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The One-Dimensional Schrödinger–Newton Equations

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Abstract. We prove an existence and uniqueness result for ground states of one-dimensional Schrödinger–Newton equations.

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1. Introduction

We consider the one-dimensional (1D) Schrödinger–Newton system

$$iu_t + u_{xx} - \gamma V u = 0, \quad V_{xx} = |u|^2 \tag{1.1}$$

which is equivalent to the nonlinear Schrödinger equation

$$iu_t + u_{xx} - \frac{\gamma}{2}(|x| * |u|^2)u = 0 \tag{1.2}$$

with nonlocal nonlinear potential

$$V(x) = \frac{1}{2}(|x| * |u|^2)(x,t) = \frac{1}{2}\int_{\mathbb{R}} |x-y||u(t,y)|^2 \,\mathrm{d}y.$$

We are interested in the existence of nonlinear bound states of the form

$$u(t,x) = \phi_\omega(x)\mathrm{e}^{-\mathrm{i}\omega t}. \tag{1.3}$$

The Schrödinger–Newton system in three space dimensions

$$iu_t + \Delta u - \gamma V u = 0, \quad \Delta V = |u|^2 \tag{1.4}$$

has a longstanding history. With γ designating appropriate positive coupling constants it appeared first in 1954, then in 1976 and lastly in 1996 for describing the quantum mechanics of a Polaron at rest by Pekar [1], of an electron trapped in its own hole by the first author [2] and of selfgravitating matter by Penrose [3]. The two-dimensional model is studied numerically in [4]. For the bound state problem there are rigorous results only for the three dimensional model. In [2] the existence of a unique ground state of the form (1.3) is shown by solving an appropriate minimization problem. This ground state solution $\phi_\omega(x), \omega < 0$ is a positive spherically symmetric strictly decreasing function. In [5] the existence of infinitely many distinct spherically symmetric solutions is proven and, in [6], a proof for the existence of anisotropic bound states is claimed. So far, there are no results for the 1D model except for its semiclassical approximation [7]. One mathematical difficulty of the 1D problem is that the Coulomb potential does not define a positive definite quadratic form (see below).

From numerical investigations of the problem we conjecture that in the attractive case $\gamma > 0$ equation (1.2) admits for each $\omega > 0$ infinitely many nonlinear bound states of the form (1.3) which means that subject to a normalization condition $\int_{\mathbb{R}} |u(t,x)|^2 \, \mathrm{d}x = N$ the model exhibits an infinite discrete energy spectrum. In the present letter, we are interested in the ground states of the model

$$u(t,x) = \phi_\omega(x) \mathrm{e}^{-i\omega t}, \quad \phi_\omega(x) > 0. \tag{1.5}$$

We prove for any $\omega > 0$ the existence of an unique spherically symmetric ground state by solving an appropriate minimization problem. We also prove the existence of an antisymmetric solution by solving the same minimization problem restricted to the class of antisymmetric functions.

2. Mathematical Framework

2.1. FUNCTIONAL SETTING

The natural function space X for the quasi-stationary problem is given by

$$X = \left\{ u : \mathbb{R} \to \mathbb{C} : \int_{\mathbb{R}} |u_x|^2 + |u|^2 + |x||u|^2 \, \mathrm{d}x < \infty \right\}. \tag{2.1}$$

Indeed, for each $u \in X$ the energy E and the particle number (or charge) N associated to (1.2) given by

$$\begin{aligned} E(u) &= \int_{\mathbb{R}} |u_x(x)|^2 \, \mathrm{d}x + \frac{\gamma}{4} \int_{\mathbb{R}} \int_{\mathbb{R}} |x-y||u(x)|^2 |u(y)|^2 \, \mathrm{d}x \mathrm{d}y \\ &= T(u) + \frac{\gamma}{2} V(u) \end{aligned} \tag{2.2}$$

and

$$N(u) = \int_{\mathbb{R}} |u(x)|^2 \, \mathrm{d}x, \tag{2.3}$$

respectively, are well-defined quantities. In particular, the energy functional $E : X \longrightarrow \mathbb{R}_0^+$ is of class C^1.

The space X is a Hilbert space and by Rellich's criterion (see, e.g. Theorem XIII.65 of [8]) the embedding $X \hookrightarrow L^2$ is compact.

2.2. SCALING PROPERTIES

If $\phi_\omega(x)$ is a solution of the stationary equation

$$-\phi_\omega''(x) + \frac{\gamma}{2}\left(\int_{\mathbb{R}} |x-y||\phi_\omega(y)|^2\,\mathrm{d}y\right)\phi_\omega(x) = \omega\phi_\omega(x), \tag{2.4}$$

then $\phi_1(x) = \omega^{-1}\phi_\omega(x/\omega^{1/2})$ solves

$$-\phi_1''(x) + \frac{\gamma}{2}\left(\int_{\mathbb{R}} |x-y||\phi_1(y)|^2\,\mathrm{d}y\right)\phi_1(x) = \phi_1(x) \tag{2.5}$$

and

$$E(\phi_\omega) = \omega^{5/2}E(\phi_1), \quad N(\phi_\omega) = \omega^{3/2}N(\phi_1). \tag{2.6}$$

In addition, by the virial theorem

$$4\omega N(\phi_\omega) = 20\,T(\phi_\omega) = 5\gamma\,V(\phi_\omega). \tag{2.7}$$

3. Ground States

3.1. EXISTENCE OF GROUND STATES

We consider the following minimization problem:

$$e_0(\lambda) = \inf\{E(u), u \in X, N(u) = \lambda\}. \tag{3.1}$$

We note that the functional $u \to E(u)$ is not convex since the quadratic form $f \to \int_{\mathbb{R}}\int_{\mathbb{R}} |x-y| f(x)\bar{f}(y)\,dxdy$ is not positive so that standard convex minimization does not apply. To see this choose, for example, $f(x) = \xi_{[0,1]}(x) - \xi_{[1,2](x)}$ where $\xi_{[a,b]}(x)$ denotes the characteristic function of the interval $[a, b]$. For finite discrete systems it has been shown that the associated matrix has only one positive eigenvalue [9], which was computed in [10].

THEOREM 3.1. *For any $\lambda > 0$ there is a spherically symmetric decreasing $u_\lambda \in X$ such that $e_0(\lambda) = E(u_\lambda)$ and $N(u_\lambda) = \lambda$.*

Proof. Let $(u_n)_n$ be a minimizing sequence for $e_0(\lambda)$, that is $N(u_n) = \lambda$ and $\lim_{n\longrightarrow\infty} E(u_n) = e_0(\lambda)$. We also may assume that $|E(u_n)|$ is uniformly bounded.

Denoting u^* the spherically symmetric-decreasing rearrangement of u we have (see e.g. Lemma 7.17 in [11])

$$T(u^*) \leq T(u), \quad N(u^*) = N(u).$$

For the potential $V(u)$ we apply the following rearrangement inequality:

LEMMA 3.2. *Let f, g be two nonnegative functions on $\mathbb{R}$, vanishing at infinity with spherically symmetric-decreasing rearrangement f^*, g^*, respectively. Let v be a nonnegative spherically symmetric increasing function. Then*

$$\int_{\mathbb{R}} \int_{\mathbb{R}} f(x) v(x-y) g(y)\, \mathrm{d}x \mathrm{d}y \geq \int_{\mathbb{R}} \int_{\mathbb{R}} f^*(x) v(x-y) g^*(y)\, \mathrm{d}x \mathrm{d}y \tag{3.2}$$

Proof. If v is bounded, $v \leq C$, then $(C-v)^* = C - v$ and by Riesz's rearrangement inequality (Lemma 3.6 in [11]) we have

$$\int_{\mathbb{R}} \int_{\mathbb{R}} f(x)(C - v(x-y)) g(y)\, \mathrm{d}x \mathrm{d}y \leq \int_{\mathbb{R}} \int_{\mathbb{R}} f^*(x)(C - v(x-y)) g^*(y)\, \mathrm{d}x \mathrm{d}y.$$

Since

$$\int_{\mathbb{R}} f(x)\, \mathrm{d}x \int_{\mathbb{R}} g(y)\, \mathrm{d}y = \int_{\mathbb{R}} f^*(x)\, \mathrm{d}x \int_{\mathbb{R}} g^*(y)\, \mathrm{d}y$$

the claim follows. If v is unbounded we define a truncation by $v_n(x) = \sup(v(x), n)$ and apply the monotone convergence theorem. □

By the preceding lemma we have

$$V(u^*) \leq V(u)$$

since $|x|$ is an increasing spherically symmetric function. Therefore we may suppose that $u_n = u_n^*$. We claim that $u_n^* \in X$. Indeed, since $|x|$ is a convex function we have

$$V(u) \geq \frac{1}{2} \int_{\mathbb{R}} \left| \int_{\mathbb{R}} (x-y)|u(y)|^2\, \mathrm{d}y \right| |u(x)|^2\, \mathrm{d}x$$

by Jensen's inequality and therefore

$$E(u_n^*) \geq T(u_n^*) + \lambda \frac{\gamma}{4} \int_{\mathbb{R}} |x||u_n^*|^2\, \mathrm{d}x$$

proving our claim. We may extract a subsequence which we denote again by $(u_n^*)_n$ such that $u_n^* \to u^*$ weakly in X, strongly in L^2 and a.e. where $u^* \in X$ is a nonnegative spherically symmetric decreasing function. Note that $u^* \neq 0$ since $N(u^*) = \lambda$. We want to show that $E(u^*) \leq \liminf_{n \longrightarrow \infty} E(u_n^*)$. Since

$$T(u^*) \leq \liminf_{n \longrightarrow \infty} T(u_n^*)$$

it remains to analyze the functional $V(u)$. First of all, we note that for spherically symmetric densities $|u(x)|^2$ we have

$$V(u)=\int_{\mathbb{R}} |x||u(x)|^2\left(\int_{-|x|}^{|x|} |u(y)|^2\,\mathrm{d}y\right)\mathrm{d}x.$$

Let

$$\eta(x)=\int_{-|x|}^{|x|} |u^*(y)|^2\,\mathrm{d}y, \quad \eta_n(x)=\int_{-|x|}^{|x|} |u_n^*(y)|^2\,\mathrm{d}y.$$

Then $\eta_n(x)\to\eta(x)$ uniformly since

$$||\eta_n(x)-\eta(x)||_\infty \le ||u_n^*-u^*||_2\,||u_n^*+u^*||_2 \le 2\sqrt{\lambda}||u_n^*-u^*||_2.$$

Now

$$V(u_n^*)-V(u^*)=\int_{\mathbb{R}}|x||u_n^*(x)|^2\big(\eta_n(x)-\eta(x)\big)\,\mathrm{d}x+\int_{\mathbb{R}}|x|\eta(x)\big(|u_n^*(x)|^2-|u^*(x)|^2\big)\,\mathrm{d}x$$

As $n\to\infty$ the first integral will tend to zero while the second will remain nonnegative since the continuous functional $\phi\to\int_{\mathbb{R}}|x|\eta(x)|\phi(x)|^2\,\mathrm{d}x$ is positive. Hence

$$V(u^*)\le \liminf_{n\longrightarrow\infty} V(u_n^*)$$

proving the theorem. □

3.2. UNIQUENESS OF GROUND STATES

As in [2] we need a strict version of the rearrangement inequality for the potential energy $V(u)$:

LEMMA 3.3. *If* $u\in X$ *and* $u^*(x)\notin\{\mathrm{e}^{\mathrm{i}\theta}u(x-a):\theta,a\in\mathbb{R}\}$, *then we have the strict inequality*:

$$V(u)>V(u^*) \tag{3.3}$$

Proof. We write $|x|=-\frac{1}{1+|x|}+\frac{|x|^2+|x|+1}{1+|x|}=-g(x)+(|x|+g(x))$ where $g(x)$ is a spherically symmetric strictly decreasing function and $g(x)+|x|$ is increasing. Then, from the strict inequality for strictly decreasing functions (see [2]) we have $V(u)>V(u^*)$. □

After suitable rescaling the solution of the minimization problem (3.6) satisfies the stationary equation (2.5) which is equivalent to the system of ordinary differential equations

$$-\phi''+\gamma V\phi=\phi, \quad V''=\phi^2. \tag{3.4}$$

Obviously, $\phi(x)>0$ for all x and after another rescaling we may assume that the pair (ϕ, V) satisfies the initial conditions $\phi(0)>0, \phi'(0)=0, V(0)=V'(0)=0$. System (3.4) is Hamiltonian with energy function given by

$$\mathcal{E}(\phi, \phi', V, V') = \phi'^2 + \phi^2 + \frac{\gamma}{2} V'^2 - \gamma V \phi^2 \tag{3.5}$$

and $\mathcal{E} = \phi^2(0)$ for any symmetric solution.

THEOREM 3.4. *The system* (3.4) *admits a unique symmetric solution* (ϕ, V) *such that* $\phi>0$ *and* $\phi\to 0$ *as* $|x|\to\infty$.

Proof. Suppose there are two distinct solutions (u_1, V_1), (u_2, V_2) having the required properties. We may suppose $u_2(0) > u_1(0)$. For $x\geq 0$ we consider the Wronskian

$$w(x) = u_2'(x)u_1(x) - u_1'(x)u_2(x).$$

Note that $w(0)=0$ and $w(x)\to 0$ as $x\to\infty$. It satisfies the differential equation

$$w' = \gamma(V_2 - V_1)u_1u_2.$$

Suppose $u_2(x) > u_1(x)$ for all $x\geq 0$. Then $V_2(x) > V_1(x)$ for all $x\geq 0$ since $(V_2-V_1)''=u_2^2-u_1^2>0$ and hence $w'(x)>0$ for all $x>0$ which is impossible. Hence there exists $\bar{x}>0$ such that $\delta(x) = u_2(x) - u_1(x) > 0$ for $x\in[0,\bar{x}[$, $\delta(\bar{x})=0$ and $\delta'(\bar{x})<0$. However, then $w(\bar{x}) = \delta'(\bar{x})u_1(\bar{x}) < 0$, but $w'(x)>0$ for all $x<\bar{x}$ which is again impossible. □

3.3. EXISTENCE OF ANTISYMMETRIC GROUND STATES

We consider the subspace X^{as} of X consisting of antisymmetric functions, i.e. of functions u such that $u(-x)=-u(x)$. Repeating the arguments of the Proof of Theorem 2.1 we prove the existence of a solution of the minimization problem

$$e_1(\lambda) = \inf\{E(u), u\in X^{as}, N(u)=\lambda\} \tag{3.6}$$

which we conjecture to be the first excited state.

THEOREM 3.5. *For any* $\lambda>0$ *there is an antisymmetric* $v_\lambda\in X$, *positive for* $x>0$ *such that* $e_1(\lambda)=E(v_\lambda)$ *and* $N(v_\lambda)=\lambda$.

Proof. We may restrict the problem to the positive half-axis with Dirichlet boundary conditions. Then

$$
\begin{aligned}
E(u) &= 2\int_0^\infty |u_x(t,x)|^2\,\mathrm{d}x + \frac{\gamma}{2}\int_0^\infty\int_0^\infty (|x-y|+|x+y|)|u(t,x)|^2|u(t,y)|^2\,\mathrm{d}x\mathrm{d}y \\
&= 2\int_0^\infty |u_x(t,x)|^2\,\mathrm{d}x + \frac{\gamma}{2}\int_0^\infty\int_0^\infty |x-y||u(t,x)|^2|u(t,y)|^2\,\mathrm{d}x\mathrm{d}y + \\
&\quad + \gamma\int_0^\infty |u(t,x)|^2\,\mathrm{d}x \int_0^\infty |x||u(t,x)|^2\,\mathrm{d}x.
\end{aligned}
$$

Let $(u_n)_n$ be a minimizing sequence for $e(\lambda)$, that is $N(u_n)=\lambda$ and $\lim_{n\longrightarrow\infty} E(u_n)=e_1(\lambda)$. We may suppose that the u_n are nonnegative on the positive half-axis. The rest of the proof follows the same lines as the Proof of Theorem 3.1. □

Remark 3.6. As in Theorem 3.4 we can show that the odd solution $\phi(x)$ of (3.4) such that $\phi(x)>0$ for all $x>0$ which corresponds to the initial conditions $\phi(0)=0, \phi'(0)>0, V(0)=V'(0)=0$ is unique.

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