(\pi, 1/2)L(\pi \times \chi, 1/2)$; we eventually need only bounds for $L(\pi \times \chi, 1/2)$, where π is essentially fixed[5] and χ varying.

The phenomenon of reduction to another case of subconvexity was noted by the first-named author in [49]. We establish this case in Theorem 5.1 (again, in all aspects); the proof generalizes [65], and we refer to the introduction of [65] and to Section 1.2 for intuition about it.

1.1.2. *Local computations.* — Let us be more precise about the local integrals that occur in, e.g., the Rankin-Selberg method. For our purpose they must be examined carefully; we are particularly interested in their *analytic* properties, i.e., how large or small they can be. This is the purpose of Part III of the paper; we deal also with the local Hecke integrals.

For example, the local integral occurring in the Rankin-Selberg method can be interpreted as a linear functional ℓ on the tensor product $\pi_1 \otimes \pi_2 \otimes \pi_3$ of three representations π_j of $GL_2(k)$. The space of such functionals is at most one dimensional; $|\ell|^2$ is therefore proportional to the Hermitian form

$$x_1 \otimes x_2 \otimes x_3 \mapsto \int_{PGL_2} \langle x_1, g.x_1 \rangle \langle x_2, g.x_2 \rangle \langle x_3, g.x_3 \rangle dg,$$

and this is of tremendous utility for the analytic theory. Similarly, when studying the Hecke integral, it is most convenient to study the Hermitian form defined by $x \mapsto \int_{A(k)} \langle a.x, x \rangle da$ where A the group of diagonal matrices (modulo the center).

It is a wonderful observation of Waldspurger (see also [33]) that global period formulas become very simple when expressed using such canonically normalized local functionals.

1.1.3. *Spectral identities of* L*-functions.* — The identity (1.2), although evident, is the keystone of our argument. Its usage in the "period" form presented above seems to have been noticed independently by the present authors in their attempt to geometrize [49], and by Bernstein and Reznikov in their work on the subconvexity problem for triple product L-functions [5].

However (1.2) also manifests itself at the level of L-functions, and looks rather striking in this guise: indeed (1.3) may be recognized as an identity between (weighted) sums

---

[5] As it turns out, the vector $\varphi_2 \in \pi_2$ depends only on π_2 (up to archimedean components, this is just the new vector), as the latter is fixed, the quantities $\langle \varphi_2 \overline{\varphi_2}, \varphi \rangle_{reg}$ decay very rapidly as the eigenvalue or level of φ increases. So, we may regard φ as essentially fixed.

of central values of triple product L-functions and (weighted) sums of "canonical" square roots of similar central values:

$$\int_{\pi_1} w(\pi_1) \mathrm{L}(\pi_1 \otimes \pi_2 \otimes \pi_3, 1/2) d\mu_{\mathrm{P}}(\pi_1)$$

$$= \int_{\pi} \tilde{w}(\pi) \sqrt{\mathrm{L}(\pi_2 \otimes \tilde{\pi}_2 \otimes \pi, 1/2)} . \sqrt{\mathrm{L}(\pi_3 \otimes \tilde{\pi}_3 \otimes \pi, 1/2)} d\mu_{\mathrm{P}}(\pi).$$

In this form, this identity was discovered already by N. Kuznetsov [44] (with π_2, π_3 Eisenstein series see also [54]) and an interesting application of this result was made by M. Jutila [40]. This period identity has—implicitly or explicitly—played an important role in the analytic theory of GL_2 forms. We refer to Section 4.5 for some more discussion of how to convert (1.2) to an identity of L-functions.

In fact, it was shown by A. Reznikov (in the archimedean setting at least), that such phenomena is not isolated, and may be systematically described through the formalism of *strong Gelfand configurations*: that is commutative diagrams of algebraic groups
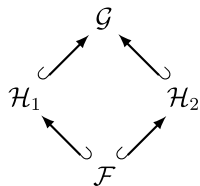
in which the pairs $(\mathcal{F}, \mathcal{H}_i)$ and $(\mathcal{H}_i, \mathcal{G})$ are strong Gelfand pairs. The present paper corresponds to the configuration $\mathcal{G} = \mathrm{GL}_2 \times \mathrm{GL}_2 \times \mathrm{GL}_2 \times \mathrm{GL}_2$, $\mathcal{H}_i = \Delta \mathrm{GL}_2 \times \Delta \mathrm{GL}_2$ with two different diagonal embeddings and $\mathcal{F} = \Delta \mathrm{GL}_2$ (diagonally embedded). As is explained in [60], such strong Gelfand configurations yield naturally to spectral identities between periods and then between L-functions. We refer to *loc. cit.* for more interesting examples of that sort.

1.1.4. *Structure of the paper; reading suggestions.* — The paper splits into five parts; the first four parts are largely independent of each other, and contain various results of independent interest. The fifth part brings together these results to prove the main theorem.

The reader may wish to skip directly to Sections 4 and 5 of the paper; the results of Sections 2 and 3 are largely technical and not particularly surprising.

– In the remainder of Sections 1, 1.2, 1.4 we consider two corollaries to our main theorem—which can both be phrased *without* L-functions—and we explain how the proofs work in these instances. Indeed, the general proof is obtained by adelizing and combining these two particular cases. The corollaries we consider are:

1. In Section 1.2, we discuss the "Burgess bound," which relates to the issue of the smallest quadratic non-residue modulo a prime q.
2. In Section 1.4, we discuss a problem in analysis on a negatively curved surface, viz.: how large can the Fourier coefficients of an eigenfunction along a closed geodesic be?

– Section 2 (viz. Sections 2.2 to 2.6) is of more general nature: we discuss a system of Sobolev norms on adelic quotients, inspired largely by work of Bernstein and Reznikov. This section exists to give a suitable language for talking about adelic equidistribution, and the norms are a somewhat cleaner version of those appearing in [65]. Some of the remarks here are of independent interest, although they are of technical nature.
– Section 3 discusses some of the analytic theory of torus-invariant functionals on a representation of GL$_2(k)$ (where k is a local field), and of trilinear functionals on representations of GL$_2(k)$.
– Section 4 discusses the global theory of torus periods on GL$_2$ and the diagonal period on GL$_2 \times$ GL$_2 \times$ GL$_2$.
– Section 5 gives the proof of Theorem 1.2 along with the important intermediary result Theorem 5.1 (a subconvex bound of L-function of character twist uniform in the character aspect).

1.2. *The Burgess bound and the geometry of* SL$_2(\mathbf{Z}) \backslash \mathbf{H}$

In this section and also in Section 1.4 we present some of the ideas of the general proof in the most down-to-earth setting as we could manage. In both of these sections, we have by and large eschewed mention of L-functions.

1.2.1. *The Burgess bound.* — Let χ be a Dirichlet character to the modulus q. It is well-known that subconvexity for the Dirichlet L-function $L(\chi, 1/2)$, in the q-aspect, is substantively equivalent to a bound of the nature:

$$(\mathbf{1.4}) \qquad \left| \sum_{i=1}^{M} \chi(m) \right| \leqslant M q^{-\delta},$$

with $M \approx q^{1/2}$ and some absolute constant $\delta > 0$.

In this question, $q^{1/2}$ is a "threshold": indeed it is rather easy to establish (1.4) when $M = q^{0.51}$ (the Polya-Vinogradov inequality). The bound (1.4), for M in the range $M \approx q^{1/2}$ was proven (in a wider range) by Burgess [12].

In the present section (see also [66])—which, we hope, will make sense to the reader without any knowledge of L-functions—we explain how (1.4) is related to an equidistribution equation on the space of lattices, and then discuss how to prove the

uniform distribution statement. A key part of the paper—Section 5.1—will implement the discussion of this section in a more general context.

To simplify that discussion, we assume for the rest of this section that q is prime and that χ is the Legendre symbol.

1.2.2. *The space of lattices.* — Put $X = SL_2(\mathbf{Z}) \backslash SL_2(\mathbf{R})$, the space of unimodular lattices in $\mathbf{R}^2$.

We say a sequence of finite subsets $S_i \subset X$ is becoming uniformly distributed if, for any $f \in C_c(X)$, we have

$$\frac{1}{|S_i|} \sum_{S_i} f \to \int_X f,$$

the latter integral being taken with respect to the unique $SL_2(\mathbf{R})$-invariant probability measure on X.

1.2.3. *Burgess bound and lattices.* — For $x \in \mathbf{R}$ consider the lattice

$$\Lambda_x = \frac{1}{\sqrt{q}} \big(\mathbf{Z}.(1, x) + \mathbf{Z}.(0, q) \big) \in X.$$

As x varies, Λ_x moves on a horocycle in X—an orbit of the group of upper triangular, unipotent matrices. This horocycle is in fact *closed*, since $\Lambda_{x+q} = \Lambda_x$.

Given $0 < \eta < 1$, let $F : X \to \mathbf{R}$ be defined by $F(L) = |L \cap [0, \eta]^2| - 1$, i.e. F counts the number of non-trivial lattice points in a small square box. A simple computation shows that for $x \in \mathbf{Z}$, $F(\Lambda_x)$ equals the number of nonzero solutions $(\alpha, \beta) \in [0, \eta\sqrt{q}]^2 \cap \mathbf{Z}^2$ to the equation

$$\beta \equiv \alpha x \bmod q.$$

It follows that

$$\frac{1}{q} \sum_{x \bmod q} F(\Lambda_x) \chi(x) = \frac{1}{q} \Big| \sum_{\beta \in [1, \eta\sqrt{q}]} \chi(\beta) \Big|^2.$$

Now, the Burgess bound would follow if we knew that the right-hand side was small. The above identity therefore relates the Burgess bound to a type of equidistribution statement: we must show that the sets

(**1.5**) $\{\Lambda_x : x \in \mathbf{Z}/q\mathbf{Z}$ is a quadratic residue$\}$,

and the similar set for quadratic nonresidues, are uniformly distributed[6] on X.

---

[6] Strictly speaking, the function F is not of compact support on X; in fact, it grows at the cusps. We shall ignore this technical detail for the purpose of explanation.
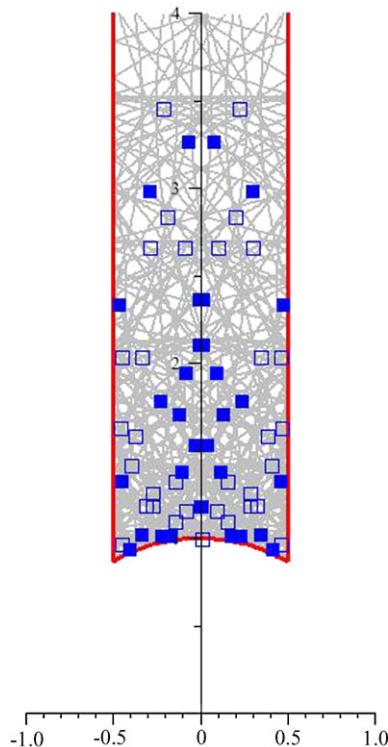
FIG. 1. — The horocycle $\frac{x}{173} + \frac{i}{173}$. Gray dots: $x \in \mathbf{R}$, plain (resp. empty) square $x \in \mathbf{Z}$ a quadratic (resp. non-quadratic) residue mod 173

Remark. — This connection between the bound (1.4), and a uniform distribution statement on the space of lattices, is not an accident: it is a special case of the connection between L-functions and automorphic forms. Indeed, the uniform distribution (1.5) encodes much more than (1.4): it encodes, at once, subconvex bounds for twists $L(\frac{1}{2}, f \times \chi)$ where f is a fixed SL$_2$($\mathbf{Z}$)-modular form (of any weight); the latter specializes to the former when f is an Eisenstein series. Our point above, however, is that the connection between (1.4) and the space of lattices can be made in an elementary way.

It is possible to visualize the desired uniform distribution statement by projecting from the space of lattices to $Y := SL_2(\mathbf{Z}) \backslash \mathbf{H}$ (at the price of losing the group actions). The lattice Λ_x projects to the class of $z_x = \frac{i}{q} + x$.

1.2.4. *Equidistribution statements.* — We shall try to establish (1.5) by first proving uniform distribution of a "bigger" set, and then refining that statement. Consider, then, the following three equidistribution statements, as $q \to \infty$:

(A) The closed horocycle $\{\Lambda_x : x \in [0, q]\}$ becomes uniformly distributed on X;
(B) $\{\Lambda_x : x \in \mathbf{Z} \cap [0, q]\}$ becomes uniformly distributed on X;

(C) $\{\Lambda_x : x \in \mathbf{Z} \cap [0, q]\}$ becomes u.d. on X, *when each point Λ_x is weighted by $\chi(x)$.*

For χ the quadratic character modulo q, we might rewrite (C) as:

(C2) $\{\Lambda_x : x \in \mathbf{Z} \cap [0, q]$, x a quadratic residue mod $q\}$ becomes u.d. on X.

We have already discussed informally, and it is true in a very precise sense, that (C) and (C2) have substantively the same content as the subconvexity result that we are aiming for. Note that (C) and (C2) seem "unnatural" at first; it seems like z_x is parameterized by an *additive* structure, i.e. $\mathbf{Z}/q\mathbf{Z}$; thus it is odd to restrict attention to a "multiplicatively" defined set. But in truth the examples of (C) and (C2) have—as we shall see—an underlying "multiplicative" symmetry; the fact that it appears additive is a reflection of the degeneration of a torus in GL_2 to a unipotent group.

Observe that (A), (B), (C2) are asserting the equidistribution of smaller and smaller sets. So what we need, besides a proof of (A)—which happens to be an old result of Peter Sarnak—is a method to pass from the equidistribution of a large set, to the equidistribution of a smaller subset. It is provided by the following easy principle (cf. [66] for a further discussion of its applicability in this type of context):

Ergodic principle **1.3.** — *Suppose a group* H *acts ergodically on a probability space* (X, μ), *and ν is an* H*-invariant measure which is dominated by some positive multiple of μ. Then ν is a scalar multiple of μ.*

By "ν is dominated by some positive multiple of μ" (written $\nu \ll \mu$) we mean that there is a constant A such that for any measurable set $\mathrm{S} \subset \mathrm{X}$, $|\nu|(\mathrm{S}) \leqslant \mathrm{A}\mu(\mathrm{S})$. Indeed, if this is so, ν is absolutely continuous with respect to μ, and thus may be expressed as $f\mu$ for some $f \in \mathrm{L}^1(\mu)$; then f must be a H-invariant function, necessarily constant by ergodicity. The principle is therefore trivial; on the other hand, its consequences in the number-theoretic context are surprising.

Using the principle, we can pass from the equidistribution of μ to the equidistribution of ν. (Of course, one needs a more quantitative form of this principle; we enunciate such a form in the context we need in Section 2.5.3.)

By applying the ergodic principle to the group $\{n(t) : t \in \mathbf{Z}\}$, we establish the implication (A) $\implies$ (B). To show that (B) $\implies$ (C) is a little more subtle, because the choice of H is not clear; we will need to pass to a covering to uncover it! We now discuss this in more detail.

1.3.1. *The entry of the adèle group and the implication* (B) $\implies$ (C). — It is well-known that the inverse limit (over the principal congruence subgroups)

$$\widetilde{\mathrm{X}} := \varprojlim_{q} \Gamma(q) \backslash \mathrm{SL}_2(\mathbf{R})$$

carries not only an action of $\mathrm{SL}_2(\mathbf{R})$ but of the much larger group $\mathrm{SL}_2(\mathbf{A_Q})$, where $\mathbf{A_Q}$ is the adele ring of $\mathbf{Q}$; there is a natural projection $\pi : \widetilde{\mathrm{X}} \to \mathrm{SL}_2(\mathbf{Z}) \backslash \mathrm{SL}_2(\mathbf{R})$.

Let ν be the measure implicit in (C), that is to say, $\nu = \sum_{x=1}^{q}(\frac{x}{q})\delta_{\Lambda_x}$. As we have commented, ν has no apparent invariance.

However, there is a closed subgroup $H^{(1)} \subset SL_2(\mathbf{A_Q})$, *and a* $H^{(1)}$-*equivariant measure* $\tilde{\nu}$ *on* $\tilde{X}$, *which projects to* ν. In other terms, the measure of (C) *acquires* invariance after lifting to the adeles. (The group $H^{(1)}$ admits a surjection onto $(\mathbf{Z}/q\mathbf{Z})^{\times}$, and this surjection is compatible with the natural action of $(\mathbf{Z}/q\mathbf{Z})^{\times}$ on the sets in (B), (C)). Now, by a suitable application of the ergodic principle on $\tilde{X}$, rather than X, we deduce that (B) $\implies$ (C).

In this way, the role of adeles in our proof is not merely to provide a convenient language, but also the group actions that we use simply do not exist at the level of $SL_2(\mathbf{Z})\backslash SL_2(\mathbf{R})$—or rather, only their shadows, the Hecke operators, are visible.

Remark. — (The adeles for dynamicists). A somewhat more intuitive way of constructing this is as follows: fix a prime p, and consider the pth Hecke operator T_p on X. It is a multi-valued function.

We can *formally* turn it into an invertible single-valued function by considering the space of sequences:

$$\tilde{X}_p = (\ldots, x_{-2}, x_{-1}, x_0, x_1, \ldots) \in X^{\mathbf{Z}}, x_{i+1} \in T_p x_i \quad \text{for all } i.$$

Then the "shift" operation $S : \tilde{X}_p \to \tilde{X}_p$ can be considered as a version of T_p that has been forced to be invertible.

It can be verified that $\tilde{X}_p$ is isomorphic to the quotient:

$$SL_2(\mathbf{Z}[p^{-1}])\backslash SL_2(\mathbf{R}) \times SL_2(\mathbf{Q}_p)/M,$$

where M is the subgroup of diagonal matrices in $SL_2(\mathbf{Q}_p)$ whose entries belong to $\mathbf{Z}_p^{\times}$. Moreover, S is identified with the right action of a diagonal matrix in $SL_2(\mathbf{Q}_p)$.

If we imitate this procedure for *all primes simultaneously*, one is naturally led to the space $\tilde{X}$.

1.4. *Geodesic restriction problems*

In this section, we present another corollary to our main results and discuss the idea of its proof, again, largely without mention of L-functions. This section is phrased in the language of analysis on a Riemannian manifold. Our discussion can be considered a variation on the sketch of proof that was already presented in our ICM article [51].

1.4.1. *Geodesic restriction problems: the results of [11].* — Let M be a Riemannian surface of finite volume, with Laplacian Δ_M, and let φ_λ be an eigenfunction of Δ_M with eigenvalue $-\lambda^2$. Let $\mathcal{G}$ be a closed geodesic of length L on M; we fix a parameterization $t \mapsto \gamma(t)$ of $\mathcal{G}$ by arc length, so that $\gamma(t + L) = \gamma(t)$.

In this section, we shall discuss the restriction of φ_λ to $\mathcal{G}$. A theorem of Burq, Gerard, and Tzvetkov [11], generalizing a result of Reznikov [59], asserts the following general bound

$$\frac{\|\varphi_\lambda\|_{L^2(\mathcal{G})}}{\|\varphi_\lambda\|_{L^2(M)}} \leqslant C(\mathcal{G}, M)\, \lambda^{1/4},$$

the constant $C(\mathcal{G}, M)$ depending only on M and the geodesic γ.

This is in fact the "worst possible behavior" as the following basic example shows: let $M = S^2$, embedded as the unit sphere in $\mathbf{R}^3$, with the induced metric. Let φ_n be the restriction to M of $(x, y, z) \mapsto (x + iy)^n$. Then φ_n is a Laplacian eigenfunction of L^2-norm $\asymp n^{-1/4}$; on the other hand, when restricted to the equatorial geodesic $z = 0$ it corresponds to the function:

$$\gamma(t) \mapsto e^{int}.$$

In particular, $\varphi_n|\gamma$ is a *single Fourier mode*, and $\|\varphi_n\|_{L^2(\mathcal{G})} \asymp n^{1/4}\|\varphi_n\|_{L^2(M)}$.

Returning to the general case, let ω be an integral multiple of $\frac{2\pi}{L}$. Consider the "ω-th" Fourier coefficient along $\mathcal{G}$, namely,

$$a(\varphi_\lambda, \omega) = \int_0^L \varphi_\lambda(\gamma(t))e^{i\omega t}dt.$$

It measures the correlation of $\varphi_\lambda|\mathcal{G}$ with a single Fourier mode. By Cauchy-Schwarz:

$$\frac{|a(\varphi_\lambda, \omega)|}{\|\varphi_\lambda\|_{L^2(M)}} \leqslant L^{1/2}C(\mathcal{G}, M)\lambda^{1/4}.$$

As the above example shows, this bound is indeed sharp.

1.4.2. *Geodesic restriction problems in the arithmetic case.* — As was explained to us by A. Reznikov, a consequence to our main result, Theorem 1.2, is that this behavior *never* occurs on an surface which is of *arithmetic type* and when φ_λ varies amongst a suitable orthogonal basis of Laplacian eigenfunctions. We give definitions of these concepts below.

In such a situation, we obtain a much stronger result: *let $\{\varphi_\lambda\}$ be a basis of Hecke-Laplace eigenfunctions. Then there is an absolute constant $\delta > 0$ such that*

(**1.6**) $$\frac{|a(\varphi_\lambda, \omega)|}{\|\varphi_\lambda\|_{L^2(M)}} \leqslant C(\mathcal{G}, M)\lambda^{-\delta}.$$

The bound on the right hand side is independent of ω, and includes the "difficult" case when ω and λ are close. Thus the Fourier coefficients of an Laplace/Hecke eigenform along a fixed geodesic decay *uniformly*: such uniformity is a direct consequence of the *hybrid* nature of the subconvex bound proven in Theorem 1.2.

If M is an arithmetic *hyperbolic* surface, then it is expected that (1.6) holds for *any* orthonormal basis of Laplace eigenforms. This comes from the fact that, in the hyperbolic case, the multiplicities of Laplace eigenvalues are expected to be small, so that any Laplace eigenfunction could be expressed as a short linear combination of Laplace/Hecke eigenfunctions.

1.4.3. *The definition of arithmetic hyperbolic manifold.* — By an *arithmetic hyperbolic manifold* we shall mean the quotient of the upper half-plane by a lattice that arises from a quaternion algebra over $\mathbf{Q}$. However, for simplicity of exposition, we restrict ourselves to a slight subclass of $(M, \mathcal{G})$; we describe this subclass here, and also translate some of our data into automorphic language. We strongly suggest skipping this section at a first reading.

Let D be a quaternion algebra over $\mathbf{Q}$, split at ∞. Let $\mathbf{G}$ be the algebraic group $GL_1(D)/\mathbf{G}_m$, so that the $\mathbf{Q}$-points of $\mathbf{G}$ are $D(\mathbf{Q})^\times/\mathbf{Q}^\times$. Let $\mathbf{A}$ resp. $\mathbf{A}_f$ be the ring of adeles resp. finite adeles of $\mathbf{Q}$. Let K_f be an open compact subgroup of $\mathbf{G}(\mathbf{A}_f)$ with the property that $\mathbf{G}(\mathbf{Q}) \cdot \mathbf{G}(\mathbf{R}) \cdot K_f = \mathbf{G}(\mathbf{A}_f)$. Writing Γ for $\mathbf{G}(\mathbf{Q}) \cap K_f$, we have a natural homeomorphism from $\Gamma \backslash \mathbf{G}(\mathbf{R})$ to $\mathbf{G}(\mathbf{Q}) \backslash \mathbf{G}(\mathbf{A})/K_f$. Since $\mathbf{G}(\mathbf{R}) \cong PGL_2(\mathbf{R})$, it acts on $\mathbf{H}^2$ (although only the connected component preserves orientation). We refer to the quotient $\Gamma \backslash \mathbf{H}^2$ as an arithmetic hyperbolic manifold.

We therefore have a projection:

$$\mathbf{G}(\mathbf{Q}) \backslash \mathbf{G}(\mathbf{A}) = \mathbf{A}^\times D(\mathbf{Q})^\times \backslash D(\mathbf{A})^\times \longrightarrow \Gamma \backslash \mathbf{H}^2.$$

For simplicity of exposition, we shall also restrict our discussion to geodesics $\mathcal{G}$ that arise as a projection of a full adelic orbit $(\mathbf{T}(\mathbf{Q}) \backslash \mathbf{T}(\mathbf{A})).g$ to $\Gamma \backslash \mathbf{H}^2$, where $\mathbf{T} \subset \mathbf{G}$ is a maximal torus and $g \in \mathbf{G}(\mathbf{A})$. (In general, such a projection is the union of $\mathcal{G}$ with finitely many closed geodesics, the number of such geodesics being the class number of a suitable quadratic order; this restriction, therefore, amounts to the requirement that this class number is 1; (1.6) however remains true without such requirement).

We now associate automorphic data to our eigenfunctions and frequencies:

– Let π_λ be the automorphic representation of $\mathbf{G}$ generated by the pull-back of φ_λ.
– The torus $\mathbf{T}(\mathbf{Q})$ is of the form $E^\times/\mathbf{Q}^\times$, where E is a real quadratic field extension of $\mathbf{Q}$.
– Associated to the character $\gamma(t) \mapsto e^{i\omega t}$ is a character of $\mathbf{T}(\mathbf{Q}) \backslash \mathbf{T}(\mathbf{A})$, and, in particular, a character ω_E of $\mathbf{A}_E^\times/E^\times$. Let π_ω be the automorphic representation of GL_2 over $\mathbf{Q}$ obtained by automorphic induction from (E, ω_E); thus, the L-function of π_ω coincides with the L-function $L(s, E, \omega_E)$.

By a result of Waldspurger, $|a(\varphi_\lambda, \omega)|^2$ is proportional to the central value of the completed L-function $\Lambda(\pi \times \pi_\omega, \frac{1}{2})$.

1.4.4. *Sketch of the proof of* (1.6). — As mentioned above, (1.6) is a consequence of Theorem 1.2. We will now outline a proof of this corollary, for arithmetic hyperbolic surfaces, in purely geometric terms.

The function $t \mapsto \varphi_\lambda(\gamma(t))$ oscillates over a length scale of size λ^{-1}. It therefore stands to reason that the most interesting case of (1.6) is when $\omega \sim \lambda$. This is indeed so, and the key step of the proof of Theorem 1.2 corresponds—in this present language—to the use of certain identities to switch from the "difficult" range when $\omega \sim \lambda$ to the "easier" range when λ is small and ω is large. *These identities are extremely specific to the arithmetic cases under consideration;* we do not know how to prove anything like (1.6) for a general hyperbolic surface M.

Notation as previous. One may construct (for "deep" number-theoretic reasons) the following auxiliary data:

- another arithmetic hyperbolic surface $\widetilde{M}$, depending only on M;
- a Laplace/Hecke eigenfunction $\tilde{\varphi}_\lambda$ on $\widetilde{M}$ of eigenvalue $-\lambda^2$;
- For each admissible ω, i.e each integral multiple of $2\pi/L$, we associate a Laplace eigenfunction θ_ω on $\widetilde{M}$ with eigenvalue $-\frac{1}{4} - \omega^2$.

moreover, whenever $\omega_1, \omega_2, \frac{\omega_1+\omega_2}{2}$ are admissible, one has

$$(\mathbf{1.7}) \qquad |a(\varphi_\lambda, \omega_1) a(\varphi_\lambda, \omega_2)|^2 \sim \left| \int_{\widetilde{M}} \tilde{\varphi}_\lambda \theta_{\frac{\omega_1+\omega_2}{2}} \theta_{\frac{\omega_1-\omega_2}{2}} \right|^2 .$$

The $\sim$ here does not indicate approximate equality, but rather, equality up to a constant that is precisely computable; it is essentially a ratio of Γ-functions.

The identity (1.7) has the remarkable feature that the left-hand side has a "quadri-linear" nature, whereas the right hand side has a "bilinear" nature. Thus the map $\varphi_\lambda \mapsto \tilde{\varphi}_\lambda$ is in no natural sense linear; rather, it is defined element-by-element over a special basis of Laplace/Hecke eigenfunctions.

1.4.5. *Number-theoretic explanation.* — Supposing $\omega_1, \omega_2, \frac{\omega_1+\omega_2}{2}$ admissible,

$$\Lambda\left(\pi \times \pi_{\omega_1}, \frac{1}{2}\right)\Lambda\left(\pi \times \pi_{\omega_2}, \frac{1}{2}\right) = \Lambda\left(\pi \times \pi_{\omega_+} \times \pi_{\omega_-}, \frac{1}{2}\right),$$
$$\omega_\pm := \frac{\omega_1 \pm \omega_2}{2}.$$

(1.7) now follows using the main result of [32]: The manifold $\widetilde{M}$ is a quotient of the unique quaternion algebra D' which is nonsplit at those places where $\epsilon_v(\pi \times \pi_{\omega_+} \times \pi_{\omega_-}) = -1$—if no D' exists, the left-hand side of (1.7) vanishes—and $\widetilde{\varphi}_\lambda, \theta_{\omega_+}, \theta_{\omega_-}$ belong to the Jacquet-Langlands transfer to D' of $\pi, \pi_{\omega_+}, \pi_{\omega_-}$.

1.4.6. *The switch of range from* $\omega \sim \lambda$ *to* $\lambda = O(1)$. — Take $\omega_1 = \omega_2 = \omega$, and apply Cauchy-Schwarz to the right-hand side of (1.7) to obtain:

$$|a(\varphi_\lambda, \omega)|^4 \lesssim \|\tilde{\varphi}_\lambda\|^2_{L^2(\widetilde{M})} \langle \theta_\omega^2, \theta_0^2 \rangle_{L^2(\widetilde{M})}.$$

We shall analyze this by expanding both θ_ω^2 and θ_0^2 into constituents. Since θ_0 is itself a Laplacian eigenfunction with eigenvalue $1/4$, all Δ-eigenfunctions that occur in the spectral expansion of θ_0^2 will have "small" eigenvalue. Carrying this out[7]

$$(\mathbf{1.8}) \qquad \langle \theta_\omega^2, \theta_0^2 \rangle = \sum_{\substack{\Delta \psi_\mu = \mu^2 \\ \mu \ll 1}} \langle \theta_\omega^2, \psi_\mu \rangle \langle \psi_\mu, \theta_0^2 \rangle,$$

where the ψ_μ-sum ranges over a basis of Hecke-Laplace eigenfunctions on $\widetilde{M}$.

It is, in fact, possible to now apply (1.7) once more to understand the term $\langle \theta_\omega^2, \psi_\mu \rangle$; however, we apply it "in the reverse direction." This shows that there exists a manifold $\check{M}$ and $\check{\psi}_\mu$ so that

$$|\langle \theta_\omega^2, \psi_\mu \rangle|^2 \sim |a(\check{\psi}_\mu, 2\omega) a(\check{\psi}_\mu, 0)|^2.$$

We have achieved our objective and *switched the range*: starting with the analysis of $a(\varphi_\lambda, \omega)$ with λ, ω essentially arbitrary, we have reduced it to the analysis of $a(\check{\psi}_\mu, 2\omega)$ where μ may be assumed small relative to ω. Although we are not done, this allows us to out-flank the most tricky case of the question: when $|\lambda - \omega| = O(1)$.

1.4.7. *The range when ω is large.* — We now discuss bounding the Fourier coefficient $a(\varphi_\lambda, \omega)$ when ω is very large compared to λ. For simplicity, let us assume in the present section that we are dealing with a *fixed* eigenfunction $\varphi = \varphi_\lambda$, and analyze the question of bounding $a(\varphi_\lambda, \omega)$ as $\omega \to \infty$.

We need to be more precise about what is necessary to prove. In this context, it is evident (by real-analyticity) that $a(\varphi_\lambda, \omega)$ decays exponentially with ω; however, this is not enough. For our previous argument we require[8]:

$$(\mathbf{1.9}) \qquad e^{\pi |\omega|/2} |a(\varphi_\lambda, \omega)| \leqslant C(\varphi_\lambda) |\omega|^{-\delta},$$

for some $\delta > 0$.

Now, $a(\varphi_\lambda, \omega)$ is the integral of $\varphi_\lambda(\gamma(t))$ against $e^{i\omega t}$. To eliminate the exponential factors, we *deform the path*: we replace $t \mapsto \gamma(t)$ by a path $t \mapsto \gamma'(t)$ so that, first of all, $\gamma'(t)$ approximates—at least locally—a horocycle; secondly, the deformed integral still determines the original integral, but is larger than it by a factor of size $e^{\pi |\omega|/2}$. Thus, to prove (1.9), we need only prove polynomial decay for the deformed integral. The deformed integral is analyzed using dynamical properties of the horocycle flow, especially mixing; it is related to the analysis in [65, 1.3.4].

---

[7] In the cases we encounter, there will be difficulties with convergence, and regularization of the following expression will be needed.

[8] The exponential factors arise from, in essence, the Γ-functions that were suppressed in (1.7).

2. Norms on adelic quotients

The classical Sobolev norms on $\mathbf{R}^n$, or on a real manifold, measure the L^p-norms of a function *together with its derivatives*. For example, let $\|f\|_{2,k}$ be defined as the sum of the L^2-norms of the first k derivatives of $f \in C^\infty(\mathbf{R}/\mathbf{Z})$, and let $S_{2,k}$ be the completion with respect to this norm. Then:

Sa. Sobolev inequality: The Sobolev norms control point-evaluation, e.g.

$$|f(0)| \ll \|f\|_{2,1}.$$

Sb. Distortion: If $h : \mathbf{R}/\mathbf{Z} \to \mathbf{R}/\mathbf{Z}$ is a diffeomorphism, then

$$\|f \circ h\|_{2,1} \leqslant (\sup_x |h'(x)|)\|f\|_{2,1}.$$

Sc. Sobolev embedding: The Sobolev norms are compact with respect to each other: the inclusion of $S_{2,k}$ into $S_{2,k'}$ is compact for $k < k'$.

Sc.* Sobolev embedding in quantitative form: if $k' \geqslant k + 2$, then the trace of $S_{2,k}$ with respect to $S_{2,k'}$ is finite. This means that, if W_k is the completion of $C^\infty(\mathbf{R}/\mathbf{Z})$ with respect to $S_{2,k}$, then the induced homomorphism $W_k \to W_{k+2}$ is trace-class.

Sd. Fourier analysis:

$$|S_{2,k}(f)|^2 \ll \int (1 + |\lambda|)^{2k}|\hat{f}(\lambda)|^2,$$

where $\hat{f}(\lambda)$ is the Fourier transform.

It is very convenient to have a system of norms on adelic quotients with corresponding properties. We shall present them in terms of a list of axiomatic properties they satisfy, before giving the definition (Section 2.3). These properties are intended to be analogous to (Sa)—(Sd) above. Prior to doing this, we need to first recall L^2-spectral decomposition (Section 2.2).

We strongly recommend that the reader *ignore the definition* of the Sobolev norms and rather *work with its properties.*

2.1. *Notation*

2.1.1. *On implicit constants.* — We use throughout the notation $A \ll B$ of Vinogradov to mean: there exists a constant c so that $A \leqslant cB$. If we write $A \ll_\delta cB$, it means that the constant c is permitted to depend on δ, and so on. We shall also use a modification of this notation: $A \ll B^\star$ means that there exist constants c_1, c_2 so that $A \leqslant c_1 B^{c_2}$.

Although the notation $A \ll B$ is generally understood to mean that the implicit constant is *absolute*, it is extremely convenient in our context to allow it to depend on

certain predetermined parameters (e.g., the number field over which we work) without explicit comment. We gather together at this point references to where these conventions are introduced, for the convenience of the reader. To wit: in Part II, constants may depend on the isomorphism class of $(\mathbf{G}, \rho)$ over the number field F; in Part III, the constants may depend on the discriminant of the local field (Section 3.1.3); in Parts IV and V, they may depend on the isomorphism class of the ground field F.

For Parts IV and V, we shall in fact make a more stringent use of the notation where we require certain implicit constants to be polynomial; see Section 4.1.5.

Later in the text we shall use indexed families of norms—the Sobolev norms $\mathcal{S}_d$. They will depend on an indexing parameter d, as the notation suggests. In a similar fashion to the Vinogradov convention, we allow ourselves to write inequalities omitting the parameter d; see Sections 2.3.1 and 3.1.9 for a further discussion of this point.

2.1.2. — Let F be a number field. We denote by ξ_F the complete ζ-function of F. It has a simple pole at 1; the residue is denoted by $\xi_F^*(1)$.

2.1.3. — Let $\mathbf{G}$ a reductive algebraic F-group. Choose a faithful representation $\mathbf{G} \hookrightarrow \mathrm{SL}(\mathrm{F}^r)$ for some $r \geqslant 1$; we shall suppose that it contains a copy of the adjoint representation. Henceforth we shall feel free to identify $\mathbf{G}$ with a matrix group by means of this embedding. In the case of GL_2, we shall fix the faithful representation to be

$$\rho : g \mapsto \begin{pmatrix} g & 0 \\ 0 & (g^t)^{-1} \end{pmatrix} \in \mathrm{SL}_4,$$

where, as usual, g^t denotes the transpose.

Let $\mathfrak{g}$ be the Lie algebra of $\mathbf{G}$; if v is a place of F, we write $\mathfrak{g}_v = \mathfrak{g} \otimes_F F_v$. We also fix a basis for $\mathfrak{g}$. In that way, we regard the adjoint embedding as a map $\mathrm{Ad} : \mathbf{G} \to \mathrm{GL}(\dim \mathfrak{g})$.

Let

$$\mathbf{X} := \mathbf{G}(\mathrm{F}) \backslash \mathbf{G}(\mathbf{A}).$$

We denote by $\mathrm{C}^\infty(\mathbf{X})$ the space of smooth functions on $\mathbf{X}$; the space $\mathbf{X}$ can be understood as an inverse limit of quotients of real Lie groups by discrete subgroups, and a smooth function simply means one that factors through a smooth function on one of these quotients.

2.1.4. — We fix a maximally F-split torus $\mathbf{T} \subset \mathbf{G}$ and, correspondingly, a minimal parabolic F-subgroup $\mathbf{B}$ containing $\mathbf{T}$.

2.1.5. — For v non-archimedean and $m \geqslant 0$ an integer, we denote by $\mathrm{K}_v[m]$ the open-compact (principal congruence) subgroup

$$\mathrm{K}_v[m] := \mathbf{G}(\mathrm{F}_v) \cap \left\{ g \in \mathrm{GL}_r(\mathcal{O}_v),\ g \equiv \mathrm{Id}_r\ (\varpi_v^m) \right\},$$

where ϖ_v is a uniformizer in F_v.

Choose, for each place v, a maximal compact subgroup K_v so that:

1. $K_v \supset K_v[0]$ when v is nonarchimedean (this implies that $K_v = K_v[0]$ for almost all v);
2. For v nonarchimedean, K_v is *special*, i.e. it is the stabilizer of a special vertex in the building of $\mathbf{G}(F_v)$.

This entails, in particular, that if P_v is the set of F_v-points of any parabolic subgroup, then $P_v K_v = \mathbf{G}(F_v)$.

There exists a constant A, depending on $\mathbf{G}$ and the chosen faithful representation, so that

$$(\mathbf{2.1}) \qquad (1 + A/q_v)^{-1} \leqslant \frac{[K_v : K_v[m_v]]}{q_v^{m_v \dim(\mathbf{G})}} \leqslant (1 + A/q_v).$$

We denote by $\underline{m} : v \mapsto m_v$ any function on the set of places of F to the non-negative integers, which is zero for almost all v. Write $\|\underline{m}\| = \prod_v q_v^{m_v}$ (we take $q_v = e = 2.718\dots$ for archimedean places); we note that

$$|\{\underline{m} : \|\underline{m}\| \leqslant N\}| = O(N).$$

For such $\underline{m}$, we set

$$K[\underline{m}] := \prod_{v \text{ finite}} K_v[m_v].$$

We also put $K = \prod_v K_v$.

2.1.6. — We now fix a normalization of left-invariant Haar measures on the various groups: the F_v and $\mathbf{A}$-points of $\mathbf{G}$, as well as any parabolic subgroup $\mathbf{P} \subset \mathbf{G}$, as well as of any Levi factor $\mathbf{M} \subset \mathbf{P}$; and finally, on K_v, for every v. These measures should have the following properties:

1. The measures on adelic points should be the product of local measures;
2. The projection $\mathbf{P}(F)\backslash\mathbf{P}(\mathbf{A}) \to \mathbf{M}(F)\backslash\mathbf{M}(\mathbf{A})$ corresponding to the decomposition $\mathbf{P} = \mathbf{MN}$ is measure-preserving (for the *left* Haar measures!)
3. For all v, the measure on K_v has mass 1. For nonarchimedean v, it is the restriction of the measure from $\mathbf{G}(F_v)$.
4. The map $\mathbf{P}(F)\backslash\mathbf{P}(\mathbf{A}) \times K \to \mathbf{G}(F)\backslash\mathbf{G}(\mathbf{A})$ should be measure-preserving (i.e., the preimage of any set has the same measure as the set).

It is not difficult to construct such measures; we shall make an explicit choice, in the case of $\mathbf{G} = \mathrm{GL}_2$, in Section 3.1.5. It is worth observing that, since we are only concerned with upper bounds in this paper, and not exact formulas, precise choices are

never of importance, so long as they remain consistent. With any such choice, it follows from (2.1) that $\|\underline{m}\|^{-\varepsilon} \ll \mathrm{vol}(K[\underline{m}])\|\underline{m}\|^{\dim(\mathbf{G})} \ll \|\underline{m}\|^{\varepsilon}$; in fact, one can replace the upper and lower bounds by constants in the case that $\mathbf{G}$ is semisimple.

Finally, put on $\mathbf{X}$ the corresponding quotient measure.

If H is any locally compact group, we define the *modular character* $\delta_H : H \to \mathbf{R}^{\times}$ via the rule $\mu(Sh) = \delta_H(h)^{-1}\mu(S)$, where μ is a left Haar measure on H, and S is any set with $\mu(S) > 0$. In other words, if $d_l h$ is a left Haar measure, then $d_l(hh') = \delta_H(h')^{-1}d_l h$.

2.1.7. — For $g \in \mathbf{G}(F_v)$, we define $\|g\| = \sup_{ij}|\rho_{ij}(g_v)|$; in the adelic case, we take the product over all places. We set $\|\mathrm{Ad}(g)\|$ to be defined as $\|g\|$, but with ρ replaced by the adjoint embedding. (The "functional" difference between these two norms lies in the fact that $\|\mathrm{Ad}(g)\|$ is invariant under the center, whereas $\|g\|$ is not.)

For $g \in \mathbf{G}(\mathbf{A})$, define the *height* by

$$\mathrm{ht}(g)^{-1} := \inf_{x \in F^r - \{0\}} \prod_v \sup_{i=1...r} |(\mathrm{Ad}(g).x)_i|_v.$$

This descends to a function on $\mathbf{X}$, and—if the center of $\mathbf{G}$ is anisotropic—the map $\mathrm{ht} : \mathbf{X} \to \mathbf{R}_{\geqslant 1}$ is proper.

Lemma. — *Fix $x_0 \in \mathbf{X}$. For any $x \in \mathbf{X}$, there exists $g \in \mathbf{G}(\mathbf{A})$ with $x_0 g = x$ and $\|\mathrm{Ad}(g)\| \leqslant \mathrm{ht}(x)^{\star}$.*

Proof. — This is well-known; see e.g. [21], footnote 15 for a proof in the case of real groups, from which the stated result is easily deduced. □

2.1.8. — Throughout this paper the phrase "π is a unitary representation of the group G" will be understood to mean that the underlying space of π is a Hilbert space and G acts by isometries on that space.[9]

Let V be a unitary representation of $\mathbf{G}(F_v)$. The space of smooth vectors V^{∞} is defined, in the case when v is nonarchimedean, as the subspace of V comprising vectors whose stabilizer is open; in the case when v is archimedean, it is that subspace for which the map $g \mapsto g.v$ defines a smooth map from $\mathbf{G}(F_v)$ to V. *It is always dense in* V.

Let V be a unitary representation of $\mathbf{G}(\mathbf{A})$. It factorizes as a tensor product of unitary representations of $\mathbf{G}(F_v)$, and we define the smooth subspace V^{∞} as the (image in V of the) tensor product of the local smooth subspaces.

2.2. *Structure of adelic quotients and the Plancherel formula*

In the present section we are going to recall the "Plancherel formula" for $L^2(\mathbf{X})$, that is to say, its decomposition into irreducible $\mathbf{G}(\mathbf{A})$-representations.

---

[9] In some contexts, unitary representation is used to mean "unitarizable", i.e., such that there exists *some* inner product. We shall always understand it to mean that we have fixed a *specific* inner product.

2.2.1. *Eisenstein series.* — There is a standard parameterization of the automorphic spectrum via the theory of "Eisenstein series" that we shall now recall. See also [1] for a résumé of the theory, and [52] for a detailed treatment.

Let $\mathcal{X}$ denote the set of pairs $(\mathbf{M}, \sigma)$, where $\mathbf{M}$ is a F-Levi subgroup of a F-parabolic subgroup, containing $\mathbf{T}$, and let σ be an irreducible subrepresentation of the space of functions on $\mathbf{M}(\mathrm{F}) \backslash \mathbf{M}(\mathbf{A})$, which is "discrete series" in the following sense: all $f \in \sigma$ are square-integrable with respect to the inner product

$$f \mapsto \|f\|_\sigma^2 = \int_{Z_{\mathbf{M}}(\mathbf{A})\mathbf{M}(\mathrm{F})\backslash\mathbf{M}(\mathbf{A})} |f|^2,$$

where $Z_{\mathbf{M}}(\mathbf{A})$ denote the center of $\mathbf{M}(\mathbf{A})$.

We can equip $\mathcal{X}$ with a measure in the following way: We write

$$\mathcal{X} = \bigsqcup_{\mathbf{M}} \mathcal{X}_{\mathbf{M}},$$

indexed by Levis containing $\mathbf{T}$. We require that for any continuous assignment of $\chi \in \mathcal{X}_{\mathbf{M}}$ to f_χ in the underlying space of χ,

$$\int_{\mathbf{M}(\mathrm{F})\backslash\mathbf{M}(\mathbf{A})} \left| \int_\chi f_\chi \, d\chi \right|^2 = \int_\chi \|f_\chi\|_\sigma^2 \, d\chi.$$

This uniquely specifies a measure $d\chi$ on $\mathcal{X}_{\mathbf{M}}$, and so also on $\mathcal{X}$.

There exists a natural equivalence relation $\sim$ on $\mathcal{X}$: declare $(\mathbf{M}, \sigma)$ and $(\mathbf{M}', \sigma')$ to be equivalent if there exists w in the normalizer of $\mathbf{T}$ with $\mathrm{Ad}(w)\mathbf{M} = \mathbf{M}'$ and $\mathrm{Ad}(w)\sigma = \sigma'$. There is a natural quotient measure on $\mathcal{X}/\sim$.

For $\chi = (\mathbf{M}, \sigma) \in \mathcal{X}$, we denote by $\mathcal{I}(\chi)$ the unitarily induced representation $\mathrm{Ind}_{\mathbf{P}(\mathbf{A})}^{\mathbf{G}(\mathbf{A})}\sigma$, where $\mathbf{P}$ is any parabolic subgroup containing $\mathbf{M}$. (Its isomorphism class is —not obviously—independent of the choice of $\mathbf{P}$.) It consists of functions $\mathbf{G}(\mathbf{A}) \to \mathrm{V}_\sigma$ (where V_σ is a vector space realizing the representation σ) satisfying the transformation property

$$f(pg) = \delta^{1/2}(p)\sigma(m_p)f(g),$$

where δ is the modular character (cf. Section 2.1.6) and $p \mapsto m_p$ the projection $\mathbf{P} \to \mathbf{M}$. We define a norm on $\mathcal{I}(\chi)$ by

$$\|f\|_{\mathrm{Eis}}^2 := \int_{\mathrm{K}} \|f(k)\|_{\mathrm{V}_\sigma}^2 \, dk,$$

where K is equipped with the Haar probability measure.

There exists a natural intertwiner (the "unitary Eisenstein series", obtained by averaging over $\mathbf{P}(\mathrm{F})\backslash\mathbf{G}(\mathrm{F})$ and analytic continuation):

$$\mathcal{I}(\chi) \overset{\mathrm{Eis}}{\to} \mathrm{C}^\infty(\mathbf{X}).$$

The map Eis is an isomorphism away from a set of parameters χ of measure zero; we call the latter the set of *singular* parameters. For instance in the case of GL$_2$ the parameters are pairs of unitary characters (χ^+, χ^-) for $\mathbf{M} = \mathrm{GL}_1 \times \mathrm{GL}_1 \hookrightarrow \mathbf{B}$ (the standard Borel) and the singular ones are the ones for which $\chi^+ = \chi^-$; for this reason, we shall define for this case a variant Eis* of Eis which is non-zero (see Section 4.1.10.)

In any case, to almost every $\chi \in \mathcal{X}$ is associated an automorphic representation— the image of $\mathcal{I}(\chi)$—denoted Eis(χ). The resulting automorphic representation depends only on the class of χ in $\mathcal{X}/\sim$. If Eis is an isomorphism—as is so away from a set of χ of measures 0—we equip Eis(χ) with a norm by requiring Eis to be an isometry. Whenever defined, we call this norm the *Eisenstein norm* on the space Eis(χ).[10]

Theorem (Langlands). — *The map $\int_{\chi \in \mathcal{X}} \mathcal{I}(\chi) \to \mathrm{L}^2(\mathbf{X})$, defined by integrating the map* Eis, *extends to an isometric isomorphism of the Hilbert space $\int_{\mathcal{X}} \mathcal{I}(\chi)\,d\chi$ and $\mathrm{L}^2(\mathbf{X})$.*

For a slightly more precise formulation, we refer to [1].

We shall call any automorphic representation that occurs as an Eis(χ), $\chi \in \mathcal{X}$, a *standard* automorphic representation. Not every automorphic representation is standard. For example, every standard automorphic representation is abstractly unitarizable. The standard automorphic representations are precisely those needed for unitary decomposition.

The set of standard automorphic representations will be denoted by $\hat{\mathrm{G}}_{\mathrm{A}ut}$ and the push-forward of the measure $d\chi$ on $\mathcal{X}$ to $\hat{\mathrm{G}}_{\mathrm{A}ut}$ will be denoted by $d\mu_\mathrm{P}$.

Remark. — The Plancherel decomposition of a function in, e.g., $\mathrm{C}_c^\infty(\mathbf{X})$ is pointwise defined; explicitly, for $\varphi \in \mathrm{C}_c^\infty(\mathbf{X})$, we have the equality of continuous functions,

$$\varphi = \int_{\chi \in \mathcal{X}} \sum_{f \in \mathcal{B}(\chi)} \langle \varphi, \mathrm{Eis}(f) \rangle \mathrm{Eis}(f)\, d\chi,$$

where $\mathcal{B}(\chi)$ is an orthogonal basis for $\mathrm{I}(\chi)$, and the right-hand side is absolutely convergent. This is not a triviality; it follows, for example, from the results of W. Müller [55]; it may be that there is a more elementary proof also.

2.2.2. *The canonical norm for* GL$_2$. — In the case of GL$_n$ for *generic standard* representations—although not necessarily cuspidal—it is possible to give a simple description of a canonical norm on the space of any standard automorphic representation. We explain for GL$_2$; the general case is obtained by replacing the role of $a(y)$ below with $g \in \mathrm{GL}_{n-1} \subset \mathrm{GL}_n$ embedded as usual.

---

[10] The terminology may be slightly misleading; if $\pi \subset \mathrm{L}^2(\mathbf{X})$ is, e.g., a cuspidal representation, then $(\mathbf{G}, \pi) \in \mathcal{X}$, and the Eisenstein norm on π is simply the restriction of the L^2-norm.

Suppose π is generic. Let $\mathcal{W}_\pi = \otimes_v \mathcal{W}_{\pi,v}$ be the Whittaker model of π. There are two natural inner products that one can equip $\mathcal{W}_{\pi,v}$ with, namely—see Section 3.1.5 for the measure normalizations—

$$(\mathbf{2.2}) \qquad \langle W_v, W_v' \rangle = \int_{F_v^\times} W_v(a(y))\overline{W_v'}(a(y))d^\times y,$$

$$\langle W_v, W_v' \rangle_{reg} = \frac{\int_{F_v^\times} W_v(a(y))\overline{W_v'}(a(y))d^\times y}{\zeta_v(1)L_v(\pi, \text{Ad}, 1)/\zeta_v(2)}.$$

The latter inner product has certain good normalization properties, e.g.

$$\langle W_v, W_v \rangle_{reg} = |W_v(1)|^2$$

for almost all v.

We define an inner product on $\mathcal{W}_\pi$, by its value on pure tensors $W = \prod_v W_v$.

$$(\mathbf{2.3}) \qquad \|W\|_{reg}^2 := \Lambda^*(\pi, \text{Ad}, 1) \times \prod_v \langle W_v, W_v \rangle_{reg};$$

where

$$\Lambda^*(\pi, \text{Ad}, 1) = \lim_{s \to 1} \frac{\Lambda(\pi, \text{Ad}, s)}{(s-1)^r},$$

with $\Lambda(\pi, \text{Ad}, s) = \prod_v L(\pi_v, \text{Ad}, s)$ denotes the completed L-function and r is taken to be the largest non-negative integer for which the limit is nonzero. The regularized value $L^*(\pi, \text{Ad}, 1)$ satisfy ([24])

$$(\mathbf{2.4}) \qquad L^*(\pi, \text{Ad}, 1) = C(\pi)^{o(1)}, \quad \text{as } C(\pi) \to \infty.$$

Finally, we define the *canonical norm* on the space of π by the rule

$$(\mathbf{2.5}) \qquad \|\varphi\|_{can}^2 = \frac{2\xi_F(2)(\text{disc}F)^{1/2}}{\xi_F^*(1)} \|W_\varphi\|_{reg}^2,$$

where $\varphi \mapsto W_\varphi$ is the usual intertwiner (4.2). The terminology is justified by

Lemma **2.2.3.** — *Suppose π generic and standard. Then, for $\varphi \in \pi$, $\|\varphi\|_{can}^2 = \|\varphi\|_{L^2(\mathbf{X})}^2$ if π is cuspidal; and $\|\varphi\|_{can}^2 = 2\xi_F(2)\|\varphi\|_{\text{Eis}}^2$ if π is Eisenstein and nonsingular.*

The verification of this equality for cusp forms is a consequence of the Rankin-Selberg method (cf. Section 4.4.2); the Eisenstein case is detailed in Section 4.1.7.

Remark. — One unfortunate consequence (perhaps unavoidable) is that the canonical norm does not always behave continuously in families. It is possible to have a family of automorphic forms $\phi(s)$ belonging to standard generic automorphic representations so that $\varphi(s) \to \varphi(0)$ pointwise on $\mathbf{X}$, but the canonical norms do not converge. This happens when the order of pole of the adjoint L-function jumps, e.g. at singular parameters.

2.3. *Norms on adelic quotients*

2.3.1. — We shall set up two families of Hilbert norms, valid for any $d \in \mathbf{R}$:

- A notion of "Sobolev norm" $\mathcal{S}_d^{\mathrm{V}}$ on any unitary $\mathbf{G}(\mathbf{A})$-representation V.
- A finer notion of "Sobolev norm" $\mathcal{S}_d^{\mathbf{X}}$ for functions on $\mathbf{X}$.

We shall follow the following convention: If $\ell \in \mathrm{V}^*$ is a functional and we write $|\ell(f)| \leqslant \mathcal{S}^{\mathrm{V}}(f)$, without a subscript d, it means *there exists a constant d, depending on only the isomorphism class of $\mathbf{G}$ over* F, *so that* $|\ell(f)| \leqslant \mathcal{S}_d^{\mathrm{V}}(f)$. In particular, ℓ is continuous in the topology defined by the family of norms $\mathcal{S}_d^{\mathrm{V}}$.

1. We would like to warn the reader that the constructions are not totally formal. Namely, some of the subtler features of the norms rely on Bernstein's uniform admissibility theorem as well as Müller's theorem [3, 55]. Indeed, one of the properties of the norms is established only for the group $\mathbf{G} = \mathrm{GL}_n$. In fact, for the purpose of the present paper, none of these deep results are important.
2. We observe that all our norms can take the value ∞. (In precise terms, we understand a norm N on a vector space V to be a function $\mathrm{N} : \mathrm{V} \to [0, \infty]$ that satisfies the usual axioms; equivalently, we could regard N to be a pair consisting of a subspace $\mathrm{W} \subset \mathrm{V}$, and a (usual, finite-valued) norm on W.)

$\mathcal{S}_d^{\mathrm{V}}$ will always take finite values on V^∞; $\mathcal{S}_d^{\mathbf{X}}$ will always take finite values on the space of compactly supported smooth functions $\mathrm{C}_c^\infty(\mathbf{X})$. We shall sometimes refer to the completion of V^∞ in the norm induced by $\mathcal{S}_d^{\mathrm{V}}$ as the *Hilbert space associated to* $\mathcal{S}_d^{\mathrm{V}}$. Similarly for $\mathcal{S}_d^{\mathbf{X}}$.

2.3.2. *The Sobolev norms on a unitary representation.* — Let V be a unitary admissible representation of either $\mathbf{G}(\mathrm{F}_v)$ (some place v) or $\mathbf{G}(\mathbf{A})$. We shall define, in both contexts, a *generalized Laplacian operator* $\Delta : \mathrm{V}^\infty \to \mathrm{V}^\infty$. This being so, we define the dth Sobolev norm via

$$(\mathbf{2.6}) \qquad \mathcal{S}_d^{\mathrm{V}}(f) := \|\Delta^d f\|_{\mathrm{V}}.$$

The Laplacian will have the property that Δ is invertible, and a suitable power of Δ^{-1} is trace class from V to itself.

2.3.3. *Local.* — Let v be a place and V an unitary admissible representation of $\mathbf{G}(\mathrm{F}_v)$. We shall make a certain (Hilbert) orthogonal decomposition $\mathrm{V} = \bigoplus_{m \geqslant 0} \mathrm{V}[m]$; roughly speaking, vectors in $\mathrm{V}[m]$ have "higher frequency" as m grows.

- If v is a finite place, we define $\mathrm{V}[m]$ to be the orthogonal complement of the $\mathrm{K}_v[m-1]$-invariants vectors inside the $\mathrm{K}_v[m]$-invariants vectors.
- If v is archimedean, fix a basis $\{\mathrm{X}_i\}$ for the Lie algebra $\mathfrak{g}_v := \mathfrak{g} \otimes_{\mathrm{F}} \mathrm{F}_v$ and let $\mathcal{C} := \sum_i (1 - \mathrm{X}_i^2)$; let $\mathrm{V}[m]$ be the direct sum of all $\mathcal{C}$-eigenspaces with eigenvalue in $[e^m, e^{m+1})$ (here $e = \exp(1) = 2.718\ldots$).

The space V[m] is finite dimensional. Write $e_v[m]$ for the projector onto V[m], and put:

$$\Delta_v = \sum_{m \geqslant 0} q_v^m e_v[m].$$

Note that $\sum_m e_v[m]$ is the identity.

2.3.4. *Global.* — In the global setting, we use, as before, the notation $\underline{m}$ for a function $v \mapsto m_v$ from places to non-negative integers, and set

$$e[\underline{m}] := \prod_v e_v[m_v], \ \Delta_{\mathbf{A}} = \sum \|\underline{m}\| e[\underline{m}].$$

Note that one has $\sum_{\underline{m}} e[\underline{m}] = 1$.

2.3.5. *Sobolev norms on* $C^\infty(\mathbf{X})$. — On $C^\infty(\mathbf{X})$ we shall introduce an increasing system of Sobolev norms $\mathcal{S}_d^{\mathbf{X}}$ which will take into account the noncompactness of the space $\mathbf{X}$. Let H be the operation of "multiplication by $1 + \mathrm{ht}(x)$." Put:

$$\mathcal{S}_d^{\mathbf{X}}(f) := \|H^d \Delta_{\mathbf{A}}^d f\|_2^2.$$

2.4. *Properties of the Sobolev norms*

This section enunciates the properties of the Sobolev norms, defined in the prior section.

2.4.1. *Properties of the unitary Sobolev norms.* — Write G for either $\mathbf{G}(\mathbf{A})$ or $\mathbf{G}(F_v)$.

S1a. (Sobolev inequality) For $f \in C(G)$ and a fixed smooth function $\omega : G \to \mathbf{C}$ of compact support with $\omega(1) = 1$,

$$|f(1)| \ll_\omega \mathcal{S}^{L^2(G)}(f \cdot \omega).$$

S1b. (Distortion property) There is a constant κ, depending only[11] on $(\mathbf{G}, \rho)$, so that, for V a unitary representation of G,

$$\mathcal{S}_d^V(gf) \ll \|\mathrm{Ad}(g)\|^{\kappa d} \mathcal{S}_d^V(f).$$

S1c. (Embedding) For each d, there exists $d' > d$ so that the trace of $\mathcal{S}_d^V$ w.r.t. $\mathcal{S}_{d'}^V$ is finite. (See [4]*Appendix for definitions; this means that the inclusion from the Hilbert space associated to $\mathcal{S}_d^V$, to the Hilbert space associated to $\mathcal{S}_{d'}^V$, is trace-class.

---

[11] If the representation ρ contains the adjoint representation, one may take $\kappa = 1$.

S1d. (Linear functionals can be bounded place-by-place.) Let $\pi = \otimes \pi_v$ be a unitary representation of $\mathbf{G}(\mathbf{A})$; let $\ell = \prod_v \ell_v, \ell_v \in \pi_v^*$ be a factorizable functional with the property that $|\ell(x_v)| \leqslant 1$ when $x_v \in \pi_v$ is spherical and of norm one.[12] Then:

$$|\ell_v| \leqslant \mathrm{A} \mathcal{S}_d^{\pi_v} \text{ for all } v \implies |\ell| \leqslant \mathrm{A}' \mathcal{S}_{d'}^{\pi}$$

where d' depends on d, and A' depends on A, d.

2.4.2. *Properties of the* $\mathbf{X}$*-Sobolev norms*

S2a. (Sobolev inequality) There exists d_0 so that $\mathcal{S}_{d_0}^{\mathbf{X}}$ majorizes L$^\infty$-norms.

S2b. (Distortion property) There exists a constant κ_2, depending only on $(\mathbf{G}, \rho)$, so that

$$\mathcal{S}_d^{\mathbf{X}}(gf) \ll \|\mathrm{Ad}(g)\|^{\kappa_2 d} \mathcal{S}_d^{\mathbf{X}}(f), f \in \mathrm{C}^\infty(\mathbf{X}), g \in \mathbf{G}(\mathbf{A}).$$

S2c. (Embedding) For each d, there exists $d' > d$ so that the trace of $\mathcal{S}_d^{\mathbf{X}}$ w.r.t. $\mathcal{S}_{d'}^{\mathbf{X}}$ is finite.

2.4.3. *Relationship between unitary Sobolev norms on* L$^2(\mathbf{X})$ *and* $\mathbf{X}$*-Sobolev norms*

S3a. We have a majorization

$$\mathcal{S}_d^{\mathrm{L}^2(\mathbf{X})} \ll \mathcal{S}_d^{\mathbf{X}}.$$

S3b.

$$\mathcal{S}_d^{\mathbf{X}}(f) \ll \mathcal{S}_{d'}^{\mathrm{L}^2(\mathbf{X})}(f), \ f \in \mathrm{L}_{cusp}^2$$

where d' depends on d, and L_{cusp}^2 is the cuspidal subspace of L$^2(\mathbf{X})$.[13]

S3c. Let ℓ a linear form on $\mathrm{C}^\infty(\mathbf{X})$. Suppose, for each standard automorphic representation π, we have $|\ell|_{\pi^\infty}| \leqslant \mathcal{S}_d^\pi$, where the unitary structure on π is the Eisenstein norm. Then

$$|\ell(v)| \ll \mathcal{S}_{d'}^{\mathbf{X}}(v),$$

whenever both sides are defined; d' depends only on d.

---

[12] If this condition is satisfied not for all v, but for all $v \notin \mathrm{T}$, where T is some fixed finite set of places, then the constant has to be replaced by $\mathrm{A}^{1+|\mathrm{T}|}$.

[13] A stronger and more natural statement is: the truncation operator $\wedge^{\mathrm{T}}$ of Arthur is continuous from the $\mathcal{S}^{\mathrm{L}^2(\mathbf{X})}$-topology to the $\mathcal{S}^{\mathbf{X}}$ topology. This statement is actually a quantitative form of [2, Lemma 1.4], which proves the same result but with the finite level fixed.

2.4.4. *Other properties.* — Suppose ℓ is a linear functional on either $C^\infty(\mathbf{X})$ or a unitary $\mathbf{G}(\mathbf{A})$-representation, and let $A(d)$ be the operator norm of ℓ with respect to $\mathcal{S}_d$.

S4a. $\log A(d)$ is convex with respect to d.[14]

S4b. Given d and $j \geqslant 1$, there exists $d' > d$ and common orthogonal bases $e_1, e_2, \ldots$ for $\mathcal{S}_d, \mathcal{S}_{d'}$ with the property that $\frac{\mathcal{S}_d(e_i)}{\mathcal{S}_{d'}(e_i)} \leqslant (1 + |i|)^{-j}$.

S4c. $\mathcal{S}_d^{\mathrm{V}}$ and $\mathcal{S}_{-d}^{\mathrm{V}}$ are self-dual.

S4d. If $\ell : \mathrm{V} \to \mathrm{W}$ is any linear functional from V to another normed vector space:

$$(\mathbf{2.7}) \qquad \ell(v) \leqslant A \|\Delta^d v\|_{\mathrm{V}} \quad (v \text{ a } \Delta\text{-eigenfunction}) \quad \Longrightarrow$$
$$|\ell(v)| \leqslant A' \mathcal{S}_{d'}^{\mathrm{V}}(v), \quad \text{all } v \in \mathrm{V}, \text{ some } d' > d.$$

S4e. $\mathcal{S}_d^{\mathbf{X}}(fg) \ll_d \mathcal{S}_d^{\mathbf{X}}(f) \mathcal{S}_d^{\mathbf{X}}(g)$.

(S4d) is a consequence of the fact that a suitable power of Δ^{-1} is trace class.

2.5. *Examples*

We shall now discuss a number of examples of using the Sobolev norms to quantify mixing or uniform distribution. The essence of all our examples is well-known. We simply want to make the point that the axioms of Sobolev norms make it simple to derive the results in great generality.

In the sequel we will use the following notation: for $\mathcal{S}_?^\star$ one of the families of Sobolev norms discussed previously, we will write $A \leqslant \mathcal{S}^\star(f)B$ to mean that there is a constant $d > 0$ so that $A \leqslant \mathcal{S}_d^\star(f)B$.

2.5.1. *Bounds for matrix coefficients and canonically normalized functionals: local bounds.* — For any place v, we denote by $\Xi_v(g)$ the Harish-Chandra spherical function on $G_v :=$ $\mathbf{G}(F_v)$, i.e. $\Xi_v(g) := \langle gv, v \rangle$ where v is the K_v-invariant function in the representation of G_v unitarily induced from the trivial character of a minimal parabolic subgroup.

It follows from [13] (as extended to reductive groups in [56]) that, if π_v is a tempered representation of $\mathbf{G}(F_v)$ one has, for any $x_1, x_2 \in \pi_v, g \in G_v$ and some constant $d \geqslant 0$ depending on $\mathbf{G}$ only

$$|\langle gx_1, x_2 \rangle| \leqslant A_v \mathcal{S}_d^{\pi_v}(x_1) \mathcal{S}_d^{\pi_v}(x_2) \Xi_v(g),$$

where $A_v = [K_v : K_v[0]]$ is equal to 1 for almost all v.[15]

Indeed, [13] implies, in fact, that $|\langle gx_1, x_2 \rangle| \leqslant [K_v : K_v[m]] \Xi_v(g)$ for $x_1, x_2 \in V[m]$. Now (2.1) and (2.7) establish the desired statement.

---

[14] This will be helpful in bounding operator norms when d is not integral, for, like the case of $\mathbf{R}^n$, the Sobolev norms are most accessible when $d \in \mathbf{Z}_{\geqslant 0}$.

[15] Indeed, if x_1, x_2 are both stabilized by a subgroup of $K_v[0]$ of index $\heartsuit$, the bound may be taken to be $\heartsuit \|x_1\| \|x_2\| \Xi_v(g)$; we shall use this later.

In the case of non-tempered representations, one has a corresponding, but weaker, bound. In this paper we will only use the special case of $\mathbf{G} = GL_2$; and moreover, π_v will always be a local constituent at v of a generic automorphic representation of $\mathbf{G}(\mathbf{A})$. In that case, one has

$$(\mathbf{2.8}) \qquad |\langle gx_1, x_2 \rangle| \leqslant A_v \mathcal{S}_d^{\pi_v}(x_1) \mathcal{S}_d^{\pi_v}(x_2) \, \Xi_v(g)^{1-2\theta},$$

for some absolute constant $\theta < 1/2$. This is due to Selberg for $F = \mathbf{Q}$, and [25] in general. As of now, it follows from the work of Kim and Shahidi [42], that (2.8) holds for $\theta \geqslant 3/26$.

It will be convenient, for the purpose of this document, to allow θ to be any number in $]3/26, 1/4[$. More precisely, any statement involving θ will be valid for any choice of θ in this interval. This notational convention will suppress ϵs at a later point. The reader may safely substitute $3/26 + \epsilon$ every time he/she sees the symbol θ.

2.5.2. *Bounds for matrix coefficients and canonically normalized functionals: global bounds.* — Let $\iota : \tilde{\mathbf{G}} \mapsto \mathbf{G}$ be the simply connected covering of $\mathbf{G}$. For $f \in C^\infty(\mathbf{X})$, set

$$(\mathbf{2.9}) \qquad \mathcal{P}f(x) = \int_{g \in \tilde{\mathbf{G}}(F) \backslash \tilde{\mathbf{G}}(\mathbf{A})} f(\iota(g)x) dg$$

for $x \in \mathbf{X}$; here dg is the invariant probability measure on $\tilde{\mathbf{G}}(F) \backslash \tilde{\mathbf{G}}(\mathbf{A})$. The endomorphism $\mathcal{P}$ realizes the orthogonal projection onto locally constant functions on $\mathbf{X}$, and is in particular L^2- and L^∞-bounded; note also that it commutes with the $\mathbf{G}(\mathbf{A})$-action (cf. [65]).

Take $g \in \mathbf{G}(\mathbf{A})$ and $f_1, f_2 \in C^\infty(\mathbf{X})$. Then there exists $\beta > 0$ and d, both depending only on the isomorphism class of $\mathbf{G}$ over F, so that

$$(\mathbf{2.10}) \qquad \left| \langle g.f_1, f_2 \rangle - \langle g.\mathcal{P}f_1, \mathcal{P}f_2 \rangle \right| \ll \|\mathrm{Ad}(g)\|^{-\beta} \mathcal{S}_d^{\mathbf{X}}(f_1) \mathcal{S}_d^{\mathbf{X}}(f_2).$$

This is a consequence of property (τ), which has been established through the work of many people; the proof was completed (together with the most difficult case) by L. Clozel [14]. Property (τ) was put in a quantitative form in [28], and the quoted statement[16] follows from Lemma 3.3 and Theorem 3.10 of *loc. cit.* Here we will need (2.10) only for the case of $\mathbf{G} = GL_2$: it follows from (2.8) that the exponent $\beta = 1/2 - \theta$ is admissible, any θ as above.

Remark. — Let us explain one of the reasons that (2.10) is so useful in the context of the present paper. It has been observed J.-L. Waldspurger [67] and greatly extended by Ichino-Ikeda [32, 33] that a wide variety of period functionals can be expressed by

---

[16] It should be noted that we could not follow the quantitative arguments of [28] in the case which relies on Clozel's work; however, their argument certainly furnishes a bound.

integrating matrix coefficients; roughly speaking, there exists a variety of pairs $(\mathbf{H} \subset \mathbf{G})$ so that, with suitable choice of measure *and suitable regularization*,

$$(\mathbf{2.11}) \qquad \left| \int_{\mathbf{H}(F)\backslash\mathbf{H}(\mathbf{A})} \varphi \; dh \right|^2 = \int_{\mathbf{H}(\mathbf{A})} \langle h.\varphi, \varphi \rangle dh.$$

The right-hand side is usually divergent, and is interpreted by a suitable regularization; in many situations, almost every local factor is equal to the local factor of a suitable L-function, which suggests a regularization involving a special value of that L-function.

In any case, (2.11) was used in [15] and [20] in order to reduce equidistribution results for $\mathbf{H}(F)\backslash\mathbf{H}(\mathbf{A})$—or translates thereof—to bounds for matrix coefficients. We shall use the same technique in Section 4.2.1 of the present paper.

We note, however, that this technique is not universally applicable, and in many instances the correct formulation of a result analogous to (2.11) remains mysterious.

2.5.3. *A quantitative form of the Ergodic principle II.* — Next, let us give a quantitative form of the Ergodic principle 1.3 from Section 1.2.

Lemma. — *Suppose* $H \subset \mathbf{G}(\mathbf{A})$ *is noncompact, and* $\chi : H \to \mathbf{C}^{\times}$ *a unitary character. Suppose that* $\mathbf{X}$ *has finite measure, i.e., that the center of* $\mathbf{G}$ *is anisotropic.*[17]

Let ν *be a (possibly signed)* χ*-equivariant measure on* $\mathbf{X}$*, i.e.* $\nu^h = \chi(h)\nu$ *for* $h \in H$*. Let* μ *be the* $\mathbf{G}(\mathbf{A})$*-invariant (Haar) probability measure, and suppose that, for some* $d \geqslant 0$*, we have the majorization:*

$$|\nu|(f) \ll \mu(f) + \epsilon \mathcal{S}_d^{\mathbf{X}}(f) \quad (f \geqslant 0).$$

Let σ *be any probability measure on* H*. Then, for any* f *with* $\mathcal{P}f = 0$*,*

$$|\nu(f) - \delta_{\chi=1}\mu(f)|^2 \ll \left(\epsilon \|\sigma\|_{d'}^2 + \|\sigma \star \check{\sigma}\|_{-\beta} \right) \mathcal{S}_{d'}^{\mathbf{X}}(f)^2.$$

Here $\delta_{\chi=1}$ *is* 1 *if* χ *is trivial and zero otherwise,*

$$\|\sigma\|_d := \int_h \|\mathrm{Ad}(h)\|^d d\sigma(h),$$

and similarly for $\|\sigma \star \check{\sigma}\|$*,* $\check{\sigma}$ *denote the pullback of* σ *by* $g \mapsto g^{-1}$*,* β *is as in* (2.10)*, and* d' *depends only on* d*.*

This is *indeed* a quantitative form of the Ergodic principle 1.3: if, e.g., $\epsilon = 0$, we see at once that the left-hand side may be made arbitrarily small to choosing σ to have large support.

---

[17] The lemma could be easily adapted to the general case by introducing a character; but we will only use it for $\mathbf{G} = \mathrm{PGL}_2$.

Proof. — We may assume that $\mu(f) = 0$. For σ a probability measure on H we set

$$f \star_\chi \sigma := \int_{\mathrm{H}} \chi(h)(h \cdot f) d\sigma(h).$$

We see:

$$|\nu(f)|^2 = |\nu(f \star_\chi \sigma)|^2 \leqslant |\nu|(|f \star_\chi \sigma|^2) \leqslant \epsilon \mathcal{S}_d^{\mathbf{X}}(|f \star_\chi \sigma|)^2 + \|f \star_\chi \sigma\|_2^2.$$

The required bound now follows from (S2b), (S4e) and from the bounds for matrix coefficients of Section 2.10. □

Remark. — In the context of the subconvexity problem for L-functions, this amounts to the *amplification method* [23, 35, 36] and the measure σ play the role of the *amplifier*.

2.5.4. *Uniform distribution of horocycles.* — As a final example, we discuss the uniform equidistribution of horospheres. A result of Peter Sarnak proves that *a closed horocycle of length* L on a hyperbolic surface becomes uniformly distributed as L approaches ∞. What we present below is an adelized version of the result for general groups, but only in the easiest version: where "closed horocycle" is replaced by (in more explicit language than what follows) "closed orbit of a maximal horospherical subgroup."

Let us fix $\mathbf{N}$ to be the unipotent radical of a minimal proper parabolic subgroup $\mathbf{B}$ of $\mathbf{G}$; for $f \in \mathrm{C}^\infty(\mathbf{X})$ let

$$f_{\mathrm{N}}(g) := \int_{\mathbf{N}(\mathrm{F}) \backslash \mathbf{N}(\mathbf{A})} f(ng) dn,$$

denote the constant term. Then for $f \in \mathrm{L}_0^2(\mathbf{X}) \cap \mathrm{C}^\infty(\mathbf{X})$, and $b \in \mathbf{B}(\mathbf{A})$

$$(\mathbf{2.12}) \qquad |f_{\mathrm{N}}(b)| \ll \delta_{\mathbf{B}}(b)^{1/2} \mathcal{S}^{\mathbf{X}}(f),$$

here by $\mathrm{L}_0^2(\mathbf{X})$, we mean the orthogonal complement of locally constant functions (i.e. the kernel of (2.9)), and $\delta_{\mathbf{B}} : \mathbf{B}(\mathbf{A}) \to \mathbf{R}_{>0}$ is the modular character (cf. Section 2.1.6).

Proof. — By Property (3c) of Sobolev norms, it is enough to verify (2.12) when f belongs to an standard automorphic representation $\pi \subset \mathrm{L}_0^2(\mathbf{X})$. The constant term is zero unless (notation of Section 2.2) $\pi = \mathrm{Eis}(\mathbf{M}, \sigma)$, where $\mathbf{M}$ is a minimal Levi subgroup. Let $\mathbf{B}$ be a parabolic subgroup with unipotent radical $\mathbf{M}$. *As an abstract unitary representation*, π is isomorphic to the space V_π of functions from $\mathbf{G}(\mathbf{A})$ to the representation space of σ, satisfying:

$$f(bg) = \sigma(b) \delta_{\mathbf{B}}(b) f(g), \quad b \in \mathbf{B}(\mathbf{A}),$$

equipped with the inner product $\int_{\mathrm{K}} \|f\|^2$. Here K is the fixed maximal compact subgroup of $\mathbf{G}(\mathbf{A})$.

Let ω be a smooth compactly supported bi-K-invariant function on $\mathbf{G}(\mathbf{A})$ so that $\omega(1) = 1$. Consider res $: V_\pi \to L^2(\mathbf{G}(\mathbf{A}))$ defined by $f \mapsto \omega f$. Apply (S1a) to see $|f(1)| \ll \mathcal{S}^{L^2(\mathbf{G}(\mathbf{A}))}(\omega f)$. The map res is L^2-bounded, and commutes with the action of the finite part of K; moreover, if $\mathcal{D}$ is any element of the universal enveloping algebra $\mathfrak{U}$ of $\mathfrak{g}$, then $\mathcal{D}\omega = \sum_i \omega_i \mathcal{D}_i$ for various $\mathcal{D}_i \in \mathfrak{U}$ and ω_i smooth, compactly supported on $\mathbf{G}(\mathbf{A})$. It follows $\mathcal{S}^{L^2(\mathbf{G}(\mathbf{A}))}(\omega f) \ll \mathcal{S}^{V_\pi}(f)$. $\qquad\square$

2.6. *Proofs concerning Sobolev norms*

In this section, which can be safely skipped in the course of reading the paper, we give the proofs that the Sobolev norms defined in Section 2.3 have the good properties indicated in that section.

The proof of the distortion properties, (S1b), (S2b) are elementary, and we omit them. Similarly we omit the proof of properties (S4a)–(S4d).

2.6.1. — We will establish (S1a), from which (S2a) follows easily. We will establish (S1a) in the adelic setting; the local setting is easier.

Let ω be as in (S1a). Take $f \in L^2(G)[\underline{m}]$, where, as usual, $\underline{m}$ is a function from places of F to non-negative integers.

The function f factors through the $\dim(\mathbf{G})[F : \mathbf{Q}]$-dimensional manifold $\mathbf{G}(\mathbf{A})/K[\underline{m}]$; this has around $\mathrm{vol}(K[\underline{m}])^{-1}$ connected components. We can apply the usual Sobolev inequality (for real manifolds) to the connected component of the identity. It shows that $f(1)$ is bounded by the L^2-norm, on the connected component of the identity, of (e.g.) the first $\dim(\mathbf{G})$ derivatives of $f\omega$.

From this it follows that:

$$|f(1)| \ll \mathrm{vol}(K[\underline{m}])^{-1} \prod_{v|\infty} \|m_v\|^{[F:\mathbf{Q}]\dim\mathbf{G}} \|f\omega\|_{L^2} \ll \|m\|^{d_0} \|f\|_{L^2}$$

where we take any $d_0 \geqslant 2[F : \mathbf{Q}]\dim(\mathbf{G})$. This, in view of (2.7), implies (S1a).

To deduce (S2a), take $f \in L^2(\mathbf{X})$. Fix $x_0 \in \mathbf{X}$. By pull-back of functions, we immediately deduce from (S1a) that $|f(x_0)| \ll \mathcal{S}_{d_0}^{L^2(\mathbf{X})}(f)$. Now, by the Lemma of Section 2.1.7, for any $x \in \mathbf{X}$, there exists $g \in \mathbf{G}(\mathbf{A})$ with $x_0 g = x$ and $\|g\| \leqslant \mathrm{ht}(x)^\star$. Therefore, the distortion properties imply that

$$(\mathbf{2.13}) \qquad |f(x)| = |f^g(x_0)| \ll \mathcal{S}_{d_0}^{L^2(\mathbf{X})}(f^g) \ll \mathrm{ht}(x)^\star \mathcal{S}_{d_0}^{L^2(\mathbf{X})}(f).$$

Thus (S2a).

2.6.2. — We prove (S3b). It suffices to check that for any $m \geqslant 1$ there exists d so that

$$(\mathbf{2.14}) \qquad |f(x)| \ll_m (\mathrm{ht}x)^{-m} \mathcal{S}_d^{L^2(\mathbf{X})}(f), \quad f \in L^2_{cusp}.$$

It suffices, by (S4d), to establish this estimate for $f \in L^2_{cusp}[\underline{m}]$.

Let $x \in \mathbf{X}$. Let $\mathbf{T}$ be as in Section 2.1.4. It is a consequence of reduction theory that there exists a compact subset $\Omega \subset \mathbf{G(A)}$ and $R > 0$ and $a \in \mathbf{T(A)}$ so that $x = \mathbf{G}(F).a\omega$, where $\omega \in \Omega$, and $a \in \mathbf{T(A)}$ is so that there exists at least one simple root α with $|\alpha(a)|_{\mathbf{A}} \geqslant \mathrm{ht}(x)^{\star}$. Let $\mathbf{U}$ be the unipotent radical of the maximal proper parabolic subgroup associated to α.

We choose a sequence $\mathbf{Z}_0 = \mathbf{Z} \subset \mathbf{Z}_1 \subset \mathbf{Z}_j = \mathbf{U}$ where $\mathbf{Z}_{j+1}/\mathbf{Z}_j$ is central in $\mathbf{U}/\mathbf{Z}_j$ for $j \geqslant -1$ (we interpret $\mathbf{Z}_{-1}$ to be trivial). Now expand f in a Fourier series along $\mathbf{Z}_0$, i.e.

$$f(x) = \sum_{\psi} f_{\psi}(x), f_{\psi}(x) := \int_{\mathbf{Z}(F)\backslash\mathbf{Z(A)}} f(ux)\psi(u)\,du;$$

the measure is the invariant probability measure, whereas the ψ sum extends on characters of $\mathbf{Z(A)}$ trivial on $\mathbf{Z}(F)$.

Only those characters ψ which are trivial on $\mathbf{Z(A)} \cap K[\underline{m}]$ contribute to this summation, and the number of such characters is bounded by $\ll \|\underline{m}\|^{\dim \mathbf{Z}}$. Let $\mathfrak{u}$ be the Lie algebra of $\mathbf{Z}(F \otimes_{\mathbf{Q}} \mathbf{R})$. To any ψ as above, we may associate its differential $d\psi \in \mathfrak{u}^*$. Fix a norm on $\mathfrak{u}^*$. Integrating by parts, we conclude that for any $n \geqslant 1$,

$$\sum_{\psi \neq 1} f_{\psi}(x) \ll_n |\alpha(a)|_{\mathbf{A}}^{-n}\|m\|^{\star} \sup_{u \in \mathbf{U(A)}, D \in \mathcal{D}} |\mathcal{D}f(ux)|,$$

where $\mathcal{D}$ is a finite set of $\mathbf{G(A)}$-invariant differential operators. Applying (2.13), we deduce that the sum is $\ll_n \mathrm{ht}(x)^{-n}\|m\|^{a(n)}\mathcal{S}_{d(n)}^{L^2(\mathbf{X})}(f)$.

It remains to estimate the term f_{ψ} for $\psi = 1$. For this, we expand it in a Fourier series along $\mathbf{Z}_1/\mathbf{Z}_0$. Proceeding in this way, we arrive inductively at the conclusion.   □

2.6.3. — We prove (S1d). First of all,

Lemma. — *Let π_v be an irreducible admissible representation of $\mathbf{G}(F_v)$. Then the image of $e_v[m]$ has dimension bounded above by $A_1 q_v^{A_2 m}$, where A_2 depends only on the isomorphism class of $\mathbf{G}$.*

This follows (for v nonarchimedean) from Bernstein's proof of uniform admissibility [3] and (for v archimedean) since any irreducible representation of a maximal compact subgroup occurs in π_v with a multiplicity bounded by its dimension (a consequence of e.g. the subrepresentation theorem; recall we interpret $q_v = e = 2.71828\ldots$ for v archimedean). We give, for illustration, Bernstein's proof for PGL$_2$; this, and the mild variant of GL$_2$ with a fixed central character, is the only case in which we shall use the result later in this paper.

Proof. — (in the case of $\mathbf{G} = \mathrm{PGL}_2$). Let $\mathcal{C}$ the characteristic function of $K[\varpi_v^m]\left(\begin{smallmatrix} \varpi_v & 0 \\ 0 & 1 \end{smallmatrix}\right)K[\varpi_v^m] \subset \mathrm{GL}_2(F_v)$. Then $\mathcal{C}$ acts on $\pi^{K[\varpi_v^m]}$, the $K[\varpi_v^m]$-fixed vectors in

π, as does the finite group $\mathcal{G} = \bar{K}_v / \bar{K}[\varpi_v^m]$. Let $\mathbf{C}[\mathcal{G}]$ be the algebra of endomorphisms of $\pi^{K[\varpi_v^m]}$ generated by $\mathcal{G}$. Let $\langle \mathcal{C} \rangle$ be the algebra of endomorphisms of $\pi^{K[\varpi_v^m]}$ generated by $\mathcal{C}$. Then $\mathbf{C}[\mathcal{G}] \cdot \langle \mathcal{C} \rangle \cdot \mathbf{C}[\mathcal{G}]$ must be the whole algebra of endomorphisms of $\pi^{K[\varpi_v^m]}$, because a corresponding fact holds for the Hecke algebra of $K[\varpi_v^m]$-bi-invariant functions on $GL_2(F_v)$, which acts irreducibly on $\pi^{K[\varpi_v^m]}$. On the other hand, $\dim\langle \mathcal{C} \rangle \leqslant \dim \pi^{K[\varpi_v^m]}$. Therefore, the dimension of $\mathbf{C}[\mathcal{G}] \cdot \langle \mathcal{C} \rangle \cdot \mathbf{C}[\mathcal{G}]$ is at most $|\mathcal{G}|^2 . \dim \pi^{K[\varpi_v^m]}$; since it acts irreducibly, its dimension is $= (\dim \pi^{K[\varpi_v^m]})^2$, yielding the result. $\qquad\square$

Now, let π be as in (S1d)—i.e. a representation of $\mathbf{G}(\mathbf{A})$—and let V be the underlying vector space. The Lemma shows that any $X \in V[\underline{m}]$ is a sum $X = \sum_i x_i$ where x_i is a pure tensor, and the number of x_is is at most $\prod_{m_v \neq 0}(A_1 q_v^{A_2 m_v}) \leqslant \|m\|^{C_2}$, where $C_2 = A_2 + \frac{\log A_1}{\log 2}$. For any such pure tensor $x_i = \bigotimes_v x_{i,v}$, we have (notation as in the statement of [S1d]):

$$|\ell(x_i)| = \left| \prod_{m_v \neq 0} \ell(x_{i,v}) \prod_{m_v = 0} \ell(x_{i,v}) \right| \leqslant A' \|m\|^{d+\epsilon} \|x\|.$$

Here we used the fact that $\prod_{m_v \neq 0} A \leqslant A' \|m\|^{\varepsilon}$, where A' can be taken to a function of A and ϵ. Thus an application of Cauchy-Schwarz yields:

$$|\ell(X)| \leqslant \|m\|^{C_2/2} A' \|m\|^{d+\epsilon} \left(\sum_i \|x_i\|^2 \right)^{1/2} \leqslant A'(\epsilon) \|\Delta_{\mathbf{A}}^{d+C_2/2+\epsilon} X\|$$

(S1d) now follows from (2.7). $\qquad\square$

Remark. — It will be useful to apply (S1d) to multilinear functionals. Suppose that we are given (e.g.) a bilinear functional $B : \pi_1 \otimes \pi_2 \to \mathbf{C}$, which factorizes as $B = \prod_v B_v$, and, for every v, $B(x_{1,v}, x_{2,v}) \leqslant A\mathcal{S}^d(x_{1,v})\mathcal{S}^d(x_{2,v})$; also $|B_v| \leqslant 1$ when both $x_{1,v}, x_{2,v}$ are spherical of norm 1.

Then $B(x_1, x_2) \ll A'\mathcal{S}^{d'}(x_1)\mathcal{S}^{d'}(x_2)$ (similar notation to (S1d)). This follows formally: apply (S1d) first to the linear functional $x \mapsto B(x, y)$, for y fixed; we see it is bounded in absolute value by $A'\mathcal{S}^{d'}(x)\mathcal{S}^d(y)$. Now apply (S1d) to the functional $y \mapsto B(x, y)$, for x fixed; note that the condition on spherical inputs is satisfied not for all v, but only for those v where x_v is spherical, this being handled by the footnote to (S1d).

2.6.4. — We prove (S2c). This proof is modeled on ideas from [4]. For $x \in \mathbf{X}$, consider the linear form on $C_c^\infty(\mathbf{X})$ given by: $\ell_x : f \mapsto \mathrm{ht}(x)^r \Delta_{\mathbf{A}}^r f$. By (S2a), ℓ_x is bounded with respect to $\mathcal{S}_{d_0}^{L^2(\mathbf{X})}$, for some $d_0 > 0$ and uniformly in x (at least after increasing d_0 as necessary). It follows that, for suitable $d > d_0$, the trace of $|\ell_x|^2$ with respect to the square of $\mathcal{S}_d^{\mathbf{X}}$ is finite and independent of x. Integrate over $x \in \mathbf{X}$ to obtain the desired conclusion.

2.6.5. — We prove (S3c); however, some preliminaries are necessary. We shall need to make use of an assertion substantively equivalent to the well-known theorem of W. Müller [55] that discrete automorphic spectrum is trace-class. That fact, in turn is closely related to the fact that the Eisenstein series on a general group have finite order. We believe that to handle the general case should be routine using the techniques of Müller's paper [55]; but we have not verified the detail, and this is why the proof is presently only for GL$_n$, where it is possible to give a direct proof of Müller's trace-class theorem—using the work of Moeglin and Waldspurger.

For any automorphic representation, set

$$\mathrm{C}_{\mathrm{Sob}}(\pi) = \inf_{f \in \pi, \|f\|_2 = 1} \|\Delta_{\mathbf{A}} f\|.$$

It might be referred to as the "Sobolev-conductor" of the representation π; as we shall see momentarily, it is bounded above and below in terms of the analytic conductor in the case of GL$_n$.

We begin with:

(**2.15**)   The number of cuspidal π with $\mathrm{C}_{\mathrm{Sob}}(\pi) \leqslant \mathrm{X}$ is at most polynomial in X

Proof. — There exists (by (S3a) and (S2c)) some d so that the inclusion of $\mathrm{L}^2(\mathbf{X})$ into the Hilbert space associated to $\mathcal{S}_d^{\mathbf{X}}$ is trace class. There exists d' so that $\mathcal{S}_{\mathbf{X}}^d$ is majorized by $\mathcal{S}_{d'}^{\mathrm{L}^2(\mathbf{X})}$ for cuspidal functions, by (S3b). Choose e_π in each cuspidal representation[18] so that $\mathcal{S}_{d'}(e_\pi) = \mathrm{C}_{\mathrm{Sob}}(\pi)^{d'} \|e_\pi\|_2$. Then $\mathcal{S}_d^{\mathbf{X}}(e_\pi) \ll \mathrm{C}_{\mathrm{Sob}}(\pi)^{d'} \|e_\pi\|_2$. The trace-class feature forces $\sum_\pi \mathrm{C}_{\mathrm{Sob}}(\pi)^{-d'} < \infty$. Thus (2.15). $\square$

With notation as in Section 2.2, we now claim:

(**2.16**)   There is $d_1 \geqslant 1$ so that $\displaystyle\int_{\chi \in \mathcal{X}} \mathrm{C}_{\mathrm{Sob}}(\mathcal{I}(\chi))^{-d_1} d\chi < \infty$.

For $\mathbf{G} = \mathrm{GL}_n$, the classification of discrete spectrum (Section 2.2.1) together with the classification of discrete series ([52]) reduces this to (2.15).

Finally, (2.16) $\implies$ (S3c). Let ℓ be a linear form as in (S3c). We express an arbitrary $f \in \mathrm{L}^2(\mathbf{X})$ as an integral $f = \int f_\chi d\chi$, where $f_\chi \in \mathcal{I}(\chi)$ (notation of Section 2.2; we have already remarked in Section 2.2.1 that the Plancherel decomposition is pointwise defined). Using (2.16) and Cauchy-Schwarz,

$$(\mathbf{2.17}) \qquad |\ell(f)|^2 = \left| \int \ell(f_\chi) d\chi \right|^2 \ll \int \mathrm{C}_{\mathrm{Sob}}(\mathcal{I}(\chi))^d \left| \ell(f_\chi) \right|^2 d\chi$$

$$\leqslant \int \|\Delta_{\mathbf{A}}^{d'} f_\chi\|^2 d\chi \ll \mathcal{S}_{d'}^{\mathrm{L}^2(\mathbf{X})}(f).$$

Let us note the following corollary, which can also be proved in other ways:

---

[18] It is not difficult to check, although it is in fact not essential, that the infimum which defines $\mathrm{C}_{\mathrm{Sob}}(\pi)$ is realized.

Corollary. — *The number of cuspidal representations on* $\mathrm{GL}_n(\mathrm{F})$ *with fixed central character and conductor* $\leqslant \mathrm{X}$ *is bounded by a power of* X.

It follows from the prior Lemma and the following Lemma due to W. T. Gan[19]

Lemma **2.6.6.** — *As* π *varies through automorphic cuspidal representations of* $\mathrm{GL}_n(\mathbf{A})$,

$$(\textbf{2.18}) \qquad \frac{\log \mathrm{C}(\pi)}{\log \mathrm{C}_{\mathrm{Sob}}(\pi)} \text{ is bounded above and below.}$$

It is of interest (from the point of view of the "Selberg-class") to determine what the correct growth rate in this corollary is; among other things, it is related to the question of the number of twists needed in converse theorems; in that connection:

Question. — *Let* $\mathrm{N}(\mathrm{X})$ *be the number of cuspidal representations on* $\mathrm{GL}_n(\mathbf{Q})$ *with analytic conductor* $\leqslant \mathrm{X}$. *Determine the asymptotic behavior of* $\mathrm{N}(\mathrm{X})$*; for example, is it true that* $\frac{\log \mathrm{N}(\mathrm{X})}{\log \mathrm{X}}$ *approaches* $n + 1$?

3. Integral representations of L-functions: local computations.

3.1. *Notations*

This part, comprising Sections 3.2–3.7, will be essentially of local nature; we collate here some notations that will be used.

3.1.1. *Local fields.* — Throughout this part, we shall work over a local field k. We shall always suppose k to be given as the completion of F at a place v; thus, we regard the place v and an isomorphism $\mathrm{F}_v \cong k$ to be given along with k.[20]

We denote by $|\cdot| = |\cdot|_v$ the absolute value on k normalized so that $|x|_{\mathbf{R}} = \max\{x, -x\}$, $|z|_{\mathbf{C}} = z.\overline{z}$ and for v non-archimedean, $|\varpi| = q^{-1}$ for ϖ an uniformizer of k with q the size of the residue field; in that later case we denote by $\mathfrak{o}$ the maximal order of k. If k is archimedean, we denote by $\deg(k)$ its degree over $\mathbf{R}$.

3.1.2. *Additive characters.* — We equip once for all each local field $k = \mathrm{F}_v$ with an additive character ψ obtained as follows: we let $e_{\mathbf{Q}} : \mathbf{A}_{\mathbf{Q}} \to \mathbf{C}^\times$ be the unique additive character which coincides with $x \to \exp(2\pi i x)$ on $\mathbf{R}$ and set $e_{\mathrm{F}} = e_{\mathbf{Q}}(\mathrm{tr}_{\mathrm{F}/\mathbf{Q}})$. We denote by ψ the restriction of e_{F} to $\mathrm{F}_v = k$. In the sequel, any local considerations involving an additive character of k will refer to that character ψ or its complex conjugate. If k is nonarchimedean, we denote its conductor by d_ψ. We set $d_\psi = 0$ if k is archimedean.

---

[19] This statement was derived from a result sketched by Gan to one the authors (A.V.), namely, the conductor of any representation of $\mathrm{GL}_n(k)$—for k a nonarchimedean local field—admitting a vector fixed under the r-th principal congruence subgroup is at most rn^2.

[20] The reason for this is that it will be helpful in handling implicit constants.

3.1.3. *Implicit constants.* — Unless otherwise specified, given some parameter ε say, the implicit constant in $\ll_\varepsilon$ will depend (of course) on the parameter ε but may also depend on the quantity q^{d_ψ} *as k varies through completions of* F. Note that $q^{d_\psi} = 1$ for almost every k.

3.1.4. *Subgroups.* — We denote by Z, N, B the usual upper triangular unipotent (resp. Borel) subgroup of GL_2 or PGL_2 and by A the diagonal subgroup with lower diagonal entry equal to 1. We denote $G = GL_2(k)$, $\bar{G}$ the quotient of G by its center and Z, N, B, A the group of k points of Z, N, B, A. For t, x, y in any ring R, we set

$$z(t) = \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix}, \qquad n(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \qquad a(y) = \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}.$$

We let K be the standard maximal compact subgroup of G, i.e. the group of integral matrices if k is nonarchimedean, and the stabilizer of the "standard" orthogonal form $(x_1, x_2) \mapsto x_1^2 + x_2^2$ or the "standard" Hermitian form $(z_1, z_2) \mapsto z_1\overline{z_1} + z_2\overline{z_2}$ if v is real or complex.

For v non-archimedean and $m \geqslant 0$ we denote by $K[m] \subset K_0[m] \subset K$ the subgroups of matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GL_2(\mathfrak{o})$ such that $a \equiv d \equiv 1$, $b \equiv c \equiv 0$ (ϖ^m), or $c \equiv 0$ (ϖ^m), or with no constraint on a, b, c, d, respectively.

3.1.5. *Measure normalizations.* — Fix an additive Haar measure on k which is self-dual with respect to the character ψ. (Note that for $k \cong \mathbf{R}$, this gives Lebesgue measure; for $k \cong \mathbf{C}$, this gives the measure $idz \wedge d\bar{z} = 2dx \wedge dy$; and for k nonarchimedean, it assigns mass $q^{-d_\psi/2}$ to the maximal compact subring of k.)

Transport it to the Haar measure dn on N via $x \mapsto n(x)$; we equip $A = \{a(y) : y \in k^\times\}$ and $Z = \{z(u) : u \in k^\times\}$ with their Haar measures $\zeta_k(1)\frac{dy}{|y|}$, $\zeta_k(1)\frac{du}{|u|}$ respectively.

The product measure $\zeta_k(1)\frac{du}{|u|}dx\zeta_k(1)\frac{dy}{|y|}$ defines the right Haar measure $d_R b$ on $B = \{z(u)n(x)a(y) : u, y \in k^\times, x \in k\}$ while the left Haar measure $d_L b$ is $|y|^{-1}d_R b$. The left and right Haar measures on the Borel of $\overline{G}$ are defined explicitly through the same parameterisations (with $\zeta_k(1)du/|u|$ removed). We denote by dk the Haar probability measure on K. The Haar measures on G, $\overline{G}$ are then given by $d_L bdk$.

The measure on G determines a Plancherel measure $d\mu_P$ on the unitary dual of G. This is normalized by the formula:

$$f(1) = \int_{\pi \in \hat{G}} \langle f, \chi_\pi \rangle_{L^2(G)} d\mu_P, \quad f \in C_c(G)$$

where χ_π is the character of $\pi \in \hat{G}_{Aut}$.

For k archimedean, we fix a norm, $\| \cdot \|_{\mathfrak{g}}$ say, on $\mathfrak{g}$ the Lie algebra of G.

3.1.6. *A measure computation.* — It will be of use, as a check, to later compare dg to the measure $d'g := d_{\mathrm{L}}b dn$ pushed forward from the map $\mathrm{B} \times \mathrm{N} \to \mathrm{G}$, $(n, b) \mapsto nwb$ (here $w = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ denote the Weyl element; recall that the image of that map is dense in G by the Bruhat decomposition). The later coincides with the measure pushed forward from $dn \times d_{\mathrm{R}}b$ on $\mathrm{N} \times \mathrm{B}$ via $(n, b) \mapsto nwb$, for both are invariant and have the same behavior near the element w. In fact,

$$(\textbf{3.1}) \qquad d'g = \iota_k dg, \quad \iota_k = q^{-d_\psi/2} \frac{\zeta_k(1)}{\zeta_k(2)}.$$

Indeed, let $f \in \mathrm{C}_c(\mathrm{G})$, and let $\bar{f}(g) = \int_{b \in \mathrm{B}} f(bg) d_{\mathrm{L}}(b)$. Then

$$\int f(g) dg = \int \bar{f}(k) dk, \quad \text{while} \quad \int f(g) d'g = \int \bar{f}(wn(x)) dx.$$

The function $\bar{f}$ has the transformation property $\bar{f}(bg) = |b|^{-1}\bar{f}(g)$ where $|b| = |n.a(y)| := |y|$; so we are reduced to computing the ratio $\frac{\int \bar{f}(k)}{\int \bar{f}(wn(x)) dx}$ for a *single* $\bar{f}$ with this property. We choose the square of the "height" function $\bar{f}(g) = \frac{|\det g|}{\|(0,1)g\|^2}$, where $\|\cdot\|$ is a K-invariant norm on k^2, which we normalize so that $\|(0, 1)\| = 1$. We are reduced, thus, to evaluating $\iota_k = \int_{x \in k} \|(1, x)\|^{-2} dx$.

1. If k is nonarchimedean, with residue field of size q,

$$\iota_k = q^{-d_\psi/2}(1 + (q-1)(q^{-2} + q^{-3} + \cdots)) = q^{-d_\psi/2}\frac{\zeta_k(1)}{\zeta_k(2)}.$$

2. If $k = \mathbf{R}$, then $\iota_k = \int \frac{dx}{1+x^2} = \pi = \frac{\Gamma_{\mathbf{R}}(1)}{\Gamma_{\mathbf{R}}(2)}$.
3. If $k = \mathbf{C}$, then $\iota_k = 2 \int \frac{dx \wedge dy}{(1+x^2+y^2)^2} = 4\pi \int_{r=0}^{\infty} \frac{r\,dr}{(1+r^2)^2} = 2\pi = \frac{\Gamma_{\mathbf{C}}(1)}{\Gamma_{\mathbf{C}}(2)}$.

3.1.7. *Representations: convention.* — In the sequel, the irreducible admissible representations of G that will occur are assumed to be local constituents of unitary automorphic representations of $\mathrm{GL}_2(\mathbf{A})$. In particular, by the work of Kim and Shahidi [42], such representations will never be complementary series with parameter $\geqslant \theta$. (cf. (2.8)).

3.1.8. *The local analytic conductor.* — Given $d = 1, 2$ and π an irreducible admissible representation of $\mathrm{GL}_d(k)$ acting on a vector space V say we define the (local) analytic conductor of π, $\mathrm{C}(\pi)$, to be

- k non-archimedean: $\mathrm{C}(\pi) = q^{f(\pi)}$ where $f(\pi)$ is the conductor of π.
- k archimedean: the L-factor of π has the form

$$\mathrm{L}(\pi, s) = \prod_{i=1}^{d} \Gamma_k(s + \mu_{\pi,i})$$

with $\Gamma_{\mathbf{R}}(s) = \pi^{-s/2}\Gamma(s/2)$, $\Gamma_{\mathbf{C}}(s) = 2(2\pi)^{-s}\Gamma(s)$ and $\mu_{\pi,i} \in \mathbf{C}$; the analytic conductor is given by

$$C(\pi) = \prod_{i=1}^{d}(2 + |\mu_{\pi,i}|)^{\deg(k)}.$$

In the sequel, the usual notation of Vinogradov $A \ll_{\pi} B$ will have the following slightly more precise meaning: there are constants $C, d \geqslant 0$ (independent of π) such that $|A| \leqslant C.C(\pi)^d.|B|$; we will then say that A is bounded by B, polynomially in π.

3.1.9. — We recall that we have defined previously (Section 2.3.2) a system of Sobolev norms $\mathcal{S}_d^{\mathrm{V}}$ on any unitary representation V of G; these norms have the property, among others, that $\mathcal{S}_d^{\mathrm{V}}(v) = \|\Delta^d v\|$ for a certain positive self-adjoint operator Δ on V. As a complement to our earlier convention 3.1.3, the indices of the Sobolev norms occurring in inequalities such as $A \leqslant \mathcal{S}^{\star}(f)B$ will always be independent of the field k.

3.1.10. *Principal series representations.* — For (ω, ω') a pair of characters of $k^{\times}$, we denote by $\mathcal{I}(\omega, \omega')$ or $\omega \boxplus \omega'$ the corresponding *principal series representation* of G which is unitarily induced from the corresponding representation of B: the L^2-space of functions f on G such that

$$f\left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}g\right) = |a/d|^{1/2}\omega(a)\omega'(d)f(g),$$

with respect to the inner product

$$\langle f, f \rangle = \int_{\mathrm{K}} |f(k)|^2 dk.$$

If ω, ω' are unitary, the resulting inner product is G-invariant and $\mathcal{I}(\omega, \omega')$ is thus a unitary representation. Such representation will be called a *unitary principal series*. In this case, for $f \in \mathcal{I}$,

$$(\mathbf{3.2}) \qquad \|f\|^2 = \int_{k \in \mathrm{K}} |f(k)|^2 = \iota_k^{-1} \int_{x \in k} |f(wn(x))|^2 dx,$$

where $\iota_k = q^{-d_{\psi}/2}\frac{\zeta_k(1)}{\zeta_k(2)}$ is as in Section 3.1.6.

Even if ω, ω' are not unitary, a principal series representation $\mathcal{I}(\omega, \omega')$ may be *unitarizable* although not, in general, with respect to the above inner product. In particular, in such cases, $\overline{\omega}^{-1} = \omega'$.

3.1.11. *Deformation.* — It will be of utility to deform induced representations. To that end we introduce the following notation: Suppose that π is induced from[21] two unitary characters ω, ω', i.e. $\pi = \mathcal{I}(\omega, \omega')$ and s is a complex number. We set

$$\pi_s := \omega|\cdot|^s \boxplus \omega'|\cdot|^{-s};$$

it is realized in the induced space $\mathcal{I}_s$. For $f \in \mathcal{I}$, we define f_s to be the vector in $\mathcal{I}_s$ whose restriction to the maximal compact K coincide with that of f.

The map $f \mapsto f_s$ is K-equivariant from π to π_s. It is not G-equivariant in general.

3.1.12. γ-*factors.* — If π is a unitary representation of $\mathrm{GL}_r(k)$, we define the γ-factor, as usual, as the ratio

$$\gamma(\pi, \psi, s) = \varepsilon(\pi, \psi, s)\frac{\mathrm{L}(\tilde{\pi}, 1-s)}{\mathrm{L}(\pi, s)}.$$

We recall that, if ψ is unramified and k non-archimedean, then

(**3.3**) $\varepsilon(\pi, \psi, s) = q^{-f_\pi(s-1/2)},$

where f_π is the local conductor of π (in the sense of [39]). When $r = 1$ and π is a character χ of $k^\times$, we obtain the γ-factor of Hecke and Tate. Recall that, for Φ a Schwarz function on k and

$$Z(\Phi, \chi, s) = \int \Phi(x)\chi(x)|x|^s d^\times x,$$

we have

(**3.4**) $Z(\widehat{\Phi}, \chi^{-1}, 1-s) = \gamma(\chi, \psi, s)Z(\Phi, \chi, s),$

where $\Phi \mapsto \widehat{\Phi}$ is the Fourier transform; also $\gamma(\chi, \psi, s)$ can be expressed formally as $\int_{y \in k} \psi(x)\chi(x)^{-1}|x|^{-s}dx$, and it has poles only when $\Re(\chi\alpha^s) = 1$.

We shall use the following estimates for the γ-factor: given $0 < \epsilon < 1/100$, one has for GL_1,

(**3.5**) $|\gamma(\chi, \psi, \sigma)| \asymp_{\epsilon, q^{d_\psi}} \mathrm{Cond}(\chi)^{-(\sigma-1/2)}, \quad (\sigma \in [\epsilon, 1/2 - \epsilon]),$

and for GL_2,

(**3.6**) $|\gamma(\pi, \psi, \sigma)| \asymp_{\epsilon, q^{d_\psi}} \mathrm{Cond}(\pi)^{-(\sigma-1/2)}, \quad (\sigma \in [3/26 + \epsilon, 1/2 - \epsilon]).$

This follows from the prior remark, the dependence of ε-factors on ψ, and the fact that $\mathrm{L}(\pi, s)$ and $\mathrm{L}(\tilde{\pi}, 1-s)$ are uniformly bounded above and below in the stated region, if k

---

[21] The choice of ω, ω is unique up to order.

is nonarchimedean; and by a direct computation (see [38, Chap. 1, Sections 5 & 6]) when k is archimedean.

For GL$_1$, the same bound remains valid (with the same proof), so long as π is unitary; we may even replace $(3/26, 1/2)$ by any compact subinterval of $[0, 1]$.

3.1.13. — Let $f \in C_c^\infty(\mathbf{R}_+)$, and consider the integral

$$G(\lambda, t) := \int_0^\infty f(x)|x|^{i\lambda} e^{itx} dx.$$

The method of stationary phase shows that the dominant contribution to this integral comes from stationary points of the phase function $\lambda \log(x) + tx$, i.e., when $x = \lambda/t$. If f is fixed, it therefore follows that $G(\lambda, t)$ is small unless λ and t are of the same order of magnitude. Moreover, stationary phase predicts that, when $\lambda \sim t$, the magnitude $|G(\lambda, t)| \sim \lambda^{-1/2}$.

A similar phenomenon occurs when we replace $\mathbf{R}$ by a nonarchimedean local field with residue field of size q; in that case, the pertinent integrals are closely related to Gauss sums. If χ is a character of conductor $C = q^{f_\chi} > 1$, and $D = q^{d_\psi}$ the discriminant of ψ, then: for $t \in k^\times$,

$$(\mathbf{3.7}) \qquad \int_{|x|=|u|} \chi(x)\psi(tx)d^\times x = \begin{cases} 0, & |u| \neq CD/|t| \\ \eta C^{-1/2}|u|^{\Re \chi}, & |u| = CD/|t| \end{cases}$$

with $|\eta| = 1$.

The following general version will be helpful:

3.1.14.

Lemma. — *Let* $A < B \in \mathbf{R}$. *Let* k *be a local field,* $\phi \in C_c^\infty(k^\times)$ *a smooth function, and* χ *a character of* k *of conductor* C *such that, writing* $|\chi(x)| = |x|^\sigma$, *one has* $A \leqslant \sigma \leqslant B$.

Set, for $t \in k^\times$,

$$G_\phi(\chi, t) = \int_k \phi(x)\psi(tx)\chi(x)dx.$$

Then:

1. *For every* $N \geqslant 0, \epsilon > 0$, *we have*:

$$|G_\phi(\chi, t)| \ll_{\phi, N, A, B, \epsilon} C^{-1/2+\epsilon} \max\left(\frac{1+|t|}{C}, \frac{C}{|t|}\right)^N.$$

2. *Fix* $\varepsilon > 0$. *Then there exists* t, *satisfying* $|\frac{\log|t|}{\log(C+1)} - 1| \leqslant \varepsilon$, *so that* $|G_\phi(\chi, t)| \gg_{\varphi, \varepsilon} C^{-1/2-2\varepsilon}$.

Proof. — The statement (2) follows from statement (1) and Plancherel formula:

$$\int_k |G_\phi(\chi, t)|^2 dt = \int_k |\phi(x)\chi(x)|^2 dx,$$

noting that the volume of the set $|t| \leqslant C$ is $\asymp C$. It suffices thereby to prove (1).

We shall give the proof of the bound $|G_\phi(\chi, t)| \ll_{\phi,N,A,B} ((1+|t|)/C)^N$. Let us suppose, first of all, that k is archimedean, and fix a basis $\{D_i\}$ of $k^\times$-invariant differential operators on $k^\times$ of degree $[k : \mathbf{R}]$. Then χ is an eigenfunction of each D_i, and there exists i so that the eigenvalue λ_i satisfies $|\lambda_i| \gg C$. Now,

$$|G_\phi(\chi, t)| \ll C^{-N} \int D_i^N(\phi(x)\psi(tx))\chi(x) \ll_{\phi,N,A,B} C^{-N}(1+|t|)^N.$$

If k is nonarchimedean, there exists a constant $c > 0$, depending only on ϕ and the discriminant of ψ, so that $\phi(x)\psi(tx)$ is invariant under $x \mapsto xy$ whenever $|t(y-1)| \leqslant c$. In particular, $G_\phi(\chi, t)$ vanishes whenever $|t|C^{-1} < c$.

The bound $|G_\phi(\chi, t)| \ll_{\phi,N,A,B} (|t|/C)^{-N}$ is similar; we replace the role of multiplicative translations and multiplicatively invariant differential operators by additive translations and additively invariant differential operators. We also reverse the roles of χ and ψ in the prior argument.

As for the bound $G_\phi(\chi, t) \ll_{\phi,N,A,B} C^{-1/2+\varepsilon}$, we may assume that χ is unitary, after replacing ϕ by $\phi|x|^\sigma$ suitable σ. Fix $\varepsilon > 0$ small. By (3.4), applied to $x \mapsto \phi(x)|x|^{1-\varepsilon}\psi(tx)$, we may write $G_\phi(\chi, t)$ as

$$\frac{\int_k \widehat{(\phi\alpha^{1-\varepsilon})}(x+t)(\chi\alpha^\varepsilon)^{-1}(x)dx}{\gamma(\chi, \psi, \varepsilon)} = \frac{\int_k \widehat{(\phi\alpha^{1-\varepsilon})}(x)\chi^{-1}(x-t)|x-t|^{-\varepsilon}dx}{\gamma(\chi, \psi, \varepsilon)}.$$

Our assertion follows from the rapid decay of $\widehat{(\phi\alpha^{1-\varepsilon})}(x)$ and (3.5). $\qquad\square$

3.2. *Whittaker models*

Let ψ be the non-trivial additive character of k described previously. Given π a generic representation of G, π admits a unique realization in a Whittaker model, which we shall denote by $\mathcal{W}(\pi, \psi)$, and thereby in a Kirillov model, which we denote by $\mathcal{K}(\pi, \psi)$. We equip the Kirillov model with that inner product given by the L^2-inner product on $k^\times$:

$$(\mathbf{3.8}) \qquad \langle W, W \rangle = \int_{k^\times} |W(y)|^2 d^\times y.$$

Unless otherwise specified, the inner product (3.8) is *always* the one that we will put on any Kirillov or Whittaker model.

Let us suppose, moreover, that π is unitary, i.e. *equipped with a* G-*invariant inner product.* Then we may choose an intertwiner $\pi \mapsto \mathcal{K}(\pi, \psi)$ which preserves inner products. Such an intertwiner is unique up to a scalar of absolute value 1, and we will regard it as having been fixed; thus we will identify π with $\mathcal{K}(\pi, \psi)$. Throughout this paper, since we are concerned only with the size of various quantities, the scalar of absolute value 1 will not prove a problem. Therefore—in the setting where π is equipped with an inner product—we will regard the following statement, for example, as meaningful:

(**3.9**)   Choose $v \in \pi$ corresponding to the element $\mathrm{W} \in \mathcal{W}(\pi, \psi) \ldots$

3.2.1. *Intertwiners from the induced model to the Whittaker model.* — It will be useful later to record these in somewhat explicit form. Notation as in Section 3.1.10, let $\pi = \mathcal{I}(\omega, \omega')$. An explicit intertwiner $\pi \to \mathcal{K}(\pi, \psi)$ is given by:

(**3.10**) $f \mapsto \mathrm{W}_f, \ \ \mathrm{W}_f(g) = \eta_k^{-1/2} \int_k f(wn(x)g)\psi(x)dx, \eta_k = \zeta_k(1)\iota_k.$

If ω, ω' are unitary, then the map (3.10) is isometric by (3.2); this was the reason for the inclusion of η_k.

Taking $g = a(y)$, one has

$$\mathrm{W}_f(a(y)) = \eta_k^{-1/2}|y|^{1/2}\omega'(y) \int_{x \in k} f(wn(x))\psi(xy)dx.$$

In general, the right-hand integral is not absolutely convergent, but it is conditionally convergent—interpreted as a limit of integrals over increasing compact $|x| \leqslant \mathrm{R}$ —if f is a smooth vector.

It follows from the above expressions, using the shorthand $\alpha(y) := |y|$, that:

(**3.11**) $\eta_k^{1/2} \int \mathrm{W}_f(y)\chi(y)d^\times y = \left(\int f(wn(x))\chi'(x)^{-1}dx \right) \gamma(\chi'^{-1}, \psi, 1)\zeta_k(1),$

where $\chi' := \chi\omega'\alpha^{1/2}$.

Remark. — It is useful to discuss convergence. Suppose ω, ω' unitary. The left hand side of (3.11) is holomorphic for $\Re(\chi) > -1/2$. On the other hand, the right-hand side integral is absolutely convergent and holomorphic in the range $-1/2 < \Re(\chi) < 1/2$.

We observe that the left-hand side of (3.11) has poles precisely when $\chi\omega = \alpha^{-1/2}$ or $\chi\omega' = \alpha^{-1/2}$. These poles arise from the asymptotics of $\mathrm{W}(y)$ as $|y| \to 0$. On the right-hand side, the integral involving f has poles at $\chi\omega' = \alpha^{1/2}$, arising from the behavior as $|x| \to 0$; this pole is cancelled by the zero of the γ-factor $\gamma(\chi^{-1}\omega'^{-1}, \psi, 1/2)$ at this point. Moreover, one has poles at $\chi\omega = \alpha^{-1/2}$ (arising from the asymptotics of f as $|x| \to \infty$) and $\chi\omega' = \alpha^{-1/2}$ (arising from the pole of γ).

3.2.2. *The Jacquet-Langlands functional equation.* — Let π be generic, not necessarily unitary. For χ a character of $F^\times$, let $\Re\chi$ denote the real part of χ ($|\chi(.)| = |\cdot|^{\Re\chi}$) and set for $W \in \mathcal{K}(\pi, \psi)$,

$$(\mathbf{3.12}) \qquad \ell^\chi(W) := \int_{k^\times} W(u)\chi(u)d^\times u.$$

This integral is absolutely convergent for $\Re(\chi) = \sigma > 0$. By the theory of Hecke and Tate, $\ell^\chi(W)$ admits a meromorphic continuation to $\widehat{k^\times}$; more precisely the ratio $\frac{\ell^\chi(W)}{L(\pi\otimes\chi,1/2)}$ is holomorphic; it satisfies the local functional equation

$$(\mathbf{3.13}) \qquad \gamma(\pi \otimes \chi, \psi, 1/2)\ell^\chi(W) = \ell^{\omega^{-1}\chi^{-1}}(\widetilde{W})$$

where $\widetilde{W} = \pi(w).W$, and ω is the central character of π.

Note we regard $\widetilde{W}$ as belonging to the Whittaker model of π; it would also be possible to incorporate the ω-twist into it and regard it as being in the Whittaker model of $\tilde\pi$, bearing in mind that $\tilde\pi \cong \omega(\det)^{-1} \otimes \pi$. In any case, the Sobolev norms of $\widetilde{W}$ and W are the same, because the element w belongs to K.

Proposition **3.2.3** (Pointwise bounds for the Whittaker function). — *For any* $N \geqslant 1$, *any* $\varepsilon > 0$ *and all* $W \in \mathcal{K}$, *there exists* $d = d(N)$ *so that*:

$$|W(y)| \ll_N \mathcal{S}_d^{\mathcal{K}(\pi,\psi)}(W) \begin{cases} |y|^{1/2-\theta}, & |y| \leqslant 1, \\ |y|^{-N}, & |y| \geqslant 1. \end{cases}$$

Proof. — The assertion for $|y| \geqslant 1$ is elementary; it is a quantification of the well-known fact that Whittaker functions decay at ∞. For example, in the real case, one may use the fact that there exists an element of the Lie algebra of G which acts on the Kirillov model by $W(y) \mapsto yW(y)$. We leave it to the reader (see also [7, (29)]). As a consequence for the assertion with $|y| > 1$, we deduce that

$$(\mathbf{3.14}) \qquad |\ell^\chi(W)| \ll S^{\mathcal{K}(\pi,\psi)}(W)$$

whenever $\Re(\chi) > 0$.

By inverse Mellin transform[22] and (3.13), we have

$$W(y) = \int_{\Re\chi=\sigma\gg1} \chi^{-1}(y)d\chi \int_{k^\times} W(u)\chi(u)d^\times u$$

$$= \int_{\Re\chi=\sigma\gg1} \chi^{-1}(y)\frac{\ell^{\chi^{-1}\omega^{-1}}(\widetilde{W})}{\gamma(\pi \otimes \chi, \psi, 1/2)}d\chi$$

---

[22] The measure on the space of χ is obtained as follows: the χ-space is a principal homogeneous space for the dual group to $k^\times$, and we transport the dual measure to that space.

$$= \int_{\Re\chi=\sigma'} \chi^{-1}(y) \frac{\ell^{\chi^{-1}\omega^{-1}}(\widetilde{W})}{\gamma(\pi\otimes\chi,\psi,1/2)} d\chi.$$

At the last stage, we have shifted the contour to $\Re\chi=\sigma'$ for $\sigma'\in(\frac{3}{26}-1/2,0)$: under our convention, the contour shift does not cross any pole of the local L-factor $L(\pi\times\chi,1/2)$, and therefore the integrand remains holomorphic.

We observe that in the nonarchimedean case

$$\int_{k^\times} \widetilde{W}(u)\chi^{-1}(u)\omega^{-1}(u)d^\times u = 0$$

as long as the conductor of $\chi\omega$ is strictly greater than any r for which $\widetilde{W}$ is $e[r]$-invariant. Applying then (S4d), (3.6), the previous bound (3.14) in the range $\Re(\chi^{-1})>0$ and the previous observations, we conclude that, for any $m\geqslant 1$,

$$|W(y)| \ll_{m,\sigma'} |y|^{-\sigma'} \mathcal{S}^{\mathcal{K}(\pi,\psi)}(W) \int_\chi C(\chi\otimes\omega)^{-m} C(\pi\otimes\chi)^{\sigma'}.$$

The inner integral may be expressed as $\int_\chi C(\chi)^{-m} C(\tilde\pi\otimes\chi)^{\sigma'}$. Since $\sigma'<0$ the inner integral is bounded independently of π by taking m large enough. (This is only for convenience; an implicit dependence on $C(\pi)$ could be absorbed into the Sobolev norm using the results of Section 2.6.5.) In the archimedean case, we reason similarly, replacing the statement "$\cdots=0$ as long as the conductor of χ is strictly greater than $\ldots$" by integration by parts[23]. $\qquad\square$

We shall need later the following variant also. Its role in our proof will be the following: we shall need, at certain points, to utilize non-unitarizable representations, and this Proposition that follows allows us to maintain control on their Whittaker functions so long as they are "close enough" to being unitary.

Proposition **3.2.4.** — *Notation as in Section* 3.1.11, *so that π is a unitary principal series, $f=f_0\in\pi=\pi_0$, and $f_s\in\pi_s$ a deformation as in Section* 3.1.11. *Let W_s be the image of f_s under the intertwiner* (3.10). *Then there exists an absolute $d\geqslant 0$ so that, for $0\leqslant\delta\leqslant 1/20$, and $|\Re(s)|\leqslant\delta$,*

$$(\mathbf{3.15}) \qquad \int_{k^\times} |W_s(a(y))|^2 \max(|y|,|y|^{-1})^\delta d^\times y \ll \mathcal{S}^\pi_{d\delta}(f)^2,$$

Proof. — The Mellin transform $\int_{k^\times} W_0(x)\chi(x)d^\times x$ is absolutely convergent for $\Re(\chi)>-1/2$, cf. Remark 3.2.1.

---

[23] In the archimedean case, there is an element $Z\in\mathfrak{g}$ (more precisely in the Lie algebra of $a(k^\times)$) of bounded norm so that $\int_{k^\times} \widetilde{W}(u)\chi^{-1}(u)d^\times u \ll C(\chi)^{-1}\int_{k^\times} Z.\widetilde{W}(u)\chi^{-1}(u)d^\times u$.

Fix any $\eta \in (0, 1/2)$; we will use it to parameterize a compact subinterval of $[0, 1]$. Take any $\beta \in (-1 + \eta, -\eta)$; by Mellin inversion and (3.11),

$$(\mathbf{3.16}) \qquad f_0(wn(x)) = \frac{\eta_k^{1/2}}{\zeta_k(1)} \int_{\Re(\chi)=\beta} \chi(x) \frac{\ell^{\chi\omega'^{-1}\alpha^{1/2}}(W_0)}{\gamma(\chi^{-1}, \psi, 0)} \, d\chi,$$

with the measure $d\chi$ is as before. We now bound $f_0(wn(x))$: we apply Proposition 3.2.3 and (3.6) to equation (3.11), and use similar reasoning to Proposition 3.2.3 to pass from a pointwise bound for the integrand in (3.16), to the integral.

We arrive at the existence of d so that for $x \in k$ and $\beta \in [-1 + \eta, -\eta]$:

$$(\mathbf{3.17}) \qquad |f_0(wn(x))| \ll_\eta \mathcal{S}_d^\pi(f_0)|x|^\beta.$$

Observe that: $f_s(wn(x)) = f_0(wn(x))q(x)^s$, where $q(x) = \|(1, x)\|^{-2}$ (notations of Section 3.1.6). Combining this with (3.17) and substituting into (3.11), we see that for $\eta \in (0, 1/2)$ there exists $d = d(\eta) \geqslant 0$ so that, whenever $|\Re(\chi)| + |2\Re(s)| \leqslant 1/2 - \eta$,

$$\left| \int W_s(y)\chi(y)d^\times y \right| \ll \mathcal{S}_d^\pi(f_0)C(\chi)^d.$$

Now fix $\beta = 1/20$. Estimate $\int |W_s(y)|^2|y|^{\pm\beta}d^\times y$ by Plancherel, yet again controlling the χ-integration as in Proposition 3.2.3. Thus, there exists d so

$$(\mathbf{3.18}) \qquad \int |W_s(y)|^2 \max(|y|, |y|^{-1})^\beta d^\times y \ll \mathcal{S}_d^\pi(f_0)^2 \quad (|\Re(s)| \leqslant \beta).$$

This is almost what we want; however, the "number of derivatives" in the Sobolev norm in (3.15) is "small," whereas (3.18) involves an unspecified number of derivatives (i.e., we have no control of d). To get around this, we interpolate.

Define for every complex s with $\Re(s) \in [0, \beta]$ and $\tau \in [-1, 1]$, a function $F_s \in L^2(k^\times)$ via the rule

$$F_s(y) = W_{\tau s}(\Delta^{-\frac{d}{\beta}s}f_0) \cdot \max(|y|, |y|^{-1})^{s/2}.$$

Here Δ denote the local Laplacian discussed in Section 2.3.3 and $W_{\tau s}(\Delta^{-\frac{d}{\beta}s}f_0)$ denotes the result of applying the intertwiner (3.10) to the representation $\pi_{\tau s}$ and the vector whose restriction to K is that of $\Delta^{-\frac{d}{\beta}s}f_0$. Also, d is as in (3.18).

Then $s \mapsto F_s$ is a holomorphic function from $\{s \in \mathbf{C} : |\Re(s)| \leqslant \beta\}$ to $L^2(k^\times)$. Moreover, independently of τ, $\|F_s\|^2 \ll \|f_0\|^2$ for $\Re(s) = 0$, while for $\Re(s) = \beta$, we have by (3.18)

$$\|F_s\|^2 \ll \mathcal{S}_d^\pi(\Delta^{-\frac{s}{\beta}d}f_0)^2 = \|\Delta^{(1-\frac{s}{\beta})d}f_0\|^2 = \|f_0\|^2;$$

hence we have $\|F_{s_0}\|^2 \ll \|f_0\|^2$ in the region $\Re(s_0) \in [0, \beta]$ (independently of τ) : for every s_0, apply the usual maximum principle to the holomorphic function $s \mapsto \langle F_s, F_{s_0} \rangle$. We have shown that for all $|\Re(s)| = \delta$, all $\tau \in [-1, 1]$,

$$\int |W_{\tau s}(f)|^2 \max(|y|, |y|^{-1})^\delta \ll \mathcal{S}^{d\delta/\beta}(f_0)^2,$$

which implies the desired assertion. $\square$

3.2.5. *Computations in the Kirillov model.* — Suppose that k is archimedean; the action of the Lie algebra of G, $\mathfrak{g}$, on $\mathcal{K}(\pi, \psi)$ is well known [38, Chap. 1 Section 5,6]. For instance, it follows from the explicit computations of [65, Section 8.1.1] that given $Z_1, \ldots, Z_m \in \mathfrak{g}$, a smooth compactly supported function $W \in C_c^\infty(k^\times) \subset \mathcal{K}(\pi, \psi)$ one has

$$\|Z_1 \cdot Z_2 \cdots Z_m \cdot W\| \ll C(\pi)^{m/\deg(k)} \mathcal{S}_{2m}^{k^\times}(W),$$

where

$$\mathcal{S}_{2m}^{k^\times}(W) := \int_{k^\times} (|y| + |y|^{-1})^{2m} \left| \sum_i \mathcal{D}_i W \right|^2 d^\times y,$$

and $\mathcal{D}_i$ ranges over a basis of $k^\times$-invariant differential operators of degree $\leqslant 2m$.

Remark. — This result is stated in [65, Lemma 8.4], except the exponent of $C(\pi)$ is given as $2m/\deg(k)$ rather than $m/\deg(k)$. however, the improved statement follows from a simple computation: if λ_π denote the Casimir eigenvalue, ν_π the eigenvalue for the action of the Lie algebra of diagonal matrices and $D := y\frac{d}{dy}$, then there are generators of the Lie algebra (independent of π) which act respectively as $\nu_\pi + D, y, y^{-1}(\lambda_\pi + (\nu_\pi + D) - (\nu_\pi + D)^2)$. Taking into account that both ν_π and $\lambda_\pi - \nu_\pi^2$ are bounded by $C(\pi)$, we are done.

By the mean-value Theorem, we have, for $\delta \in \mathbf{R}$, $0 < \delta < 1$:

$$(\textbf{3.19}) \qquad \| \exp(\delta Z).W - W \|_\pi \ll \delta \sup_{\delta' \in [0, \delta]} \| \exp(\delta' Z) \cdot Z \cdot W \| = \delta \|Z.W\|$$

$$\ll \delta C(\pi)^{1/\deg(k)} \mathcal{S}_2^{k^\times}(W).$$

Moreover, all the implicit constants in the above inequalities may be taken to depend continuously on Z_i or on Z. In particular, they may be taken to be uniform when Z is restricted to a fixed compact set.

3.2.6. *Some norms related to Whittaker models.* — Let π_2, π_3 be two generic irreducible representations of G with π_3 isomorphic to a unitary principal series $\mathcal{I}_3$ (see Section 3.1.10); let χ_i, $i = 2, 3$ denote their central characters. Let $L^2(N\backslash G; \chi_2\chi_3)$ be the L^2-space of N-equivariant functions on G which transform by $\chi_2\chi_3$ under multiplication by the center, with respect to the inner product

$$\langle \varphi, \varphi \rangle_{N\backslash G} = \int_{N\backslash \bar{G}} |\varphi(g)|^2 dg.$$

Lemma **3.2.7.** — *The* $\mathrm{GL}_2(k)$-*equivariant map*

$$J = J_{\pi_2, \pi_3} : (W, f) \in \mathcal{W}(\pi_2) \times \mathcal{I}_3 \mapsto Wf$$

defines an isometry from $\pi_2 \otimes \pi_3$ *into* $L^2(N\backslash G; \chi_2\chi_3)$.

Proof. — It suffices to check that it preserves inner products of pure tensors. Take (W, f), (W', f'), and observe that—using the Iwasawa decomposition—

$$(\textbf{3.20}) \qquad \langle J(W, f), J(W', f') \rangle_{N\backslash G} = \int_{y \in k^\times} \int_{k \in K} W(a(y)k)\overline{W'(a(y)k)} f(k)\overline{f'(k)} d^\times y.$$

Since the integral $\int_{y \in k^\times} W(a(y)k)\overline{W'(a(y)k)}$ equals the inner product $\langle W, W' \rangle$ *for any* $k \in K$, the right-hand side factors as a product $\langle W, W' \rangle_{\pi_2} \langle f, f' \rangle_{\pi_3}$. We are done. $\square$

Lemma **3.2.8.** — *Suppose k non-archimedean. Let*

$$\pi_2 = \chi_2^+ \boxplus \chi_2^-, \qquad \pi_3 = \chi_3^+ \boxplus \chi_3^-$$

be unramified principal series representations with π_3 *tempered (i.e. the characters* $\chi_3^{\pm}$ *are unitary). We assume moreover that the additive character* ψ *is unramified.*

Let W *be a spherical vector, in the Whittaker model of* π_2 *and* f *spherical in the induced model of* π_3. *Let* $\pi_2(s), \pi_3(t)$ *and* $W_s \in \pi_2(s), f_t \in \pi_3(t)$ *be deformations of respectively* W, f *parameterized by complex parameters s and t. Here* W_s *is obtained, as before, by applying the intertwiner* (3.10) *to the* $f_2(s)$, *where* $f_2 \in \pi_2$ *is the preimage of* W *under the intertwiner* (3.10).

We denote the map J_{π_2, π_3} *by* J *and* $J_{\pi_2(s), \pi_3(t)}$ *by* $J_{s,t}$. *For* $0 \leqslant \delta < 1/10$, $|s|, |t| \leqslant \delta/2$, *and* $u \in G$ *one has*

$$(\textbf{3.21}) \qquad \frac{\langle u.J(W \otimes f), J_{s,t}(W_s \otimes f_t) \rangle_{N\backslash G}}{\|W\|^2 \|f\|^2} \ll \|\mathrm{Ad}(u)\|^{\theta + \delta - 1}.$$

Proof. — We may assume that $W(1) = f(1) = 1$.

We shall reduce the bound (3.21) to a bound where the representation π_2, π_3 are unitary, although π_2 is not tempered. In fact, we claim that:

$$(\textbf{3.22}) \qquad |\langle u.J(W \otimes f), J_{s,t}(W_s \otimes f_t) \rangle_{N\backslash G}| \leqslant 2\langle u.J(W_{\theta+\delta}^0 \otimes f^0), J(W_{\theta+\delta}^0 \otimes f^0) \rangle_{N\backslash G};$$

where π_2 replaced by the complementary series $|\cdot|^{\theta+\delta} \boxplus |\cdot|^{-\theta-\delta}$ with parameter $\theta + \delta$ and π_3 replaced by $1 \boxplus 1$; $W^0_{\theta+\delta}, f^0$ denote the spherical vectors in the corresponding models of the respective representations, normalized as above. Then by Lemma 3.2.7, the latter integral factors as

$$\langle u.\mathrm{J}(W^0_{\theta+\delta} \otimes f^0), \mathrm{J}(W^0_{\theta+\delta} \otimes f^0)\rangle_{\mathrm{N}\backslash\mathrm{G}}$$

$$= \langle u.W^0_{\theta+\delta}, W^0_{\theta+\delta}\rangle \langle u.f^0, f^0\rangle$$

$$\ll \|\mathrm{Ad}(u)\|^{\theta+\delta-1}\|W^0_{\theta+\delta}\|^2\|f^0\|^2 \ll \|\mathrm{Ad}(u)\|^{\theta+\delta-1}\|W\|^2\|f\|^2$$

as follows from bounds for matrix coefficients and a computation of the norm in the Whittaker model.

As for the claim: we note that by the Cartan decomposition and K-invariance, we may assume that u is of the form $a(t_0)$ with $0 < |t_0| \leqslant 1$; again by the Cartan decomposition

$$\langle u.\mathrm{J}(W \otimes f), \mathrm{J}_{s,t}(W_s \otimes f_t)\rangle_{\mathrm{N}\backslash\mathrm{G}} = \int_{k^\times}\int_{\mathrm{K}} u.(Wf)(a(y)k)\overline{W_s f_t}(a(y))|y|^{-1}dk d^\times y.$$

Fix a unitary character χ, and consider the space of functions F on G satisfying $F(z(t)n(x)gk) = \chi(t)\psi(x)F(g)$, $k \in \mathrm{K}$. This space is equipped with a natural inner product, namely, that arising from $\mathrm{L}^2(\mathrm{N}\backslash\bar{\mathrm{G}})$. This inner product is a Hermitian form in the values $(F(a(\varpi^\alpha)))_{\alpha\in\mathbf{Z}}$. In particular, the matrix coefficient $\langle uF_1, F_2\rangle$ may be expressed:

$$\langle u.F_1, F_2\rangle = \int_{k^\times}\int_{\mathrm{K}} u.F_1(a(y)k)\overline{F_2}(a(y))|y|^{-1}dk d^\times y$$

$$= \sum_{\alpha,\beta\in\mathbf{Z}} c^u_{\alpha,\beta,\chi}F_1(a(\varpi^\alpha))\overline{F_2(a(\varpi^\beta))}.$$

Moreover the $c^u_{\alpha,\beta,\chi}$ satisfy $|c^u_{\alpha,\beta,\chi}| \leqslant c^u_{\alpha,\beta,1}$. It follows that if $\bar{F}_i$ $(i = 1, 2)$ as above, are invariant under the center and satisfy $|F_i(a(\varphi^\alpha))| \leqslant \bar{F}_i(a(\varphi^\alpha))$ for every α, then also $|\langle F_1, F_2\rangle_{\mathrm{N}\backslash\mathrm{G}}| \leqslant \langle \bar{F}_1, \bar{F}_2\rangle_{\mathrm{N}\backslash\mathrm{G}}$.

We claim that $|\mathrm{J}_{s,t}(W_s \otimes f_t)a(\varpi^\alpha)|$ and $|\mathrm{J}(W \otimes f)(a(\varpi^\alpha)|$ are both bounded by $2\mathrm{J}_{\theta,\delta}(W^0_{\theta+\delta} \otimes f^0(a(\varpi^\alpha))$, the latter quantity being non-negative. This will complete the proof of the claim (3.22).

By choice of data, all quantities vanish when $\alpha < 0$. When $\alpha \geqslant 0$, they may be explicitly evaluated. Our assertion amounts to the following:

$$(\textbf{3.23}) \qquad \left(\sum_{\substack{k\in[-n,n]\\2|(k-n)}} q^{k(\theta+\delta/2)}\right)q^{n\delta/2} \leqslant 2\left(\sum_{\substack{k\in[-n,n]\\2|(k-n)}} q^{k(\theta+\delta)}\right), \quad n \geqslant 0, q \geqslant 2 \text{ and } \delta, \theta \geqslant 0.$$

$\square$

3.3. *Bounds for the normalized Hecke functionals*

Let χ be a character of $k^\times$, and π an irreducible unitary representation of G with central character ω. It is known that the space of functionals ℓ^χ on π satisfying

$$(\mathbf{3.24}) \qquad \ell^\chi \left(\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} v \right) = \chi(a).\omega\chi^{-1}(d)\, \ell^\chi(v)$$

is at most one-dimensional. Our goal is to normalize an element in this space (using the unitary structure on π) up to a scalar of absolute value 1. Once this is done, we shall study the analytic properties of the resulting "normalized" functional.

3.3.1. *The normalized Hecke functional. —* As before, *we fix once and for all an identification of π with $\mathcal{K}(\pi, \psi)$ which carries the unitary structure on π to the unitary structure on $\mathcal{K}(\pi, \psi)$ given by (3.8).* This being understood, define a functional ℓ^χ on the Kirillov model $\mathcal{K}(\pi, \psi)$—and therefore on π—via the rule of (3.12) (interpreted by meromorphic continuation in general). In the case χ is the trivial character, we denote ℓ^χ simply as ℓ; thus $\ell(W) = \int_{k^\times} W(y)d^\times y$. The resulting functional satisfies (3.24).

We have thus normalized a functional in the one-dimensional space of solutions to (3.24), up to a scalar of absolute value 1; this ambiguity arises from the ambiguity in choosing the isometry between π and its Kirillov model.

Equivalently, the functional may be expressed in terms of matrix coefficients: one has

$$\ell^\chi(v)\overline{\ell^{\overline{\chi}^{-1}}(v)} = \int_{k^\times} \langle a(y)v, v \rangle \chi(y)d^\times y,$$

as follows from (3.8). In particular, if χ is unitary, the *associated Hermitian form $w^\chi := |\ell^\chi|^2$* depends *only* on the unitary structure on π—i.e., there is not even an ambiguity of absolute value 1—and can be expressed thus:

$$(\mathbf{3.25}) \qquad h^\chi(v) = \int_{k^\times} \langle a(y)v, v \rangle \chi(y)d^\times y, \quad v \in \pi.$$

Lemma **3.3.2** ([65], Lemma 11.7). *— Suppose k non-archimedean; let v the new vector, and let r be the conductor of χ. Then*:

$$\ell^\chi(n(\varpi^{-r})v) = \begin{cases} \mathrm{L}(\pi \otimes \chi, 1/2) & r = 0 \\ \eta & r > 0 \end{cases}$$

where η has absolute value $q^{-r/2}(1 - q^{-1})$.

Lemma **3.3.3.** *— Suppose k archimedean. There is $v \in \pi$, with the property that $\mathcal{S}_d^\pi(v) \ll_d \mathrm{C}(\pi)^{2d}$ for every d, and so that, for any $\varepsilon > 0$ there is $\mathrm{T} \in k$ with $\frac{\log |\mathrm{T}|}{\log \mathrm{C}(\chi)} \in [1 - \varepsilon, 1 + \varepsilon]$ so that,*

$$|\ell^\chi(n(\mathrm{T})v)| \gg_\varepsilon \mathrm{C}(\chi)^{-1/2-\varepsilon}.$$

Moreover, $\ell^{\chi_s}(n(\mathrm{T})v) \ll_\epsilon \mathrm{C}(\chi)^{-1/2+\epsilon}$ for any s, the implicit constant uniform when $\Re(s)$ remains in a compact set.

Proof. — We take $v = \mathrm{W}$ to be (considered in the Kirillov model of π) a smooth bump function supported in the ball of radius $1/10$ centered at $y = 1$. (For concreteness, fix once and for all such a function both in the case of **R** and **C**. The implicit constants in the result will depend on this choice.)

The bounds on Sobolev norms are a consequence of the comments in Section 3.2.5. Noting that,

$$\ell^\chi(n(\mathrm{T})v) = \int_k \mathrm{W}(y)\psi(\mathrm{T}y)\chi(y)d^\times y,$$

the assertions concerning upper and lower bounds for ℓ^χ follow from Lemma 3.1.14. $\square$

3.4. *Normalized trilinear functionals*

Let π_i, for $1 \leqslant i \leqslant 3$, be three unitary irreducible generic representations of G, the product of whose central characters (χ_i, $i = 1, 2, 3$) is 1. In the sequel, we denote by ω the product $\chi_1\chi_2$. It is known [46, 57, 58] that the space of trilinear invariant functionals

$$\pi_1 \otimes \pi_2 \otimes \pi_3 \to \mathbf{C}$$

is at most dimension 1. Again, we wish to normalize one up to scalars of absolute value 1, and study the analytic properties of the resulting "normalized" functional.

3.4.1. *Integration of matrix coefficients.* — We take the following normalization (cf. [32, 61, 67]).

$$(\mathbf{3.26}) \qquad |\mathrm{L_W}(x_1, x_2, x_3)|^2 = \int_{g \in \tilde{\mathrm{G}}} \langle gx_1, x_1 \rangle \langle gx_2, x_2 \rangle \langle gx_3, x_3 \rangle dg.$$

By (2.8) together with the convention Section 3.1.7, the integral is convergent, and, indeed, bounded by a constant multiple of $\prod_{i=1}^3 \mathcal{S}^{\pi_i}(x_i)$. It is true, although not obvious, that the right-hand side is positive. By direct computation (see [32]),

$$(\mathbf{3.27}) \qquad \frac{|\mathrm{L_W}|^2}{\prod_i \|x_i\|^2} = \zeta_k(2)^2 \frac{\mathrm{L}(\pi_1 \otimes \pi_2 \otimes \pi_3, \frac{1}{2})}{\prod_{i=1}^3 \mathrm{L}(\pi_i, \mathrm{Ad}, 1)}$$

if k is nonarchimedean and all vectors are unramified.

Another functional arises naturally from Rankin-Selberg integrals. Suppose that $\pi_3 \simeq \chi_3^+ \boxplus \chi_3^-$, is a principal series representation ($\chi_3^+ \chi_3^- = \chi_3$) realized in the model $\mathcal{I}_3$, and that π_i $i = 1, 2$ are realized in the respective Whittaker models $\mathcal{W}(\pi_1, \psi)$ and $\mathcal{W}(\pi_2, \overline{\psi})$. (More precisely, we follow the convention of fixing once and for all an

equivariant isometry between π_i and their respective Whittaker models, equipped with the inner product (3.8); thus, for instance, we will regard a linear functional on the Whittaker model $\mathcal{W}(\pi_i)$ as a linear functional on π_i.)

Then, for $W_i \in \mathcal{W}(\pi_i), f_3 \in \mathcal{I}_3$, we put

$$(\mathbf{3.28}) \qquad L_{RS}(W_1, W_2, f_3) = \zeta_k(1)^{1/2} \int_{N\backslash\bar{G}} W_1 W_2 f_3,$$

$$= \zeta_k(1)^{1/2} \int_K \int_{k^\times} W_1(a(y)k) W_2(a(y)k) f_3(a(y)k) |y|^{-1} d^\times y \, dk.$$

If W_i, f_3 are spherical, and k nonarchimedean, then

$$(\mathbf{3.29}) \qquad \frac{L_{RS}(W_1, W_2, f_3)}{W_1(1) W_2(1) f_3(1)} = \zeta_k(1)^{1/2} \frac{L(\pi_1 \otimes \pi_2 \otimes \chi_3^+, 1/2)}{L(\chi_3^+/\chi_3^-, 1)}.$$

If we specialize to $\chi_3^\pm = |\cdot|^{\pm 1/2}$, $W_2 = \overline{W_1}$ then $L_{RS} = \zeta_k(1)^{1/2} \|W_1\|^2 \int_K f(k) dk$, giving in particular for spherical vectors

$$\frac{\|W_i\|^2}{|W_i(1)|^2} = \zeta_k(2)^{-1} L(\pi_1 \times \overline{\pi}_i, 1) = \frac{L(\pi_i, \mathrm{Ad}, 1)}{\zeta_k(2)} \zeta_k(1).$$

We conclude that, in the unramified case,

$$\frac{|L_{RS}|^2}{\prod_{i=1}^3 \|x_i\|^2} = \zeta_k(2)^2 \frac{L(\pi_1 \otimes \pi_2 \otimes \pi_3, \frac{1}{2})}{\prod_{i=1}^3 L(\pi_i, \mathrm{Ad}, 1)}.$$

Lemma **3.4.2.** — *Let* $L = L_{RS}$ *or* L_W *be defined as in either* (3.26) *or* (3.28). *Then for* $x_1 \in \pi_1, x_3 \in \pi_3$:

$$(\mathbf{3.30}) \qquad \|x_1\|^2 \|x_3\|^2 = \int_{\pi_2} d\mu_P(\pi_2) \sum_{x_2 \in \mathcal{B}(\pi_2)} |L(x_1, x_2, x_3)|^2,$$

where $d\mu_P(\pi_2)$ *is the Plancherel measure on the unitary dual of* G, *and* $\mathcal{B}(\pi_2)$ *is an orthonormal basis for* π_2. *In particular,* $|L_{RS}|^2 = |L_W|^2$ *whenever* π_2 *is tempered.*

The proof that follows is inspired by joint work of one of the authors (A.V.) with Y. Sakellaridis.

Proof. — The validity of (3.30) for $L = L_W$ is an immediate consequence of the Plancherel formula.

The validity of (3.30), where $L = L_{RS}$ (3.28) follows from the first Lemma of Section 3.2.6 together with the following: it is known (the Whittaker-Plancherel theorem) that for $F \in L^2(N\backslash\bar{G})$,

$$(\mathbf{3.31}) \qquad \|F\|_{L^2(N\backslash\bar{G})}^2 = \zeta_k(1) \int_\pi d\mu_P(\pi) \sum_{x \in \mathcal{B}(\pi)} |\langle F, W_x \rangle|^2,$$

where, for each π, we choose an orthonormal basis $\mathcal{B}_\pi$ and a unitary intertwiner from π to its Whittaker model $\mathcal{W}(\pi)$. In fact, the key point in the validity of (3.31) is the equality, for $W_i \in \mathcal{W}(\pi)$ between $\int_{x \in k} \psi(x) \langle n(x) W_1, W_2 \rangle$ and $\zeta_k(1) W_1(1) \overline{W_2(1)}$. This follows from the usual Fourier inversion formula, taking into account that the Haar measure on k was chosen to be self-dual w.r.t. k.

The second assertion (the coincidence of the two definitions of $|L|^2$ when π_2 is tempered) follows from uniqueness of unitary decomposition, applied to $\pi_1 \otimes \pi_3$, together with the fact that the support of the Plancherel measure is the full tempered spectrum, and from a continuity argument. $\qquad\square$

Remark. — It may also be deduced that $|L_{\mathrm{RS}}|^2$ and $|L_W|^2$ coincide even when not all π_i are tempered. To do this, write $\Pi = \bigotimes_{i=1}^3 \pi_i$. Both $|L_W|^2$ and $|L_{\mathrm{RS}}|^2$ are Hermitian forms on Π and can be polarized to maps $\Pi \otimes \tilde{\Pi} \to \mathbf{C}$. These polarized maps vary holomorphically in parameters, and therefore their coincidence may be extended from tempered representations by analytic continuation.

Remark. — To give a trilinear functional on $\pi_1 \otimes \pi_2 \otimes \pi_3$ is equivalent to describing an equivariant intertwiner $\pi_1 \otimes \pi_3 \to \tilde{\pi}_2$. Dualizing, we denote by I_{π_2} the functional $I_{\pi_2} : \pi_1 \otimes \pi_3 \to \pi_2$ that arises from the normalized trilinear functional on $\pi_1 \otimes \tilde{\pi}_2 \otimes \pi_3$. In explicit terms,

$$L(x_1 \otimes x_2 \otimes x_3) = \langle I_{\pi_2}(x_1 \otimes x_3), x_2 \rangle,$$

and one has

$$(\mathbf{3.32}) \qquad \| I_{\pi_2}(x_1 \otimes x_3) \|^2 = \sum_{x_2 \in \mathcal{B}(\pi_2)} |L(x_1 \otimes x_2 \otimes x_3)|^2.$$

It is often convenient to think or phrase arguments in terms of I_{π_2}, instead of L. In the sequel, the notation I_π will designate the intertwiner associated with $L_{\mathrm{RS}}(\star, \pi, \star)$ or $L_W(\star, \pi, \star)$.

It should also be noted that Bernstein and Reznikov made extensive use of a local study of such trilinear functionals in their beautiful papers.

3.5. *Bounds for trilinear functionals, I: soft methods*

In this section we present some "soft" upper bounds on the normalized trilinear functionals. In fact, the entire section is not strictly necessary for the present paper, since we reprove the bounds by brute force in later sections. However, *under the Ramanujan conjecture*, one does not need the brute-force computations, so the reader may prefer to focus on this section at first.

Our main result is Lemma 3.5.3. It bounds the Sobolev norms of $I_{\pi_3}(x_1 \otimes x_2)$ in terms of a certain norm $\| \cdot \|_{\mathcal{U}}$ on the space of π_1 and π_2, assuming that all three representations are tempered.

For the rest of this section, we will assume that the local field k is *non-archimedean*; the proof in the archimedean case is similar.

3.5.1. — Suppose V is any unitary G-representation. We define[24]

$$\|x\|_{\mathcal{U}} = \left(\int_{k \in K} |\langle kx, x \rangle|^2 \right)^{1/4},$$

where K is endowed with Haar probability measure.

Lemma. — $\|x\|_{\mathcal{U}}$ *is a norm.*

Proof. — Let $v \mapsto v^{K}$ be the averaging operator over K. Then $\|x\|_{\mathcal{U}}^2 = \|(x \otimes x)^K\|_{V \otimes V}$. Now, Cauchy-Schwarz demonstrates that

$$(\mathbf{3.33}) \qquad \|(x \otimes y)^K\|_{V \otimes V}^2 \leqslant \|(x \otimes x)^K\|_{V \otimes V} \|(y \otimes y)^K\|_{V \otimes V}.$$

Since $(x + y) \otimes (x + y) = x \otimes x + x \otimes y + y \otimes x + y \otimes y$, we conclude.   □

Lemma. — *There exists $d < d' < 0$ so that*

$$\mathcal{S}_d^{V} \leqslant \| \cdot \|_{\mathcal{U}} \leqslant \mathcal{S}_{d'}^{V}$$

on any unitary representation.

We omit the proof, for this is never used in the present paper.

Lemma. — *Suppose $g \in G$; then $\|gx\|_{\mathcal{U}} \leqslant C(g)\|x\|_{\mathcal{U}}$, where $C(g)$ is a continuous function of g; we may take $C = [KgK : K]^{1/4}$.*

Proof. — Write $U = gKg^{-1} \cap K$; we equip it with the restriction of the measure from K. Then $\int_U |\langle ugx, gx \rangle|^2 \leqslant \|x\|_{\mathcal{U}}^4$. Thus

$$(\mathbf{3.34}) \qquad \|(gx \otimes gx)^{U}\|^2 \leqslant [K : U]\|x\|_{\mathcal{U}}^4,$$

where, once again, $w \mapsto w^{U}$ denotes the projection onto U-invariants. Clearly, $\|(gx \otimes gx)^K\| \leqslant \|(gx \otimes gx)^U\|$ (averaging decreases norms!) and we are done since $[K : U] = [KgK : K]$.   □

---

[24] If K were a compact Abelian group and $x \in L^2(K)$, this coincides with the Gowers U^4-norm.

Lemma **3.5.2.** — *For x_1, $x_2 \in \pi_1$, π_2 and for any $d \geqslant 0$ we have*:

$$(\textbf{3.35}) \qquad \mathcal{S}_d^{\pi_3}(\mathrm{I}_{\pi_3}(x_1 \otimes x_2)) \ll_d \mathcal{S}_{d'}^{\pi_1}(x_1)\mathcal{S}_{d'}^{\pi_2}(x_2),$$

where d' depend on d only.

Proof. — We have seen after (3.26) that there exists an absolute d_0 so that

$$(\textbf{3.36}) \qquad \langle \mathrm{I}_{\pi_3}(x_1 \otimes x_2), x_3 \rangle \ll \prod_{i=1}^{3} \mathcal{S}_{d_0}^{\pi_i}(x_i),$$

which shows, in particular, that $\mathcal{S}_{-d_0}^{\pi_3}(\mathrm{I}_{\pi_3})$ is bounded by a quantity as in the right-hand side of (3.35). However, we wish the same result where $-d_0$ is replaced by an arbitrary positive quantity. To establish this, observe first that, by equivariance, that $x_i \in \pi_i[m] \implies \mathrm{I}_{\pi_3}(x_1 \otimes x_2) \in \pi_3[m]$. It follows, then, that a bound of the form (3.35) is true when $x_1 \in \pi_1[j]$, $x_2 \in \pi_2[k]$ for any j, k. Now apply (2.7) (twice: first fix x_1, and then fix x_2) to conclude the same result for arbitrary x_1, x_2. $\qquad\square$

Lemma **3.5.3.** — *Suppose π_i are all tempered. For any $\eta > 0$, set $\mathrm{B}_\eta = \{g \in \mathrm{G} : |\Xi(g)| \geqslant \eta\}$. Then there exists an absolute constant $d_0 \geqslant 0$ (independent of k) so that*:

$$\mathcal{S}_{-d_0}^{\pi_3}(\mathrm{I}_{\pi_3}(x_1 \otimes x_2)) \ll_\varepsilon (\mathrm{vol}\mathrm{B}_\eta)^\varepsilon \|x_1\|_\mathcal{U}\|x_2\|_\mathcal{U} + \eta^{1-\varepsilon}\mathcal{S}^{\pi_1}(x_1)\mathcal{S}^{\pi_2}(x_2).$$

The reader should ignore the second term at a first reading; the content is that $\mathrm{I}_{\pi_3}(x_1 \otimes x_2)$ is bounded "up to ϵs" by $\|x_1\|_\mathcal{U}\|x_2\|_\mathcal{U}$.

Proof. — It is sufficient to show that there exists $n \geqslant 0$ so that, whenever x_3 is invariant under an open compact subgroup $\mathrm{U} \subset \mathrm{K}$, $|\mathrm{L}(x_1, x_2, x_3)| \leqslant [\mathrm{K} : \mathrm{U}]^n \cdot \mathrm{RHS}$, where RHS is the right-hand side of the stated bound.

We endow both U and K with Haar probability measure. By a modification of (3.33), $\|(x_1 \otimes x_2)^\mathrm{U}\|^2 \leqslant [\mathrm{K} : \mathrm{U}]\|x_1\|_\mathcal{U}^2\|x_2\|_\mathcal{U}^2$. Writing $\prod_i \langle gx_i, x_i \rangle$ as

$$\langle g(x_1 \otimes x_2)^\mathrm{U}, (x_1 \otimes x_2)^\mathrm{U} \rangle \langle gx_3, x_3 \rangle,$$

we deduce from bounds on matrix coefficients (see Section 2.5.1, esp. footnote):

$$\prod_i \langle gx_i, x_i \rangle \leqslant \begin{cases} \Xi(g)^2[\mathrm{K} : \mathrm{U}]^2\|x_1\|_\mathcal{U}^2\|x_2\|_\mathcal{U}^2\|x_3\|^2, \\ \Xi(g)^3 \prod_i \mathcal{S}^{\pi_i}(x_i)^2. \end{cases}$$

To analyze $|\mathrm{L}(x_1, x_2, x_3)|$, we split the g-integral into B_η and its complement, applying the former bound on B_η and the latter on its complement. This leads to the stated bound; the bounds on $\Xi(g)$ are obtained using only the fact that

$$\int_{g \in \mathrm{G}} |\Xi(g)|^{2+\varepsilon} \ll_\varepsilon 1. \qquad\qquad\square$$

Remark. — When π_3 is not necessarily tempered, an analysis of the prior argument shows that, if x_i are invariant by subgroups U_i, then $|L(x_1, x_2, x_3)|$ is bounded above by

$$(\mathbf{3.37}) \qquad [K:U_3]q^{m\theta/2}\|x_1\|_{\mathcal{U}}\|x_2\|_{\mathcal{U}}\|x_3\| + \prod_i [K:U_i]^{1/2}q^{(\theta-1/2)(m+1)}\|x_1\|\|x_2\|\|x_3\|$$

3.6. *Choice of test vectors in the trilinear form*

In this section, we estimate L and I_{π_3} for specifically chosen test vectors.

Proposition **3.6.1.** — *Let π_1, π_2 be two generic unitary representations of $\mathrm{GL}_2(k)$. Suppose that π_3 is the principal series representation $1 \boxplus \chi_3$, where χ_3^{-1} is the product of central characters of π_1, π_2.*

Write the analytic conductors $C_i = C(\pi_i)$. Then, for any $\epsilon > 0$, there exists vectors $x_i \in \pi_i$ with the following properties:

- *$\|x_i\| = 1$;*
- *For $d \geqslant 0$, $\mathcal{S}_d^{\pi_2}(x_2) \ll_{d,\pi_2} 1$;*
- *For $d \geqslant 0$, $\mathcal{S}_d^{\pi_3}(x_3) \ll_{d,\pi_1,\pi_2} 1$;*
- *$\|x_3\|_{\mathcal{U}} \ll_{\pi_2,\epsilon} C_1^{-1/4+\epsilon}$;*
- *$|L(x_1, x_2, x_3)| \gg_{\pi_2,\epsilon} C_1^{-1/2-\epsilon}$;*
- *If k is non-archimedean and all the π_i are unramified, we may take x_1, x_2, x_3 to be the new vectors. In this case,*

$$\frac{|L(x_1, x_2, x_3)|^2}{\prod \|x_i\|^2} = \zeta_k(2)^2 \frac{L(\pi_1 \otimes \pi_2 \otimes \pi_3, \frac{1}{2})}{\prod_i L(1, \pi_i, \mathrm{Ad})}.$$

The rest of this section indicates the choice of test vector and the proof of the assertions. The unramified case has already been noted.

3.6.2. *Case of k nonarchimedean.* — Write f to be the larger of the valuations of the conductors of π_1, π_2. We set:

1. $x_1 \in \pi_1$ to be the new vector in the Whittaker model $\mathcal{W}(\pi_1, \psi)$, normalized to have norm 1;
2. $x_2 \in \pi_2$ to be the new vector in the Whittaker model $\mathcal{W}(\pi_2, \overline{\psi})$, normalized to have norm 1;
3. $x_3 \in \pi_3$ to be the vector in the principal series representation corresponding to the function $f_3 \in \mathcal{I}_3$ of norm 1 which upon restriction to K, is supported on the compact subgroup $K_0[f]$, and is a $K_0[f]$-eigenvector: explicitly

$$(\mathbf{3.38}) \qquad f_{3|K} : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \mathrm{vol}(K_0[f])^{-1/2}\chi_3(d)1_{K_0[f]}.$$

The ratio $\frac{|W(1)|}{\|W\|}$, when W is the new vector, is well-known (e.g. [64, Section 7]), and in particular is absolutely bounded above and below at nonarchimedean places; the assertion on $\mathcal{S}_{d_i}^{\pi_i}(x_i)$ $(i = 2, 3)$ follows from the fact that x_i is invariant under $K[f_i]$, with f_i the conductor of π_i.

Any K-translate of x_3 is either orthogonal or proportional to x_3, and therefore

$$\|x_3\|_{\mathcal{U}} = \mathrm{vol}(K_0[f])^{1/4} \ll_{\pi_2} C_1^{-1/4}.$$

The lower bound for L follows from (3.28) taking into account that the function $k \mapsto W_1(a(y)k)W_2(a(y)k)f_3(k)$ is supported on $K_0[f]$ and is $K_0[f]$-invariant:

$$
\begin{aligned}
L(x_1, x_2, x_3) &= \int_K \int_{k^\times} W_1(a(y)k)W_2(a(y)k)f_3(k)|y|^{-1/2} d^\times y\, dk \\
&= \mathrm{vol}(K_0[f])^{1/2} \int_{k^\times} W_1(a(y))W_2(a(y))|y|^{-1/2} d^\times y \gg C_1^{-1/2}.
\end{aligned}
$$

Thus in the non-archimedean case, the proposition is valid with $\epsilon = 0$.

3.6.3. *Case of k archimedean: the tempered case.* — We assume that π_2 is *tempered*. Let φ be a smooth function on $k^\times$, with $\varphi(1) = 1$, which is supported in a fixed compact neighbourhood of the identity and so that $\int |\varphi|^2 d^\times y = 1$ and $\int |\varphi|^2 |y|^{-1/2} d^\times y = 1$. Fix $\delta, M > 0$; later, we will choose δ to be "small", to an extent depending on the parameter ϵ chosen in Proposition 3.6.1, whereas M will be a "large" but absolute constant.

We set

(**3.39**) $\qquad C := C_2^M \max(C_1, C_2)^{1+\delta}.$

Intuitively, the reader should think of C as being "a tiny bit larger than C_1." It will be convenient to define

(**3.40**) $\qquad \mathcal{K}_C := \{g \in K : |g - k| \leqslant C^{-1} \text{ for some } k \in K \cap B\}$

where $|g - k|$ simply denotes the largest absolute value of any matrix entry of $g - k$. Thus $\mathcal{K}_C$ is a small neighbourhood of $K \cap B$ whose volume, with respect to the Haar measure on K, is $\asymp C^{-1}$.

If C is large enough, a matrix $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in K$ belongs to $\mathcal{K}_C$ iff $|c| \leqslant C^{-1}$; indeed, if this is so, $|b| = |c|$ and $|a| = |d| = \sqrt{1 - |c|^2}$, from where we see that $|g - k| \leqslant C^{-1}$, where k is the diagonal matrix with entries $a/|a|, d/|d|$.

The vectors x_1, x_2, x_3 are as follows:

1. $x_1 \in \pi_1$ is $y \mapsto \varphi(y)$ in the Kirillov model $\mathcal{K}(\pi_1, \psi)$: $W_1(a(y)) = \varphi(y)$ in the Whittaker model.
2. $x_2 \in \pi_2$ is $\overline{\varphi(y)}$ in the Kirillov model $\mathcal{K}(\pi_2, \overline{\psi})$: $W_2(a(y)) = \overline{\varphi}(y)$ in the Whittaker model.

3. We give an explicit construction of $x_3 \in \pi_3$ below: for the moment it is sufficient to say that correspond to a function $f_3 \in \mathcal{I}_3$ which upon restriction to K is a smooth bump function around $K \cap B$, supported in $\mathcal{K}_C$.

We advise the reader to skip the details of the explicit construction below; the properties just mentioned are all that is needed to follow the proof.

Definition **3.6.4** (Construction of x_3). — *Let ϕ be a (non-negative) bump function on $\mathbf{R}$, taking value 1 at zero, and supported in $|x| \leqslant \delta_0$. Set $\Psi_3(x, y) = \phi_1(x)\phi_2(y)$, where*

$$(\textbf{3.41}) \qquad \phi_1(x) = \phi(C|x|), \qquad \phi_2(y) = \chi_3(y).\phi(|y| - 1).$$

Then, for sufficiently small (absolute) $\delta_0 > 0$, the restriction to K of

$$(\textbf{3.42}) \qquad f(g) = |\det g|^{1/2} \int_{k^\times} \Psi_3((0, t)g)\chi_3^{-1}(t)|t|d^\times t$$

is supported in $\mathcal{K}_C$, and has $\|f\|_{L^2(K)} \sim C^{-1/2}$. We take $f_3 = \frac{f}{\|f\|_{L^2(K)}}$, and $x_3 \in \pi_3$ the vector corresponding to f_3.

Proof. — For $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in K$, we note that $\Psi_3(ct, dt) \neq 0$ implies (for sufficiently small δ_0) that $|dt| \geqslant (9/10)^2$ so $|t| \geqslant (9/10)^2$ (since $|d| \leqslant 1$) and then $|c| \ll C^{-1}$. From this we deduce—for sufficiently values of δ_0—that $f_{3|K}$ is supported in $\mathcal{K}_C$.

Moreover, $f(g) = \zeta_k(1)\chi_3(d) \int_t \phi(C|ct|)\phi(|dt| - 1)dt$. From this expression, it is easy to verify $\|f\|_{L^2(K)} \sim C^{-1/2}$; later on, we shall also use the fact that, at least if M is sufficiently large,

$$(\textbf{3.43}) \qquad \left| \int_{\mathcal{K}_C} f(k) \right| \geqslant \frac{1}{2} \int_{\mathcal{K}_C} |f(k)|dk \asymp C^{-1}.$$

The point in this inequality is that $\omega(d)$ remains close to 1 so long as $g \in \mathcal{K}_C$. It follows that corresponding inequalities hold for f_3, but with the last expression replaced by $C^{-1/2}$.

We remark for later usage that, if v is nonarchimedean and we take

$$(\textbf{3.44}) \qquad \phi_1 = \text{vol}(K_0[f])^{-1/2}1_{\varpi^f \mathfrak{o}}, \qquad \phi_2 = \omega^{-1}.1_{\mathfrak{o}^\times},$$

then the function f_3 defined by (3.42) coincides with that defined in (3.38).[25]

The assertions concerning $\|x_1\|$, $\|x_2\|$ are immediate; the assertions concerning the Sobolev norms of x_1, x_2 follow from discussion of Section 3.2.5.

---

[25] By construction, the integral belongs to $\mathcal{I}_3$ so it is sufficient to check the equality for $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in K$. In that case the right hand side equals

$$\int_{k^\times} \Psi_3(ct, dt)\chi_3^{-1}(t)|t|d^\times t.$$

Note that $\max(|ct|, |dt|) = |t|$ since $g \in K$; so the integrand is nonvanishing only when $|t| = 1$. In that case it is nonvanishing only if $d \in \mathfrak{o}^\times$ and $c \in \varpi^f \mathfrak{o}$, in which case the integral equals $\text{vol}(K_0[f])^{-1/2}\chi_3(d)$ proving our claim.

By (3.28), we have

$$(\textbf{3.45}) \qquad \mathrm{L}(x_1, x_2, x_3) = \int_{\mathrm{K}} \ell(k.\mathrm{W}_1, k.\mathrm{W}_2) f_3(k) dk$$

where $\ell(\mathrm{W}, \mathrm{W}')$ denote the (Borel-equivariant) functional on $\mathcal{W}(\pi_1, \psi) \otimes \mathcal{W}(\pi_2, \overline{\psi})$ given by

$$\ell(\mathrm{W}, \mathrm{W}') := \int_{y \in k^{\times}} \mathrm{W}(a(y))\mathrm{W}'(a(y))|y|^{-1/2}d^{\times}y.$$

Write

$$\ell(k.\mathrm{W}_1, k.\mathrm{W}_2) = \ell(\mathrm{W}_1, \mathrm{W}_2) + \ell(\mathrm{W}_1, k.\mathrm{W}_2 - \mathrm{W}_2)$$
$$+ \ell(k.\mathrm{W}_1 - \mathrm{W}_1, \mathrm{W}_2) + \ell(k.\mathrm{W}_1 - \mathrm{W}_1, k.\mathrm{W}_2 - \mathrm{W}_2)$$

We need to bound the terms only for $k \in \mathcal{K}_{\mathrm{C}}$; they are individually bounded thus:

1. We have set things up so that $\ell(\mathrm{W}_1, \mathrm{W}_2) \geqslant 0.9$;
2. By Cauchy-Schwarz and (3.19),[26]

$$|\ell(\mathrm{W}_1, k.\mathrm{W}_2 - \mathrm{W}_2)| \ll \|\mathrm{W}_1\| \|k.\mathrm{W}_2 - \mathrm{W}_2\|$$
$$\ll \|\mathrm{W}_1\| \mathrm{C}^{-1/\deg(k)} \mathrm{C}_2^{1/\deg(k)}$$
$$\ll_{\pi_2} \mathrm{C}^{-1/\deg(k)}.$$

3. Similarly we have $|\ell(k.\mathrm{W}_1 - \mathrm{W}_1, \mathrm{W}_2)| \ll \|\mathrm{W}_2\| \mathrm{C}^{-1/\deg(k)} \mathrm{C}_1^{1/\deg(k)}$.
4. As for the last term: set

$$\mathrm{W}(y) = (k.\mathrm{W}_1 - \mathrm{W}_1)(a(y)), \qquad \mathrm{W}'(y) = (k.\mathrm{W}_2 - \mathrm{W}_2)(a(y)).$$

Given $\varepsilon > 0$ small enough, we have, since π_2 *is tempered*, by Proposition 3.2.3,

$$\left| \int \mathrm{W}(y)\mathrm{W}'(y)|y|^{-1/2}d^{\times}y \right|$$
$$\leqslant \left(\int |\mathrm{W}(y)|^2 |y|^{-\varepsilon} d^{\times}y \right)^{1/2} \left(\int |\mathrm{W}'(y)|^2 |y|^{-1+\varepsilon} d^{\times}y \right)^{1/2}$$
$$\ll \mathcal{S}^{\pi_2}(\mathrm{W}') \left(\|\mathrm{W}\| + \|\mathrm{W}\|^{1-\mathrm{O}(\varepsilon)} \mathcal{S}^{\pi_1}(\mathrm{W})^{\mathrm{O}(\varepsilon)} \right)$$
$$\ll (\mathrm{C}_1/\mathrm{C})^{(1-d\varepsilon)/\deg(k)} \mathrm{C}_1^{d'\varepsilon}$$

for some $d' \geqslant 0$. Here, to bound $\int |\mathrm{W}(y)|^2 |y|^{-\varepsilon} d^{\times}y$, we split into $|y| \geqslant 1$, where the bound $\|\mathrm{W}\|$ suffices, and $|y| \leqslant 1$, which gives—by Proposition 3.2.3 and interpolation—a term $\|\mathrm{W}\|^{1-\mathrm{O}(\varepsilon)} \mathcal{S}^{\pi_1}(\mathrm{W})^{\mathrm{O}(\varepsilon)}$; we also used $\mathcal{S}^{\pi_2}(\mathrm{W}') \ll_{\pi_2} 1$, as well as (3.19).

---

[26] Using the fact that $y \mapsto \mathrm{W}_i(a(y))$ is supported on a fixed compact set of $k^{\times}$.

It follows by taking M large enough, we will have $|\ell(k.W_1, k.W_2) - 1| \leqslant \frac{1}{100}$ for all $k \in \mathcal{K}_C$. On the other hand, $|\int_{\mathcal{K}_C} f_3(k)| \geqslant \frac{1}{2} \int_{\mathcal{K}_C} |f_3(k)|$. Therefore,

$$L(x_1, x_2, x_3) \gg \left| \int_{\mathcal{K}_C} f_3(k) \right| \gg_{\pi_2} C_1^{-1/2-\delta/2}.$$

We choose $\delta \leqslant \epsilon$ to conclude.

3.6.5. *The non-tempered case.* — If π_2 is *not tempered*, then π_2 is an unramified principal series representation and may be written into the form $\pi_2^0 \cdot | \cdot |^{it}$ with $t \in \mathbf{R}$ and $C(\pi_2^0) \leqslant 9$. Let W_2^0 be the K-invariant vector in the Whittaker model of π_2^0, normalized to have L^2-norm 1. Since the set of possibilities for the isomorphism class of π_2^0 is precompact, there exists a finite set $\mathcal{F}$ of smooth non-negative function φ, each compactly supported in $[1/2, 2]$ and with $\varphi_i(1) = 1$, with the following property:

For any such π_2, there exists $\varphi \in \mathcal{F}$ such that:

$$\int_{k^\times} \varphi_i(y) W_2^0(a(y)) |y|^{-1/2} d^\times y \gg 1$$

where the constant implied is absolute.

Let π_2^0 be as above, and let $\varphi \in \mathcal{F}$ be the corresponding function. We take x_1 to be the vector corresponding to $y \mapsto \varphi(y) y^{-it}$ in the Kirillov model of π_1; take x_2 to be the vector corresponding to $W_2(y)|y|^{it}$ in the Kirillov model of $\pi_2 = \pi_2^0 | \cdot |^{it}$. In the previous notations, we have by definition

$$\ell(W_1, W_2) \gg 1$$

Now for $k \in \operatorname{supp}(f_{3|K})$ we have by K-invariance of W_2,

$$\ell(k.W_1, k.W_2) = \ell(W_1, W_2) + \ell(k.W_1 - W_1, W_2)$$

Since the term $\ell(kW_1 - W_1, W_2)$ is bounded by $O((C_1/C)^{1/\deg(k)} C_2^m)$, for some $m \geqslant 1$, we deduce, just as in the prior argument, that

$$L(x_1, x_2, x_3) \gg C_1^{-1/2-\delta/2}. \qquad \qquad \square$$

3.7. *Bounds for trilinear functionals, II: computations*

In this section we shall establish the following result, *which can be replaced by the results of Section 3.5 under the assumption of the Ramanujan-Petersson conjecture.* Indeed central bound (3.46) follows from Lemma 3.5.3 if π is tempered. The reader may, indeed, wish to omit the present section at a first reading, because it contains "ugly" computations.

Proposition **3.7.1.** — *Let π be a generic unitary representation with trivial central character. Let π_1, π_2 be two generic unitary representations and $\epsilon > 0$.*

We denote the product of the central characters of π_1, π_2 by χ_3^{-1}. Let $\pi_3 = 1 \boxplus \chi_3$ be the corresponding unitary principal series representation.

Let $x_3 \in \pi_3$ be the vector constructed in Proposition 3.6.1—*applied to $\pi_1, \pi_2, \pi_3, \epsilon$; let $\tilde{x}_3$ be the conjugate vector in the contragredient representation $\tilde{\pi}_3 = 1 \boxplus \omega$ (i.e., the vector representing the functional $v \mapsto \langle v, x_3 \rangle$).*

  – *For π, π_j all unramified, and $x \in \pi$ the new vector of norm 1, one has*

$$\frac{|\mathrm{L}(x, x_3, \tilde{x}_3)|^2}{\|x\|^2 \|x_3\|^2 \|\tilde{x}_3\|^2} = \zeta_k(2)^2 \frac{\mathrm{L}(\pi \otimes \pi_3 \otimes \tilde{\pi}_3, \frac{1}{2})}{\mathrm{L}(\pi, \mathrm{Ad}, 1)\mathrm{L}(\pi_3, \mathrm{Ad}, 1)\mathrm{L}(\tilde{\pi}_3, \mathrm{Ad}, 1)}.$$

  – *In general, there is an absolute constant $d_0 \geqslant 0$ (independent of k) such that:*

$$(\textbf{3.46}) \qquad |\mathrm{L}(x, x_3, \tilde{x}_3)|^2 \ll_{\epsilon, \pi_2} \mathrm{C}_1^{d_0 \epsilon} \mathrm{C}(\chi_3)^{-1/2} \left(\frac{\mathrm{C}_1}{\mathrm{C}(\chi_3)} \right)^{\theta - 1/2} \mathcal{S}_{d_0}^\pi(x) \quad (x \in \pi).$$

Averaging (3.46) over an orthogonal basis $\mathcal{B}(\pi)$ of π we obtain

Corollary. — *With the previous notations, if π, π_j are all unramified, one has*

$$\frac{\|\mathrm{I}_\pi(x_3 \otimes \tilde{x}_3)\|^2}{\|x_3\|^2 \|\tilde{x}_3\|^2} = \zeta_k(2)^2 \frac{\mathrm{L}(\pi \otimes \pi_3 \otimes \tilde{\pi}_3, \frac{1}{2})}{\mathrm{L}(\pi, \mathrm{Ad}, 1)\mathrm{L}(\pi_3, \mathrm{Ad}, 1)\mathrm{L}(\tilde{\pi}_3, \mathrm{Ad}, 1)}.$$

In general, there is an absolute constant $d_0 \geqslant 0$ (independent of k) so that:

$$(\textbf{3.47}) \qquad \mathcal{S}_{-d_0}^\pi(\mathrm{I}_\pi(x_3 \otimes \tilde{x}_3)) \ll_{\epsilon, \pi_2} \mathrm{C}_1^{d_0 \epsilon} \mathrm{C}(\chi_3)^{-1/2} \left(\frac{\mathrm{C}_1}{\mathrm{C}(\chi_3)} \right)^{\theta - 1/2}.$$

We observe that the proof is parallel in nonarchimedean and archimedean cases, and we could indeed have given an integrated treatment. The unramified assertion has already been discussed.

Proof. — Let $\mathrm{C} = \mathrm{C}_2^{\mathrm{M}} \max(\mathrm{C}_1, \mathrm{C}_2)^{1+\delta}$, where M, δ are as in the prior section (cf. (3.39); recall that δ is chosen depending on ϵ, whereas M is an absolute constant) when k is archimedean, and $\mathrm{C} = q^f$ when k nonarchimedean; here f is the larger of the conductors of π_1 and π_2. Let $\mathcal{K}_{\mathrm{C}}$ be as in (3.40) when v is archimedean, and $\mathrm{K}_0[f]$ when v is nonarchimedean.

We realize π and π_3 in their respective Whittaker models, i.e., fix isometric intertwiners between π, π_3 and these models. Similarly, we realize $\tilde{\pi}_3$ in the induced model $\tilde{\mathcal{I}}_3$. Denote by W, W_3 and $\tilde{f}_3$ the corresponding vectors: W_3 is obtained from $f_3 \in \mathcal{I}_3$ by means of the intertwiner (3.10). By (3.28) and (3.43), we have

$$\mathrm{L}(x, x_3, \tilde{x}_3) = \int_{\mathrm{K}} \int_{k^\times} \mathrm{W}(a(y)\kappa) \mathrm{W}_3(a(y)\kappa) \tilde{f}_3(\kappa) |y|^{-1/2} d^\times y \, d\kappa$$

$$\ll C^{-1/2} \max_{\kappa \in \mathcal{K}_C} \int_{k^\times} |W(a(y)\kappa)W_3(a(y)\kappa)||y|^{-1/2} d^\times y$$

$$\ll C^{-1/2} \mathcal{S}^\pi(W) \max_{\kappa \in \mathcal{K}_C} \int_{k^\times} |W_3(a(y)\kappa)||y|^{-\theta}(1+|y|)^{-N} d^\times y,$$

where we applied Proposition 3.2.3 for the last inequality. In the nonarchimedean case, it suffices to specialize to $\kappa = 1$, since W_3 is in any case $\mathcal{K}_C$-invariant.

We now apply the bounds of the next Lemma to conclude. Loosely speaking, it says that the Whittaker function W_3 is peaked near $y \sim C(\chi_3)/C$ and it takes a value of size ~ 1 there. The reason for this can already be discerned from (3.10): $W_3(y)$ is, approximately, the Fourier transform of a oscillating function restricted to an interval of the shape $|y| > C$. The "frequency" of the function near the endpoints of this interval are approximately $C(\chi_3)/C$. $\qquad\square$

Lemma **3.7.2.** — *Notations as above, for any* $M, \varepsilon > 0$, *one has*

$$|W_3(a(y)\kappa)| \ll_{M,\varepsilon,\psi,\pi_2} (C/|y|)^\varepsilon \left(\frac{C}{C(\chi_3)}\right)^{1/2} |y|^{1/2}\left(1 + \frac{|y|}{C(\chi_3)/C}\right)^{-M}.$$

The proof of Lemma 3.7.2 is, regrettably, an explicit calculation. Let us note that

$$\eta_k^{1/2} W_3(g) = |\det g|^{1/2} \int_{k^\times} \int_k \Psi_3((t, tx)g)\psi(x)|t|\chi_3^{-1}(t) dx d^\times t,$$

where Ψ is as in Definition 3.6.4 in the archimedean case; in particular $\Psi_1(x, y) = \phi_1(x)\phi_2(y)$, where ϕ_i are as in (3.41); in the nonarchimedean case the same is true with ϕ_i as in (3.44); finally η_k is as in (3.10), and is a harmless constant. In particular $\eta_k^{1/2} W_3(a(y))$ may be expressed as

$$y^{1/2} \int_{k^\times} \int_k \phi_1(tx)\phi_2(ty)\psi(x)|t|\chi_3^{-1}(t) dx d^\times t.$$

Proof. — (of Lemma 3.7.2—nonarchimedean case.) It suffices to check $\kappa = 1$.

$$\eta_k^{1/2} W_3(a(y)) = \text{vol}(K_0[f])^{-1/2}|y|^{1/2} \int_{k^\times} 1_{\varpi^f \mathfrak{o}}(yt)|t| d^\times t \int_k \chi_3(x) 1_{\mathfrak{o}^\times}(tx)\psi(x) dx$$

$$\asymp C^{1/2}|y|^{1/2} \int_{|t||y|C \leqslant 1} |t| d^\times t \int_{|x||t|=1} \chi_3(x)\psi(x) dx.$$

If χ_3 is ramified, this equals, in absolute value, $\asymp_\psi (C/C_\chi)^{1/2}|y|^{1/2}$ when $|y| \leqslant \frac{DC(\chi_3)}{C}$, where $D = q^{d_\psi}$, and 0 otherwise (cf. Section 3.1.13). On the other hand, if χ_3 is unramified, it is zero for $|y| > qD/C$, and otherwise is bounded by $\ll_\psi C^{1/2}|y|^{1/2}\log(1 + \frac{qD}{|y||C|})$. $\square$

Proof. — (of Lemma 3.7.2—archimedean case.) We prove the bound first when $\kappa = \mathrm{I}d$ as it is notationally more pleasant; we will comment on the necessary modifications afterwards.

We have

$$(\mathbf{3.48}) \qquad \mathrm{W}_3(a(y)) \asymp \mathrm{C}^{1/2}|y|^{1/2} \int_{k^\times} d^\times t \phi(\mathrm{C}|t||y|)|t| \int_k \chi_3(x)\phi(|tx| - 1)\psi(x)dx.$$

Set $\mathrm{G}(t) = |t| \int_k \chi_3(x)\phi(|tx| - 1)\psi(x)dx = \chi_3^{-1}(t) \int_k \phi(|x| - 1)\psi(x/t)\chi_3(x)dx$; then by Lemma 3.1.14, one has for any $\mathrm{N} \geqslant 0$ and $\varepsilon > 0$

$$|\mathrm{G}(t)| \ll_{\mathrm{N},\varepsilon,\phi} \mathrm{C}(\chi_3)^{-1/2+\varepsilon} \min\left(\mathrm{C}(\chi_3)|t|, \frac{1 + |t|^{-1}}{\mathrm{C}(\chi_3)}\right)^{\mathrm{N}}.$$

We deduce that $|\mathrm{W}_3(a(y)|$ is bounded, up to an implicit constant, by

$$(\mathbf{3.49}) \qquad \mathrm{C}^{1/2}|y|^{1/2} \int_{k^\times} \phi(\mathrm{C}|t||y|)\mathrm{G}(t)d^\times t \ll_{\varepsilon,\pi_2,\mathrm{N}} \mathrm{C}^\varepsilon \left(\frac{\mathrm{C}}{\mathrm{C}(\chi_3)}\right)^{1/2} \frac{|y|^{1/2-\varepsilon}}{(1 + \frac{|y|}{\mathrm{C}(\chi_3)/\mathrm{C}})^{\mathrm{N}}}.$$

For general $\kappa = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathcal{K}_\mathrm{C}$, replace Ψ_3 by the function $\kappa.\Psi_3$ with

$$\kappa.\Psi_3 \colon (x, y) \mapsto \Psi_3((x, y)\kappa) = \Psi_3(ax + cy, bx + dy)$$

$$= \phi(\mathrm{C}|ax + cy|)\phi(|bx + dy| - 1).$$

In (3.48) we then need to replace $\phi(\mathrm{C}|t||y|)|t| \int_k \chi_3(x)\phi(|tx| - 1)\psi(x)dx$ by

$$|t| \int_k \phi(\mathrm{C}|aty + ctx|)\phi(|bty + dtx| - 1)\psi(x)\chi_3(x)dx$$

$$= \chi_3^{-1}(t) \int_k \phi(\mathrm{C}|aty + cx|)\phi(|bty + dx| - 1)\psi(x/t)\chi_3(x)dx$$

Observe that since $|b|, |c| \ll \mathrm{C}^{-1}$, $|a| - 1, |d| - 1 \ll \mathrm{C}^{-1}$, the integral is zero unless $|ty| \ll \mathrm{C}^{-1}$, while the x variable satisfies $\big||x| - 1\big| \leqslant \delta_0 + \mathrm{O}(\mathrm{C}^{-1})$.

The above proof carries on *mutatis mutandis*. At the first stage, we compute the x-integral, noting that $x \mapsto \phi(\mathrm{C}|aty + cx|)\phi(|bty + dx| - 1)$ satisfies similar smoothness bounds to $\phi(|x| - 1)$, since $c|\mathrm{C}| \ll 1$. Then (3.49) uses in addition only the fact that we may restrict to $|ty| \ll \mathrm{C}^{-1}$, and so also goes through. $\qquad\square$

4. Integral representations of L-functions: global computations

We now turn to global aspects of the analysis of standard L-functions on GL(2) (Section 4.2) and of the triple product L-function (Section 4.4).

4.1. *Notation*

4.1.1. *Subgroups.* — Henceforth F is a number field, $\mathbf{A} := \mathbf{A}_F$ and we often denote $|\cdot|$ for $|\cdot|_{\mathbf{A}}$. For $\mathbf{G} = GL_2$ or PGL_2, we set $\mathbf{X_G} = \mathbf{G}(F)\backslash\mathbf{G}(\mathbf{A})$. We will often write simply $\mathbf{X}$ for $\mathbf{X}_{PGL_2}$.

For the various usual subgroups of $\mathbf{G}$ we adopt the notations and parameterizations of Section 3.1.4. We denote by K the usual maximal open compact subgroup of $\mathbf{G}(\mathbf{A})$.

We put $H := A(\mathbf{A}) = a(\mathbf{A}^\times)$ and

$$H^{(1)} = A(\mathbf{A}^{(1)}) = \{a(y) : y \in \mathbf{A}^\times : |y|_{\mathbf{A}} = 1\} \subset H.$$

H and $H^{(1)}$ are *closed, non-compact subgroups* of $GL_2(\mathbf{A})$. By an abuse of notation, we regard them as closed, non-compact subgroups of $PGL_2(\mathbf{A})$. We let Y be the H-orbit of the identity coset in $\mathbf{X}_{PGL_2}$. Then Y carries a H-invariant measure of infinite volume.

4.1.2. *Measures.* — We adopt on each of the groups the product of those measures specified in Section 3.1.5.

Although not strictly necessary, it is a useful check to be aware of the relation with a Tamagawa measure. A routine computation shows that, for the measures on PGL_2,

$$(\textbf{4.1}) \qquad dg = (\mathrm{disc} F)^{-1} \xi_F(2) d^\tau g,$$

where $d^\tau g$ is the Tamagawa measure. In particular, the volume of the quotient $\mathbf{X}$ is $2\xi_F(2)(\mathrm{disc} F)^{1/2}$.

4.1.3. *Additive characters and Fourier expansion.* — Let $\psi = e_F = \bigotimes_v \psi_v$ be the non-trivial additive character of $\mathbf{A}/F$ given as $e_F(.) = e_{\mathbf{Q}}(\mathrm{tr}_{F/\mathbf{Q}}(.))$ with $e_{\mathbf{Q}}$ the unique additive character of $\mathbf{A}$ whose restriction to $\mathbf{R}$ coincide with $e^{2\pi i x}$.

For $f(g)$ a function on $\mathbf{X}_{GL_2}$, we define its constant term f_N and Whittaker function W_f by

$$(\textbf{4.2}) \qquad f_N(g) := \int_{\mathbf{A}/F} f(n(x)g)dx, \ W_f(g) = \int_{\mathbf{A}/F} \psi(x)f(n(x)g)dx$$

and one has the Fourier expansion

$$(\textbf{4.3}) \qquad f(g) = f_N(g) + \sum_{y \in F^\times} W_f(a(y)g).$$

4.1.4. *Representations.* — In this section, π will be a standard automorphic representation on GL(2), and $C(\pi)$ will be its analytic conductor, i.e. if $\pi = \bigotimes \pi_v$, then $C(\pi) = \prod_v C(\pi_v)$, the right-hand quantities as in Section 3.1.8. If π is such, then π carries a canonical inner product, and we shall always regard it as a unitary representation with respect to this inner product; we fix moreover an inner product on each π_v compatibly with this choice.

Therefore, if we say, "let $f = \bigotimes f_v \in \pi = \bigotimes \pi_v$," we always have

$$\langle f, f \rangle = \prod_v \langle f_v, f_v \rangle.$$

Remark. — Here is a note of warning in connection with this terminology. Suppose $\pi = \bigotimes \pi_v$ is a standard automorphic representation. The map $f \mapsto W_f$ intertwines π with $\bigotimes_v \mathcal{W}(\pi_v)$. This map is *not* an isometry when π is endowed with the canonical inner product, and each Whittaker model is endowed with the norm (3.8). See Section 2.2.2.

4.1.5. *Infinite product notation.* — We denote by $\Lambda(\pi, s)$ resp. $\mathrm{L}(\pi, s)$ the completed L-function of π, resp. the L-function omitting archimedean factors.

Let us adopt the following notational conventions, to be held in force through the rest of the paper:

- First of all, if π is an automorphic representation, we shall understand the notation $\mathrm{A} \ll_\pi \mathrm{B}$ to mean "there exists absolute constants a, b so that $|\mathrm{A}| \leqslant a\mathrm{C}(\pi)^b |\mathrm{B}|$".
- We continue with the already established convention: if we write $|\ell(f)| \ll \mathcal{S}(f)$, we mean that there exists a constant d, depending only on $[\mathrm{F} : \mathbf{Q}]$, so that $|\ell(f)| \ll \mathcal{S}_d(f)$.
- (Regularizing products over places.) Suppose that $\Lambda_1, \Lambda_2(s)$ are (completed) global L-function with local factors $\mathrm{L}_{1v}(s), \mathrm{L}_{2v}(s)$; let s_0 be so that $\mathrm{L}_{1v}(s_0) \neq 0$ and $\mathrm{L}_{2v}(s_0) \neq 0$ for all v, and so that Λ_1 is holomorphic at s_0; and suppose that E_v is a function on places of F with the property that $\mathrm{E}_v = \mathrm{L}_{1v}(s_0)/\mathrm{L}_{2v}(s_0)$ for almost all v.

  Put $\Lambda^*(s_0)$ to be $\Lambda_1(s_0)/\Lambda_2^*(s_0)$, where $\Lambda_2^*(s_0)$ is the first nonvanishing Laurent coefficient of Λ_2 at $s = s_0$. In particular, $\Lambda^*(s_0) = \Lambda_1(s_0)/\Lambda_2(s_0)$ if Λ_2 is holomorphic and nonvanishing at $s = s_0$; we define $\mathrm{L}^{\mathrm{S},*}(s_0)$ in the same way, omitting the factors at $v \in \mathrm{S}$. We shall then write:

$$\prod_v^* \mathrm{E}_v \stackrel{\mathrm{def}}{=} \Lambda^*(s_0) \prod_v \frac{\mathrm{E}_v}{\mathrm{L}_{1v}(s_0)/\mathrm{L}_{2v}(s_0)} = \mathrm{L}^{\mathrm{S},*}(s) \prod_{v \in \mathrm{S}} \mathrm{E}_v \prod_{v \notin \mathrm{S}} \frac{\mathrm{E}_v}{\mathrm{L}_{1v}(s_0)/\mathrm{L}_{2v}(s_0)},$$

  where the second equality holds for any finite set of places S; we shall often use it with S the set of all archimedean places.

4.1.6. *Eisenstein series.* — Given two characters χ^+, χ^- of $\mathrm{F}^\times \backslash \mathbf{A}^\times$ whose product is unitary, we denote by $\mathcal{I}(\chi^+, \chi^-)$ or $\chi^+ \boxplus \chi^-$ the representation of GL$_2(\mathbf{A})$ unitarily induced from the corresponding representation on $\mathbf{B}(\mathbf{A})$: the L^2-space of functions f on GL$_2(\mathbf{A})$ such that

$$f\left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} g \right) = |a/d|_{\mathbf{A}}^{1/2} \chi^+(a) \chi^-(d) f(g), \qquad \langle f, f \rangle = \int_{\mathrm{K}} |f(k)|^2 dk.$$

Given f in such a space, we denote by $E(f) = \mathrm{Eis}(f)$ the corresponding Eisenstein series (defined by analytic continuation, in general), i.e.

$$\mathrm{Eis}(f) := \sum_{\mathrm{B}(\mathrm{F})\backslash \mathrm{GL}_2(\mathrm{F})} f(\gamma g).$$

For s a complex parameter and $\pi = \mathcal{I}(\chi^+, \chi^-)$ we set

$$(\textbf{4.4}) \qquad \pi_s = \pi(s) = \mathcal{I}(\chi^+|\cdot|_\mathbf{A}^s, \chi^-|\cdot|_\mathbf{A}^{-s})$$

and for $f \in \pi$ we define $f_s = f(s)$ to be the unique function in $\pi(s)$ whose restriction to K coincide with f.

4.1.7. *Fourier coefficients of Eisenstein series.* — Given an Eisenstein series, $\mathrm{Eis}(f)$, its constant term is given by:

$$(\textbf{4.5}) \qquad \mathrm{Eis}(f)_\mathrm{N}(g) = f(g) + \int_\mathbf{A} f(wn(x)g)dx$$

The Eisenstein series has a pole at the point $\chi^+/\chi^- = |\cdot|_\mathbf{A}$ coming from its constant term; the residue, with respect to the coordinate s as in (4.4), is given by

$$(\textbf{4.6}) \qquad \prod_v^* \int f_v(wn(x))dx = \frac{1}{2}(\mathrm{discF})^{-1/2}\frac{\xi_\mathrm{F}^*(1)}{\xi_\mathrm{F}(2)}\int_{k\in\mathrm{K}} f(k);$$

with $\xi_\mathrm{F}^*(1) := \mathrm{res}_{s=1}\xi_\mathrm{F}(s)$; see (3.2) and discussion thereafter.

The other Fourier coefficients of Eisenstein series are determined by the Whittaker function:

$$W_{\mathrm{Eis}(f)}(g) = \int_{\mathrm{F}\backslash\mathbf{A}} f(wn(x)g)\psi(x)dx = \prod_v W_{f,v},$$

$$(\textbf{4.7}) \qquad W_{f,v}(g) = \int_{x\in\mathrm{F}_v} f_v(wn(x)g)\psi_v(x)dx.$$

$$W_{f,v}(a(y)) = |y|^{1/2}\chi^-(y)\int_{x\in\mathrm{F}_v} f_v(wn(x))\psi_v(xy)dx.$$

Let us recall that in Section 2.2.1, we have discussed two possible norms on Eisenstein series: the Eisenstein norm $\langle\mathrm{Eis}(f), \mathrm{Eis}(f)\rangle_{\mathrm{Eis}} = \langle f, f\rangle$ and the canonical norm $\langle\mathrm{Eis}(f), \mathrm{Eis}(f)\rangle_{can}$ formed out of the inner products of the local Whittaker functions $W_{f,v}$ (cf. (2.5) and (2.3)). In view of Section 3.1.10 and the remark in Section 3.1.6,

we have the following relation—if $\Lambda(\pi, \mathrm{Ad}, 1)$ has a simple pole—(recall that $\eta_{k_v} = q_v^{-d_{\psi_v}/2}\zeta_v(1)^2/\zeta_v(2)$)

$$(\mathbf{4.8}) \qquad \langle \mathrm{Eis}(f), \mathrm{Eis}(f) \rangle_{can}$$

$$= \Lambda^*(\pi, \mathrm{Ad}, 1)\frac{2\xi_F(2)(\mathrm{disc}F)^{1/2}}{\xi_F^*(1)}\prod_v \frac{\langle f_v, f_v \rangle \eta_{kv}}{\zeta_v(1)L_v(\pi, \mathrm{Ad}, 1)/\zeta_v(2)}$$

$$= 2\xi_F(2)\langle f, f \rangle = 2\xi_F(2)\langle \mathrm{Eis}(f), \mathrm{Eis}(f) \rangle_{Eis}.$$

4.1.8. *The intertwiner operator.* — The map $\mathrm{M}: f \mapsto \int_{x \in \mathbf{A}} f(wn(x)g)dx$ that occurs in (4.5) is the *standard intertwining operator*. It is an isometry on the "unitary axis," i.e. when the characters $\chi^{\pm}$ are unitary. We shall need at several points to study its analytic behavior off the analytic axis.

$$\mathrm{M} = \frac{\Lambda(\chi^+/\chi^-, 0)}{\epsilon(\chi^+/\chi^-, 0)\Lambda(\chi^+/\chi^-, 1)}\prod \bar{\mathrm{M}}_v,$$

$$\bar{\mathrm{M}}_v := \frac{\epsilon(\chi_v^+/\chi_v^-, \psi_v, 0)L(\chi_v^+/\chi_v^-, 1)}{L(\chi_v^+/\chi_v^-, 0)}\mathrm{M}_v.$$

Here $\mathrm{M}_v: \mathcal{I}(\chi_v^+, \chi_v^-) \mapsto \mathcal{I}(\chi_v^-, \chi_v^+), f \mapsto \int_{x \in F_v} f(wn(x)g)dx$; the operator $\bar{\mathrm{M}}_v$ has the advantage of being holomorphic in χ_v^+, χ_v^-. In view of the functional equation, the global correction factor $\frac{\Lambda(\chi^+/\chi^-, 0)}{\epsilon(\chi^+/\chi^-, 1)\Lambda(\chi^+/\chi^-, 1)}$ has absolute value 1 (it is the "scattering matrix") although it may not be equal to 1.

By Section 3.1.6, $\bar{\mathrm{M}}_v$ takes the spherical vector with value 1 on K, to the spherical vector with value 1 on K, for almost all v. More generally, it preserves norms up to a scalar depending only on ψ: [26, Section 4].

Lemma. — *Let* $f_v \in \mathcal{I}(\chi_v^+, \chi_v^-)$. *Suppose that* $|s| \leqslant \delta_0$, *and the deformation* $f_{v,s}$ ($s \in \mathbf{C}$) *is as in Section 3.1.11. Then there exists d so that*:

$$(\mathbf{4.9}) \qquad \sup_{k \in \mathrm{K}}|\bar{\mathrm{M}}_v f_{v,s}(k)| \ll \mathcal{S}_d^{\mathcal{I}}(f), \int_{k \in \mathrm{K}}|\bar{\mathrm{M}}_v f_{v,s}(k)|^2 \ll \mathcal{S}_{d\delta_0}(f_v)^2.$$

Proof. — Because of (3.11), and with notation as contained there, $\bar{\mathrm{M}}_v f(1)$ is proportional to $\bar{\ell}^{\chi'}(W_f)$, where $\chi' = \alpha^{-1/2}/\chi^-$, and where $\bar{\ell}^{\chi}$ is the "normalized" functional $\frac{\int W(y)\chi(y)d^{\times}y}{L(1/2, \pi \otimes \chi)}$, for $W \in \mathcal{W}(\pi, \psi)$. More precisely, $\bar{\mathrm{M}}_v f(1) = \Upsilon \bar{\ell}^{\chi'}(f)$, where

$$\Upsilon = \zeta_k(1)^{-1}\eta_k^{1/2}\frac{L_v(1, \chi_v^+/\chi_v^-)\epsilon(0, \psi_v, \chi_v^+/\chi_v^-)\zeta_v(1)}{\epsilon(\alpha, \psi, 0)}.$$

Applying the local functional equation shows that $\bar{\mathrm{M}}_v f_v(1)$ is given by:

$$\bar{\ell}^{\chi''}(\mathrm{W}_f) \frac{\Upsilon}{\epsilon(\pi, \chi', 1/2)}, \chi'' = \alpha^{1/2}/\chi^+.$$

The local bound $\sup_{k \in \mathrm{K}_v} |\bar{\mathrm{M}}_v f_s(k)| \ll \mathcal{S}_{d_0}^{\mathcal{I}}(f)$, now follows from Proposition 3.2.3 together with the following observation: the smallest eigenvalue of the local Laplacian Δ_v on $\mathcal{I}(\chi_v^+, \chi_v^-)$ is bounded below by a positive power of $(\mathrm{Cond}_v(\chi^+) + \mathrm{Cond}_v(\chi^-))$. This (a simpler form of Lemma 2.18) allows us to absorb dependencies on $\chi^\pm$ into the Sobolev norm.

To obtain the stated bound for $\int_{k \in \mathrm{K}} |\bar{\mathrm{M}}_v f_{v,s}(k)|^2$, we interpolate, taking into account the equality: $\|\bar{\mathrm{M}}_v f_{v,s}\|_{\mathrm{L}^2(\mathrm{K}_v)} \asymp_\psi \|f_v\|_{\mathrm{L}^2(\mathrm{K}_v)}$ whenever $\Re(s) = 0$. This argument is analogous to the interpolation in the proof of Proposition 3.2.4. $\qquad\square$

We now give a global analogue of this statement:

Lemma **4.1.9.** — *Let* $f \in \mathcal{I}(\chi^+, \chi^-)$. *Suppose that* $|s| \leqslant \delta_0$, *and the deformation* f_s ($s \in \mathbf{C}$) *is as in* (4.4). *Then there exists d so that*

$$(\textbf{4.10}) \qquad \sup_{k \in \mathrm{K}} |\mathrm{M}f_s(k)| \ll \mathcal{S}_d^{\mathcal{I}}(f), \qquad \int_{k \in \mathrm{K}} |\mathrm{M}f_s(k)|^2 \ll \mathcal{S}_{d\delta_0}^{\mathcal{I}}(f)^2.$$

Proof. — An application of (S1d) to the previous Lemma gives the first statement. From this, it follows that $\int_{k \in \mathrm{K}} |\mathrm{M}f_s(k)|^2 dk$ is bounded by $\mathcal{S}_{d_0}^{\mathcal{I}}(f)$. In order to deduce the second statement, we observe that $\int_{k \in \mathrm{K}} |\mathrm{M}f_s(k)|^2 dk = \|f\|_{\mathcal{I}}^2$ when $\Re(s) = 0$, and then interpolate, as in the proof of Proposition 3.2.4. $\qquad\square$

4.1.10. *The Eisenstein series at a singular parameter.* — It will also be of interest to consider the Eisenstein series associated to $\chi^+ = \chi^- = 1$. At this point, the map $f \mapsto (g \mapsto \int_{x \in \mathbf{A}} f(wn(x)g)dx)$ is the negative of the identity map, and accordingly the Eisenstein intertwiner vanishes to order one. Write Eis^* for the (normalized) intertwiner from $\mathcal{I}(1, 1)$ to $\mathrm{C}^\infty(\mathbf{X})$, given by:

$$f \mapsto \xi_{\mathrm{F}}(1 + 2s)\mathrm{Eis}(f_s)|_{s=0}.$$

For $f = \prod_v f_v$ factorizable, let us define $\langle \mathrm{Eis}^*(f), \mathrm{Eis}^*(f) \rangle_{can}$ via (2.5) and (2.3).

Then, if $f \in \mathcal{I}(1, 1)$, then the constant term

$$(\textbf{4.11}) \qquad \mathrm{Eis}^*(f)_{\mathrm{N}}(a(y)g)$$

$$= -\frac{1}{2}\xi_{\mathrm{F}}^*(1)\frac{d}{ds}|_{s=0} \left(\int_{x \in \mathbf{A}} |y|^{1/2+s} f_s(wn(x)g)dx + |y|^{1/2-s} f_s(g) \right)$$

$$= \frac{1}{2}\xi_{\mathrm{F}}^*(1)(2f(g)|y|^{1/2}\log y - \mathrm{M}^* f(g)|y|^{1/2})$$

where $\mathrm{M}^* f$ is the derivative, at $s = 0$, of $f_s + \int_{x \in \mathbf{A}} f_s(wn(x)g)dx$; as usual, this needs to be interpreted by a process of analytic continuation.

Trivial bounds on M^* follow from bounds on intertwining operators in Section 4.1.8 (in particular, one may bound the derivative of an analytic function by bounding its value on a small circle around the point of interest). This shows that, for every φ belonging to the space of $\mathrm{Eis}^*(\mathcal{I}(1, 1))$, we have

$$(\mathbf{4.12}) \qquad \varphi_{\mathrm{N}}(a(y)g) \ll |y|^{1/2} \log|y| \mathcal{S}^{\pi}(\varphi),$$

Now take $f = f^0$ to be the spherical vector in $\mathcal{I}(1, 1)$, normalized so that $f^0|\mathrm{K} = 1$. The function $\mathrm{M}^* f^0$ is continuous, and thus bounded on compact sets; it follows from (4.11) that there exists X_0 so that

$$(\mathbf{4.13}) \qquad \mathrm{Eis}^*(f^0)(x) \gg \mathrm{ht}(x) \log \mathrm{ht}(x), \quad \mathrm{ht}(x) \geqslant \mathrm{X}_0.$$

4.2. *Hecke integrals on* PGL$_2$

In this section we shall study the Hecke integral for the standard L-function on GL$_2$, and we shall give some bounds on the Hecke integral of a translate of a given vector, using the method described in Section 2.5.2, Remark.

4.2.1. *The Hecke-Jacquet-Langlands integral.* — We recall in this section the integral representation for the standard L-function on GL$_2$, following Jacquet and Langlands. Let χ be a character of $\mathbf{A}^{\times}/\mathrm{F}^{\times}$; the integral

$$(\mathbf{4.14}) \qquad \ell^{\chi}(\varphi) := \int_{\mathbf{A}^{\times}/\mathrm{F}^{\times}} (\varphi - \varphi_{\mathrm{N}})(a(y)) \chi(y) d^{\times}y$$

defines a functional on the space of any standard generic automorphic representation π. Indeed, it is absolutely convergent if π is cuspidal, and, in general, can be interpreted by analytic continuation in the χ variable.

It was observed by Hecke, and generalized by Jacquet and Langlands, that the period ℓ^{χ} is very closely related to the standard L-function: if $\varphi = \bigotimes \varphi_v$ is a pure tensor in π, and we factorize the associated Whittaker function (see (4.2)) as $\mathrm{W}_{\varphi} = \prod_v \mathrm{W}_{\varphi,v}$, one has:

$$(\mathbf{4.15}) \qquad \ell^{\chi}(\varphi) = \prod_v^* \ell^{\chi_v}(\mathrm{W}_{\varphi,v}) \left(= \Lambda(\pi \otimes \chi, 1/2) \prod_v \frac{\ell^{\chi_v}(\mathrm{W}_{\varphi,v})}{\mathrm{L}(\pi_v \otimes \chi_v, 1/2)} \right),$$

for $\ell^{\chi_v}(\mathrm{W}_{\varphi,v})$ the local Hecke functionals defined in Section 3.3. The verification of (4.15) is "unfolding." It is valid for π Eisenstein, even in the singular case.

We will use also use the following "canonically normalized" expression, given in terms of matrix coefficients and which follows from Sections 3.25 and 2.2.2) (cf. the Remark in Section 2.5.2): for χ a unitary character, and π standard,

$$(\mathbf{4.16}) \qquad \frac{|\ell^\chi(\varphi)|^2}{\langle \varphi, \varphi \rangle_{can}} = \left(\frac{2\xi_F(2)(\mathrm{disc}F)^{1/2}}{\xi_F^*(1)} \right)^{-1} \prod_v^* \frac{h^\chi(\varphi_v)}{\langle \varphi_v, \varphi_v \rangle}$$

where almost all local factors in the last product are equal to $\frac{|L_v(\pi \otimes \chi, 1/2)|^2}{\zeta_v(1)L_v(\pi, \mathrm{Ad}, 1)/\zeta_v(2)}$. Again, this formula is valid for π Eisenstein, even at singular points; of course, at singular points, the regularization implicit in $\prod^*$ involves taking a higher residue of $L(s, \pi, \mathrm{Ad})$ at $s = 1$.

It should be noted that the right-hand sides of (4.16) and (4.15) make sense for all χ, even when the left-hand sides must be interpreted by analytic continuation.

4.2.2. *Bounds for the Hecke integral of the translate of a function*

Lemma. — *Let* π *be a generic automorphic representation of* PGL_2. *For* $\varphi \in \pi$, *any* $g \in \mathrm{PGL}_2(\mathbf{A})$, *and unitary* χ: *there is an absolute constant* d *so that*

$$|\ell^\chi(g.\varphi)| \ll_{\pi, \chi} \mathrm{disc}(Yg)^{-\frac{1-2\theta}{32}} \mathcal{S}_d^\pi(\varphi).$$

Here $\mathrm{disc}(Yg)$ *is the discriminant of the adelic torus orbit* Yg *as defined in* [20]—*see below—and the unitary structure on* π *is the canonical norm.*

The paper [20, 4.1–4.2] attaches to the adelic torus orbit Yg local and global discriminants, denoted, respectively, as $\mathrm{disc}_v(Yg)$ and $\mathrm{disc}(Yg)$. Although most of that paper deals with the case of adelic points of anisotropic tori, the definition is perfectly applicable to the split adelic torus H. For our purposes, it is enough to know that, for the special case $g_0 = n(T)$, $|T|_{\mathbf{A}} \geqslant 1$ we have:

$$(\mathbf{4.17}) \qquad \mathrm{disc}_v(Yg) \asymp \max(1, |T|_v^2), \mathrm{disc}(Yg) \asymp \prod_v \max(1, |T|_v^2) \geqslant |T|_{\mathbf{A}}^2.$$

Remark. — The exponent $-\frac{1-2\theta}{32}$ is certainly not best possible: it is taken from the general computations of *loc. cit.*, which are in no way optimal for PGL_2 (see for instance [15] for better bounds of similar integrals.) What is important to us, in this paper, is that this exponent is negative.

Proof. — By (4.16) and (2.8), we have:

$$|\ell^{\chi_v}(g_v \cdot \varphi_v)|^2 = \int_{F_v^\times} \langle a(y)g.\varphi_v, g.\varphi_v \rangle \chi_v(y) d^\times y$$

$$= \int_{F_v^\times} \langle g^{-1}a(y)g.\varphi_v, \varphi_v \rangle \chi_v(y) d^\times y$$

$$\ll \mathcal{S}^{\pi_v}(\varphi_v)^2 \int_{F_v^\times} \Xi_v(g^{-1}a(y)g)^{1-2\theta} d^\times y$$

$$\ll \mathrm{disc}_v(Yg)^{-\frac{1-2\theta}{16}} \mathcal{S}^{\pi_v}(\varphi_v)^2.$$

The bound in the last step is carried out, in a more general setting in [20, Lemma 9.14], and we do not reproduce it here.

Write $L_v = L(\pi_v \otimes \chi_v, 1/2)/L(\pi_v, \mathrm{Ad}, 1)^{1/2}$. We are going to apply Property (1d) of Sobolev norms to $\prod \ell'_v$, where ℓ'_v is the "normalized" functional on π_v so that $|\ell'_v|^2 = \frac{|\ell^{\chi_v}|^2}{|L_v|^2}$. It enjoys the following properties:

1. For any place for which $\varphi_v \otimes \chi_v$ is spherical, $|\ell'(\varphi_v)|^2 = \langle \varphi_v, \varphi_v \rangle = \mathcal{S}_0^{\pi_v}(\varphi_v)^2$.
2. There exists A, d_0 so that, for any v, the operator norm of ℓ' w.r.t. $\mathcal{S}_{d_0}^{\pi_v}$ is $\leqslant A|L_v|^{-1}\mathrm{disc}_v(Yg)^{-\frac{1-2\theta}{32}}$.
3. There exists an absolute constant C so that, for nonarchimedean v, $|L_v|^{-1} \leqslant C$.

Thus property (S1d) applied to $\ell' := \prod_v \ell'_v$ shows that, for $\varphi \in \pi$,

$$\frac{|\ell'(\varphi)|^2}{\mathcal{S}_{d'}^\pi(\varphi)^2} \ll_\varepsilon \mathrm{disc}(Yg)^{-\frac{1-2\theta}{16}+\varepsilon} \prod_{v|\infty} |L_v|^{-2},$$

for some $d > 0$. Applying trivial or convexity bounds for all local factors or L-functions, we arrive at:

$$\frac{|\ell^\chi(\varphi)|^2}{\mathcal{S}^\pi(\varphi)^2} \ll_{\pi,\chi,\varepsilon} \mathrm{disc}(Yg)^{-\frac{1-2\theta}{16}},$$

as required. $\qquad\square$

This Lemma admits the following mild refinement where we improve the χ-dependence at the implicit cost of weakening the π-dependence:

*Lemma **4.2.3.*** — *Notation as in the prior Lemma, for any $N \geqslant 0$, there is $d = d(N)$ so that*

$$|\ell^\chi(g.\varphi)| \ll_\pi (\mathrm{Cond}(\chi))^{-N}\mathrm{disc}(Yg)^{-\frac{1-2\theta}{32}} \mathcal{S}_d^\pi(\varphi).$$

Proof. — By (2.7), it suffices to prove such an assertion for $\varphi \in \pi[\underline{m}]$. (One also verifies that the implicit dependence of A' on A in (2.7) is independent of π). The assertion for $\varphi \in \pi[\underline{m}]$ follows by integration by parts. $\qquad\square$

4.3. *Regularization*

In this section, we define a regularization process to define integrals of non-necessarily decaying functions on $\mathbf{X}_{\mathrm{PGL}_2} = \mathrm{PGL}_2(F)\backslash\mathrm{PGL}_2(\mathbf{A})$. Such a regularization was given by Zagier [69]; for our purposes, an alternate way of defining it using convolution will be of use.

4.3.1. — For motivational purposes, we consider first a toy example, namely, the corresponding situation on $\mathbf{R}_{>0}$ (one could even consider the case of the integers). We shall regard $\mathbf{R}_{>0}$ as a multiplicative group in what follows.

A *finite* function on $\mathbf{R}_{>0}$ is one whose translates, under the action of the multiplicative translations:

$$\tau_y f(x) = f(xy)$$

span a finite dimensional representation; equivalently, it is a linear combination of functions $x^{\alpha}(\log x)^b$, for $\alpha \in \mathbf{C}$ and $b \in \mathbf{N}$.

We call a finite function *admissible* if the exponent $\alpha = 0$ never occurs. A more intrinsic and useful way of formulating this is: f is admissible if the span $\langle \tau_y f \rangle$ of all multiplicative translates does not contain the trivial representation of $\mathbf{R}_{>0}$.

Let V_+ be the space of continuous functions on $\mathbf{R}_{>0}$ of rapid decay as $x \to \infty$ (i.e. $|f(x)| \ll_N |x|^{-N}$ for all N); let V_- be the space of rapid decay as $x \to 0$; and let $V_0 = V_+ \cap V_-$. Let V be the space of all continuous functions f on $\mathbf{R}_{>0}$ so that there exists *admissible* finite functions f_1, f_2 so that $f - f_1 \in V_+, f - f_2 \in V_-$. Clearly $V_0 \subset V$. The following Lemma is well-known:

Lemma **4.3.2.** — *There exists a unique functional* (*the "regularized integral"*) *on* V *which extends integration* $f \mapsto \int f(x) \frac{dx}{x}$ *on* V_0, *and is invariant under multiplicative translation.*

This functional may be defined in multiple ways:

1. Given $f \in V$, we may find finite collections $y_i \in \mathbf{R}_{>0}$, $c_i \in \mathbf{C}$ so that $f' := \sum c_i \tau_{y_i} f \in V_0$, and so that $\sum c_i \neq 0$. We then define the regularized integral of f to be $\frac{\int f'(x) \frac{dx}{x}}{\sum c_i}$.
2. Consider $\int_{1/T}^{T} f(x) \frac{dx}{x}$; this has the form $g(T) + h(T)$, where $g(T)$ is an admissible finite function, and $h(T)$ has a limit as $T \to \infty$. We define the integral to be $\lim_{T \to \infty} h(T)$.
3. Consider $F_+(w) = \int_1^{\infty} f(x) x^w \frac{dx}{x}$; define similarly $F_-(w)$. The functions $F_{\pm}(w)$ are convergent for $\mp \Re(w) \gg 1$; they extend to meromorphic functions on the plane, and are holomorphic at $w = 0$. We define the integral to be $F_+(0) + F_-(0)$.

The regularized integral of any admissible finite function is zero. This follows, without computation, because of invariance under multiplicative translation. Thus, if there exists finite function f_0 so that $f - f_0 \in V_0$, then the regularized integral of f equals $\int (f - f_0)(x) \frac{dx}{x}$.

The remarks of this section adapt without change to define a regularized integral on $\mathbf{A}^{\times}/F^{\times}$.

4.3.3. — Given a function Φ on $GL_2(F)\backslash GL_2(\mathbf{A})$, with unitary central character χ (i.e. which transform by χ under translation by $Z(\mathbf{A})$), we say that Φ is of controlled increase if there exists a function

$$f : N(\mathbf{A})A(F)\backslash GL_2(\mathbf{A}) \to \mathbf{C},$$

spanning a finite-dimensional space under the translation action of $GL_2(\mathbf{A})$, and with central character χ, so that, for every $N \geqslant 0$

$$(\mathbf{4.18}) \qquad \Phi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} y_1 & 0 \\ 0 & 1 \end{pmatrix} k\right) = f\left(\begin{pmatrix} y_1 & 0 \\ 0 & 1 \end{pmatrix} k\right) + O(|y_1|^{-N}) \quad \text{as } |y_1| \to \infty.$$

In other terms, the difference is of rapid decay. In more explicit terms, there must exist a finite collection of functions $\chi_i : \mathbf{A}^\times/F^\times \to \mathbf{C}$ ($i \in I$), each finite under the left translation action of $\mathbf{A}^\times/F^\times$, as well as a corresponding collection of K-finite functions $\mathcal{K}_i : K \to \mathbf{C}$, so that the left-hand side is well-approximated by $\sum \chi_i(y_1)\mathcal{K}_i(k)$.

A basic example to bear in mind is any sum or product of Eisenstein series.

The expression f is uniquely determined. We denote it by Φ_N^*; it need not coincide with the true constant term of Φ. The set of *exponents* of Φ (or of Φ_N^*)—denoted S_Φ — is the set of characters of $\mathbf{A}^\times/F^\times$ which are (generalized) eigenvalues for the translation action of $\mathbf{A}^\times/F^\times$ on the space spanned by Φ_N^* and its translates. Of course if $\Phi_N^* = 0$ (Φ is of rapid decay) we set $S_\Phi = \emptyset$.

Example **4.3.4.** — *If $\chi^+ \neq \chi^-$, the exponents of a (non-zero) Eisenstein series $E \in \mathrm{Eis}(\mathcal{I}(\chi^+, \chi^-))$ are $\{\chi^+|\cdot|^{1/2}, \chi^-|\cdot|^{1/2}\}$. If $\chi^+ = \chi^- = \chi$ the same is true for $E \in \mathrm{Eis}^*(\mathcal{I}(\chi, \chi))$; the exponent $\chi|\cdot|^{1/2}$ is now a generalized eigenvalue, i.e., has multiplicity. If E_s denote the Eisenstein series on the modular group with eigenvalue $1/4 - s^2$, the exponents of $E_{s_1}E_{s_2}E_{s_3}$ is the set of characters $|\cdot|^{3/2+s}$, where $s = \pm s_1 \pm s_2 \pm s_3$.*

In the sequel, we will identify the complex numbers $\mathbf{C}$ with a subset of $\widehat{\mathbf{A}^\times/F^\times}$ via $z \mapsto |\cdot|^z$. For instance, given $S \subset \widehat{\mathbf{A}^\times/F^\times}$, we say that "$1 \in S$" when S contains the character $x \mapsto |x|$. We will also use an additive notation: given two such subsets S_1, S_2 we denote by $S_1 + S_2$ the set of pairwise products of the characters of S_1 and S_2. Of course $S_1 + \emptyset = \emptyset$.

The operation $\Phi \mapsto \Phi_N^*$ is multiplicative: given two functions Φ_1, Φ_2 with exponents S_1, S_2, $(\Phi_1\Phi_2)^* = \Phi_{1,N}^*\Phi_{2,N}^*$ and $\Phi_1\Phi_2$ has exponents in $S_1 + S_2$. Finally, the complex conjugate $\overline{\Phi}$ has exponents in $\bar{S}$ (i.e. the set of conjugates of those characters in S).

The set of characters whose real part is $1/2$ (the real part being defined by $|\chi(.)| = |\cdot|_{\mathbf{A}}^{\Re(\chi)}$) will be called the *unitary axis*: this is the set of exponents of the automorphic Eisenstein series.

4.3.5. *Regularized integral and regularized innerproduct.* — Let $\mathcal{V}_S$ be the vector space of smooth functions that are of controlled increase with trivial central character and whose exponents belong to S; and $\mathcal{V}$ the union of $\mathcal{V}_S$, where S is taken through all finite subsets of characters that do not contain any character of square $|\cdot|^2$ (i.e., any quadratic twist of $|\cdot|$).

Lemma. — *There's a unique* $\mathrm{PGL}_2(\mathbf{A})$*-invariant functional on* $\mathcal{V}$ *extending integration on* $\mathrm{L}^1(\mathbf{X}_{\mathrm{PGL}_2}) \cap \mathcal{V}$.

Proof. — One definition, due to Zagier, is given as follows: let $\mathbf{E} = \mathrm{Eis}(\Phi_N^*)$ be the Eisenstein series induced from all exponents of Φ that are of real part $> 1/2$ (or suitable derivatives thereof) and define

$$\int_{\mathbf{X}_{\mathrm{PGL}_2}}^{reg} \Phi := \int_{\mathbf{X}_{\mathrm{PGL}_2}} (\Phi - \mathbf{E}).$$

The right hand side makes sense, for $\Phi - \mathbf{E}$ lies in L^1. Since it is clear on representation-theoretic grounds that the regularized integral of $\mathbf{E}$ must be zero (because the exponents of Φ do not contain 1), the uniqueness follows. $\square$

As a corollary of the previous Lemma, we can define the *regularized inner product* for Φ_1, Φ_2 of controlled increase with the same central character and such that $1 \notin S_1 + \overline{S_2}$:

$$\langle \Phi_1, \Phi_2 \rangle_{reg} = \int_{\mathbf{X}_{\mathrm{PGL}_2}}^{reg} \Phi_1 \overline{\Phi_2}.$$

4.3.6. *Regularization via convolution with measures.* — Here is an alternate definition that will be, in fact, more suited for our purposes. (It also works better in higher rank.)

For every place v, we may choose a compactly supported measure μ_v on $\mathrm{PGL}_2(F_v)$ with the property that, for $\Phi \in \mathcal{V}_S$, $\Phi \star \mu_v \in \mathrm{L}^1$. If $\int \mu_v \neq 0$, then $\Phi \mapsto \frac{\int (\Phi \star \mu_v)}{\int \mu_v}$ defines a functional such as in the Lemma: this functional is independent of v or μ_v, as we see by choosing a different place w and measure μ_w', and noting that $\star \mu_v$ and $\star \mu_w'$ commute. This functional is $\prod_{v' \neq v} \mathrm{PGL}_2(F_v)$-invariant; since v is arbitrary, it is $\mathrm{PGL}_2(\mathbf{A})$-invariant. Other definitions utilize truncation or related ideas; the disadvantage of these is that the $\mathrm{PGL}_2(\mathbf{A})$-invariance is less clear.

4.3.7. *Explicit choice of a regularizing measure.* — The following special case will occur:

Let v be a finite place with a residue field of size q_v. Suppose that Φ is of controlled increase and unramified at v, and all $\chi \in S_\Phi$ have real part 1. Suppose that φ is a cusp form unramified at v.

Then one may choose a K$_v$-bi-invariant (signed) measure μ_v so that $\Phi \star \mu_v$ has rapid decay, so that the total mass of $|\mu_v|$ is at most $4^{|S_\Phi|}$, and so that $\varphi \star \mu_v = \lambda \varphi$, $|\lambda| \geqslant (1 - \frac{2q_v^\theta}{q_v+1})^{|S_\Phi|}$. In other words, we may "kill the growth of Φ whilst only wounding φ;" it should be noted that here the set $|S_\Phi|$ is counted "with multiplicity."

Indeed, there exists a finite function $f = \sum_{\chi \in S_\Phi} f_\chi$ on $N(\mathbf{A})A(F) \backslash GL_2(\mathbf{A})$ so that $\Phi - f$ is rapidly decaying, as in (4.18). The standard Hecke operator $T_v := 1_{K_v a(\varpi_v) K_v}$ acts on each f_χ by a (generalized) eigenvalue $\lambda_v(\chi)$, satisfying $q_v + 1 \geqslant |\lambda_v| \geqslant q_v - 1$ in absolute value; on the other hand, T_v acts on φ by an eigenvalue that is at most $2q_v^\theta$.

The measure $(1 - \frac{T_v}{\lambda_v(\chi)})$ therefore annihilates f_χ, and has total mass $(1 + \frac{q_v+1}{|\lambda_v(\chi)|}) \in$ [2, 4]. It acts on φ by an eigenvalue that is $\geqslant 1 - \frac{2q_v^\theta}{q_v+1}$. Take μ_v to be the convolution of these measures, for all $\chi \in S_\Phi$.

4.3.8. *Regularized IP formula.* — A simple form of the Plancherel formula is the following: if Φ_1, Φ_2 are functions on $\mathbf{X}_{PGL_2}$ with rapid decay then

$$\langle \Phi_1, \Phi_2 \rangle = \int_{\pi \in \widehat{PGL_2}^{Aut}} \langle \Pi_\pi \Phi_1, \Pi_\pi \Phi_2 \rangle d\mu_P(\pi)$$

where Π_π denote the orthogonal projection on the space of π.

We describe now a version of that formula for functions Φ_1, Φ_2 on $\mathbf{X}_{PGL_2}$ of controlled increase. Let π be a standard automorphic representation of PGL$_2$, and $\Phi \in \mathcal{V}_S$. If S does not intersect the unitary axis and π is *generic* (i.e. not equal to a quadratic character) then, for $\mathcal{B}(\pi)$ an orthonormal basis of π, we set

$$(\mathbf{4.19}) \qquad \Pi_\pi \Phi = \sum_{\varphi \in \mathcal{B}(\pi)} \langle \Phi, \varphi \rangle_{reg} \varphi \in \pi.$$

Likewise, we define similarly $\Pi_\pi \Phi$ for any π which is *nongeneric* (i.e. one-dimensional), so long as S does not contain any exponent whose square is $|\cdot|^2$.

Proposition. — *Given Φ_1 and Φ_2 of controlled increase with exponents of real part $> 1/2$. Let* S$_1$, S$_2$ *denote the respective sets of exponents. Suppose that* S$_1$, S$_2$ *are disjoint and that* S$_1 \cup$ S$_2 \cup$ S$_1 + \overline{S_2}$ *does not contain any character whose square is $|\cdot|^2$, then*:

$$(\mathbf{4.20}) \qquad \langle \Phi_1, \Phi_2 \rangle_{reg} = \int_\pi \langle \Pi_\pi \Phi_1, \Pi_\pi \Phi_2 \rangle d\mu_P(\pi) + \langle \Phi_1, \mathcal{E}_2 \rangle_{reg} + \langle \mathcal{E}_1, \Phi_2 \rangle_{reg}$$

where Π_π is the regularized *projection onto the space of automorphic π, and*

$$\mathcal{E}_i = Eis((\Phi_i)_N^*)$$

We will call the additional contribution, $\langle \Phi_1, \mathcal{E}_2 \rangle_{reg} + \langle \mathcal{E}_1, \Phi_2 \rangle_{reg}$, in the regularized Plancherel formula the *degenerate contribution*.

Remark. — If Φ_1 is of rapid decay, the formula continues to hold with $\langle \mathcal{E}_1, \Phi_2 \rangle_{reg} = 0$

Proof. — Firstly our assumptions insure that all the terms of (4.20) are well defined. Moreover since the exponents of Φ_1 and Φ_2 are $> 1/2$ and not of the form $\chi.|\cdot|$ with χ quadratic, the representations underlying $\mathcal{E}_i$ have no subquotient isomorphic to a standard automorphic representation, and thus $\Pi_\pi \mathcal{E}_i = 0$ for $i = 1, 2$ and all π. Similarly $\langle \mathcal{E}_1, \mathcal{E}_2 \rangle_{reg} = 0$ by our assumption that S_1, S_2 are disjoint and have real parts larger than $1/2$. So it is enough to check, with $\bar{\Phi}_i = \Phi_i - \mathcal{E}_i$, that

$$\langle \bar{\Phi}_1, \bar{\Phi}_2 \rangle_{reg} = \int_\pi \langle \Pi_\pi \bar{\Phi}_1, \Pi_\pi \bar{\Phi}_2 \rangle d\mu_P(\pi)$$

but then $\bar{\Phi}_i$ belongs to $L^2(\mathbf{X}_{PGL_2})$. $\qquad\square$

4.4. *(Regularized) triple products*

In this section we establish the following: Suppose that π_i are generic standard automorphic representations, at least one of which is Eisenstein. Then, for each factorizable vector $\varphi_i = \bigotimes_v \varphi_{i,v}$, we have:

$$(\mathbf{4.21}) \qquad \frac{\left| \int_{\mathbf{X}} \varphi_1 \varphi_2 \varphi_3(g) dg \right|^2}{\|\varphi_1\|_{can}^2 \|\varphi_2\|_{can}^2 \|\varphi_3\|_{can}^2} = \frac{1}{8(\text{discF})} \prod_v^* \frac{\zeta_v(1)}{\zeta_v(2)^3} \frac{|L_W(\varphi_{1,v}, \varphi_{2,v}, \varphi_{3,v})|^2}{\prod_{i=1}^3 \langle \varphi_{i,v}, \varphi_{i,v} \rangle}$$

where the local factors are equal at almost all places, to

$$L_v := \frac{\zeta_v(1)}{\zeta_v(2)} \frac{L(\frac{1}{2}, \pi_1 \otimes \pi_2 \otimes \pi_3)}{\prod_{i=1}^3 L(1, \text{Ad}, \pi_i)}.$$

If all π_i are Eisenstein, the integration on the left-hand side is to be understood by regularization.

From this, we deduce—by summation through an orthogonal basis of π_1, with respect to the canonical norm—the following:

$$(\mathbf{4.22}) \qquad \frac{\|\Pi_{\pi_1}(\overline{\varphi_2 \varphi_3})\|^2}{\|\varphi_2\|_{can}^2 \|\varphi_3\|_{can}^2} = \frac{1}{8}(\text{discF})^{-1} \prod_v^* \frac{\zeta_v(1)}{\zeta_v(2)^3} \frac{\|I_{\pi_{1,v}}(\varphi_{2,v} \otimes \varphi_{3,v})\|^2}{\langle \varphi_{2,v}, \varphi_{2,v} \rangle \langle \varphi_{3,v}, \varphi_{3,v} \rangle},$$

with a.e. local factor equal to L_v above.

Before we embark on the proof, we note that it is sufficient—by a continuity argument—to treat the case where no π_i is a singular Eisenstein series (i.e. of the form $\chi \boxplus \chi$). For instance, let us consider the case when π_1, π_2 are cuspidal, $\pi_3 = 1 \boxplus 1$ and let us take a family $\varphi_3(t) \in \pi_3(t)$, where $\pi_3(t) = |\cdot|^t \boxplus |\cdot|^{-t}$ deforming φ_3 (i.e., $\varphi_3(t) \to \varphi_3(0)$ pointwise as $t \to 0$). Then the left-hand side and right-hand side of (4.21), denoted $L(t)$

and R(t) respectively, do not necessarily depend continuously on t when $t = 0$. However, both L*(1, Ad, $\pi_3(t)$)L(t) and L*(1, Ad, $\pi_3(t)$)R(t) extend to continuous functions around $t = 0$, and are equal for $t \neq 0$. If we denote by LR their common limit, then L(0) and R(0) are both given by L*(1, Ad, π_3)LR, and are therefore equal.

4.4.1. *Upper bounds.* — Unfortunately, we shall use the beautiful formula (4.21) only for upper bounds. We now explicate the bounds that are derived from it.

The local factors of (4.22) are equal, to L$_v := \frac{\zeta_v(1)}{\zeta_v(2)^3} \frac{L(\frac{1}{2}, \pi_1 \otimes \pi_2 \otimes \pi_3)}{\prod_{i=1}^3 L(1, \mathrm{Ad}, \pi_i)}$, at almost all places. Observe that—taking into account bounds towards the Ramanujan conjecture—L$_v$ is absolutely bounded above and below at nonarchimedean v. Let S be the set of places where the local factor is not equal to L$_v$, together with all archimedean places, and suppose we are supplied with the estimate

$$\frac{\mathcal{S}_{-d}^{\pi_{3,v}}(\mathrm{I}_{\pi_{3,v}}(x_1 \otimes x_2))^2}{\langle x_1, x_1 \rangle \langle x_2, x_2 \rangle} \leqslant \mathrm{B}_v$$

for $v \in$ S. We conclude—inserting $\Delta_{\mathbf{A}}^{-d}$ and applying the uniqueness of trilinear functionals—

$$(\mathbf{4.23}) \qquad \frac{\|\Delta_{\mathbf{A}}^{-d} \Pi_{\pi_1}(\overline{\varphi_2 \varphi_3})\|^2}{\|\varphi_2\|_{can}^2 \|\varphi_3\|_{can}^2} \ll \mathrm{A}^{-|\mathrm{S}|} \frac{\mathrm{L}(\pi_1 \otimes \pi_2 \otimes \pi_3, \frac{1}{2})}{\prod_{i=1}^3 \mathrm{L}^*(\mathrm{Ad}, \pi_i, 1)} \prod_{v \in \mathrm{S}} \mathrm{B}_v,$$

where A is an absolute constant.

Let us note that: given d, there exists d' so that:

$$(\mathbf{4.24}) \qquad \int_{\pi \text{ generic}} \mathcal{S}_d(\Pi_\pi(\varphi_2 \varphi_3)) d\mu_{\mathrm{P}}(\pi) \ll \mathcal{S}_{d'}(\varphi_2) \mathcal{S}_{d'}(\varphi_3)$$

(the Sobolev norms are relative to the canonical inner product.) Note that this is easy if π_2, π_3 are cuspidal; in that case it can be deduced directly from (S3b). In the remaining cases, by virtue of the bounds of Section 2.6.5, it suffices to check that for π generic and any $d \geqslant 0$

$$(\mathbf{4.25}) \qquad \mathcal{S}_d(\Pi_\pi(\varphi_2 \varphi_3)) \ll \mathcal{S}_{d'}(\varphi_2) \mathcal{S}_{d'}(\varphi_3)$$

for some d' depending on d only.

For this we appeal to the prior formula. The bound on B$_v$ is supplied by (3.5.2); we use also the fact that L($\pi \otimes \pi_2 \otimes \pi_3, 1/2$) is bounded polynomially in the C(π_i) and that L(π_i, Ad, 1) = C(π_i)$^{o(1)}$, $i = 1, 2$. This yields a bound as in (4.25), but only for factorizable $\varphi_2 = \bigotimes_v \varphi_{2,v}$, $\varphi_3 = \bigotimes_v \varphi_{3,v}$, and where the bound on the right-hand side is instead

$$\mathrm{C}(\pi_1)^{\mathrm{A}} \mathrm{C}(\pi_2)^{\mathrm{A}} \prod_v \mathcal{S}_d(\varphi_{2,v}) \mathcal{S}_d(\varphi_{3,v}).$$

We now apply (S1d) (see also Remark 2.6.3), together with (2.18) to absorb $C(\pi_1)^A$, $C(\pi_2)^A$ into the Sobolev norms. Since we did not prove (2.18), we draw attention to the fact that we do not need this last step in any application; it would be fine to retain the factor $C(\pi_1)^A C(\pi_2)^A$.

4.4.2. *The Rankin-Selberg integral.* — In this section, we shall prove the main result, Equation (4.21), under the assumption that either π_1 or π_2 are cuspidal. In this case, we may proceed by the usual Rankin-Selberg method.

We recall here the definition and basic properties of the Rankin/Selberg integral: let π_i, $i = 1, 2, 3$ be generic automorphic representations of $\mathrm{GL}_2(\mathbf{A})$ such that the product of their central characters $\chi_1\chi_2\chi_3$ is trivial. We assume that π_3 is Eisenstein, (say $\pi_3 = \chi_3^+ \boxplus \chi_3^-$ for a pair of characters satisfying $\chi_3^+ \cdot \chi_3^- = \chi_3$) and π_1 is cuspidal.

The (absolutely convergent) integral

$$\mathrm{L}(\varphi_1, \varphi_2, \varphi_3) := \int_{\mathbf{X}} \varphi_1\varphi_2\varphi_3(g)dg, \quad \varphi_i \in \mathrm{V}_{\pi_i}, \ i = 1, \ldots, 3,$$

defines a linear functional on the space of the representation $\pi_1 \otimes \pi_2 \otimes \pi_3$. If the φ_i are factorisable vectors, so that $\mathrm{W}_{\varphi_i} = \prod_v \mathrm{W}_{i,v}$, and $\varphi_3 = \mathrm{Eis}(f_3)$, with $f_3 = \bigotimes_v f_{3,v}$, the Rankin-Selberg unfolding method yields the following factorization (if $\chi_3^+ \neq \chi_3^-$)

$$(\mathbf{4.26}) \qquad \int_{\mathbf{X}} \varphi_1\varphi_2\varphi_3(g)dg = \prod_v^* \frac{\mathrm{L}_{\mathrm{RS},v}(\mathrm{W}_{1,v}, \mathrm{W}_{2,v}, f_{3,v})}{\zeta_v(1)^{1/2}}$$

where (see (3.28))

$$\frac{\mathrm{L}_{\mathrm{RS},v}(\mathrm{W}_{1,v}, \mathrm{W}_{2,v}, f_{3,v})}{\zeta_v(1)^{1/2}} = \int_{\mathrm{N}(\mathrm{F}_v)\backslash\mathrm{PGL}_2(\mathrm{F}_v)} \mathrm{W}_{1,v}\mathrm{W}_{2,v}f_{3,v}(g)dg;$$

and, for almost every v,

$$(\mathbf{4.27}) \qquad \frac{\mathrm{L}_{\mathrm{RS},v}(\mathrm{W}_{1,v}, \mathrm{W}_{2,v}, f_{3,v})}{\mathrm{W}_{1,v}(1)\mathrm{W}_{2,v}(1)f_{3,v}(1)} = \zeta_v(1)^{1/2}\frac{\mathrm{L}_v(\pi_1 \otimes \pi_2 \otimes \chi_3^+, 1/2)}{\mathrm{L}_v(\chi_3^+/\chi_3^-, 1)}.$$

Taking residue at the pole point $\chi_3^+ = |\cdot|^{1/2}$, $\varphi_2 = \overline{\varphi}_1$, and using (4.6), yields:

$$\langle \varphi_1, \varphi_1 \rangle_{\mathbf{X}} = 2(\mathrm{disc}\mathrm{F})^{1/2}\frac{\xi_{\mathrm{F}}(2)}{\xi_{\mathrm{F}}^*(1)} \prod_v^* \langle \mathrm{W}_{1,v}, \mathrm{W}_{1,v} \rangle$$

with $\langle \mathrm{W}_{1,v}, \mathrm{W}_{1,v} \rangle = \zeta_k(1)\mathrm{L}(\pi, \mathrm{Ad}, 1)/\zeta_k(2)$ for a.e. v. This proves (2.5) in the cuspidal case.

From these remarks, (4.26), together with the definition of the canonical norm in Section 2.2.2, we deduce that:

$$(\textbf{4.28}) \qquad \frac{\left|\int_{\mathbf{X}} \varphi_1 \varphi_2 \varphi_3(g) dg\right|^2}{\|\varphi_1\|_{can}^2 \|\varphi_2\|_{can}^2} = \frac{1}{4} (\mathrm{discF})^{-1} \prod_v^* \frac{\zeta_v(1)}{\zeta_v(2)^2} \frac{|\mathrm{L}_{\mathrm{RS},v}(W_{1,v}, W_{2,v}, f_{3,v})|^2}{\prod_{i=1}^2 \langle W_{i,v}, W_{i,v} \rangle}$$

with almost all factors equal to $\frac{\mathrm{L}(\pi_{1,v} \otimes \pi_{2,v} \otimes \pi_{3,v}, \frac{1}{2})}{\zeta_v^{-1}(1) \prod_{i=1}^3 \mathrm{L}(\pi_i, \mathrm{Ad}, 1)}$. Note that, since π_3 is Eisenstein, the product would not converge without the inclusion of the $\zeta_v(1)$ factor. Now (4.21)—in the case where one of π_1, π_2 is cuspidal, and π_3 is not isomorphic to $1 \boxplus 1$—follows, taking into account the equality between L_{RS} and L_{W} that was already established in Section 3.4.1, together with the relation (4.8) between $\prod_v \langle f_v, f_v \rangle$ and the canonical norm on π_3.

4.4.3. *A regularized triple product.* — We discuss now the situation when *all* π_i are Eisenstein, i.e.

$$\pi_i = \mathrm{Eis}(\mathcal{I}(\chi_i^+, \chi_i^-)), \ \varphi_i = \mathrm{Eis}(f_i),$$

we the $\chi_i^\pm$ are unitary. There are two equivalent definitions of the regularized integral $\pi_1 \otimes \pi_2 \otimes \pi_3 \to \mathbf{C}$; we define them and prove their equivalence:

Set

$$(\textbf{4.29}) \qquad \mathrm{L}_a : \varphi_1 \otimes \varphi_2 \otimes \varphi_3 \mapsto \int_{\mathbf{X}_{\mathrm{PGL}_2}}^{reg} \varphi_1 \varphi_2 \varphi_3,$$

and set $\mathrm{L}_c : \pi_1 \times \pi_2 \times \pi_3 \to \mathbf{C}$ to be the value at $s = 0$ of the meromorphic continuation (from $\Re(s) \gg 1$) of the following expression:

$$(\textbf{4.30}) \qquad \mathrm{L}_s : \varphi_1 \otimes \varphi_2 \otimes \mathrm{Eis}(f_3) \mapsto \int_{\mathbf{N}(\mathbf{A})\mathrm{A}(\mathrm{F})\backslash\mathrm{PGL}_2(\mathbf{A}_{\mathrm{F}})} ((\varphi_1 \varphi_2)_{\mathrm{N}} - (\varphi_1)_{\mathrm{N}}(\varphi_2)_{\mathrm{N}}) f_3(s) dg.$$

Note that the later expression is convergent for $\Re s \gg 1$ due to the rapid decay of $(\varphi_1 \varphi_2)_{\mathrm{N}} - (\varphi_1)_{\mathrm{N}}(\varphi_2)_{\mathrm{N}}$ and unfolds to

$$(\textbf{4.31}) \qquad s \mapsto \int_{\mathbf{N}(\mathbf{A})\mathrm{A}(\mathbf{A})\backslash\mathrm{PGL}_2(\mathbf{A})} W_{\varphi_1} W_{\varphi_2} f_3(s) dg$$

which extends to an holomorphic function in a neighborhood of $s = 0$.

Lemma. — $\mathrm{L}_a = \mathrm{L}_c$. *Moreover, if we write* L *for the common value of these expressions,*

$$(\textbf{4.32}) \qquad \frac{|\mathrm{L}(\varphi_1, \varphi_2, \varphi_3)|^2}{\|\varphi_1\|_{can}^2 \|\varphi_2\|_{can}^2 \|\varphi_3\|_{can}^2} = \frac{1}{8} (\mathrm{discF})^{-1} \prod_v^* \frac{\zeta_v(1)}{\zeta_v(2)^3} \frac{|\mathrm{L}_v(\varphi_{1,v}, \varphi_{2,v}, \varphi_{3,v})|^2}{\prod_{i=1}^3 \langle \varphi_{i,v}, \varphi_{i,v} \rangle}$$

In this way, we have established (4.21) in the remaining case also.

Proof. — Considering the central characters χ_i, $i = 1, 2, 3$ fixed, the pairs of characters (χ_i^+, χ_i^-) $i = 1, 2, 3$ such that $\chi_i^+ \chi_i^- = \chi_i$ will be referred as the *parameters*. The set of parameters has the structure of a 3-dimensional complex manifold with infinitely many connected components.

The reasoning by which we derived (4.21) in the "at least one π_i cuspidal" case may be applied to L_c, at least on an open set of parameters which intersects every connected component. By analytic continuation this shows that $|L_c|^2$ is indeed given by (4.21) everywhere.

To establish $L_a = L_c$, we consider the parameters in a given connected component: in other words, assuming the characters $\chi_i^\pm$, $i = 1, 2, 3$ unitary, we consider any analytic deformation

$$\varphi_1(s_1) \otimes \varphi_2(s_2) \otimes \varphi_3(s_3) \in \pi_1(s_1) \otimes \pi_2(s_2) \otimes \pi_3(s_3)$$

indexed by the complex parameters $(s_1, s_2, s_3) \in \mathbf{C}^3$. Clearly, the integral (4.30) extends (via (4.31)) to an holomorphic function on an open subset of $\mathbf{C}^3$ (containing $(0, 0, 0)$)

$$(\mathbf{4.33}) \qquad (s_1, s_2, s_3) \mapsto \int_{\mathbf{N(A)A(F)\backslash PGL_2(A_F)}} ((\varphi_1(s_1)\varphi_2(s_2))_N - (\varphi_1(s_1))_N (\varphi_2(s_2))_N) f_3(s_3) dg.$$

Let $\mathbf{E}(s_1, s_2)$ be the Eisenstein series formed out of the exponents of the product $\Phi := \varphi_1(s_1).\varphi_2(s_2)$ which are of real part $> 1/2$. Explicitly: let S_Φ be the set of exponents of Φ; let $\Phi_{N, >1/2}^*$ be the part of Φ_N^* that corresponds to exponents $\chi \in S_\Phi$ with $\Re(\chi) > 1/2$, and let $\mathbf{E}(s_1, s_2) := \mathrm{Eis}(\Phi_{N, >1/2}^*)$, interpreted by analytic continuation if there exists $\chi \in S_\Phi$ with $\Re(\chi) \in (1/2, 1)$.

The map $(s_1, s_2) \mapsto \mathbf{E}(s_1, s_2)$ defines a meromorphic function on an open subset of $\mathbf{C}^2$; in an open subset of $\mathbf{C}^3$, one has

$$\int_{\mathbf{X}_{PGL_2}}^{reg} \varphi_1(s_1)\varphi_2(s_2)\varphi_3(s_3) = \int^{reg} (\varphi_1(s_1)\varphi_2(s_2) - \mathbf{E}(s_1, s_2))\varphi_3(s_3).$$

Considering Fourier expansions, one see that, given any $N > 1$, one has

$$(\mathbf{4.34}) \qquad \varphi_1(s_1)\varphi_2(s_2)(x) - \mathbf{E}(s_1, s_2)(x) \ll_N \mathrm{ht}(x)^{-N}, \quad x \in \mathbf{X}_{PGL_2},$$

as long as $\Re s_1 \gg_N \Re s_2 \gg_N 1$.

Therefore, there is an open subset of $\mathbf{C}^3$ (in which $s_3 > 1/2$) so that the previous integral is absolutely convergent. It unfolds to

$$(\mathbf{4.35}) \qquad \int_{\mathbf{N(A)A(F)\backslash PGL_2(A)}} ((\varphi_1(s_1)\varphi_2(s_2))_N - \mathbf{E}(s_1, s_2)_N) f_3(s_3) dg.$$

Indeed, (4.34) implies that the constant term $((\varphi_1(s_1)\varphi_2(s_2))_{\mathrm{N}} - \mathbf{E}(s_1, s_2)_{\mathrm{N}}$ is *bounded*, and from this one justifies the unfolding process.

The definitions (4.33) and (4.35) are *a priori* convergent in different regions and they cannot be compared directly. Nonetheless, they coincide on the intersection of their domains of holomorphic continuation. Indeed, there exists a nonempty open set of parameters, intersecting every connected component of either domain of holomorphic continuation, so that both (4.33) and (4.35) can be defined by regularizing the integral over $\mathbf{N}(\mathbf{A})\mathrm{A}(\mathrm{F})\backslash\mathrm{PGL}_2(\mathbf{A})$; this being done, their difference vanishes by invariance of the regularized integral.

To be more specific, we have, whenever $|s_1| + |s_2| \leqslant \mathrm{A}$, the bound

$$(\varphi(s_1)\varphi(s_2))_{\mathrm{N}}(a(y)k) \ll \max(|y|, |y|^{-1})^{1+\mathrm{A}},$$

with bounds of similar nature for $\varphi(s_1)_{\mathrm{N}}\varphi(s_2)_{\mathrm{N}}$ and also $\mathbf{E}(s_1, s_2)_{\mathrm{N}})$. Therefore, if $\Re(s_3) \gg_{\mathrm{A}} 1$, and we write out the integrals (4.35) and (4.33) in the Iwasawa decomposition, we obtain functions of $a(y)k$ which are integrable in the region $|y| \leqslant 1$. By contrast, in the region $|y| \geqslant 1$ they are asymptotic to sums of finite functions of y, i.e., functions whose translates span a finite-dimensional vector space; these may be regularized as in Section 4.3.1, and our conclusion follows.[27] $\qquad\qquad\square$

4.5. *An example*: $\mathrm{F} = \mathbf{Q}$ *at full level*

For $\mathrm{F} = \mathbf{Q}$, we have in particular $\xi_{\mathrm{F}}^*(1) = \mathrm{disc}\mathrm{F} = 1$. Let us describe some explicit consequences of the foregoing remarks. In particular, we elaborate on the remark, contained in the introduction, that the "evident" identity (1.2) gives rise to an identity between families of L-functions.

4.5.1. — Let $\mathcal{M}$ be the set of even Maass cusp forms on $\mathrm{SL}_2(\mathbf{Z})\backslash\mathbf{H}$. Let $\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$. We shall use Λ to denote the completed L-function. For $\varphi \in \mathcal{M}$ define

$$\eta_\varphi(x, y) = \Lambda(\varphi, 1/2 + x + y)\Lambda(\varphi, 1/2 + x - y),$$

the corresponding definition for an Eisenstein series $\mathrm{E}_s \in \mathrm{Eis}(|\cdot|^s, |\cdot|^{-s})$ is:

$$\eta(s, x, y) = \xi(1/2 + s + x + y)\xi(1/2 - s + x + y)\xi(1/2 + s + x - y)$$
$$\times \xi(1/2 - s + x - y);$$

The quotient of $\mathbf{X}$ by the maximal compact K may be identified with the quotient of $\mathrm{SL}_2(\mathbf{Z})\backslash\mathbf{H}$, by $z \mapsto -\bar{z}$. The measure of the quotient is $\frac{\pi}{3}$, i.e. it arises from $2\frac{dx\,dy}{y^2}$ (but if we work on $\mathrm{SL}_2(\mathbf{Z})\backslash\mathbf{H}$, it is simply $\frac{dx\,dy}{y^2}$).

---

[27] The reader may wish to consider the following simpler example of this reasoning: the characteristic function of $[0, 1]$ and the characteristic function of $[1, \infty]$ have Mellin transforms, respectively, $\frac{1}{s}$ ($s > 0$) and $-\frac{1}{s}$ ($s < 0$). However, the fact that the meromorphic extensions are negative to each other can be deduced without computation: the sum of $f_1 + f_2$ is the constant function, and its Mellin transform in any regularized sense must be zero wherever defined, by invariance.

For $s \in \mathbf{C}^\times$ let E_s resp. E_s^* be the result of applying the Eisenstein intertwiner to the vector $f \in |\cdot|^s \boxplus |\cdot|^{-s}$ whose restriction to K is 1 resp. $\xi(1 + 2s)$. Then E_s and E_s^* descend to functions on $SL_2(\mathbf{Z})\backslash\mathbf{H}$; their constant terms are

$$y^{1/2+s} + \frac{\xi(2s)}{\xi(1+2s)}y^{1/2-s} \quad \text{resp. } \xi(1+2s)y^{1+2s} + \xi(1-2s)y^{1-2s}.$$

For $s \in i\mathbf{R}^\times$, we have by (4.8), $\langle E_s, E_s \rangle_{can} = 2\xi(2) = \pi/3$ while

$$\langle E_s^*, E_s^* \rangle_{can} = 2\xi(2)\xi(1+2s)\xi(1-2s) \to \infty, \quad s \to 0$$

By (4.21), if $\psi \in \mathcal{M}$ or is an Eisenstein series,

$$\langle E_{t_1}^* E_{t_2}^*, \psi \rangle^2 = \frac{\|\psi\|_{can}^2}{2\xi(2)} \frac{\eta_\psi(t_1, t_2)^2}{\Lambda^*(1, \mathrm{Ad}, \psi)}$$

For $t_1 \neq \pm t_2 \in i\mathbf{R}$:

$$(\mathbf{4.36}) \qquad E_{t_1}^* E_{t_2}^* = \sum_{t_1' = \pm t_1, t_2' = \pm t_2} \frac{\xi(1+2t_1')\xi(1+2t_2')}{\xi(2+2t_1'+2t_2')} E_{1/2+t_1'+t_2'}$$

$$+ \sum_{\varphi \in \mathcal{M}} \frac{\eta_\varphi(t_1, t_2)}{\sqrt{2\xi(2)L(\mathrm{Ad}, \psi, 1)}\|\varphi\|_{can}}\varphi$$

$$+ \frac{1}{2\pi}\int_{s=0}^{i\infty} \frac{\eta(s, t_1, t_2)}{\xi(1+2s)\xi(1-2s)}E_s^*$$

The first and third term on the right-hand side are, of course, intimately related, through a process of contour-shifting; indeed, if we compute the constant term of the right-hand side, one finds that (after shifting contours) the residues of the third term match exactly, and indeed cancel, part of the constant term of the first term.

It is interesting to substitute the point i ("period over a non-split torus with class number one"). Noting that,

$$E_s^*(i) = 2^{s+3/2}\xi_{\mathbf{Q}(i)}(1/2 + s),$$

the resulting formula relates $\xi_{\mathbf{Q}(i)}(1/2 + t_1)\xi_{\mathbf{Q}(i)}(1/2 + t_2)$ with $\xi_{\mathbf{Q}(i)}(1 \pm t_1 \pm t_2)$ and integrals over the critical line. The contribution of cusp forms may be expressed in terms of Fourier coefficients of half-integral weight.

4.5.2. *The associativity formula.* — Equation (4.36) expresses a "multiplication table" for forms; this is of course constrained by the associativity of multiplication. These constraints lead (among other things) to identities generalizing that of Kuznetsov:

Set

$$f_1(t_1, t_2, t_3, t_4) = \frac{\xi(1+2t_3)\xi(1+2t_4)}{\xi(2+2t_3+2t_4)}\eta(t_1, t_2, 1/2 + t_3 + t_4)$$

and take $\Delta(t_1, t_2, t_3, t_4) = \sum_{\pm,\pm}(f(t_1, t_2, \pm t_3, \pm t_4) + \sum_{\pm,\pm} f(t_3, t_4, \pm t_1, \pm t_2)$. Then the function Σ defined by:

$$\Sigma = \int_{s=0}^{i\infty} \frac{ds}{2\pi} \frac{\eta(s, t_1, t_2)\eta(s, t_3, t_4)}{\xi(1+2s)\xi(1-2s)} + \frac{3}{\pi} \sum_{\varphi\in\mathcal{M}} \frac{\eta_\varphi(t_1, t_2)\eta_\varphi(t_3, t_4)}{\Lambda(\varphi, \mathrm{Ad}, 1)}$$

$$+ \Delta(t_1, t_2, t_3, t_4)$$

is *invariant under all permutations of coordinates.* This is a spectral identity between families of L-functions; a version was first discovered by Kuznetsov.

4.5.3. *The Motohashi formula.* — One may also take the period of (4.36) over a split torus, i.e., integrate $\int_{y\in\mathbf{R}_+} \mathrm{E}_{t_1}(a(y))\mathrm{E}_{t_2}(a(y))d^\times y$; the integral does not converge, but can be regularized as in Section 4.3.1.

The resulting formula relates, on the left-hand side,

$$\int_{\nu\in i\mathbf{R}} \xi(1/2 + t_1 + \nu)\xi(1/2 - t_1 + \nu)\xi(1/2 + t_2 + \lambda - \nu)\xi(1/2 - t_2 + \lambda - \nu)$$

to, on the right hand side,

$$\sum_{\mathcal{M}} \Lambda(1/2 + t_1 + t_2, \varphi)\Lambda(1/2 + t_1 - t_2, \varphi)\Lambda(1/2 + \lambda, \varphi)$$

$$+ \text{ Eisenstein and degenerate terms}$$

This is an example of a formula of Motohashi [53]; it is perhaps most interesting to let $t_1, t_2, \lambda = 0$.

The possibility of thus deriving the Motohashi formula was remarked in [51] in Section 4.3.3 together with the remark that the divergence of the resulting integrals would "cause considerable technical difficulty." The method of using regularized integrals thus settles that issue in a satisfactory way. We would like to remark that it is necessary to be wary of the following point: the regularized integral *cannot* be interchanged naively with the continuous integral over the Eisenstein series that occurs on the right of (4.36). In "commuting" the two, we obtain an extra factor of $\frac{\eta(-1/2-\lambda, t_1, t_2)}{\xi(2+2\lambda)} + \frac{\eta(-1/2+\lambda, t_1, t_2)}{\xi(2-2\lambda)}$.

4.5.4. *Commentary.* — For the purpose of analytic number theory the above formulas, although beautiful, are insufficient; one needs a family with more flexibility of "test functions." This extra flexibility is provided by the work of Motohashi and Kuznetsov. From the point of view of the method above, this can be obtained by varying the choice of test vector in the representations underlying $\mathrm{E}_{t_1}^*$ and $\mathrm{E}_{t_2}^*$.

The associativity formula, and that of Motohashi, have played an important—though not always explicit—role in analytic number theory on GL$_2$. We invite the reader to see "shadows" of these formulas hiding in various other papers on the subject.

The primary advantage of the formalism above seems to be that it generalizes immediately to ramified settings. One may, for instance, replace $\mathrm{E}^{*}_{t_1}$ by an Eisenstein series in the representation $\chi \boxplus 1$ to get a new formula; or one may replace $\mathbf{Q}$ by a number field. It is a very interesting question to investigate more deeply the "test function" version of such formulas in a general setting, and to study the integral transforms—both archimedean and p-adic—that intervene.

Such general formulas would probably lead to, among other things, a good exponent in the subconvexity theorem. In the present paper we have taken a "short cut" to subconvexity.

5. Proof of the theorems

5.1. *Subconvexity of character twists*

In the present section we shall prove the following theorem which is a special case of our main result, Theorem 1.1, but which is also necessary in its proof.

We continue with notation as in Section 4.1.

Theorem 5.1. — *There are absolute constants $\delta, \mathrm{A} \geqslant 0$ such that for F a number field, π an automorphic representation of $\mathrm{GL}_2(\mathbf{A}_\mathrm{F})$, and χ a unitary (Hecke) character of $\mathrm{F}^\times \backslash \mathbf{A}_\mathrm{F}^\times$,*

$$\mathrm{L}\left(\pi \times \chi, \frac{1}{2}\right) \ll_{\mathrm{F},\pi} \mathrm{C}(\chi)^{1/2-\delta},$$

where $\mathrm{C}(\chi)$ denotes the analytic conductor, and our conventions about $\ll$ are as in Section 4.1.5. In particular, $\mathrm{L}(\chi, \frac{1}{2}) \ll_{\mathrm{F}} \mathrm{C}(\chi)^{1/4-\delta}$.

The proof follows the description of Section 1.2; we try below to give a "translation" of the adelic steps to the steps in Section 1.2.

Let us recall that over $\mathbf{Q}$, the first result in the direction of Theorem 5.1 are due to Good [27] (in the t-aspect) and to Duke-Friedlander-Iwaniec in the conductor aspect [17]. Over general number fields the conductor aspect was obtained by the second author [65] and Diaconu-Garrett for the t-aspect [16]. The various methods used in these papers, although superficially rather different and having different strengths and weaknesses, nonetheless are closely linked; they are all, in various ways, related to versions of the identity described in Section 4.5.3. In particular, it is possible to redo the proof below in a way that is much closer to the proof of Theorem 1.1, i.e. removing the ergodic theory and substituting explicit spectral expansions.

5.1.1. *Notation.* — Let $\mathrm{H}^{(1)} = \{a(y) : |y| = 1\}$. Given χ a character as in the Theorem, we define the following signed measure μ_χ supported on a closed orbit of $\mathrm{H}^{(1)}$

on **X**:

$$(\mathbf{5.1}) \qquad \mu_\chi(\varphi) = \int_{F^\times \backslash \mathbf{A}^{(1)}} \varphi(a(y))\chi(y)d^\times y, \quad \mu = \mu_1.$$

Let us note that (5.1) makes sense for any function on $B(F)\backslash PGL_2(\mathbf{A})$, i.e. μ_χ has a canonical lifting $\tilde{\mu}_\chi$ to that space. Given $g \in GL_2(\mathbf{A})$, we denote by μ_χ^g the translate of μ_χ by g (i.e. $\mu_\chi^g(\varphi) = \mu_\chi(g.\varphi)$), and similarly define $\tilde{\mu}_\chi^g$. For $t \in \mathbf{R}_{>0}$, we pick $y_t \in \mathbf{A}^\times$ such that $|y_t|_\mathbf{A} = t$.

We shall be primarily interested in the translates μ_χ^g when g is of the form $a(y)n(T)$, for $T \in \mathbf{A}$.

In trying to get some geometric intuition for these measures, we suggest that the reader bear in mind the following simple example: $F = \mathbf{Q}$, $T = ``p \in \mathbf{Q}_p \hookrightarrow \mathbf{A}"$, the projection of $\mathrm{supp}(\mu_1^{a(q^{-1})n(T)})$ to the space of lattices is the set $\{\Lambda_x : x \in (\mathbf{Z}/q\mathbf{Z})^\times\}$. Here Λ_x are as in Section 1.2; recall in particular that $\Lambda_x = n(x)\Lambda_1$. Therefore, in classical terms, the measures $\mu_\chi^{a(y)n(T)}$ will correspond to *certain orbits of the discrete horocycle flow*; however, the *signs* of the measure μ_χ encode arithmetic information (e.g., whether or not x is a quadratic residue).

5.1.2. *A sketch of the proof in the simplest case:* $F \neq \mathbf{Q}$ *and* π *cuspidal.* — Unfortunately, there are two cases which introduce extra notation and small complexities: $F = \mathbf{Q}$, and π Eisenstein. Thus we advise the reader to read carefully the present subsection, where we sketch the proof in the case when neither of these complications exist.

The reasoning has the following three stages. Fix $\varepsilon > 0$ and $\kappa \in (0, 1)$ and write, for typographical simplicity, $Q = C(\chi)$.

Step 1. Theorem 5.1 $\Longleftarrow$ equidistribution result for μ_χ^g, for suitable g:

Claim 1: There's $\varphi \in \pi$, $T = (T_v)_v \in \mathbf{A}$ with $|T| \gg Q^{1-\varepsilon}$ and $t \in [Q^{-1-\kappa}, Q^{-1+\kappa}]$:

$$Q^{-1/2-\varepsilon}|L(\pi \otimes \chi, 1/2)| \ll_{\varepsilon,\kappa,\pi} |\mu_\chi^{a(y_t)n(T)}(\varphi)| + Q^{-\kappa/2}.$$

Step 2. Section 4.2.2 proves equidistribution of translates of μ_χ in the case $\chi = 1$.

Claim 2: For some absolute $\delta > 0$, and t as above

$$\mu^{a(y_t)n(T)}(f) \ll \mathcal{S}^\mathbf{X}(f)(|t|_\mathbf{A}^{1/2} + |T|_\mathbf{A}^{-\delta}),$$
$$f \in L_0^2(\mathbf{X}_{PGL_2}) \text{ and smooth}$$

Step 3. An application of Section 1.3 allows us to bound $\mu_\chi^{a(y_t)n(T)}$ starting from Claim 2.

Claim 3: $|\mu_\chi^{a(y_t)n(\mathrm{T})}(\varphi)| \ll_\pi Q^{-\delta'}\mathcal{S}^{\mathbf{X}}(\varphi)$. The utility of $\mathrm{F} \neq \mathbf{Q}$ comes in here: we take advantage of the fact that we can find many pairs of distinct prime ideals with the same norm. We use this to construct a suitable measure σ with which to apply Section 2.5.3, with σ supported entirely on the group $\mathrm{H}^{(1)}$ and commuting with $n(\mathrm{T})$[28]. When $\mathrm{F} = \mathbf{Q}$ one can only make such a measure supported "near" $\mathrm{H}^{(1)}$.

In terms of the discussion in Section 1.2, and in particular Section 1.2.4, Step 1 amounts to the remark that (C) or (C2) implies subconvexity and Step 2 amounts to the implication (B) $\implies$ (C).

5.1.3. *The general proof.* — For $\mathrm{T} \in \mathbf{A}$ and h a smooth compactly supported function on $\mathbf{R}_{>0}$ we define the measure

$$\mu_{\chi,h}^{n(\mathrm{T})}(\varphi) := \int_{\mathbf{R}_{>0}} h(t)\chi(y_t)\mu_\chi^{a(y_t)n(\mathrm{T})}(\varphi)d^\times t.$$

Lemma **5.1.4** (Variant of Claim 1). — *Notation as in the theorem, set $Q := \mathrm{C}(\chi)$. For any $\varepsilon > 0, \kappa \in {]}0,1{[}$, there exists a smooth vector $\varphi \in \pi$, $\mathrm{T} = (\mathrm{T}_v)_v \in \mathbf{A}$ and a smooth, non-negative, bounded by 1, function on $\mathbf{R}_{>0}$, $h = h_{Q,\kappa}$ say, supported in the interval $[Q^{-1-\kappa}, Q^{-1+\kappa}]$ such that*

$$(5.2) \qquad Q^{-1/2-\varepsilon}|\mathrm{L}(\pi \otimes \chi, 1/2)| \ll_{\varepsilon,\kappa,\pi} |\mu_{\chi,h}^{n(\mathrm{T})}(\varphi)| + Q^{-\kappa/2}$$

(if π is not cuspidal, replace $\mu_\chi(\varphi)$ by $\tilde{\mu}_\chi(\varphi - \varphi_{\mathrm{N}})$) and moreover:

1. $\mathrm{T}_v = 0$ *unless v is archimedean or χ is ramified at v;*
2. $\frac{\log |\mathrm{T}|_{\mathbf{A}}}{\log Q} \in [1 - \varepsilon, 1 + \varepsilon]$ *if $Q \gg_\varepsilon 1$;*
3. $\mathcal{S}_d^\pi(\varphi) \ll_{d,\pi} 1$ *for any d. In particular, if π is cuspidal, then, for any d, $\mathcal{S}_d^{\mathbf{X}}(\varphi) \ll_{d,\pi} 1$.*

Proof. — For finite v, we take $\mathrm{W}_{\varphi,v} \in \mathcal{W}_{\pi_v}$ to be the new vector and $\mathrm{T}_v = \varpi_v^{-r}$, where r is the conductor of π_v, just as in Lemma 3.3.2. For archimedean v, we choose $\mathrm{W}_{\varphi,v}$ and $\mathrm{T}_v \in \mathrm{F}_v$ according to Lemma 3.3.3. Put $\mathrm{T} = (t_v)_v \in \mathbf{A}$ and let $\varphi \in \pi$ be the preimage of $\otimes\mathrm{W}_{\varphi,v}$ under the canonical intertwiner from π to its Whittaker model.

The third assertion $\mathcal{S}_d^\pi(\varphi) \ll_{d,\pi} 1$, follows since $\mathcal{S}_d(\mathrm{W}_{\varphi,v}) \ll \mathrm{Cond}(\pi_v)^{\mathrm{N}(d)}$, for some $\mathrm{N}(d)$ depending on d—noted in Lemma 12.3 in the archimedean case, and immediate in the nonarchimedean case—and from Section 2.2.2.

It follows then from Lemma 3.3.2, Lemma 3.3.3, and the results of Section 4.2 that

$$(5.3) \qquad \mathrm{L}(\pi \otimes \chi, 1/2) \ll_\varepsilon (\mathrm{C}(\pi)Q)^\varepsilon Q^{1/2} \int_{\mathrm{F}^\times \backslash \mathbf{A}^\times} \varphi(a(y)n(\mathrm{T}))\chi(y)d^\times y$$

---

[28] Let K be a parameter to be chosen as a *fixed* positive power of Q. Consider all finite primes of $\mathbf{Q}$ that are contained in [K, 2K], are split in F and above which χ is unramified. The number of such primes p is then $\gg_{\mathrm{F}} \mathrm{K}/\log \mathrm{K}$. Above each such prime p, let $v_1(p), v_2(p)$ be two *distinct* places. Let σ be the probability measure on $\mathrm{H}^{(1)}$ which is the average of the Dirac measures at the $a(\varpi_{v_1(p)}^{-1}\varpi_{v_2(p)})$ for all such primes $p \in [\mathrm{K}, 2\mathrm{K}]$.

We need to pass from this statement to the desired property (3) by truncating the range of the y-integral. That is carried out using similar reasoning to [65, Lemma 11.9]; it can be considered the geometric form of the approximate functional equation. Let h be a fixed smooth function on $\mathbf{R}_{>0}$ with values in $[0, 1]$, which is 1 on $(0, 1]$, 0 on $[2, \infty)$. Take $A = Q^{-\kappa - 1}$ and $h_A : t \to h(t/A)$.

Write as a shorthand

$$f(t) = \chi(y_t)\mu_\chi^{a(y_t)n(\mathrm{T})}(\varphi),$$

so that the integral on the right-hand side of (5.3) is given by $\int_0^\infty f(t)d^\times t$, and, more generally, the Mellin transform $\mathrm{F}(s) = \int f(t)t^s d^\times t$ is given by $\mathrm{F}(s) = \ell^{\chi|\cdot|^s}(n(\mathrm{T})\varphi)$.

Given $\varepsilon > 0$ small, we will need to bound $\mathrm{F}(s)$ for $\Re s = -1/2 - \varepsilon$. We claim that

(**5.4**) $|\mathrm{F}(s)| \ll_{\pi, \varepsilon} (1 + |s|)^2 Q^{1/2 + 3\varepsilon}$ when $\Re(s) = -1/2 - \varepsilon$.

Crudely, the L-function in front contributes Q, whereas the ramified factors contribute $Q^{-1/2}$, and the rest is of size 1.

To be more precise, let S denote the (finite) set of places where either v is infinite or φ_v is not spherical, or $\mathrm{T}_v \neq 0$, or ψ is ramified. Then $\mathrm{F}(s)$ may be expressed as:

$$\mathrm{L}^{(\mathrm{S})}(\pi \otimes \chi, 1/2 + s) \prod_{v \in \mathrm{S}} \ell^{\chi_v |\cdot|_v^s}(n(\mathrm{T}_v)\mathrm{W}_{v,\varphi}) \prod_{v \notin \mathrm{S}} \frac{\ell^{\chi_v |\cdot|_v^s}(n(\mathrm{T}_v)\mathrm{W}_{v,\varphi})}{\mathrm{L}(\pi_v \otimes \chi_v, 1/2 + s)}.$$

The product over $v \notin \mathrm{S}$ is equal to 1. For $v \in \mathrm{S}$, each factor is bounded by $\ll_{\pi, \varepsilon} \mathrm{Cond}(\chi_v)^{-1/2 + \varepsilon}$, as follows from Lemma 3.3.2 and Lemma 3.3.3. Finally, by the convexity bound together with bounds towards Ramanujan[29]

$$\mathrm{L}^{(\mathrm{S})}(\pi \otimes \chi, s) \ll_\varepsilon (1 + |s|)^2 \mathrm{C}(\pi \otimes \chi)^{1/2 + \varepsilon}   (\Re s = -\varepsilon).$$

Noting that $|\mathrm{S}| = o(\log(\mathrm{C}(\pi \otimes \chi) + \mathrm{disc}(\mathrm{F})))$ as $\mathrm{C}(\pi \otimes \chi) \to \infty$, and since $\mathrm{C}(\pi \otimes \chi) \ll \mathrm{C}(\pi)\mathrm{C}(\chi)^2$, we get (5.4).

By Mellin inversion (H being the Mellin transform of h),

$$\int_{\mathbf{R}_{>0}} h_A(t)f(t)d^\times t = \left| A^{1/2 + \varepsilon} \int_{\Re(s) = -1/2 - \varepsilon} \mathrm{H}(-s)\mathrm{F}(s)\frac{ds}{2\pi i} \right| \ll_{\pi, \varepsilon, h} Q^{\frac{-\kappa}{2} + \varepsilon}$$

The effective content of this statement is that the range $t \lesssim Q^{-\kappa - 1}$ contributes very little to the integral $\int_0^\infty f(t)d^\times t$. A corresponding analysis yields the same statement for the range $t \geqslant Q^{\kappa - 1}$, namely:

$$\int_{\mathbf{R}_{>0}} (1 - h_{Q^{\kappa - 1}}(t))\chi(y_t)\mu_\chi^{a(y_t)n(\mathrm{T})}(\varphi)d^\times t \ll_{\pi, \varepsilon, h} Q^{-\kappa/2 + \varepsilon}.$$

We take $h = h_{Q^{\kappa - 1}} - h_{Q^{-\kappa - 1}}$ to conclude.

---

[29] More precisely, it is necessary to bound each factor $\mathrm{L}(\pi_v \otimes \chi_v, 1/2 + s)^{-1}$. Suppose v is finite and that ψ is unramified at v. If π_v is tempered, this is bounded by $(1 + q_v^\varepsilon)^2$. If π_v fails to be tempered, then π_v is a twist of a spherical representation, as is $\pi_v \otimes \chi_v$. If the local L-factor is not identically 1, then necessarily $\pi_v \otimes \chi_v$ is spherical. Since $v \in \mathrm{S}$, it must be that π_v was ramified at v; the claimed bound follows.

The following will be useful later: If we replace h by any translate $y \mapsto h(y\omega)$, where ω remains within (say) the set $[1/4, 4]$, then (5.2) remains valid. This is evident from the above proof, replacing e.g. $Q^{\kappa-1}$ by $\omega Q^{\kappa-1}$. $\square$

Proposition **5.1.5** (Variant of Claim 2). — *Take $g = a(t)n(T)$ for $t \in \mathbf{R}_{>0} \hookrightarrow \mathbf{A}^\times$, $T = (T_v)_v \in \mathbf{A}$. For $f \in \mathrm{L}_0^2(\mathbf{X}_{\mathrm{PGL}_2})$ and smooth, one has*

$$\mu^g(f) \ll \mathcal{S}^{\mathbf{X}_{\mathrm{PGL}_2}}(f)(|t|_{\mathbf{A}}^{1/2} + |T|_{\mathbf{A}}^{-\delta})$$

for some absolute $\delta > 0$.

Proof. — By property (3c) of Sobolev norms, it is sufficient to prove that for any automorphic representation π equipped with its canonical norm, and $f \in \pi$ a smooth function, we have the inequality

$$(\mathbf{5.5}) \qquad |\mu^g(f)| \ll \mathcal{S}^\pi(f)(|t|_{\mathbf{A}}^{1/2} + |T|_{\mathbf{A}}^{-\delta}).$$

We may decompose $\mu^g(f)$ as $\tilde{\mu}^g(f_{\mathrm{N}}) + \tilde{\mu}^g(f - f_{\mathrm{N}})$.

By (2.12) the first term is bounded by $\mathcal{S}^\pi(f)|t|_{\mathbf{A}}^{1/2}$ (and is even zero if π is cuspidal). The second term equals $\int_{\Re s = 0} \ell^{|\cdot|^s}(g.f)ds$, by inverse Mellin transform; here $\ell^{|\cdot|^s}(f)$ is the linear functional given in (4.15) and associated with the character $|\cdot|_{\mathbf{A}}^s$. Applying Lemma 4.2.3, together with (4.17), we see that $(1 + |s|)^4 \ell^{|\cdot|^s}(f)$ is bounded above by $\mathcal{S}^\pi(f)|T|_{\mathbf{A}}^{-\delta}$, with $\delta > 0$ absolute, and our result follows. $\square$

5.1.6. *The cuspidal case.* — We now prove Claim 3 and conclude the proof of Theorem 5.1 for π cuspidal. We advise the reader to first consider Section 5.1.2 which gives a somewhat cleaner version of the proof in the case $F \neq \mathbf{Q}$.

Let t, T, φ, h be as in Lemma 5.1.4. We need to show that $|\mu_{\chi,h}^{n(T)}(\varphi)|$ decays with Q. The basic idea is this: since $\mu_{\chi,h}^{n(T)}$ is χ-equivariant under the subgroup of elements of $H^{(1)}$ which commute with $n(T)$, we can reduce this to the corresponding fact for $\chi = 1$, already known by Proposition 5.1.5, using the ergodic principle (Section 1.3).

Noting that $\mu^{a(y_t)n(T)}$ is orthogonal to all one-dimensional automorphic representations on PGL_2 except the constants, Proposition 5.1.5 implies that for f non-negative on $\mathbf{X}_{\mathrm{PGL}_2}$,

$$(\mathbf{5.6}) \qquad |\mu_{1,h}^{n(T)}(f)| \leqslant \int h \int_{\mathbf{X}_{\mathrm{PGL}_2}} f + \epsilon \mathcal{S}^{\mathbf{X}_{\mathrm{PGL}_2}}(f), \quad \text{with } \epsilon \ll_\varepsilon Q^\varepsilon(Q^{\frac{\kappa-1}{2}} + Q^{-\delta}),$$

for any $\varepsilon > 0$. Here κ is as in the statement of Lemma 5.1.4.

Let σ be the averaged sum of the Dirac measures which are supported at

$$a(\varpi_v^{-1})a(\varpi_{v'}) \in a(\mathrm{F}_v)a(\mathrm{F}_{v'}) \subset H$$

for (v, v') ranging over pairs of distinct non-archimedean places at which χ is unramified and for which $q_v, q_{v'}$ are contained in [K, 2K] (to be choose later; in any case K $\leqslant$ Q.) The number of such pairs is $\gg$ K^2/log^2 K; σ is not supported on H$^{(1)}$ but rather on

$$\mathrm{H}^{([1/4,4])} = \{a(y),\ y \in \mathbf{A}^\times,\ |y| \in [1/4, 4]\}.$$

Since $\|\mathrm{Ad}(a(\varpi_v^{-1})a(\varpi_{v'}))\| \asymp \mathrm{K}^2$, we see (notation of Section 2.5.3):

$$\|\sigma\|_d \ll \mathrm{K}^{2d}, \|\sigma \star \check{\sigma}\|_\theta \ll (\log \mathrm{K})^2 \big(\mathrm{K}^{-2} + \mathrm{K}^{-1-2\theta} + \mathrm{K}^{-4\theta}\big).$$

The measure $\mu_{\chi,h}(\varphi)$ is not exactly invariant under the substitution $\varphi \mapsto \varphi \star_\chi \sigma$ but almost: indeed since the support of σ commute with $n(\mathrm{T})$ one has

$$\mu_{\chi,h}(\varphi \star_\chi \sigma) = \mu_{\chi,h\star\eta}(\varphi),$$

where η denote the average of the Dirac measures at $q_v/q_{v'}$ on $\mathbf{R}_{>0}$ for (v, v') as above. Therefore, the reasoning of Section 2.5.3 shows that:

$$(\mathbf{5.7}) \qquad |\mu_{\chi,h\star\eta}(\varphi)|^2 \ll_{\varepsilon,\pi} \mathrm{Q}^\varepsilon (\mathrm{K}^{4d}(\mathrm{Q}^{\frac{\kappa-1}{2}} + \mathrm{Q}^{-\delta}) + \mathrm{K}^{-2} + \mathrm{K}^{-1-2\theta} + \mathrm{K}^{-4\theta})$$

Note that $|\varphi \star_\chi \sigma|^2$ descends to $\mathbf{X}_{\mathrm{PGL}_2}$, and so it was admissible to apply (5.6) to it.

It follows from the (the last line of the) proof of Lemma 5.1.4 that (5.2) holds with h replaced by $h \star \eta$ (which is supported on $[\frac{1}{4}\mathrm{Q}^{-1-\kappa}, 4\mathrm{Q}^{-1+\kappa}]$):

$$(\mathbf{5.8}) \qquad \mathrm{Q}^{-1/2-\varepsilon}\mathrm{L}(1/2, \pi \otimes \chi) \ll_{\varepsilon,\pi,\kappa} |\mu_{\chi,h\star\eta}(\varphi)| + \mathrm{Q}^{-\kappa/2}$$

Taking K to be a suitable small power of Q and combining (5.7) and (5.8), we conclude. □

5.1.7. *The Eisenstein case.* — In this section, we prove Theorem 5.1 for π noncuspidal by utilizing a simple regularization. By factorization, it suffices to consider the case where $\pi = 1 \boxplus 1$ is induced from two trivial characters.

Let φ be as furnished by Lemma 5.1.4. Let k be a fixed smooth compactly supported function on $\mathbf{G}(\mathbf{A})$ of integral 1. Fix a sufficiently large parameter X $\geqslant$ 1 (to be a fixed power of Q), and split $\varphi = \varphi_1 + \varphi_2$, where

$$\varphi_1(g) := \wedge^{\mathrm{X}}\varphi \star k,\ \varphi_2 = (\varphi - \wedge^{\mathrm{X}}\varphi) \star k,$$

say. Thus φ_1 is of rapid decay high in the cusp, whereas φ_2 is supported high in the cusp. Let us observe,

1. $|\varphi_2(x)| \ll \mathcal{S}^\pi(\varphi)\mathrm{ht}(x)^{1/2}\log \mathrm{ht}(x)$.
2. $\left|\int_{\mathbf{X}} \varphi_i\right| \ll \mathcal{S}^\pi(\varphi)\mathrm{X}^{-1/4}\ i = 1, 2$.
3. For every d, there exists N(d) so that $\mathcal{S}_d^{\mathbf{X}}(\varphi_1) \ll \mathrm{X}^{\mathrm{N}(d)}\mathcal{S}^\pi(\varphi)$.

The first property follows from (4.12). The second property follows from the first, since φ_2 is supported in $\mathrm{ht}(x) \gg \mathrm{X}$ and $\int \varphi_1 + \int \varphi_2 = 0$. To verify the third property, it suffices to check that (for any $\phi \in \pi$ and any $m \geqslant 1$) we have the bound

$$| \wedge^{\mathrm{X}} \phi(x)| \ll_m \mathrm{X}^{\mathrm{A}(m)} \mathrm{ht}(x)^{-m} \mathcal{S}^{\pi}(\phi).$$

By (4.3) together with (4.12), it suffices to verify that for $y \in \mathbf{A}^{\times}$,

$$\sum_{\alpha \in \mathrm{F}^{\times}} \mathrm{W}_{\phi}(a(\alpha y)) \ll_m |y|^{-m} \mathcal{S}^{\pi}(\phi).$$

This follows from Proposition 3.2.3, together with the definition of the norm on π given by (2.5). It is necessary only to observe that

$$\sum_{\alpha \in \mathrm{F}^{\times}} \prod_v \max(|\alpha y|_v, 1)^{-n} \ll_n |y|^{-n'}.$$

Now, let $g = a(y_t)n(\mathrm{T})$, where t, T are as in Lemma 5.1.4. We need to bound

$$\tilde{\mu}_{\chi}^g(\varphi - \varphi_{\mathrm{N}}) = \mathrm{A} + \mathrm{B} + \mathrm{C} + \mathrm{D},$$

where

- $\mathrm{A} = \mu_{\chi}^g(\varphi_1 - \frac{1}{\mathrm{vol}(\mathbf{X})} \int_{\mathbf{X}} \varphi_1)$ is bounded as in the prior argument by

$$\mathrm{Q}^{\varepsilon}(\mathrm{K}^{2d}(\mathrm{Q}^{\frac{\kappa-1}{4}} + \mathrm{Q}^{-\delta/2}) + \mathrm{K}^{-1} + \mathrm{K}^{-1/2-\theta} + \mathrm{K}^{-2\theta}).$$

- $\mathrm{B} = \mu_{\chi}^g(\frac{1}{\mathrm{vol}(\mathbf{X})} \int \varphi_1)$ is bounded by $\mathcal{S}^{\pi}(\varphi)\mathrm{X}^{-1/4}$;
- $\mathrm{C} = \tilde{\mu}_{\chi}^g(\varphi_{\mathrm{N}})$ is bounded, in view of (4.12), by $|t|^{1/2-\varepsilon}\mathcal{S}^{\pi}(\varphi) = \mathrm{Q}^{\frac{\kappa-1}{2}+\varepsilon}\mathcal{S}^{\pi}(\varphi)$.
- $\mathrm{D} = \mu_{\chi}^g(\varphi_2)$.

Now let us bound D. Let φ° be the spherical vector of norm 1 in the representation π. Decompose $\varphi^{\circ} = \varphi_1^{\circ} + \varphi_2^{\circ}$ just as above. By (4.13), $\varphi_2^{\circ} \geqslant 0$. By *loc. cit.*, it is also true that—for X sufficiently large—$|\varphi_2| \ll \mathcal{S}^{\pi}(\varphi)\varphi_2^{\circ}$. Therefore,

$$(\mathbf{5.9}) \qquad |\mathrm{D}| \leqslant \mathcal{S}^{\pi}(\varphi)(|\mu^g(\varphi_1^{\circ})| + |\mu^g(\varphi^{\circ})|) \ll_{\varepsilon} \mathcal{S}^{\pi}(\varphi)\left(\mathrm{X}^{-1/4} + \mathrm{X}^{\mathrm{A}}(|t|_{\mathbf{A}}^{1/2} + |\mathrm{T}|_{\mathbf{A}}^{-\delta})\right)$$

for some absolute constant A; we applied (5.5) to the function φ°, and Proposition 5.1.5 to the function $\varphi_1^{\circ} - \int_{\mathbf{X}_{\mathrm{PGL}_2}} \varphi_1^{\circ}$, observing that $|\int_{\mathbf{X}_{\mathrm{PGL}_2}} \varphi_1^{\circ}| \ll_{\varepsilon} \mathrm{X}^{-1/4}$. We conclude that:

$$\mathrm{Q}^{-1/2}\mathrm{L}(\chi, 1/2)^2 = \mathrm{Q}^{-1/2}\mathrm{L}(\pi \otimes \chi, 1/2)$$
$$\ll_{\varepsilon} \mathrm{Q}^{\varepsilon}\left(\mathrm{Q}^{-\frac{\kappa}{2}} + \mathrm{X}^{-1/4} + \mathrm{K}^{-\theta} + \mathrm{X}^{\mathrm{A}}\mathrm{K}^{2d}(\mathrm{Q}^{\frac{\kappa-1}{4}} + \mathrm{Q}^{-\delta/2})\right)$$

and we conclude by an appropriate choice of the parameters. $\qquad\square$

5.2. *Subconvex bounds for Rankin/Selberg L-functions*

In this section we prove Theorem 1.2. Let π_1, π_2 be the two automorphic representations of GL$_2(\mathbf{A})$ considered there and let χ_1, χ_2 denote their respective central characters.

5.2.1. — Fix a parameter $z \in i\mathbf{R}$ whose modulus will be a negative power of $C(\pi_1)$ to be determined later. We shall prove our result on certain slight constraints on the parameters of π_i so as to stay away from singular Eisenstein series. The general case will reduce to this constrained case, as we explain now.

If π_2 is not cuspidal, it will be sufficient to take $\pi_2 = 1 \boxplus |\cdot|^z$. Let π_3 be the Eisenstein series $1 \boxplus \chi_3$ where $\chi_3 = (\chi_1\chi_2)^{-1}$; we shall suppose that χ_3^2 is not of the form $|\cdot|^{2it}$ for $|t| < |z|/3$. Under such constraints we will establish the bound[30]

$$L\left(\pi_1 \otimes \pi_2 \otimes \pi_3, \frac{1}{2}\right) \ll_{\pi_2} C(\pi_1 \otimes \pi_2)^{1/2-\delta}|z|^{-8},$$

for some absolute $\delta > 0$.

This implies our main result: from Theorem 5.1, we may assume that π_1 is cuspidal. In that case, we wish a subconvex bound for $L(\pi_1 \otimes \pi_2, 1/2)$ with either π_2 cuspidal or $\pi_2 = 1 \boxplus 1$ (which yield a subconvex bound for $L(\pi_1, 1/2)$). We take z such that $|z| = C(\pi_1)^{-\delta/9}$, and apply the previous bound to a triple (π_1, π_2', π_3) where π_2' equals π_2 or $\pi_2 \otimes |\cdot|^z$ if π_2 cuspidal, and equals $1 \boxplus |\cdot|^z$ if not, while $\pi_3 = 1 \boxplus (\chi_1\chi_2')^{-1}$, choices being made so that (π_1, π_2', π_3) fulfills the above constraints. A subconvex bound follows for $L(\pi_1 \otimes \pi_2, 1/2)$ since, by convexity, we have, for any $t \in [-1, 1]$,

$$|L(\pi_1, 1/2) - L(\pi_1, 1/2 + it)| \ll C(\pi_1)^{1/4+o(1)}|t|,$$

$$|L(\pi_1 \otimes \pi_2, 1/2) - L(\pi_1 \otimes \pi_2, 1/2 + it)| \ll_{\pi_2} C(\pi_1)^{1/2+o(1)}|t|.$$

5.2.2. *Choice of the test vectors.* — Factorize $\pi_i = \bigotimes_v \pi_{i,v}$, and choose unitary structures on each $\pi_{i,v}$ so that the product coincides with the canonical norm on π_i. Given $\epsilon > 0$ small and $i = 1, 2, 3$, let $\varphi_1 \in \pi_1$, $\varphi_2 \in \pi_2$, $E \in \pi_3$ be the tensor product of the test vectors $(\varphi_{1,v})_v$, $(\varphi_{2,v})_v$, $(E_{3,v})_v$ constructed in Proposition 3.6.1 for each place v (applied with the parameter ϵ).

The canonical norm of φ_1, φ_2, E are easy to estimate: they are, by definition, the product of the norms of these local vectors, and are therefore equal to 1; moreover we have for any d,

$$\mathcal{S}_d(\varphi_2) \ll_{\pi_2} 1.$$

---

[30] Of course the exponent 8 is not optimal for specific configuration (e. g. if π_2 is cuspidal) but this will be sufficient for our present needs.

Applying (4.21), we have, from the hypotheses made in the above section,

$$(\textbf{5.10}) \qquad \frac{L(\pi_1 \otimes \pi_2, 1/2)}{C(\pi_1)^{1/2+\epsilon}} \ll_{\epsilon, \pi_2} |L^*(\pi_2, \mathrm{Ad}, 1)L^*(\pi_3, \mathrm{Ad}, 1)| \left| \int_{\mathbf{X}} \varphi_1 \varphi_2 E(g) dg. \right|$$

$$\ll_{\epsilon, \pi_2} |z|^{-4} \langle \varphi_1, \overline{\varphi_2 E} \rangle.$$

Set $Q := C(\pi_1)C(\pi_2)$. We want to check that $\langle \varphi_1, \overline{\varphi_2 E} \rangle \ll_{\pi_2} Q^{-\delta}$. Roughly, we use Cauchy-Schwarz to bound the square of this quantity by $\langle \varphi_2 E, \varphi_2 E \rangle$. The later equals $\langle \varphi_2 \overline{\varphi_2}, E\overline{E} \rangle$ which may be decomposed along the automorphic representations of $\mathrm{PGL}_2(\mathbf{A})$. It turns out that the contribution of each of these is small *except* for one-dimensional representations. Naively speaking, their contribution would be $\|\varphi_2\|^2 \|E\|_{can}^2$; the truth is a little more complicated because we need to use regularization. In any case, to reduce this "large" term, we use Friedlander-Iwaniec's amplification method.

5.2.3. *The amplification method.* — We choose a real signed measure σ compactly supported on $\mathrm{GL}_2(\mathbf{A}_f)$ which satisfies

1. $u \in \mathrm{supp}(\sigma) \implies \|u\| \leqslant K$ where $K > 0$ is some parameter which will be chosen to be a small fixed positive power of Q.
2. $\mathrm{supp}(\sigma)$ commutes with $\mathrm{GL}_2(F_v)$ at all archimedean places v and at all places v for which either π_1 or π_2 is not spherical or the chosen additive character ψ is ramified.[31]
3. $(\varphi_2 E) \star \check{\sigma}$ is of rapid decay on $\mathbf{X}$.
4. Let $|\sigma|$ denote the total variation measure. Then the total mass of $|\sigma|$ is bounded above by K^B, for some absolute constant B. Moreover, with $|\sigma|^{(2)} = |\sigma| \star |\check{\sigma}|$, one has for any $\gamma > 1/2$

$$\int \|\mathrm{Ad}(u)\|^{-\gamma} d|\sigma|^{(2)}(u) \leqslant K^{-\eta}, \quad \text{for some } \eta = \eta(\gamma) > 0.$$

5. $\varphi_1 \star \sigma = \lambda_1.\varphi_1$ with $\lambda_1 \gg_{F,\varepsilon} Q^{-\varepsilon}$ for any $\varepsilon > 0$.

The construction of such a measure (which is inspired by [18, 19]) is given in [65, Section 4.1], except for the remark that $(\varphi_2 E) \star \check{\sigma}$ is of rapid decay. However, this follows by convolving with a measure as in the remark of Section 4.3.7; one may choose the place v so that $q_v \ll (\log Q)^2$.

By property (5) of σ, stated above and Cauchy-Schwarz, we have

$$|\lambda_1|^2 \left| \langle \varphi_1, \overline{\varphi_2 E} \rangle \right|^2 = \left| \langle \varphi_1 \star \sigma, \overline{\varphi_2 E} \rangle \right|^2 = \left| \langle \varphi_1, \overline{(\varphi_2.E) \star \check{\sigma}} \rangle \right|^2$$

$$\leqslant \langle (\varphi_2.E) \star \check{\sigma}, (\varphi_2.E) \star \check{\sigma} \rangle;$$

---

[31] I.e. the places dividing the discriminant of F.

Thus far, the integrals considered are convergent. However, we shall now expand the integral implicit in convolving with $\check{\sigma}$; at this point, we need to make use of regularized integrals. This is possible since, for $u \in \mathrm{supp}(|\sigma|^{(2)})$, the set of exponents of $\varphi_2^u \overline{\varphi_2} \overline{\mathrm{E}^u} \mathrm{E}$ is either $\emptyset$ (if π_2 is cuspidal) or is comprised of characters with real part equal to 2. Noting that

$$(\mathbf{5.11}) \qquad \langle \varphi_2^u . \mathrm{E}^u, \varphi_2 . \mathrm{E} \rangle_{reg} = \langle \varphi_2^u \overline{\varphi_2}, \overline{\mathrm{E}^u} \mathrm{E} \rangle_{reg}$$

we obtain that

$$(\mathbf{5.12}) \qquad |\lambda_1|^2 \left| \langle \varphi_1, \overline{\varphi_2 \mathrm{E}} \rangle \right|^2 \leqslant \int \left| \langle \varphi_2^u \overline{\varphi_2}, \overline{\mathrm{E}^u} \mathrm{E} \rangle_{reg} \right| d|\sigma|^{(2)}(u)$$

Now, applying properties (1) and (4) of σ to (5.12), we conclude that it suffices to prove:

Proposition **5.2.4.** — *For* $u \in \mathrm{GL}_2(\mathbf{A})$, *put* $\Phi_{2,u} := \varphi_2^u \overline{\varphi_2}$ *and* $\Phi_{3,u} := \overline{\mathrm{E}^u} \mathrm{E}$. *For* $u \in \mathrm{supp}(|\sigma|^{(2)})$

$$|\langle \Phi_{2,u}, \Phi_{3,u} \rangle_{reg}| \ll_{\pi_2, \epsilon} |z|^{-4} \|\mathrm{Ad}(u)\|^{-\gamma} + \|u\|^{\mathrm{A}} \mathrm{Q}^{-\delta}$$

for some absolute positive constants $\gamma > 1/2$ *and* $\delta, \mathrm{A} > 0$.

The proof will follow. Roughly, we evaluate the right-hand side via the regularized Plancherel formula in Section 5.2.7; we handle the spectral sum (cuspidal and Eisenstein) in Section 5.2.8, and we bound the "degenerate" terms that arise in Section 5.2.9.

5.2.5. *Good and bad places.* — Fixing now $u \in \mathrm{supp}(|\sigma|^{(2)})$, we make the following definitions of "good" and "bad" places:

Let R be the set of finite places where $\varphi_{2,v}, \mathrm{E}_{3,v}$ are spherical and where $u_v = 1$; let S be the set of places where $u_v \neq 1$: by the choice made in Section 5.2.2 S consists of finite places, and the data $\varphi_{2,v}, \mathrm{E}_{3,v}, \psi_v$ are all unramified for $v \in \mathrm{S}$; let T denote the complement of $\mathrm{R} \cup \mathrm{S}$.

Observe that $|\mathrm{S}| + |\mathrm{T}| = o(\log \mathrm{C}(\pi_1) + \log \mathrm{C}(\pi_2))$, the bound on $|\mathrm{S}|$ arising from property (1) of the signed measure σ. This property will be used to control a product, over $v \in \mathrm{S} \cup \mathrm{T}$, of "implicit constants."

5.2.6. *Deformation.* — In this section, we shall deform π_2, π_3 in such a way that we will be able to apply the regularized Plancherel formula. These deformations will be parameterized by $s \in \mathbf{C}$ (for π_2) and $t \in \mathbf{C}$ (for π_3). We shall fix rather specific deformations of the vectors $\varphi_2 \in \pi_2, \mathrm{E} \in \pi_3$. We shall also fix factorizations into local constituents of various vectors that will arise.

Write $\pi_3 = 1 \boxplus \chi_3$ and let $\pi_3(t) := |\cdot|^t \boxplus \chi_3 |\cdot|^{-t}$, for $t \in \mathbf{C}$, be the one-parameter deformation of π_3 as described in Section 4.1.6. We choose f_3 so that $\mathrm{E} = \mathrm{Eis}(f_3)$. We denote the corresponding deformation of E by $\mathrm{E}(t)$: $\mathrm{E}(t) = \mathrm{Eis}(f_3(t))$.

If $\pi_2 = 1 \boxplus |\cdot|^z$, we consider the deformation $(|\cdot|^s \boxplus |\cdot|^{z-s}, \varphi_2(s))$ of (π_2, φ_2): after choosing f_2 so that $\varphi_2 = \mathrm{Eis}(f_2)$, we set $\varphi_2(s) = \mathrm{Eis}(f_2(s))$. (By convention, if π_2 is cuspidal we regard $(\pi_2(s), \varphi(s))$ as being *constant*.) Note that $\|\mathrm{E}(t)\|_{can}$ and $\|\varphi_2(s)\|_{can}$ are both constant by Lemma 2.2.3.

Factorizing $f_2 = c_{2,f} \prod_v f_{2,v}$ and $f_3 = c_{3,f} \prod_v f_{3,v}$, where $\|f_{2,v}\| = \|f_{3,v}\| = 1$ for all v; then

$$(\textbf{5.13}) \qquad |c_{2,f}| \ll 1, \; |c_{3,f}| \ll 1.$$

Let $\mathrm{W}_{\varphi_2(s)}$ be the image of $\varphi_2(s)$ under the Whittaker intertwiner. We may then factorize

$$\mathrm{W}_{\varphi_2(s)} = c_{2,\mathrm{W}}(s) \prod_v \mathrm{W}_{\varphi_2,v}(s),$$

where we take $\mathrm{W}_{\varphi_2,v}(s)$ to be the image of $f_2(s)$ under the intertwiner (3.10) for $v \in \mathrm{S} \cup \mathrm{T}$; and, for $v \in \mathrm{R}$, we normalize so that $\mathrm{W}_{\varphi_2,v}(s)$ takes the value 1 at 1. In a similar way, we factorize $\mathrm{W}_{\varphi_3(t)} = c_{3,\mathrm{W}}(t) \prod_v \mathrm{W}_{\varphi_3,v}(t)$, the normalizations being identical. Notice that $\|\mathrm{W}_{\varphi_2,v}(s)\| = \|\mathrm{W}_{\varphi_3,v}(t)\| = 1$ for $v \in \mathrm{S} \cup \mathrm{T}$, the intertwiner (3.10) being isometric.

The constant $c_{2,\mathrm{W}}(s)$ may be evaluated by computing norms. By Section 2.2.2

$$\langle \varphi_2, \varphi_2 \rangle_{can} \asymp |c_{2,\mathrm{W}}(s)|^2 \Lambda^*(\pi_2(s), \mathrm{Ad}, 1) \prod_{v \in \mathrm{S} \cup \mathrm{T}} \frac{1}{\zeta_v(1) \mathrm{L}(\pi_{2,v}(s), \mathrm{Ad}, 1)/\zeta_v(2)}$$

and therefore, for any $\varepsilon > 0$,

$$(\textbf{5.14}) \qquad |c_{2,\mathrm{W}}(s)|^2 \ll_\varepsilon \frac{\mathrm{C}(\pi_1)^\varepsilon \mathrm{C}(\pi_2)^\varepsilon}{\mathrm{L}^*(\pi_2(s), \mathrm{Ad}, 1)},$$

and similarly for $c_{3,\mathrm{W}}(t)$.

5.2.7. *Application of the regularized Plancherel formula.* — We note that $\Phi_{2,u}$ and $\Phi_{3,u}$ descend to functions on $\mathrm{PGL}_2(\mathbf{A})$. Throughout this section we regard u as fixed; set (we have suppressed the dependence on u for typographical simplicity)

$$\Phi_2(s) = \varphi_2^u \overline{\varphi_2(s)}, \qquad \Phi_3(t) = \overline{\mathrm{E}^u} \mathrm{E}(t),$$

$$\mathcal{E}_2(s) = \mathrm{Eis}(\varphi_{2\mathrm{N}}^u \overline{\varphi_{2\mathrm{N}}(s)}), \qquad \mathcal{E}_3(t) = \mathrm{Eis}(\overline{\mathrm{E}_\mathrm{N}^u} \mathrm{E}_\mathrm{N}(t)).$$

– The set of exponents of $\Phi_2(s)$ is $\mathrm{S}_2(s) = \{|\cdot|^{1+\bar{s}}, |\cdot|^{1-\bar{s}}, |\cdot|^{1+z+\bar{s}}, |\cdot|^{1+\bar{z}-\bar{s}}\}$ (or the empty set if π_2 is cuspidal).
– The set of exponents of $\Phi_3(t)$ is $\mathrm{S}_3(t) = \{|\cdot|^{1+t}, |\cdot|^{1-t}, \chi_3|\cdot|^{1-t}, \overline{\chi_3}|\cdot|^{1+t}\}$.

In particular, $(s, t) \mapsto \langle \Phi_2(s), \Phi_3(t) \rangle_{reg}$ defines an (s, t)-anti-holomorphic function in a neighbourhood of $(0, 0)$ in $\mathbf{C}^2$. We will bound the function $\langle \Phi_2(s), \Phi_3(t) \rangle_{reg}$ at the point $(0, 0)$ by analyzing it along a suitable non-empty subset $\mathcal{N} \subset \mathcal{D}$ containing $(0, 0)$ in its closure. The key point is that we should choose $\mathcal{N}$ so that (4.20) is applicable, and also so that $\pi_2(t), \pi_3(s)$ do not equal $1 \boxplus 1$ for $(t, s) \in \mathcal{N}$. For explicitness, take

$$(\mathbf{5.15}) \qquad \mathcal{N} = \{(s, t) = (t/2, t), \ t \in i\mathbf{R} : 0 < |t| < |z|/6\}.$$

By choice of $\mathcal{N}$, we can apply the regularized Plancherel formula (4.20). Moreover, for π belonging to the finite spectrum ($\pi = \chi$, $\chi^2 = 1$) and $(s, t) \in \mathcal{N}$ one has, again by invariance, $\Pi_\pi(\Phi_3(t)) = 0$. (In other words, there is no $\mathbf{G(A)}$-equivariant functional from the tensor product $\overline{\pi_3} \otimes \pi_3(t)$ to a one-dimensional $\mathbf{G(A)}$-representation). Therefore, for $(s, t) \in \mathcal{N}$, we have

$$(\mathbf{5.16}) \qquad \langle \Phi_2(s), \Phi_3(t) \rangle_{reg} = \langle \Phi_2(s), \mathcal{E}_3(t) \rangle_{reg} + \langle \mathcal{E}_2(s), \Phi_3(t) \rangle_{reg}$$
$$+ \int_{\pi \text{ generic}} \langle \Pi_\pi(\Phi_2(t)), \Pi_\pi(\Phi_3(s)) \rangle d\mu_{\mathrm{P}}(\pi),$$

where Π_π is defined in (4.19).

We have already observed that $(s, t) \mapsto \langle \Phi_2(s), \Phi_3(t) \rangle_{reg}$ defines an (s, t)-antiholomorphic function in a neighbourhood of $(0, 0)$, and, in particular, a continuous function on $\mathcal{N}$; since the same is true of the function

$$(s, t) \mapsto \int_{\pi \text{ generic}} \langle \Pi_\pi(\Phi_2(t)), \Pi_\pi(\Phi_3(s)) \rangle d\mu_{\mathrm{P}}(\pi),$$

it follows from (5.16) that

Lemma. — The degenerate term

$$(s, t) \in \mathcal{N} \mapsto \langle \Phi_2(s), \mathcal{E}_3(t) \rangle_{reg} + \langle \mathcal{E}_2(s), \Phi_3(t) \rangle_{reg}$$

extends to a continuous function on $\mathcal{N}$.

5.2.8. *Bounding the generic term*

Lemma. — For $(s, t) \in \mathcal{N}$, the last term of (5.16) is $\ll_{\pi_2} \|u\|^{\mathrm{A}} Q^{-\delta}$, for absolute $\delta > 0$ and A.

In fact, we need this bound only for $(s, t) = (0, 0)$, and the reader is welcome to make this substitution in what follows.

Proof. — Let d_0 be the larger than any of the (absolute) Sobolev degrees occurring in Lemmata 3.5.3 and 3.7.1 as well as the degree of (3.36). For any generic standard

automorphic representation of PGL_2, the inner product $\left| \langle \Pi_\pi(\Phi_2(s)), \Pi_\pi(\Phi_3(t)) \rangle \right|$ is bounded above by $\mathcal{S}^\pi_{d_0}(\Phi_2(s)) \mathcal{S}^\pi_{-d_0}(\Phi_3(t))$ (as a short cut we write $\mathcal{S}^\pi_d(\Phi)$ for $\mathcal{S}^\pi_d(\Pi_\pi(\Phi)))$, by the duality between the Sobolev norms in question . It will be sufficient to verify that

$$\mathcal{S}^\pi_{-d_0}(\Phi_3(t)) \ll_{\pi_2,\pi,\epsilon} \|u\|^{\mathrm{B}} Q^{-\delta},$$

for absolute constants $\delta > 0$ and B; indeed, for any constant $\mathrm{B} > 0$, one has

$$\int_\pi \mathrm{C}(\pi)^{\mathrm{B}} \mathcal{S}^\pi_{d_0}(\Phi_2(s)) \ll_{\mathrm{B}} \int_\pi \mathcal{S}^\pi_{d_0+\mathrm{O(B)}}(\Phi_2(s)) \ll_{\pi_2,\mathrm{B}} 1,$$

the latter inequality following from (4.24).

    We shall apply the results of Section 4.4.1 especially (4.23). In the notations of that section, we utilize the corollary to Lemma 3.7.1[32] to see—since $\mathrm{C}(\chi_3) \ll_{\pi_2} \mathrm{C}(\pi_1)$ and $t \in i\mathbf{R}$—

$$\mathrm{B}_v \ll_{\epsilon,\pi_2} \mathrm{C}(\pi_{1,v})^{d'\varepsilon} \mathrm{C}(\chi_{3,v})^{-1} \left(\frac{\mathrm{C}(\pi_{1,v})}{\mathrm{C}(\chi_{3,v})} \right)^{2\theta-1} \quad (v \in \mathrm{T})$$

for some absolute $\delta > 0$, $d' > 0$. Note that u_v does not contribute, since $u_v = 1$ for $v \in \mathrm{T}$. On the other hand, we have

(**5.17**) $\mathrm{B}_v \ll \|u_v\|^{\mathrm{A}}$ for $v \in \mathrm{S}$

and some unspecified constant A: this follows from the "trivial" bound (3.36) (see the sentence following that equation).

    We take into account also the global subconvex bound of Theorem 5.1; it implies $\mathrm{L}(\overline{\pi_3} \otimes \pi_3 \otimes \pi, 1/2) \ll_\pi \mathrm{C}(\chi_3)^{1-\delta}$. Thus, by the results of Section 4.4.1,

$$\mathcal{S}^\pi_{-d_0}(\Pi_\pi(\Phi_3(t)) \ll_{\pi_2,\pi,\epsilon} \mathrm{C}(\chi_3)^{-\delta} \left(\frac{\mathrm{C}(\pi_1)}{\mathrm{C}(\chi_3)} \right)^{2\theta-1} \|u\|^{\mathrm{A}} \|\mathrm{E}\|^2_{can} \|\mathrm{E}(t)\|^2_{can}.$$

Now, $\|\mathrm{E}(t)\|_{can} = \|\mathrm{E}\|_{can} = 1$. Using again that $\mathrm{C}(\chi_3) \ll_{\pi_2} \mathrm{C}(\pi_1)$, we conclude the proof of the Lemma.   □

5.2.9. *Bounding the degenerate term*

        Lemma. — For $(s, t) \in \mathcal{N}$ we have, for any $\varepsilon > 0$

(**5.18**) $\left| \langle \Phi_2(s), \mathcal{E}_3(t) \rangle_{reg} + \langle \mathcal{E}_2(s), \Phi_3(t) \rangle_{reg} \right| \ll_{\pi_2,\epsilon,\varepsilon} \mathrm{Q}^\varepsilon |z|^{-4} \|\mathrm{Ad}(u)\|^{-\gamma}$

for some absolute $\gamma > 1/2$.

---

[32] If π_v is tempered, e.g., if π is Eisenstein or under the Ramanujan-Peterson conjecture we may, more simply, apply Lemma 3.5.3.

Proof. — We start with the first portion of degenerate term: $\langle \Phi_2(s), \mathcal{E}_3(t) \rangle_{reg}$. Let us consider the linear functional on

$$\Pi = \Pi(s,t) := \pi_2 \otimes \overline{\pi}_2(\overline{s}) \otimes \pi_3 \otimes \overline{\pi}_3(\overline{t}) \to \mathbf{C}$$

defined by the rule

$$H = H_{s,t} : (\varphi_2, \varphi_2', \varphi_3, \varphi_3') \mapsto \int_{\mathbf{X}_{PGL_2}}^{reg} \varphi_2 \varphi_2' \mathrm{Eis}(\varphi_{3,N}\varphi_{3,N}').$$

Thus, $\langle \Phi_2(s), \mathcal{E}_3(t) \rangle = H(\varphi_2^u, \overline{\varphi_2(s)}, E^u, \overline{E(t)})$; also, expanding the constant terms $\varphi_{3,N}$, $\varphi_{3,N}'$, we may express $H_{s,t}$ as a sum of four terms

$$H_{s,t} = \sum_{\pm,\pm} H_{s,t}^{\pm\pm},$$

where, if we realize π_2 in its Whittaker model and π_3 in the principal series model,

(**5.19**) $$H_{s,t}^{++}(\varphi_2^u, \overline{\varphi_2(s)}, \mathrm{Eis}(f_3)^u, \overline{\mathrm{Eis}(f_3(t))}) = c \prod_v^* H_v^{++},$$

$$c = c_{2,W}\overline{c_{2,W}(s)}|c_{3,f}|^2, \; H_v^{++} := \int_{N(F_v)\backslash PGL_2(F_v)} W_{2,v}^{u_v} \overline{W_{2,v}(s)} f_{3,v}^{u_v} \overline{f_{3,v}(t)}.$$

The other terms (e.g. $H_{s,t}^{+-}$, H_v^{+-}) are defined similarly, by introducing standard intertwining operators M (see Section 4.1.8) in front of the f_3's. The second portion of the degenerate $\langle \mathcal{E}_2(s), \Phi_3(t) \rangle_{reg}$, admits analogous expansion, of the shape $\sum_{\pm,\pm} J_{s,t}^{\pm\pm}$ and the evaluation of each $J_{s,t}^{\pm\pm}$ is entirely similar to that of $H_{s,t}^{\pm\pm}$. We will examine in detail two terms: H^{++} and J^{--}. We then complete the proof of the Lemma in Section 5.2.12.

5.2.10. *Local bounds for* H. — We shall need bounds for H_v^{++} as well as for its partial derivatives w.r.t. s, t on the unitary axes, i.e. when s, t are purely imaginary. In fact, bounds for the partial derivatives will be derived from bounds for H_v^{++} in a small neighborhood of the unitary axes; however, for clarity, we begin by explaining the bounds for H_v^{++} when s, t are on the unitary axes.

The evaluation of H_v^{++}, for $v \in R$, is the theory of local Rankin-Selberg integrals for GL$_2$ (cf. (4.27)):

(**5.20**) $$H_v^{++} = \frac{L_v(\pi_2 \otimes \tilde{\pi}_2(\overline{s}) \otimes |\cdot|^{\overline{t}}, 1)}{\zeta_v(2 + 2\overline{t})}$$

For $v \in S$, the vectors are spherical and the additive character ψ_v is unramified; we apply Lemma 3.2.7 to conclude:

(**5.21**) $$|H_v^{++}| \ll \|\mathrm{Ad}(u_v)\|^{-\gamma}, \quad \gamma = 1 - \theta > 1/2.$$

For v in T we have again by Lemma 3.2.7

$$(\mathbf{5.22}) \qquad |\mathrm{H}_v^{++}| \leqslant \|f_{3,v}\mathrm{W}_{2,v}\|_{\mathrm{N}_v\backslash\mathrm{G}_v} \|f_{3,v}(t)\mathrm{W}_{2,v}(s))\|_{\mathrm{N}_v\backslash\mathrm{G}_v}$$

$$= \|\mathrm{W}_{2,v}\|\|\mathrm{W}_{2,v}(s)\|\|f_{3,v}\|\|f_{3,v}(t)\| = \|f_{3,v}\|^2\|\mathrm{W}_{2,v}\|\|\mathrm{W}_{2,v}(s)\| = 1.$$

More generally, for any fixed $i, j \geqslant 0$, $v \in \mathrm{S} \cup \mathrm{T}$ and any $\varepsilon > 0$, we have also

$$(\mathbf{5.23}) \qquad |\partial_s^i \partial_t^j \mathrm{H}_v^{++}| \ll_{i,j,\varepsilon} \mathrm{C}(\pi_{2,v})^\varepsilon \|\mathrm{Ad}(u_v)\|^{\varepsilon-\gamma};$$

It is to verify (5.23) that we consider H_v^{++} off the unitary axes. The function H_v^{++} being antiholomorphic in s, t, it suffices by Cauchy's formula to bound it in a small neighbourhood of $(s, t) = (0, 0)$. The required bound follows, for $v \in \mathrm{S}$, from Lemma 3.2.8; for $v \in \mathrm{T}$, it follows by a reasoning similar to the previous one that for $|\Re(s)| + |\Re(t)| \leqslant \varepsilon/2$,

$$|\mathrm{H}_v^{++}(s, t)|^2 \leqslant \int |\mathrm{W}_{2,v}(s)(a(y))|^2 \max(|y|, |y|^{-1})^\varepsilon d^\times y \ll_\varepsilon \mathcal{S}_{d\varepsilon}^{\pi_{2,v}}(\mathrm{W}_{2,v})^2$$

$$\ll_\varepsilon \mathrm{C}(\pi_{2,v})^{d'\varepsilon},$$

where we applied Proposition 3.2.4, and d, d' are absolute constants; we also used the bound for $\mathcal{S}(\mathrm{W}_{2,v})$ given in Proposition 3.6.1.

The same bounds apply to all the terms $\mathrm{H}_v^{\pm,\pm}$. We give an example of how to handle the intertwining operators that intervene, in relation to the J-term.

5.2.11. *The* J *terms.* — For the corresponding terms $\mathrm{J}_v^{\pm\pm}$ a bound similar to (5.23) applies. We shall study the term J_v^{--} to make this reasoning clear; it will also give us the opportunity to explain the reasoning involving intertwining operators.

The second degenerate term $\langle \mathcal{E}_2(s), \Phi_3(t) \rangle_{reg}$ is zero unless π_2 is Eisenstein, so we shall suppose that π_2 is Eisenstein. Corresponding to (5.19) we have the identity:

$$\mathrm{J}_{s,t}^{--}(\varphi_2^u, \overline{\varphi_2(s)}, \mathrm{Eis}(f_3)^u, \overline{\mathrm{Eis}(f_3(t))}) = c' \prod_v^* \mathrm{J}_v^{--},$$

$$|c'| = |c_2\overline{c_2}c_{3,\mathrm{W}}\overline{c_{3,\mathrm{W}}(t)}|, \quad \mathrm{J}_v^{--} := \int_{\mathrm{N}(\mathrm{F}_v)\backslash\mathrm{PGL}_2(\mathrm{F}_v)} \mathrm{W}_{3,v}^{u_v} \overline{\mathrm{W}_{3,v}(t)} \bar{\mathrm{M}}_v f_{2,v}^{u_v} \overline{\bar{\mathrm{M}}_v f_{2,v}(s)};$$

let us remind the reader that $f_2 = \otimes_v f_{2,v}$ is so that $\varphi_2 = \mathrm{Eis}(f_2)$, $\varphi_2(s) = \mathrm{Eis}(f_2(s))$ and that $\bar{\mathrm{M}}_v$ is as defined in Section 4.1.8. Recall also that $\bar{\mathrm{M}}_v$ is isometric for every v (at least up to a factor depending only on ψ_v).

The unramified evaluation is similar to the prior one.

For $v \in \mathrm{S}$, the vectors are spherical and the additive character ψ_v is unramified; we apply Lemma 3.2.8 to conclude:

$$(\mathbf{5.24}) \qquad |\mathrm{J}_v^{--}| \ll \|\mathrm{Ad}(u_v)\|^{-\gamma}, \quad \gamma = 1 - \theta > 1/2.$$

For v in T we have again by Lemma 3.2.7

$$|J_v^{--}| \leqslant 1.$$

We need, again, a bound on derivatives. For fixed $i, j \geqslant 0$, $v \in S \cup T$ and any $\varepsilon > 0$, we have also

$$(\mathbf{5.25}) \qquad |\partial_s^i \partial_t^j J_v^{--}| \ll_{i,j,\varepsilon} \left(C(\pi_{1,v}) C(\pi_{2,v}) \right)^\varepsilon \|\mathrm{Ad}(u_v)\|^{\varepsilon - \gamma}.$$

For this, just as before, we bound J_v^{--} for (s, t) in a small neighbourhood of $(0, 0)$; assuming that $|\Re(s)| + |\Re(t)| \leqslant \varepsilon/2$, we find, for $v \in T$, using Cauchy-Schwarz and Lemma 3.2.7 that

$$|J_v^{--}|^2 \leqslant \int_{F_v^\times \times K} |W_{3,v}(a(y)k)|^2 \max(|y|, |y|^{-1})^\varepsilon |\bar{M}_v f_{2,v,s}(k)|^2 d^\times y \, dk.$$

The bound (5.25) follows the bounds for $\bar{M}_v$ furnished in Section 4.1.8 and the same reasoning as at the end of Section 5.2.10. For $v \in S$ the bound follows from Lemma 3.2.8.

5.2.12. *Putting it all together.* — The degenerate term

$$\langle \Phi_2(s), \mathcal{E}_3(t) \rangle_{reg} + \langle \mathcal{E}_2(s), \Phi_3(t) \rangle_{reg} \quad (s, t) \in \mathcal{N}$$

is given by $\sum_{\pm,\pm} H_{s,t}^{\pm,\pm} + J_{s,t}^{\pm\pm}$ and, although the individual terms $H_{s,t}^{\pm\pm}, J_{s,t}^{\pm\pm}$ may not be regular in a neighbourhood of $(s, t) = (0, 0)$, we have seen in the lemma preceding Section 5.2.8 that their sum is. In particular,

$$t \mapsto \langle \Phi_2(t/2), \mathcal{E}_3(t) \rangle_{reg} + \langle \mathcal{E}_2(t/2), \Phi_3(t) \rangle_{reg}$$

defines an antiholomorphic function in the complex disc $|t| < 0.1$.

Each quantity $H_{t/2,t}^{\pm\pm}, J_{t/2,t}^{\pm\pm}$ is the product of: a constant c as in (5.19); a partial L-function at places outside $S \cup T$, given e.g. by (5.20); and local factors at $S \cup T$.

- The constant c satisfies $c \ll_{\pi_2, \varepsilon} C(\pi_1)^\varepsilon$ by (5.13) and (5.14).
- The local factors at $S \cup T$, for $t \in i\mathbf{R}$, $|t| < 0.1$, are bounded, along with their derivatives by (5.21), (5.22) and (5.23) and the corresponding bounds for J. In particular, the product of such factors is bounded by $Q^\varepsilon \|\mathrm{Ad}(u)\|^{\varepsilon - \gamma}$.
- The partial L-functions in question extend to meromorphic functions of the t-variable with a pole at $t = 0$ of order at most 4. For $t \in i\mathbf{R}$ bounded away from 0 these L-functions are bounded by

$$\ll_\varepsilon |z|^{-4} C(\pi_2)^\varepsilon C(\pi_3)^\varepsilon$$

  for any $\varepsilon > 0$, and at $t = 0$ the terms of bounded order in their Laurent expansion satisfies the same bound.

It follows that for $(s, t) \in \mathcal{N}$, the degenerate contribution is bounded by

$$\ll_\varepsilon Q^\varepsilon |z|^{-4} \|\mathrm{Ad}(u)\|^{-\gamma}, \ \gamma > 1/2,$$

as required.   □

This lemma together with the lemma of Section 5.2.8 conclude the proof of Proposition 5.2.4, hence of Theorem 1.2.

Acknowledgements

REFERENCES

1. J. ARTHUR, Eisenstein series and the trace formula, in *Automorphic Forms, Representations and* L*-functions, Part 1 (Proc. Sympos. Pure Math., XXXIII, Oregon State Univ., Corvallis, Ore., 1977)*, pp. 253–274, Am. Math. Soc., Providence, 1979.

2. J. ARTHUR, A trace formula for reductive groups. II. Applications of a truncation operator. *Compos. Math.*, **40** (1980), 87–121.

3. I. N. BERNŠTEĬN, All reductive 𝔭-adic groups are of type I. *Funkc. Anal. Prilozh.*, **8** (1974), 3–6.

4. J. Bernstein and A. Reznikov, Sobolev norms of automorphic functionals, *Int. Math. Res. Not.*, **40** (2002), 2155–2174.

5. J. Bernstein and A. Reznikov, Subconvexity bounds for triple L-functions and representation theory, arXiv:math/0608555v1, 2006.

6. V. Blomer, Rankin-Selberg L-functions on the critical line, *Manusc. Math.*, **117** (2005), 111–133.

7. V. Blomer and G. Harcos, The spectral decomposition of shifted convolution sums, *Duke Math. J.*, **144** (2008), 321–339.

8. V. Blomer and G. Harcos, Hybrid bounds for twisted L-functions, *J. Reine Angew. Math.*, **621** (2008), 53–79.

9. V. Blomer, G. Harcos, and Ph. Michel, Bounds for modular L-functions in the level aspect, *Ann. Sci. École Norm. Supér. (4)*, **40** (2007), 697–740.

10. C. J. Bushnell and G. Henniart, An upper bound on conductors for pairs, *J. Number Theory*, **65** (1997), 183–196.

11. N. Burq, P. Gérard, and N. Tzvetkov, Restrictions of the Laplace-Beltrami eigenfunctions to submanifolds, *Duke Math. J.*, **138** (2007), 445–486.

12. D. A. Burgess, On character sums and L-series. II, *Proc. Lond. Math. Soc. (3)*, **13** (1963), 524–536.

13. M. Cowling, U. Haagerup, and R. Howe, Almost L^2 matrix coefficients, *J. Reine Angew. Math.*, **387** (1988), 97–110.

14. L. Clozel, Démonstration de la conjecture τ, *Invent. Math.*, **151** (2003), 297–328.

15. L. Clozel and E. Ullmo, Équidistribution de mesures algébriques, *Compos. Math.*, **141** (2005), 1255–1309.

16. A. Diaconu and P. Garrett, *Subconvexity bounds for automorphic L-functions for GL(2) over number fields*, preprint (2008).

17. W. Duke, J. Friedlander, and H. Iwaniec, Bounds for automorphic L-functions, *Invent. Math.*, **112** (1993), 1–8.

18. W. Duke, J. Friedlander, and H. Iwaniec, Bounds for automorphic L-functions. II, *Invent. Math.*, **115** (1994), 219–239.

19. W. Duke, J. B. Friedlander, and H. Iwaniec, The subconvexity problem for Artin L-functions, *Invent. Math.*, **149** (2002), 489–577.

20. M. Einsiedler, E. Lindenstrauss, Ph. Michel, and A. Venkatesh, The distribution of periodic torus orbits on homogeneous spaces: Duke's theorem for cubic fields, *Ann. Math.*, to appear (2007), arXiv:0903.3591.

21. M. Einsiedler, E. Lindenstrauss, Ph. Michel, and A. Venkatesh, Distribution of periodic torus orbits on homogeneous spaces I, *Duke Math. J.*, **148** (2009), 119–174.

22. É. Fouvry and H. Iwaniec, A subconvexity bound for Hecke L-functions, *Ann. Sci. École Norm. Supér. (4)*, **34** (2001), 669–683.

23. J. Friedlander and H. Iwaniec, A mean-value theorem for character sums, *Mich. Math. J.*, **39** (1992), 153–159.

24. J. Hoffstein and P. Lockhart, Coefficients of Maass forms and the Siegel zero, with an appendix by Dorian Goldfeld, Hoffstein and Daniel Lieman, *Ann. Math. (2)*, **140** (1994), 161–181.

25. S. Gelbart and H. Jacquet, A relation between automorphic representations of GL(2) and GL(3), *Ann. Sci. École Norm. Supér. (4)*, **11** (1978), 471–542.

26. S. Gelbart and H. Jacquet, Forms of GL(2) from the analytic point of view, in *Automorphic forms, representations and L-functions, Part 1 (Proc. Sympos. Pure Math., XXXIII, Oregon State Univ., Corvallis, Ore., 1977)*, pp. 213–251, Am. Math. Soc., Providence, 1979.

27. A. Good, The square mean of Dirichlet series associated with cusp forms, *Mathematika*, **29** (1982), 278–295.

28. A. Gorodnik, F. Maucourant, and H. Oh, Manin's and Peyre's conjectures on rational points and adelic mixing, *Ann. Sci. École Norm. Supér. (4)*, **41** (2008), 383–435.

29. D. R. Heath-Brown, Hybrid bounds for Dirichlet L-functions, *Invent. Math.*, **47** (1978), 149–170.

30. D. R. Heath-Brown, *Convexity bounds for L-function*, preprint (2008), arXiv:0809.1752.

31. G. Harcos and Ph. Michel, The subconvexity problem for Rankin-Selberg L-functions and equidistribution of Heegner points. II, *Invent. Math.*, **163** (2006), 581–655.

32. A. Ichino, Trilinear forms and the central values of triple product L-functions, *Duke Math. J.*, **145** (2008), 281–307.

33. A. Ichino and T. Ikeda, On the periods of automorphic forms on special orthogonal groups and the Gross-Prasad conjecture, *Geom. Funct. Anal.*, **19** (2010), 1378–1425.

34. A. Ivić, On sums of Hecke series in short intervals, *J. Théor. Nr. Bordx.*, **13** (2001), 453–468.

35. H. Iwaniec, The spectral growth of automorphic L-functions, *J. Reine Angew. Math.*, **428** (1992), 139–159.

36. H. Iwaniec, Harmonic analysis in number theory, in *Prospects in Mathematics (Princeton, NJ, 1996)*, pp. 51–68, Am. Math. Soc., Providence, 1999.

37. H. Iwaniec and P. Sarnak, Perspectives on the analytic theory of L-functions, GAFA 2000 (Tel Aviv, 1999), *Geom. Funct. Anal.* (2000), Special Volume, pp. 705–741.

38. H. JACQUET and R. P. LANGLANDS, *Automorphic Forms on* GL(2), Lecture Notes in Mathematics, vol. 114, Springer, Berlin, 1970.

39. H. JACQUET, I. I. PIATETSKI-SHAPIRO, and J. SHALIKA, Conducteur des représentations du groupe linéaire, *Math. Ann.*, **256** (1981), 199–214.

40. M. JUTILA, The twelfth moment of central values of Hecke series, *J. Number Theory*, **108** (2004), 157–168.

41. M. JUTILA and Y. MOTOHASHI, Uniform bound for Hecke L-functions, *Acta Math.*, **195** (2005), 61–115.

42. H. KIM, Functoriality for the exterior square of GL$_4$ and the symmetric fourth of GL$_2$, with appendix 1 by Dinakar Ramakrishnan and appendix 2 by Kim and Peter Sarnak, *J. Am. Math. Soc.*, **16** (2003), 139–183.

43. E. KOWALSKI, Ph. MICHEL, and J. VANDERKAM, Rankin-Selberg L-functions in the level aspect, *Duke Math. J.*, **114** (2002), 123–191.

44. N. V. KUZNETSOV, Sums of Kloosterman sums and the eighth power moment of the Riemann zeta-function, in *Number Theory and Related Topics (Bombay, 1988)*, Tata Inst. Fund. Res. Stud. Math., vol. 12, pp. 57–117, Tata Inst. Fund. Res., Bombay, 1989.

45. J. LIU and Y. YE, Subconvexity for Rankin-Selberg L-functions of Maass forms, *Geom. Funct. Anal.*, **12** (2002), 1296–1323.

46. H. Y. LOKE, Trilinear forms of $\mathfrak{gl}_2$, *Pac. J. Math.*, **197** (2001), 119–144.

47. W. LUO, Z. RUDNICK, and P. SARNAK, On the generalized Ramanujan conjecture for GL(n), in *Automorphic Forms, Automorphic Representations, and Arithmetic (Fort Worth, TX, 1996)*, Proc. Sympos. Pure Math., vol. 66, pp. 301–310, Am. Math. Soc., Providence, 1999.

48. T. MEURMAN, On the order of the Maass L-function on the critical line, in *Number Theory, Vol. I (Budapest, 1987)*, Colloq. Math. Soc. János Bolyai, vol. 51, pp. 325–354, North-Holland, Amsterdam, 1990.

49. Ph. MICHEL, The subconvexity problem for Rankin-Selberg L-functions and equidistribution of Heegner points, *Ann. Math. (2)*, **160** (2004), 185–236.

50. Ph. MICHEL, Analytic number theory and families of automorphic L-functions, in *Automorphic Forms and Applications (Park City, UT, 2002)*, IAS/Park City Math. Ser., vol. 12, pp. 179–296, Am. Math. Soc., Providence, 2007.

51. Ph. MICHEL and A. VENKATESH, Equidistribution, L-functions and ergodic theory: on some problems of Yu. Linnik, in *International Congress of Mathematicians, vol. II*, pp. 421–457, Eur. Math. Soc., Zürich, 2006.

52. C. MOEGLIN and J.-L. WALDSPURGER, *Spectral Decomposition and Eisenstein Series*, Cambridge Tracts in Mathematics, vol. 113, Cambridge University Press, Cambridge, 1995. Une paraphrase de l'Écriture [A paraphrase of Scripture].

53. Y. MOTOHASHI, *Spectral Theory of the Riemann Zeta-Function*, Cambridge Tracts in Mathematics, vol. 127, Cambridge University Press, Cambridge, 1997.

54. Y. MOTOHASHI, A functional equation for the spectral fourth moment of modular Hecke L-functions, in *Proceedings of the Session in Analytic Number Theory and Diophantine Equations*, Bonner Math. Schriften, vol. 360, p. 19, 2003.

55. W. MÜLLER, The trace class conjecture in the theory of automorphic forms II, *Geom. Funct. Anal.*, **8** (1998), 315–355.

56. H. OH, Uniform pointwise bounds for matrix coefficients of unitary representations and applications to Kazhdan constants, *Duke Math. J.*, **113** (2002), 133–192.

57. A. I. OKSAK, Trilinear Lorentz invariant forms, *Commun. Math. Phys.*, **29** (1973), 189–217.

58. D. PRASAD, Trilinear forms for representations of GL(2) and local ϵ-factors, *Compos. Math.*, **75** (1990), 1–46.

59. A. REZNIKOV, *Norms of geodesic restrictions for eigenfunctions on hyperbolic surfaces and representation theory*, preprint, 2004, arXiv:math/0403437v2.

60. A. REZNIKOV, Rankin-Selberg without unfolding and bounds for spherical Fourier coefficients of Maass forms, *J. Am. Math. Soc.*, **21** (2008), 439–477.

61. Y. SAKELLARIDIS and A. VENKATESH, *Periods and harmonic analysis on spherical varieties*, preprint, 2010.

62. P. SARNAK, L-functions, in *Proceedings of the International Congress of Mathematicians, vol. I* (*Berlin, 1998*), Documenta Mathematica (Extra volume), pp. 453–465.

63. P. SARNAK, Estimates for Rankin-Selberg L-functions and quantum unique ergodicity, *J. Funct. Anal.*, **184** (2001), 419–453.

64. A. VENKATESH, Large sieve inequalities for GL(n)-forms in the conductor aspect, *Adv. Math.*, **200** (2006), 336–356.

65. A. VENKATESH, Sparse equidistribution problems, period bounds, and subconvexity, *Ann. Math.*, to appear, 2006.

66. A. VENKATESH, Notes on effective equidistribution, *Pisa/CMI summer school 2007*, unpublished, 2007.

67. J.-L. WALDSPURGER, Sur les valeurs de certaines fonctions L automorphes en leur centre de symétrie, *Compos. Math.*, **54** (1985), 173–242.

68. H. Weyl, Zur abschätzung von $\zeta(1 + it)$, *Math. Z.*, **10** (1921), 88–101.

69. D. Zagier, The Rankin-Selberg method for automorphic functions which are not of rapid decay, *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, **28** (1981), 415–437.

P. M.
Ecole Polytechnique Fédérale de Lausanne,
Lausanne, Switzerland
philippe.michel@epfl.ch

and

Université Montpellier II,
Montpellier, France

A. V.
Stanford University,
Stanford, USA
akshay@math.stanford.edu