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# Constructive Logic with Strong Negation is a Substructural Logic. II

**Abstract.** The goal of this two-part series of papers is to show that constructive logic with strong negation  $\mathbf{N}$  is definitionally equivalent to a certain axiomatic extension  $\mathbf{NFL}_{ew}$  of the substructural logic  $\mathbf{FL}_{ew}$ . The main result of Part I of this series [41] shows that the equivalent variety semantics of  $\mathbf{N}$  (namely, the variety of Nelson algebras) and the equivalent variety semantics of  $\mathbf{NFL}_{ew}$  (namely, a certain variety of  $\mathbf{FL}_{ew}$ -algebras) are term equivalent. In this paper, the term equivalence result of Part I [41] is lifted to the setting of deductive systems to establish the definitional equivalence of the logics  $\mathbf{N}$  and  $\mathbf{NFL}_{ew}$ . It follows from the definitional equivalence of these systems that constructive logic with strong negation is a substructural logic.

*Keywords:* Constructive logic, strong negation, substructural logic, Nelson algebra,  $\mathcal{FL}_{ew}$ -algebra, residuated lattice.

## 1. Introduction

Let  $\Sigma[\mathbf{IPC}]$  denote the Hilbert-style presentation of Blok and Pigozzi [6, Example 2.2.2] of the intuitionistic propositional calculus  $\mathbf{IPC}$  over the language type  $\Lambda[\mathbf{IPC}] := \{\wedge, \vee, \rightarrow, \neg, \mathbf{0}, \mathbf{1}\}$ , where  $\wedge, \vee, \rightarrow$  are binary logical connectives,  $\neg$  is a unary logical connective, and  $\mathbf{0}$  and  $\mathbf{1}$  are nullary logical connectives respectively. *Constructive logic with strong negation*, denoted  $\mathbf{N}$ , is the deductive system over the language type  $\Lambda[\mathbf{N}] := \Lambda[\mathbf{IPC}] \cup \{\sim\}$ , where  $\sim$  is a unary logical connective (called the *strong negation*), determined by the axioms and inference rules of  $\Sigma[\mathbf{IPC}]$  together with the axioms [42]:

$$\begin{array}{ll} \sim p \rightarrow (p \rightarrow q) & \sim(p \vee q) \leftrightarrow (\sim p \wedge \sim q) \\ \sim(p \rightarrow q) \leftrightarrow (p \wedge \sim q) & \sim(\neg p) \leftrightarrow p \\ \sim(p \wedge q) \leftrightarrow (\sim p \vee \sim q) & \sim(\sim p) \leftrightarrow p. \end{array}$$

(Here we are abbreviating  $(p \rightarrow q) \wedge (q \rightarrow p)$  by  $p \leftrightarrow q$ .) By [34, Chapter XII],  $\mathbf{N}$  is strongly and regularly algebraisable in the sense of [15]. The study of constructive logic with strong negation has been pursued extensively in the literature [34, 42, 37]; for a brief discussion and overview, see Wójcicki [47, Section 5.3.0].

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Let **FL** denote the sequent system of Galatos *et al.* [19, Section 2.1.3], over the language  $\Lambda[\mathbf{FL}] := \langle \wedge, \vee, *, \backslash, /, \mathbf{0}, \mathbf{1} \rangle$  of type  $\langle 2, 2, 2, 2, 2, 0, 0 \rangle$ , obtained from the Gentzen sequent calculus **LJ** by deleting all the structural rules together with the logical rules for implication, and then adding rules for the division connectives  $\backslash$  and  $/$  and the fusion connective  $*$ .<sup>1</sup> The *full Lambek calculus*, also denoted **FL**, is the deductive system determined by the sequent system **FL** in the sense that for any set of formulas  $\Gamma \cup \{\varphi\} \subseteq \text{Fm}_{\Lambda[\mathbf{FL}]}$ ,  $\Gamma \vdash_{\mathbf{FL}} \varphi$  if and only if  $\{(\triangleright\psi) : \psi \in \Gamma\} \vdash^{\mathbf{FL}} (\triangleright\varphi)$ . (Here  $S' \vdash^{\mathbf{FL}} s$  if there is a proof in **FL** of the sequent  $s$  from the set of sequents  $S'$ , while the auxiliary symbol  $\triangleright$  denotes the separator of an arbitrary sequent  $\psi_1, \dots, \psi_n \triangleright \varphi$ .) By [18, Theorem 3.2], **FL** is strongly algebraisable in the sense of [15]. For studies of **FL**, see [30, 18, 19].

Let  $(e)$ ,  $(c)$ ,  $(i)$ , and  $(o)$  denote the structural rules of exchange, contraction, left weakening, and right weakening respectively, as given in [19, Section 2.1.1]. For  $S \subseteq \{e, c, i, o\}$ , let **FL<sub>S</sub>** denote the extension of **FL** obtained by adjoining the structural rules  $\{(s) : s \in S\}$  to **FL**. (Following the practice of [19], we abbreviate the combination  $\{i, o\} \subseteq S$  by  $w$ .) Recall that, in the presence of the exchange rule, the formulas  $\varphi \backslash \psi$  and  $\psi / \varphi$  are provably equivalent (in the sense of [19, Section 2.1.2]) for all  $\varphi, \psi \in \text{Fm}_{\Lambda[\mathbf{FL}]}$  [19, Lemma 2.3]. When  $e \in S$ , therefore, we fix the language type of **FL<sub>S</sub>** as  $\{\wedge, \vee, *, \Rightarrow, \mathbf{0}, \mathbf{1}\}$ , where  $\Rightarrow$  is a binary logical connective. Thus the *full Lambek calculus with exchange and weakening*, in symbols **FL<sub>ew</sub>**, is the deductive system over the language  $\Lambda[\mathbf{FL}_{ew}] := \langle \wedge, \vee, *, \Rightarrow, \mathbf{0}, \mathbf{1} \rangle$  of type  $\langle 2, 2, 2, 2, 0, 0 \rangle$  determined by the sequent system **FL<sub>ew</sub>** ( $= \mathbf{FL}_{eio}$ ).<sup>2</sup> By [18, Theorem 3.3, Theorem 3.4] **FL<sub>ew</sub>** is strongly and regularly algebraisable in the sense of [15]. For studies of **FL<sub>ew</sub>**, see in particular [28, 23, 29, 30, 18, 19].

The aim of this two-part series of papers is to show that constructive logic with strong negation is definitionally equivalent to the axiomatic extension **NFL<sub>ew</sub>** of the deductive system **FL<sub>ew</sub>** by the axioms

$$\begin{aligned} \sim \sim p &\Rightarrow p && \text{(Double Negation)} \\ (p \wedge (q \vee r)) &\Rightarrow ((p \wedge q) \vee (p \wedge r)) && \text{(Distributivity)} \\ (p \Rightarrow (p \Rightarrow (p \Rightarrow q))) &\Rightarrow (p \Rightarrow (p \Rightarrow q)) && \text{(3-potency)} \\ ((p \Rightarrow (p \Rightarrow q)) \wedge (\sim q \Rightarrow (\sim q \Rightarrow \sim p))) &\Rightarrow (p \Rightarrow q) && \text{(Nelson)}. \end{aligned}$$

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<sup>1</sup>Following Girard [20], throughout this paper the structural rules comprise the exchange, (left, right) weakening, and contraction rules. In particular, neither identity nor cut count as a structural rule.

<sup>2</sup>For a sequent system for **FL<sub>ew</sub>** over the language type  $\Lambda[\mathbf{FL}_{ew}]$  see Kowalski and Ono [23, Section 1, p. 9].

(Here we are abbreviating  $p \Rightarrow \mathbf{0}$  by  $\sim p$ .)

The proof of this result is in two parts, with one part per paper. In Part I of this series [41] it was shown that the equivalent variety semantics of  $\mathbf{N}$  (namely, the variety  $\mathcal{N}$  of Nelson algebras [34, Chapter V]) and the equivalent variety semantics of  $\mathbf{NFL}_{ew}$  (namely, a certain variety  $\mathcal{NFL}_{ew}$  of  $\mathbf{FL}_{ew}$ -algebras) are term equivalent. For a précis of Part I [41], see Section 2.2 below. In this paper, we lift the term equivalence result of Part I [41] to the setting of deductive systems to establish the definitional equivalence of the logics  $\mathbf{N}$  and  $\mathbf{NFL}_{ew}$ . From the definitional equivalence of these systems we obtain the desired corollary that constructive logic with strong negation is a substructural logic.

The main result of this paper is

THEOREM 1.1.

1. The map  $\delta : \Lambda[\mathbf{FL}_{ew}] \rightarrow \text{Fm}_{\Lambda}[\mathbf{N}]$  defined by

$$\begin{aligned} p \wedge q &\mapsto p \wedge q \\ p \vee q &\mapsto p \vee q \\ p * q &\mapsto \sim(p \rightarrow \sim q) \vee \sim(q \rightarrow \sim p) && (*_{\text{def}}) \\ p \Rightarrow q &\mapsto (p \rightarrow q) \wedge (\sim q \rightarrow \sim p) && (\Rightarrow_{\text{def}}) \\ \mathbf{0} &\mapsto \mathbf{0} \\ \mathbf{1} &\mapsto \mathbf{1} \end{aligned}$$

is an interpretation of  $\mathbf{NFL}_{ew}$  in  $\mathbf{N}$ .

2. The map  $\varepsilon : \Lambda[\mathbf{N}] \rightarrow \text{Fm}_{\Lambda}[\mathbf{FL}_{ew}]$  defined by

$$\begin{aligned} p \wedge q &\mapsto p \wedge q \\ p \vee q &\mapsto p \vee q \\ p \rightarrow q &\mapsto p \Rightarrow (p \Rightarrow q) && (\rightarrow_{\text{def}}) \\ \neg p &\mapsto p \Rightarrow (p \Rightarrow \mathbf{0}) && (\neg_{\text{def}}) \\ \sim p &\mapsto p \Rightarrow \mathbf{0} && (\sim_{\text{def}}) \\ \mathbf{0} &\mapsto \mathbf{0} \\ \mathbf{1} &\mapsto \mathbf{1} \end{aligned}$$

is an interpretation of  $\mathbf{N}$  in  $\mathbf{NFL}_{ew}$ .

3. The interpretations  $\delta$  and  $\varepsilon$  are mutually inverse.

Hence the deductive systems  $\mathbf{N}$  and  $\mathbf{NFL}_{ew}$  are definitionally equivalent.

A deductive system  $\mathbf{S}$  over a language type  $\Lambda$  is said to be *Fregean* if the relativised interderivability relation  $\dashv\vdash_{\mathbf{S}}^T$  ( $T$  a theory of  $\mathbf{S}$ ) is a congruence relation on the formula algebra  $\mathbf{Fm}_{\Lambda}$ . A logic  $\mathbf{S}$  is said to be *non-Fregean* if it is not Fregean. A *substructural logic over  $\mathbf{FL}_S$* ,  $S \subseteq \{e, c, i, o\}$ , is a deductive system  $\mathbf{S}$  that is definitionally equivalent to a non-Fregean extension of  $\mathbf{FL}_S$ . For a justification of this definition, see Section 3 below.

The main result of this series of papers is

**THEOREM 1.2.** *Constructive logic with strong negation is a substructural logic over  $\mathbf{FL}_{ew}$ .*

The following example illustrates Theorems 1.1 and 1.2.

**EXAMPLE 1.3.** *Classical constructive logic with strong negation*, in symbols  $\mathbf{N}_c$ , is the axiomatic extension of  $\mathbf{N}$  by the Peirce law  $((p \rightarrow q) \rightarrow p) \rightarrow p$ . Let  $N_c := \{0, a, 1\}$  and consider the operations  $\wedge, \vee, \rightarrow, \neg$ , and  $\sim$  defined on  $N_c$  by means of the following tables:

$\wedge$	0	$a$	1	$\vee$	0	$a$	1	$\rightarrow$	0	$a$	1	$\neg$		$\sim$	
0	0	0	0	0	0	$a$	1	0	1	1	1	0	1	0	1
$a$	0	$a$	$a$	$a$	$a$	$a$	1	$a$	1	1	1	$a$	1	$a$	$a$
1	0	$a$	1	1	1	1	1	1	0	$a$	1	1	0	1	0

By Rasiowa [34, Chapter V§3] the algebra  $\mathbf{N}_c := \langle N_c; \wedge, \vee, \rightarrow, \neg, \sim, 0, 1 \rangle$  is, to within isomorphism, the unique 3-element Nelson algebra, and by a well known observation of Vakarelov [42, Theorem 10],  $\mathbf{N}_c$  is the deductive system determined by the logical matrix  $\langle \mathbf{N}_c; \{\mathbf{1}^{\mathbf{N}_c}\} \rangle$ .

Let  $\mathbf{N}_c^\delta$  denote the  $\{\wedge, \vee, *, \Rightarrow, 0, 1\}$ -term reduct of  $\mathbf{N}_c$ , where  $\delta$  is the map of Theorem 1.1(1) above (more precisely, of Theorem 2.1(1) below). It is readily verified that the operations of  $\mathbf{N}_c^\delta$  have tables:

$\wedge$	0	$a$	1	$\vee$	0	$a$	1	$*$	0	$a$	1	$\Rightarrow$	0	$a$	1
0	0	0	0	0	0	$a$	1	0	0	0	0	0	1	1	1
$a$	0	$a$	$a$	$a$	$a$	$a$	1	$a$	0	0	$a$	$a$	$a$	1	1
1	0	$a$	1	1	1	1	1	1	0	$a$	1	1	0	$a$	1

From direct inspection of these tables, it is easy to see that  $\mathbf{N}_c^\delta$  is term equivalent to the unique (to within isomorphism) 3-element Wajsberg algebra  $\mathbf{WA}_2 := \langle \{0, a, 1\}; \Rightarrow, \sim, 1 \rangle$ . (For information about Wajsberg algebras, see [5, Section 1, pp. 562–564].) It follows that  $\mathbf{N}_c$  is definitionally equivalent to the deductive system determined by the logical matrix  $\langle \mathbf{WA}_2; \{\mathbf{1}^{\mathbf{WA}_2}\} \rangle$ ,

*viz.*, the three-valued logic  $\mathbf{L}_3$  of Łukasiewicz [24].<sup>3</sup> This explains the well known result of Vakarelov [42, Theorem 11] asserting that the axiomatic expansion of classical propositional logic by strong negation is definitionally equivalent to  $\mathbf{L}_3$ .<sup>4</sup> ■

The remainder of this paper is devoted to establishing Theorem 1.1. After attending to numerous preliminaries in Section 2, we give necessary and sufficient conditions in Section 3 for a deductive system to be a substructural logic over  $\mathbf{FL}_S$  (in the sense of this paper). Section 4 is devoted to establishing a sufficient condition for two regularly algebraisable deductive systems to be definitionally equivalent. This condition allows us to lift the term equivalence result of Part I [41] directly to the setting of deductive systems in this paper. In Section 5 we present a Hilbert-style axiomatisation of  $\mathbf{NFL}_{ew}$  and combine the technical results of Section 4 with the main result of Part I [41] to conclude that the deductive systems  $\mathbf{N}$  and  $\mathbf{NFL}_{ew}$  are definitionally equivalent. From the definitional equivalence of  $\mathbf{N}$  and  $\mathbf{NFL}_{ew}$ , we finally obtain the desired corollary that constructive logic with strong negation is a substructural logic.

All the proofs of Part I of this series [41], together with the proofs of two lemmas of this paper (Lemmas 5.1 and 5.5), were obtained with the assistance of the automated reasoning program PROVER9 [26], using the method of proof sketches [46]. PROVER9 is a resolution-based theorem prover for first-order logic with equality that has been shown to be particularly useful in the investigation of (quasi-) equational theories where standard semantic methods cannot readily be applied. For examples of the application of automated reasoning to a wide range of problems in equational logic, see in particular [25].

For the sake of completeness, the automated proofs for Lemmas 5.1 and 5.5 of this paper are included in Appendix A. The website accompanying this series [40] contains the full set of automated proofs supporting both this work and Part I of this series [41].

## 2. Preliminaries

In this section we fix some terminology and notation that will be used throughout this paper (Section 2.1); recapitulate the main result of Part I of

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<sup>3</sup>By Blok and Pigozzi [5, Corollary 3.9], the variety generated by  $\mathbf{WA}_2$  is a discriminator variety. Hence, this example also clarifies the characterisation of discriminator varieties of Nelson algebras given in [38, Corollary 5.3].

<sup>4</sup>This situation is called ‘strange’ by Vakarelov in [43, Section 1, p. 394].

this series [41] (Section 2.2); describe the notion of definitional equivalence exploited in this paper (Sections 2.3–2.4); and summarise some elements of the theory of regularly algebraisable logics (Sections 2.5–2.6).

## 2.1. Terminology and notation

We adhere to the terminology and notation introduced in Part I of this series [41]. In particular,  $\mathbf{X} := \{v_i : i \in \omega\}$  is a countably infinite set of *variables*. Generally we find it convenient to write  $p, q, r$  [resp.  $x, y, z$ ] etc., possibly with subscripts, as metavariables ranging over  $\mathbf{X}$  in a logical [resp. algebraic] context. As in Part I [41], for typographical convenience we often denote the application of the function  $f$  to  $a$  by  $a^f$ . Given a set  $A$ ,  $\wp(A)$  denotes the power set of  $A$ .

Let  $\Lambda$  be a language type. A  $\Lambda$ -*formula*, or *formula* for short, is an element of the universe  $\mathbf{Fm}_\Lambda(\mathbf{X})$  of the absolutely free algebra  $\mathbf{Fm}_\Lambda(\mathbf{X})$  of type  $\Lambda$  generated by  $\mathbf{X}$ . Occasionally we write formulas using Polish prefix notation. We identify the  $n$ -ary logical connective  $c \in \Lambda$  with the formula  $c^{\mathbf{Fm}_\Lambda}(v_0, \dots, v_{n-1})$  [21, Section 1.1.3, p. 8]. A  $\Lambda$ -*substitution*, or more briefly *substitution*, is an endomorphism of the formula algebra  $\mathbf{Fm}_\Lambda(\mathbf{X})$ . By the freeness of  $\mathbf{Fm}_\Lambda(\mathbf{X})$ , we identify any substitution with its restriction to  $\mathbf{X}$ .

Let  $\mathcal{K}$  be a quasivariety and let  $\mathbf{A} \in \mathcal{K}$ . A  $\mathcal{K}$ -*congruence* on  $\mathbf{A}$  is any congruence  $\theta$  on  $\mathbf{A}$  such that  $\mathbf{A}/\theta \in \mathcal{K}$ . The set of all  $\mathcal{K}$ -congruences on  $\mathbf{A}$  is denoted  $\text{Con}_{\mathcal{K}} \mathbf{A}$ . For  $a, b \in A$ ,  $\Theta_{\mathcal{K}}^{\mathbf{A}}(a, b)$  denotes the principal  $\mathcal{K}$ -congruence on  $\mathbf{A}$  generated by  $a, b$ . We drop all instances of the subscript when  $\mathcal{K}$  is a variety.

A *constant term* of a quasivariety  $\mathcal{K}$  is a term  $t(x_0, \dots, x_{n-1})$  in the language of  $\mathcal{K}$  having the property that  $\mathcal{K} \models t(x_0, \dots, x_{n-1}) \approx t(y_0, \dots, y_{n-1})$ , where the  $y_0, \dots, y_{n-1}$  are new variables distinct from  $x_0, \dots, x_{n-1}$ .  $\mathcal{K}$  is said to be *pointed* if it has a constant term. By [15, Section 1.5, p. 39] every pointed quasivariety is term equivalent to a quasivariety over a language type with a distinguished constant (*i.e.*, nullary operation) symbol  $\mathbf{1}$ . In the sequel we always distinguish a constant term in every pointed quasivariety and assume that  $\mathbf{1}$  denotes this distinguished constant term.

## 2.2. Nelson algebras and Nelson $\mathbf{FL}_{ew}$ -algebras

A *Nelson algebra* is an algebra  $\langle A; \wedge, \vee, \rightarrow, \neg, \sim, 0, 1 \rangle$  of type  $\langle 2, 2, 2, 1, 1, 0, 0 \rangle$  where  $\langle A; \wedge, \vee, \sim, 0, 1 \rangle$  is a De Morgan algebra [2, Chapter XI] and the following identities are satisfied [10]:

$$(x \wedge \sim x) \wedge (y \vee \sim y) \approx x \wedge \sim x \tag{N1}$$

$$x \rightarrow x \approx \mathbf{1} \tag{N2}$$

$$(x \rightarrow y) \wedge (\sim x \vee y) \approx \sim x \vee y \tag{N3}$$

$$x \wedge (\sim x \vee y) \approx x \wedge (x \rightarrow y) \tag{N4}$$

$$(x \rightarrow y) \wedge (x \rightarrow z) \approx x \rightarrow (y \wedge z) \tag{N5}$$

$$(x \wedge y) \rightarrow z \approx x \rightarrow (y \rightarrow z) \tag{N6}$$

$$\neg x \approx x \rightarrow \mathbf{0}. \tag{N7}$$

Clearly the class  $\mathcal{N}$  of all Nelson algebras is equationally definable. Informally, a Nelson algebra may be understood as a De Morgan algebra  $\langle A; \wedge, \vee, \sim, 0, 1 \rangle$  structurally enriched with a certain weak implication operation  $\rightarrow$  generalising relative pseudocomplementation [13, Section 3]. For studies of Nelson algebras, see [34, 42, 37, 13].

A *residuated lattice* is an algebra  $\langle A; \wedge, \vee, *, \backslash, /, 1 \rangle$  of type  $\langle 2, 2, 2, 2, 2, 0 \rangle$  where  $\langle A; \wedge, \vee \rangle$  is a lattice (with lattice ordering  $\leq$ ),  $\langle A; *, 1 \rangle$  is a monoid, and the equivalences  $a * b \leq c$  if and only if  $b \leq a \backslash c$  if and only if  $a \leq c / b$  are identically satisfied. A residuated lattice  $\mathbf{A}$  is said to be *commutative* if it satisfies the identity  $x * y \approx y * x$ , *contractive* if  $a \leq a * a$  for all  $a \in A$ , and *integral* if  $a \leq 1$  for all  $a \in A$ . By [8, Proposition 4.1] the class of residuated lattices is a variety.

An *FL-algebra*  $\langle A; \wedge, \vee, *, \backslash, /, 0, 1 \rangle$  is a residuated lattice with distinguished element  $0 \in A$ . It is easy to see an FL-algebra is commutative if and only if it satisfies the identity  $x / y \approx y \backslash x$  [18, Section 2, p. 282]. For this reason we fix the language type of the variety of commutative FL-algebras (and its subvarieties) as  $\{\wedge, \vee, *, \Rightarrow, \mathbf{0}, \mathbf{1}\}$ , where  $\Rightarrow$  is a binary operation symbol. Thus an *FL<sub>eci</sub>-algebra*  $\langle A; \wedge, \vee, *, \Rightarrow, 0, 1 \rangle$  is a commutative, contractive, integral residuated lattice with distinguished element  $0 \in A$ . An *FL<sub>ew</sub>-algebra*  $\langle A; \wedge, \vee, *, \Rightarrow, 0, 1 \rangle$  is a commutative, integral residuated lattice with distinguished element  $0 \in A$  where  $0 \leq a$  for all  $a \in A$ . For studies of FL<sub>ew</sub>-algebras, see [28, 23, 29, 30].

A *Nelson FL<sub>ew</sub>-algebra* is an FL<sub>ew</sub>-algebra satisfying the identities:

$$\sim \sim x \approx x \tag{DN}$$

$$(x \vee y) \wedge (x \vee z) \approx x \vee (y \wedge z) \tag{D7}$$

$$(x \wedge y) \vee (x \wedge z) \approx x \wedge (y \vee z) \tag{D8}$$

$$x \Rightarrow (x \Rightarrow (x \Rightarrow y)) \approx x \Rightarrow (x \Rightarrow y) \tag{E2}$$

$$(x \Rightarrow (x \Rightarrow y)) \wedge (\sim y \Rightarrow (\sim y \Rightarrow \sim x)) \approx x \Rightarrow y \tag{N}$$

where  $\sim x$  abbreviates the term  $x \Rightarrow \mathbf{0}$ . By [41, Section 2.4] the class  $\mathcal{NFL}_{ew}$  of all Nelson FL<sub>ew</sub>-algebras is a variety.

The main result of Part I of this series [41] states

**THEOREM 2.1.** [41, Theorem 1.1]

1. The map  $\delta : \Lambda[\mathbf{FL}_{ew}] \rightarrow \mathbf{Fm}_{\Lambda}[\mathbf{N}]$  defined by

$$\begin{aligned} x \wedge y &\mapsto x \wedge y \\ x \vee y &\mapsto x \vee y \\ x * y &\mapsto \sim(x \rightarrow \sim y) \vee \sim(y \rightarrow \sim x) && (*_{\text{def}}) \\ x \Rightarrow y &\mapsto (x \rightarrow y) \wedge (\sim y \rightarrow \sim x) && (\Rightarrow_{\text{def}}) \\ \mathbf{0} &\mapsto \mathbf{0} \\ \mathbf{1} &\mapsto \mathbf{1} \end{aligned}$$

is an interpretation of  $\mathcal{NFL}_{ew}$  in  $\mathcal{N}$ .

2. The map  $\varepsilon : \Lambda[\mathbf{N}] \rightarrow \mathbf{Fm}_{\Lambda}[\mathbf{FL}_{ew}]$  defined by

$$\begin{aligned} x \wedge y &\mapsto x \wedge y \\ x \vee y &\mapsto x \vee y \\ x \rightarrow y &\mapsto x \Rightarrow (x \Rightarrow y) && (\rightarrow_{\text{def}}) \\ \neg x &\mapsto x \Rightarrow (x \Rightarrow \mathbf{0}) && (\neg_{\text{def}}) \\ \sim x &\mapsto x \Rightarrow \mathbf{0} && (\sim_{\text{def}}) \\ \mathbf{0} &\mapsto \mathbf{0} \\ \mathbf{1} &\mapsto \mathbf{1} \end{aligned}$$

is an interpretation of  $\mathcal{N}$  in  $\mathcal{NFL}_{ew}$ .

3. The interpretations  $\delta$  and  $\varepsilon$  are mutually inverse.

Hence the varieties of Nelson algebras and Nelson  $FL_{ew}$ -algebras are term equivalent.

### 2.3. $k$ -deductive systems

Let  $\Lambda$  be a language type and let  $1 \leq k < \omega$ . A  $k$ -formula is an element of the Cartesian product  $\mathbf{Fm}_{\Lambda}^k$ . We denote  $k$ -formulas using lowercase boldface Greek letters  $\varphi, \psi, \dots$ , except when  $k = 1$ , where we write simply  $\varphi, \psi, \dots$ . Given a substitution  $\sigma : \mathbf{Fm}_{\Lambda} \rightarrow \mathbf{Fm}_{\Lambda}$  and a  $k$ -formula  $\varphi := \langle \varphi_0, \dots, \varphi_{k-1} \rangle$ , we write variously  $\sigma\varphi$  or  $\sigma(\varphi)$  for  $\langle \sigma(\varphi_0), \dots, \sigma(\varphi_{k-1}) \rangle$ . For  $\Gamma \subseteq \mathbf{Fm}_{\Lambda}^k$  we write  $\sigma(\Gamma)$  for  $\{\sigma(\varphi) : \varphi \in \Gamma\}$ .

A  $k$ -deductive system is a pair  $\mathbf{S} := \langle \Lambda, \vdash_{\mathbf{S}} \rangle$ , where  $\vdash_{\mathbf{S}} \subseteq \wp(\mathbf{Fm}_{\Lambda}^k) \times \mathbf{Fm}_{\Lambda}^k$ , and the following conditions are satisfied for all  $\Gamma, \Delta \subseteq \mathbf{Fm}_{\Lambda}^k$  and  $\varphi \in \mathbf{Fm}_{\Lambda}^k$  [6, Definition 3.1]:



1.  $\varphi \in \Gamma$  implies  $\Gamma \vdash_{\mathbf{S}} \varphi$ ;
2.  $\Gamma \vdash_{\mathbf{S}} \varphi$  and  $\Delta \vdash_{\mathbf{S}} \psi$  for every  $\psi \in \Gamma$  implies  $\Delta \vdash_{\mathbf{S}} \varphi$ ;
3.  $\Gamma \vdash_{\mathbf{S}} \varphi$  implies  $\Gamma' \vdash_{\mathbf{S}} \varphi$  for some finite  $\Gamma' \subseteq \Gamma$ ;
4.  $\Gamma \vdash_{\mathbf{S}} \varphi$  implies  $\sigma(\Gamma) \vdash_{\mathbf{S}} \sigma(\varphi)$  for every substitution  $\sigma$ .

A *deductive system* is a 1-deductive system.

Let  $\mathbf{S}$  be a  $k$ -deductive system. The relation  $\vdash_{\mathbf{S}}$  is called the *consequence relation* of  $\mathbf{S}$ . The *consequence operator* associated with  $\vdash_{\mathbf{S}}$  is the map  $\text{Cns} : \wp(\text{Fm}_{\Lambda}^k) \rightarrow \wp(\text{Fm}_{\Lambda}^k)$  given by  $\text{Cns}(\Gamma) := \{\varphi \in \text{Fm}_{\Lambda}^k : \Gamma \vdash_{\mathbf{S}} \varphi\}$ . A set  $T \subseteq \text{Fm}_{\Lambda}^k$  is called an  *$\mathbf{S}$ -theory* (briefly, a *theory*) if  $T \vdash_{\mathbf{S}} \varphi$  implies  $\varphi \in T$ , for each  $\varphi \in \text{Fm}_{\Lambda}^k$ . The set of all theories of  $\mathbf{S}$  is denoted  $\text{Th } \mathbf{S}$ . For  $\Gamma, \Delta \subseteq \text{Fm}_{\Lambda}^k$ , the notation  $\Gamma \vdash_{\mathbf{S}} \Delta$  abbreviates ‘ $\Gamma \vdash_{\mathbf{S}} \varphi$  for all  $\varphi \in \Delta$ ’, while  $\Gamma \dashv\vdash_{\mathbf{S}} \Delta$  abbreviates ‘both  $\Gamma \vdash_{\mathbf{S}} \Delta$  and  $\Delta \vdash_{\mathbf{S}} \Gamma$ ’. For a systematic exposition of the theory of  $k$ -deductive systems, see Blok and Pigozzi [4, 6].

#### 2.4. Definitional equivalence for $k$ -deductive systems

Let  $\mathbf{A} := \langle A; c^{\mathbf{A}} \rangle_{c \in \Lambda}$  be an algebra of type  $\Lambda$ , and let  $F \subseteq A^k$  for  $k \geq 1$ . A congruence  $\theta$  on  $\mathbf{A}$  is said to be *compatible* with  $F$  if  $\langle a_0, \dots, a_{k-1} \rangle \in F$  and  $a_i \theta b_i$  ( $i = 0, \dots, k-1$ ) imply  $\langle b_0, \dots, b_{k-1} \rangle \in F$ . The *Leibniz congruence on  $\mathbf{A}$  over  $F$*  is the largest congruence on  $\mathbf{A}$  compatible with  $F$ . In symbols,

$$\Omega^{\mathbf{A}} F := \bigvee \{ \theta \in \text{Con } \mathbf{A} : \theta \text{ is compatible with } F \}.$$

We write simply  $\Omega$  for  $\Omega^{\text{Fm}_{\Lambda}}$ . For a survey of the operator  $\Omega^{\mathbf{A}} F$  in abstract algebraic logic, see [16].

For a  $k$ -dimensional deductive system  $\mathbf{S}$ , the *Tarski congruence*  $\tilde{\Omega}(\mathbf{S})$  is the largest congruence on the formula algebra that is compatible with every theory of  $\mathbf{S}$ . In symbols,

$$\tilde{\Omega}(\mathbf{S}) := \bigcap \{ \Omega T : T \in \text{Th } \mathbf{S} \}.$$

For studies of the Tarski congruence in (second-order) abstract algebraic logic see [17, 15].

Let  $\Lambda_1$  and  $\Lambda_2$  be two language types, and let  $\alpha$  be a map from  $\Lambda_1$  to  $\text{Fm}_{\Lambda_2}$ . The *standard extension* of  $\alpha$  is the function  $\bar{\alpha} : \text{Fm}_{\Lambda_1} \rightarrow \text{Fm}_{\Lambda_2}$  defined recursively based on the complexity of terms by:

$$\begin{aligned} (v_i)^{\bar{\alpha}} &= v_i, \\ (c\varphi_0, \dots, \varphi_{n-1})^{\bar{\alpha}} &= \llbracket \varphi_0^{\bar{\alpha}}, \dots, \varphi_{n-1}^{\bar{\alpha}} \rrbracket c^{\alpha} \end{aligned}$$

where  $v_i$  is a variable,  $c \in \Lambda_1$  is an  $n$ -ary connective,  $\varphi_0, \dots, \varphi_{n-1}$  are  $\Lambda_1$ -formulas, and  $\llbracket \varphi_0, \dots, \varphi_{n-1} \rrbracket$  is the surjective substitution that takes values  $\varphi_i$  on  $v_i$  for  $i = 0, \dots, n-1$ , and takes value  $v_i$  on  $v_{i+n}$  [21, Section 2.1.1, p. 48]. The map  $\bar{\alpha}$  extends to  $k$ -formulas in the natural way on defining  $\varphi^{\bar{\alpha}} := \langle \varphi_0^{\bar{\alpha}}, \dots, \varphi_{k-1}^{\bar{\alpha}} \rangle$  for all  $\varphi := \langle \varphi_0, \dots, \varphi_{k-1} \rangle \in \text{Fm}_{\Lambda_1}^k$  and  $\Gamma^{\bar{\alpha}} := \{\varphi^{\bar{\alpha}} : \varphi \in \Gamma\}$  for all  $\Gamma \subseteq \text{Fm}_{\Lambda_1}^k$ .

Let  $\mathbf{S}_1 := \langle \Lambda_1, \vdash_{\mathbf{S}_1} \rangle$  and  $\mathbf{S}_2 := \langle \Lambda_2, \vdash_{\mathbf{S}_2} \rangle$  be two  $k$ -dimensional deductive systems. A mapping  $\alpha : \Lambda_1 \rightarrow \text{Fm}_{\Lambda_2}$  is said to be an *interpretation* of  $\mathbf{S}_1$  in  $\mathbf{S}_2$  if it satisfies the following two conditions [21, Definition 2.5]:

(DE-1)  $\langle c^\alpha, \mu c^\alpha \rangle \in \tilde{\Omega}(\mathbf{S}_2)$  for all connectives  $c$  of  $\Lambda_1$  with arity  $n$  and substitutions  $\mu$  of  $\Lambda_2$  that fix the first  $n$  variables;

(DE-2) If  $\Gamma \vdash_{\mathbf{S}_1} \varphi$  then  $\Gamma^{\bar{\alpha}} \vdash_{\mathbf{S}_2} \varphi^{\bar{\alpha}}$  for all  $\Gamma \subseteq \text{Fm}_{\Lambda_1}^k$  and  $\varphi \in \text{Fm}_{\Lambda_1}^k$ .

Let  $\alpha$  be an interpretation of  $\mathbf{S}_1$  in  $\mathbf{S}_2$ , and  $\beta$  an interpretation of  $\mathbf{S}_2$  in  $\mathbf{S}_1$ . We say that  $\alpha$  and  $\beta$  are *mutually inverse* if  $\langle \varphi, \varphi^{\bar{\alpha}\bar{\beta}} \rangle \in \tilde{\Omega}(\mathbf{S}_1)$  and  $\langle \psi, \psi^{\bar{\beta}\bar{\alpha}} \rangle \in \tilde{\Omega}(\mathbf{S}_2)$  for all  $\varphi \in \text{Fm}_{\Lambda_1}$  and  $\psi \in \text{Fm}_{\Lambda_2}$ . The deductive systems  $\mathbf{S}_1$  and  $\mathbf{S}_2$  are said to be *definitionally equivalent* if there are interpretations  $\alpha$  of  $\mathbf{S}_1$  in  $\mathbf{S}_2$  and  $\beta$  of  $\mathbf{S}_2$  in  $\mathbf{S}_1$  that are mutually inverse [21, Definition 2.14].<sup>5</sup>

The notion of definitional equivalence for  $k$ -deductive systems presented here is due to Gyuris [21]. For alternative notions of definitional equivalence with applicability to abstract algebraic logic see [47, 32, 12]. For a comparison between the notion of definitional equivalence presented here and the notion of equipollence [12] due to Caleiro and Gonçalves, see [39].

## 2.5. Regularly algebraisable logics

Let  $\mathbf{S}$  be a deductive system over a language type  $\Lambda$ . Recall from [15, Section 1.4, p. 36] that a finite set  $\{\Delta_0, \dots, \Delta_{m-1}\}$  of  $\Lambda$ -formulas in two variables is a *finite system of equivalence formulas* for  $\mathbf{S}$  if for any  $n$ -ary connective  $c \in \Lambda$  and any set of  $\Lambda$ -formulas  $\{\varphi_k : k = 0, \dots, n-1\} \cup \{\psi_k : k = 0, \dots, n-1\} \cup \{\varphi, \psi, \chi\}$  the following conditions hold for  $j = 0, \dots, m-1$ :

(ALG1)  $\vdash_{\mathbf{S}} \varphi \Delta_j \varphi^6$

(ALG2)  $\varphi, \{\varphi \Delta_i \psi : i = 0, \dots, m-1\} \vdash_{\mathbf{S}} \psi$

(ALG3)  $\{\varphi \Delta_i \psi : i = 0, \dots, m-1\} \vdash_{\mathbf{S}} \psi \Delta_j \varphi$

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<sup>5</sup>For a discussion of the distinction between definitional equivalence as described in this paper, and the more familiar notion in algebraic logic of deductive equivalence, see Blok and Pigozzi [6, Note 4.1].

<sup>6</sup>To simplify notation, we are writing  $\varphi \Delta_j \varphi$  for  $\Delta_j(\varphi, \varphi)$ , etc., here and in the sequel.

(ALG4)  $\{\varphi \Delta_i \psi : i = 0, \dots, m-1\}, \{\psi \Delta_i \chi : i = 0, \dots, m-1\} \vdash_{\mathbf{S}} \varphi \Delta_j \chi$

(ALG5)  $\{\varphi_k \Delta_i \psi_k : i = 0, \dots, m-1; k = 0, \dots, n-1\} \vdash_{\mathbf{S}}$   
 $c(\varphi_0, \dots, \varphi_{n-1}) \Delta_j c(\psi_0, \dots, \psi_{n-1})$ .

$\mathbf{S}$  is said to be *regularly algebraisable* if it has a finite system of equivalence formulas and in addition the following conditions hold for  $j = 0, \dots, m-1$ :

(ALG6)  $\varphi, \psi \vdash_{\mathbf{S}} \varphi \Delta_j \psi$ .

By [15, Theorem 28], every regularly algebraisable logic is algebraisable in the sense of Blok and Pigozzi [3]. For studies of regularly algebraisable logics, see [34, 14, 15].

Let  $\mathbf{S}$  be a regularly algebraisable deductive system over a language type  $\Lambda$  with finite system of equivalence formulas  $\{\Delta_j : j = 0, \dots, m-1\}$ . Then there exists a unique quasivariety  $\text{Alg Mod}^* \mathbf{S}$  of algebras of type  $\Lambda$ , and a constant term  $\mathbf{1} := \Delta_j(x, x)$  of  $\text{Alg Mod}^* \mathbf{S}$ , such that the following conditions hold for any  $\Gamma \cup \{\varphi, \psi\} \subseteq \text{Fm}_{\Lambda}$ :<sup>7</sup>

(EQV1)  $\Gamma \vdash_{\mathbf{S}} \varphi$  if and only if  $\{\psi \approx \mathbf{1} : \psi \in \Gamma\} \models_{\text{Alg Mod}^* \mathbf{S}} \varphi \approx \mathbf{1}$

(EQV2)  $\varphi \approx \psi \iff \models_{\text{Alg Mod}^* \mathbf{S}} \{\varphi \Delta_j \psi \approx \mathbf{1} : j = 0, \dots, m-1\}$ .

(Here  $\Gamma \models_{\text{Alg Mod}^* \mathbf{S}} \Gamma'$  abbreviates ' $\Gamma \models_{\text{Alg Mod}^* \mathbf{S}} \Gamma'$  and  $\Gamma' \models_{\text{Alg Mod}^* \mathbf{S}} \Gamma$ '.) The class  $\text{Alg Mod}^* \mathbf{S}$  is called the *equivalent quasivariety semantics* of  $\mathbf{S}$ . For any presentation of  $\mathbf{S}$  by a set of axioms  $Ax$  and (proper) inference rules  $Ru$ , the equivalent quasivariety  $\text{Alg Mod}^* \mathbf{S}$  is determined by the following collection of identities and quasi-identities [15, Theorem 30]:

(AX-1)  $\varphi \approx \mathbf{1}$ , for each  $\varphi \in Ax$

(AX-2)  $\psi_0 \approx \mathbf{1}$  and ... and  $\psi_{p-1} \approx \mathbf{1}$  implies  $\varphi \approx \mathbf{1}$   
 for each inference rule  $\langle \psi_0, \dots, \psi_{p-1}, \varphi \rangle \in Ru$

(AX-3)  $\Delta_0(x, y) \approx \mathbf{1}$  and ... and  $\Delta_{m-1}(x, y) \approx \mathbf{1}$  implies  $x \approx y$ .

The remarks of this section extend in a natural way to deductive systems that are *algebraisable* in the sense of Blok and Pigozzi [3]. For details, see [3, 6, 14]. For all other terminology and notation of abstract algebraic logic not specified either above or in the sequel see Czelakowski and Pigozzi [15] and Blok and Pigozzi [3, 6].

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<sup>7</sup>By [44, Theorem 3.2.4, p. 182],  $\text{Alg Mod}^* \mathbf{S} \models \Delta_i(x, x) \approx \Delta_{i'}(y, y)$  for all  $0 \leq i, i' \leq m-1$ . Hence  $\Delta_j(x, x)$  is a constant term of  $\text{Alg Mod}^* \mathbf{S}$  as claimed.

## 2.6. 1-assertional logics

Let  $\mathcal{K}$  be a pointed quasivariety over a language type  $\Lambda$ . The **1-assertional logic** of  $\mathcal{K}$ , in symbols  $\mathbf{S}^{\text{ASL}} \mathcal{K}$ , is the deductive system from sets of  $\Lambda$ -terms to  $\Lambda$ -terms determined by the equivalence [15, Corollary 33]:

$$\Gamma \vdash_{\mathbf{S}^{\text{ASL}} \mathcal{K}} \varphi \quad \text{if and only if} \quad \{\psi \approx \mathbf{1} : \psi \in \Gamma\} \models_{\mathcal{K}} \varphi \approx \mathbf{1}$$

for all  $\Gamma \cup \{\varphi\} \subseteq \text{Fm}_{\Lambda}$ .<sup>8</sup> For studies of assertional logics see [14, 15, 7].

A pointed quasivariety  $\mathcal{K}$  is said to be *relatively point regular* if, whenever  $\mathbf{A} \in \mathcal{K}$  and  $\theta, \phi \in \text{Con}_{\mathcal{K}} \mathbf{A}$  with  $\mathbf{1}^{\mathbf{A}}/\theta = \mathbf{1}^{\mathbf{A}}/\phi$ , we have that  $\theta = \phi$ . The following result of Czelakowski and Pigozzi [15] exhibits a one-one correspondence between regularly algebraisable logics and relatively point regular quasivarieties.

**THEOREM 2.2.** [15, Corollary 35]

1. *Every regularly algebraisable deductive system  $\mathbf{S}$  is the 1-assertional logic of a unique relatively point regular quasivariety, namely its equivalent quasivariety semantics. In symbols,  $\mathbf{S} = \mathbf{S}^{\text{ASL}} \text{Alg Mod}^* \mathbf{S}$ .*
2. *Every relatively point regular quasivariety  $\mathcal{K}$  is the equivalent quasivariety semantics of a unique regularly algebraisable deductive system, namely its 1-assertional logic. In symbols,  $\mathcal{K} = \text{Alg Mod}^* \mathbf{S}^{\text{ASL}} \mathcal{K}$ .*

## 3. Substructural logics over FL

In this section we briefly criticise the notion of substructural logic over  $\mathbf{FL}_S$  ( $S \subseteq \{e, c, i, o\}$ ) presented in [18, 19] from the perspective of algebraic and non-classical logic, propose an alternative definition, and characterise (in the sense of this paper) the substructural logics over  $\mathbf{FL}_S$ .

According to Galatos and Ono [18, Section 3.1, p. 285], and Galatos *et al.* [19, Section 2.1.4], a substructural logic over  $\mathbf{FL}_S$  is a theory of  $\mathbf{FL}_S$  closed under substitutions, or equivalently, the set of theorems of an axiomatic extension of  $\mathbf{FL}_S$ . This definition is unorthodox in that:

- Deductive systems are viewed as sets of formulas and *not* as consequence relations. The study of substructural logics over  $\mathbf{FL}$  in the sense of [18, 19] thereby amounts to an investigation, in the framework of the

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<sup>8</sup>Since  $\mathcal{K}$  is closed under the formation of ultraproducts,  $\mathbf{S}^{\text{ASL}} \mathcal{K}$  is finitary and hence is a deductive system in the sense of this paper.

Blok-Pigozzi theory of algebraisable logics [3], of the *axiomatic* extensions of **FL** via an examination of the *subvarieties* of the variety of FL-algebras. But in full generality, the study of an algebraisable deductive system **S** is tantamount to an investigation of the *extensions* of **S** via an examination of the *subquasivarieties* of its equivalent quasivariety semantics. (A justification for these remarks is given prior to the statement of Corollary 3.2 below.) Thus the definition of substructural logic over  $\mathbf{FL}_S$  due to [18, 19] is in a sense unduly restrictive.

- There is nothing that prohibits a logic having *all* the structural rules from being substructural. Indeed, the classical propositional calculus is a substructural logic over  $\mathbf{FL}_{ecw}$  in the sense of [18, 19], as Galatos and Ono explicitly point out in [18, p. 279]. But, as Restall [35, p. 1] asserts, “Substructural logics [should] focus on the behaviour and presence — or more suggestively, the *absence* — of *structural rules*” [italics Restall’s].<sup>9</sup> Thus the definition of substructural logic over  $\mathbf{FL}_S$  due to [18, 19] is in a sense overly generous.

Let **S** be a deductive system over a language type  $\Lambda$ . An *extension* of **S** is any system  $\mathbf{S}' := \langle \Lambda, \vdash_{\mathbf{S}'} \rangle$  over the same language type  $\Lambda$  such that  $\Gamma \vdash_{\mathbf{S}} \varphi$  implies  $\Gamma \vdash_{\mathbf{S}'} \varphi$  for all  $\Gamma \cup \{\varphi\} \subseteq \mathbf{Fm}_\Lambda$ .  $\mathbf{S}'$  is said to be *axiomatic* if it can be obtained by adjoining new axioms to **S** only. By Blok and Pigozzi [3, Corollary 4.9], any extension of a (regularly) algebraisable deductive system is itself (regularly) algebraisable.

A deductive system **S** over a language type  $\Lambda$  is said to be *Fregean* if, for every  $T \in \text{Th } \mathbf{S}$ , the relativised interderivability relation  $\dashv\vdash_{\mathbf{S}}^T$  defined for all  $\varphi, \psi \in \mathbf{Fm}_\Lambda$  by

$$\varphi \dashv\vdash_{\mathbf{S}}^T \psi \quad \text{if and only if} \quad T, \varphi \vdash_{\mathbf{S}} \psi \text{ and } T, \psi \vdash_{\mathbf{S}} \varphi$$

is a congruence relation on  $\mathbf{Fm}_\Lambda$  [15, Definition 59]. **S** is *non-Fregean* if it is not Fregean. For studies of Fregean logics, see [17, 14, 15].

The discussion heading this section leads us to the following definition. A *substructural logic over*  $\mathbf{FL}_S$ ,  $S \subseteq \{e, c, i, o\}$ , is a deductive system **S** that is definitionally equivalent to a non-Fregean extension of  $\mathbf{FL}_S$ . The next result shows the notion of substructural logic over  $\mathbf{FL}_S$  used in this paper appropriately captures the notion of a substructural logic over **FL** as an extension of **FL** lacking some or all of the structural rules.

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<sup>9</sup>For further support for this point of view see e.g. Došen [36, p. 6].

**THEOREM 3.1.** *An extension  $\mathbf{S}$  of  $\mathbf{FL}$  is Fregean if and only if it is an axiomatic extension of  $\mathbf{FL}_{eci}$ .<sup>10</sup>*

**PROOF.** It is clear that any axiomatic extension of  $\mathbf{FL}_{eci}$  is Fregean. For the converse, suppose  $\mathbf{S}$  is a Fregean extension of  $\mathbf{FL}$ . Because  $\mathbf{S}$  is Fregean and algebraisable with theorems, from Czelakowski and Pigozzi [15, Theorem 61] we have that  $\mathbf{S}$  is regularly algebraisable. Since  $\mathbf{S}$  is regularly algebraisable,  $p \approx \mathbf{1}$  is a single defining equation for  $\mathbf{S}$  in the sense of Blok and Pigozzi [3, Definition 2.2]. This implies that  $\mathbf{S}$  is an extension of the deductive system  $\mathbf{FL}_i$ .

Observe next that  $\{p \setminus q\}$  is a protoequivalence system for  $\mathbf{S}$  in the sense of Czelakowski and Pigozzi [15, Section 1.4, p. 32]. Since  $\wedge$  is a conjunction formula for  $\mathbf{S}$  in the sense of [15, Section 2.2, p. 57], and  $\mathbf{S}$  is Fregean and algebraisable with theorems, from [15, Theorem 64] we have that  $\mathbf{S}$  has the uniterm deduction-detachment theorem (in the sense of [15, Definition 38]) with uniterm deduction-detachment system  $\{p \setminus (p \wedge q)\}$ . Because  $\mathbf{S}$  is an extension of  $\mathbf{FL}_i$ , the formulas  $\varphi \setminus (\varphi \wedge \psi)$  and  $\varphi \setminus \psi$  are provably equivalent (in the sense of [19, Section 2.1.2]) over  $\mathbf{S}$ . Therefore  $\{p \setminus q\}$  is also a uniterm deduction-detachment system for  $\mathbf{S}$ . This suffices to guarantee that  $\mathbf{S}$  is an extension of  $\mathbf{FL}_{eci}$ .

It remains only to observe that  $\mathbf{S}$  is an axiomatic extension of  $\mathbf{FL}_{eci}$ . Because  $\wedge$  is a conjunction formula for  $\mathbf{S}$ , the deductive system  $\mathbf{S}$  has the property of conjunction in the sense of Font and Jansana [17, Definition 2.45]. Since  $\mathbf{S}$  is Fregean and algebraisable with theorems, from Font and Jansana [17, Corollary 4.32] we have that  $\mathbf{S}$  is strongly algebraisable (*i.e.*,  $\text{Alg Mod}^* \mathbf{S}$  is a variety). The claim that  $\mathbf{S}$  is an axiomatic extension of  $\mathbf{FL}_{eci}$  now follows, because  $\mathbf{S}$  is regularly algebraisable. ■

A pointed quasivariety  $\mathcal{K}$  is said to be *relatively congruence orderable* if, for every  $\mathbf{A} \in \mathcal{K}$  and all  $a, b \in A$ ,  $\Theta_{\mathcal{K}}^{\mathbf{A}}(a, \mathbf{1}^{\mathbf{A}}) = \Theta_{\mathcal{K}}^{\mathbf{A}}(b, \mathbf{1}^{\mathbf{A}})$  implies  $a = b$ .  $\mathcal{K}$  is said to be *Fregean* if it is both relatively point regular and relatively congruence orderable [15, Definition 85]. For studies of Fregean quasivarieties in general algebra, see [31, 1, 22].

By [27, Corollary 1.3.5], there exists a lattice anti-isomorphism from the lattice of extensions of an algebraisable deductive system  $\mathbf{S}$  onto the lattice of subquasivarieties of  $\text{Alg Mod}^* \mathbf{S}$ , which moreover maps each extension of  $\mathbf{S}$  to its equivalent quasivariety. Combining these remarks with Theorem 2.2, Theorem 3.1, and Czelakowski and Pigozzi [15, Theorem 86] yields

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<sup>10</sup>The deductive system  $\mathbf{FL}_{eci}$  is definitionally equivalent to Johansson's minimal logic [34, Chapter XI], [47, Section 2.7]. For a discussion, see [19, Section 2.3.8].

the following corollary, which is due independently to the first author and to N. Galatos (unpublished).

**COROLLARY 3.2.** *A quasivariety of FL-algebras is Fregean if and only if it is a variety of FL<sub>eci</sub>-algebras.*<sup>11</sup>

For recent results related to Theorem 3.1 and Corollary 3.2, see Bou *et al.* [9, Section 4].

#### 4. Definitional equivalence for regularly algebraisable logics

In this section we give a sufficient condition for two regularly algebraisable logics to be definitionally equivalent (Theorem 4.6).

Let  $\mathcal{K}$  be a quasivariety over a language type  $\Lambda$  axiomatised by a set of identities *Id* and a set of quasi-identities *QId*. Recall from Czelakowski and Pigozzi [15, Definition 2] or Blok and Pigozzi [6, Section 3.3.2] that the *applied equational logic* determined by  $\mathcal{K}$ , in symbols  $\mathbf{S}^{\text{EQL}}\mathcal{K}$ , is the 2-dimensional deductive system presented by the following collection of axioms and inference rules:

$$\text{(EQ-1)} \quad \langle p, p \rangle$$

$$\text{(EQ-2)} \quad \frac{\langle p, q \rangle}{\langle q, p \rangle}$$

$$\text{(EQ-3)} \quad \frac{\langle p, q \rangle, \langle q, r \rangle}{\langle p, r \rangle}$$

$$\text{(EQ-4)} \quad \frac{\langle p_0, q_0 \rangle, \dots, \langle p_{n-1}, q_{n-1} \rangle}{\langle c(p_0, \dots, p_{n-1}), c(q_0, \dots, q_{n-1}) \rangle} \text{ for each } c \in \Lambda \text{ of arity } n$$

$$\text{(EQ-5)} \quad \langle \varphi, \psi \rangle \text{ for every identity } \forall \bar{x}(\varphi \approx \psi) \in \text{Id}$$

$$\text{(EQ-6)} \quad \frac{\langle \chi_0, \zeta_0 \rangle, \dots, \langle \chi_{n-1}, \zeta_{n-1} \rangle}{\langle \varphi, \psi \rangle} \text{ for every quasi-identity}$$

$$\forall \bar{x}(\chi_0 \approx \zeta_0 \text{ and } \dots \text{ and } \chi_{n-1} \approx \zeta_{n-1} \text{ implies } \varphi \approx \psi) \in \text{QId}.$$

Applied equational logics have the following

**THEOREM 4.1** (Completeness theorem). [6, Theorem 3.9] *Let  $\mathcal{K}$  be a quasivariety over a language type  $\Lambda$  and let  $\Gamma \cup \{ \langle \varphi_0, \varphi_1 \rangle \} \subseteq \text{Fm}_{\Lambda}^2$ . Then*

$$\{ \langle \psi_0, \psi_1 \rangle : \langle \psi_0, \psi_1 \rangle \in \Gamma \} \vdash_{\mathbf{S}^{\text{EQL}}\mathcal{K}} \langle \varphi_0, \varphi_1 \rangle \text{ if and only if} \\ \{ \psi_0 \approx \psi_1 : \langle \psi_0, \psi_1 \rangle \in \Gamma \} \models_{\mathcal{K}} \varphi_0 \approx \varphi_1.$$

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<sup>11</sup>The variety of FL<sub>eci</sub>-algebras is term equivalent to the variety of generalised Heyting algebras. For a discussion, see [19, Section 2.3.8] or [9, Section 4].

Let  $\mathbf{S}$  be a deductive system. The following useful technical lemma of Czelakowski and Pigozzi [15] asserts that the  $\text{Alg Mod}^* \mathbf{S}$ -congruences on the formula algebra are precisely the Leibniz congruences.

LEMMA 4.2. [15, Lemma 12] *Let  $\mathbf{S}$  be a deductive system over a language type  $\Lambda$ . Then  $\text{Con}_{\text{Alg Mod}^* \mathbf{S}} \mathbf{Fm}_\Lambda = \{\Omega T : T \in \text{Th } \mathbf{S}\}$ .*

For an applied equational logic  $\mathbf{S}$ ,  $\tilde{\Omega}(\mathbf{S})$  has a particularly transparent description:

LEMMA 4.3. [21, Proposition 1.26] *Let  $\mathbf{S}$  be an applied equational logic. Then  $\tilde{\Omega}(\mathbf{S}) = \text{Cn}_{\mathbf{S}}(\emptyset)$ .*

The next result, due to Gyuris, shows that the notion of definitional equivalence for deductive systems generalises the notion of term equivalence for quasivarieties described in Part I of this series [41, Section 2.1].

PROPOSITION 4.4. [21, Proposition 2.17] *Let  $\mathcal{K}_1$  and  $\mathcal{K}_2$  be two quasivarieties over language types  $\Lambda_1$  and  $\Lambda_2$ . Let  $\mathbf{S}_1 := \mathbf{S}^{\text{EQL}} \mathcal{K}_1$  and  $\mathbf{S}_2 := \mathbf{S}^{\text{EQL}} \mathcal{K}_2$  be the applied equational logics determined by  $\mathcal{K}_1$  and  $\mathcal{K}_2$  respectively. Then  $\mathbf{S}_1$  and  $\mathbf{S}_2$  are definitionally equivalent if and only if  $\mathcal{K}_1$  and  $\mathcal{K}_2$  are term equivalent. In particular, if  $\mathcal{K}_1$  and  $\mathcal{K}_2$  are term equivalent with interpretations  $\alpha : \Lambda_1 \rightarrow \text{Fm}_{\Lambda_2}$  and  $\beta : \Lambda_2 \rightarrow \text{Fm}_{\Lambda_1}$  then  $\mathbf{S}_1$  and  $\mathbf{S}_2$  are definitionally equivalent with the same mutually inverse interpretations.*

In Theorem 4.6 below, we lift the right-to-left direction of Proposition 4.4 to the setting of regularly algebraisable logics. But first, a technical lemma.

LEMMA 4.5. *Let  $\mathcal{K}$  be a relatively point regular quasivariety over a language type  $\Lambda$ . If  $\varphi \equiv \psi \pmod{\tilde{\Omega}(\mathbf{S}^{\text{EQL}} \mathcal{K})}$  for  $\Lambda$ -formulas  $\varphi, \psi$ , then  $\varphi \equiv \psi \pmod{\tilde{\Omega}(\mathbf{S}^{\text{ASL}} \mathcal{K})}$ .*

PROOF. Suppose  $\varphi \equiv \psi \pmod{\tilde{\Omega}(\mathbf{S}^{\text{EQL}} \mathcal{K})}$ . By Lemma 4.3,  $\vdash_{\mathbf{S}^{\text{EQL}} \mathcal{K}} \langle \varphi, \psi \rangle$ . By the completeness theorem for applied equational logics, therefore, we have that  $\mathcal{K} \models \varphi \approx \psi$ , whence  $\mathbf{Fm}_\Lambda / \theta \models \varphi \approx \psi$  for all  $\theta \in \text{Con}_{\mathcal{K}} \mathbf{Fm}_\Lambda$ . Since  $\text{Con}_{\text{Alg Mod}^* \mathbf{S}^{\text{ASL}} \mathcal{K}} \mathbf{Fm}_\Lambda = \text{Con}_{\mathcal{K}} \mathbf{Fm}_\Lambda$  (by Theorem 2.2), we conclude that  $\varphi \equiv \psi \pmod{\theta}$  for all  $\theta \in \text{Con}_{\text{Alg Mod}^* \mathbf{S}^{\text{ASL}} \mathcal{K}} \mathbf{Fm}_\Lambda$ . By Lemma 4.2, therefore,  $\varphi \equiv \psi \pmod{\Omega T}$  for all  $T \in \text{Th } \mathbf{S}^{\text{ASL}} \mathcal{K}$ . Thus  $\varphi \equiv \psi \pmod{\bigcap \{\Omega T : T \in \text{Th } \mathbf{S}^{\text{ASL}} \mathcal{K}\}}$ , which is to say  $\varphi \equiv \psi \pmod{\tilde{\Omega}(\mathbf{S}^{\text{ASL}} \mathcal{K})}$  as claimed. ■

THEOREM 4.6. *Let  $\mathbf{S}_1$  and  $\mathbf{S}_2$  be two regularly algebraisable deductive systems over language types  $\Lambda_1$  and  $\Lambda_2$ . Let  $\mathcal{K}_1$  and  $\mathcal{K}_2$  be the relatively  $\mathbf{1}^{\mathcal{K}_1}$ -regular and relatively  $\mathbf{1}^{\mathcal{K}_2}$ -regular quasivarieties comprising the equivalent quasivariety semantics of  $\mathbf{S}_1$  and  $\mathbf{S}_2$  respectively. Suppose  $\mathcal{K}_1$  and  $\mathcal{K}_2$  are term*



equivalent with interpretations  $\alpha : \Lambda_1 \rightarrow \text{Fm}_{\Lambda_2}$  and  $\beta : \Lambda_2 \rightarrow \text{Fm}_{\Lambda_1}$  such that  $(\mathbf{1}^{\mathcal{K}_1})^\alpha = \mathbf{1}^{\mathcal{K}_2}$  and  $(\mathbf{1}^{\mathcal{K}_2})^\beta = \mathbf{1}^{\mathcal{K}_1}$ . Then  $\mathbf{S}_1$  and  $\mathbf{S}_2$  are definitionally equivalent with the same mutually inverse interpretations.

PROOF. By Proposition 4.4,  $\mathbf{S}^{\text{EQL}} \mathcal{K}_1$  and  $\mathbf{S}^{\text{EQL}} \mathcal{K}_2$  are definitionally equivalent with mutually inverse interpretations  $\alpha : \Lambda_1 \rightarrow \text{Fm}_{\Lambda_2}$  and  $\beta : \Lambda_2 \rightarrow \text{Fm}_{\Lambda_1}$ . Throughout the proof we make implicit use of this observation.

Let  $c$  be an  $n$ -ary basic connective of  $\Lambda_1$  and  $\mu$  a substitution of  $\Lambda_2$  that fixes the first  $n$  variables. By (DE-1),  $\langle c^\alpha, \mu c^\alpha \rangle \in \tilde{\Omega}(\mathbf{S}^{\text{EQL}} \mathcal{K}_2)$ , so by Lemma 4.5,  $\langle c^\alpha, \mu c^\alpha \rangle \in \tilde{\Omega}(\mathbf{S}^{\text{ASL}} \mathcal{K}_2)$ . By Theorem 2.2(1), we conclude that  $\langle c^\alpha, \mu c^\alpha \rangle \in \tilde{\Omega}(\mathbf{S}_2)$ . Observe next that for any  $\Gamma \cup \{\varphi\} \subseteq \text{Fm}_{\Lambda_1}$ ,

$$\begin{aligned}
 \Gamma \vdash_{\mathbf{S}_1} \varphi & \text{ iff } \Gamma \vdash_{\mathbf{S}^{\text{ASL}} \mathcal{K}_1} \varphi && \text{by Theorem 2.2} \\
 & \text{ iff } \{\psi \approx \mathbf{1}^{\mathcal{K}_1} : \psi \in \Gamma\} \models_{\mathcal{K}_1} \varphi \approx \mathbf{1}^{\mathcal{K}_1} \\
 & \text{ iff } \{\langle \psi, \mathbf{1}^{\mathcal{K}_1} \rangle : \psi \in \Gamma\} \vdash_{\mathbf{S}^{\text{EQL}} \mathcal{K}_1} \langle \varphi, \mathbf{1}^{\mathcal{K}_1} \rangle && \text{by Theorem 4.1} \\
 \text{only if } & \{\langle \psi^{\bar{\alpha}}, (\mathbf{1}^{\mathcal{K}_1})^{\bar{\alpha}} \rangle : \psi \in \Gamma\} \vdash_{\mathbf{S}^{\text{EQL}} \mathcal{K}_2} \langle \varphi^{\bar{\alpha}}, (\mathbf{1}^{\mathcal{K}_1})^{\bar{\alpha}} \rangle && \text{by (DE-2)} \\
 & \text{ iff } \{\langle \psi^{\bar{\alpha}}, \mathbf{1}^{\mathcal{K}_2} : \psi \in \Gamma \rangle\} \vdash_{\mathbf{S}^{\text{EQL}} \mathcal{K}_2} \langle \varphi^{\bar{\alpha}}, \mathbf{1}^{\mathcal{K}_2} \rangle \\
 & \text{ iff } \{\psi^{\bar{\alpha}} \approx \mathbf{1}^{\mathcal{K}_2} : \psi \in \Gamma\} \models_{\mathcal{K}_2} \varphi^{\bar{\alpha}} \approx \mathbf{1}^{\mathcal{K}_2} && \text{by Theorem 4.1} \\
 & \text{ iff } \Gamma^{\bar{\alpha}} \vdash_{\mathbf{S}^{\text{ASL}} \mathcal{K}_2} \varphi^{\bar{\alpha}} \\
 & \text{ iff } \Gamma^{\bar{\alpha}} \vdash_{\mathbf{S}_2} \varphi^{\bar{\alpha}} && \text{by Theorem 2.2.}
 \end{aligned}$$

This shows that  $\alpha$  is an interpretation of  $\mathbf{S}_1$  in  $\mathbf{S}_2$ . A similar argument verifies that  $\beta$  is an interpretation of  $\mathbf{S}_2$  in  $\mathbf{S}_1$ .

Since  $\langle \varphi, \varphi^{\bar{\alpha}\bar{\beta}} \rangle \in \tilde{\Omega}(\mathbf{S}^{\text{EQL}} \mathcal{K}_1)$  for any  $\varphi \in \text{Fm}_{\Lambda_1}$ , we have that  $\langle \varphi, \varphi^{\bar{\alpha}\bar{\beta}} \rangle \in \tilde{\Omega}(\mathbf{S}^{\text{ASL}} \mathcal{K}_1)$  by Lemma 4.5. By Theorem 2.2(1),  $\langle \varphi, \varphi^{\bar{\alpha}\bar{\beta}} \rangle \in \tilde{\Omega}(\mathbf{S}_1)$ . A similar argument establishes  $\langle \varphi, \varphi^{\bar{\beta}\bar{\alpha}} \rangle \in \tilde{\Omega}(\mathbf{S}_2)$  for any  $\varphi \in \text{Fm}_{\Lambda_2}$ . Hence the interpretations  $\alpha$  and  $\beta$  are mutually inverse. This completes the proof that  $\mathbf{S}_1$  and  $\mathbf{S}_2$  are definitionally equivalent.  $\blacksquare$

## 5. $\mathbf{N}$ is a substructural logic over $\mathbf{FL}_{ew}$

In this section we complete the proofs of Theorems 1.1 and 1.2. We give a (Hilbert-style) axiomatisation of a certain deductive system  $\mathbf{H}$ , and show that  $\mathbf{H}$  is  $\mathbf{FL}_{ew}$  (Lemma 5.4). We present  $\mathbf{NFL}_{ew}$  as an axiomatic extension of  $\mathbf{H}$ , and verify that its equivalent variety semantics is  $\mathcal{NFL}_{ew}$  (Corollary 5.6). From the term equivalence of the varieties  $\mathcal{NFL}_{ew}$  and  $\mathcal{N}$  (Theorem 2.1), we conclude that the deductive systems  $\mathbf{NFL}_{ew}$  and  $\mathbf{N}$  are definitionally equivalent (Theorem 1.1). It follows from this observation that  $\mathbf{N}$  is a substructural logic over  $\mathbf{FL}_{ew}$  (Theorem 1.2).

Let  $\mathbf{H}$  denote the deductive system over the language type  $\Lambda[\mathbf{FL}_{ew}]$  presented by the following collection of axioms and inference rules:<sup>12</sup>

$$(p \Rightarrow q) \Rightarrow ((q \Rightarrow r) \Rightarrow (p \Rightarrow r)) \quad (\text{A1})$$

$$(p \Rightarrow (q \Rightarrow r)) \Rightarrow (q \Rightarrow (p \Rightarrow r)) \quad (\text{A2})$$

$$p \Rightarrow (q \Rightarrow p) \quad (\text{A3})$$

$$p \Rightarrow (q \Rightarrow (p * q)) \quad (\text{A4})$$

$$(p \Rightarrow (q \Rightarrow r)) \Rightarrow ((p * q) \Rightarrow r) \quad (\text{A5})$$

$$(p \wedge q) \Rightarrow p \quad (\text{A6})$$

$$(p \wedge q) \Rightarrow q \quad (\text{A7})$$

$$(p \Rightarrow q) \Rightarrow ((p \Rightarrow r) \Rightarrow (p \Rightarrow (q \wedge r))) \quad (\text{A8})$$

$$p \Rightarrow (p \vee q) \quad (\text{A9})$$

$$q \Rightarrow (p \vee q) \quad (\text{A10})$$

$$(p \Rightarrow r) \Rightarrow ((q \Rightarrow r) \Rightarrow ((p \vee q) \Rightarrow r)) \quad (\text{A11})$$

$$\mathbf{1} \quad (\text{A12})$$

$$\mathbf{0} \Rightarrow p \quad (\text{A13})$$

$$p, p \Rightarrow q \vdash_{\mathbf{H}} q. \quad (\text{MP})$$

LEMMA 5.1. *The following rules of inference are derived rules of  $\mathbf{H}$ :*

$$p \Rightarrow q, q \Rightarrow p, r \Rightarrow s, s \Rightarrow r \vdash_{\mathbf{H}} (p \wedge r) \Rightarrow (q \wedge s)$$

$$p \Rightarrow q, q \Rightarrow p, r \Rightarrow s, s \Rightarrow r \vdash_{\mathbf{H}} (p \vee r) \Rightarrow (q \vee s).$$

PROOF. See Appendix A. ■

LEMMA 5.2. *The deductive system  $\mathbf{H}$  is regularly algebraisable with finite system of equivalence formulas  $\{p \Rightarrow q, q \Rightarrow p\}$ .*

PROOF. The proof of Raftery and van Alten [33, Proposition 2] shows that  $\mathbf{H}$  satisfies Conditions (ALG1), (ALG4), and (ALG6). Condition (ALG3) holds for  $\mathbf{H}$  trivially, while Condition (ALG2) follows from modus ponens. By the proof of [33, Proposition 2] again,  $\mathbf{H}$  satisfies Condition (ALG5) with respect to the connectives  $\Rightarrow$  and  $*$ . Further, Lemma 5.1 suffices to guarantee that  $\mathbf{H}$  satisfies Condition (ALG5) with respect to the connectives  $\wedge$  and  $\vee$ .

---

<sup>12</sup>The axioms and inference rules (A1)–(A13) and (MP) comprise a Hilbert-style presentation of  $\mathbf{FL}_{ew}$  (see Lemma 5.4 below). For other Hilbert-style axiomatisations of  $\mathbf{FL}_{ew}$ , see Ono and Komori [28] and van Alten and Raftery [45]. Both these alternative axiomatisations enjoy the separation theorem. In contrast, the presentation of  $\mathbf{FL}_{ew}$  given here lacks the separation theorem, but is convenient for applications.

Of course, Condition (ALG5) holds vacuously for  $\mathbf{H}$  with respect to the connectives  $\mathbf{0}$  and  $\mathbf{1}$ . Thus  $\mathbf{H}$  is regularly algebraisable with finite system of equivalence formulas  $\{p \Rightarrow q, q \Rightarrow p\}$ . ■

By Condition (EQV1),  $\text{Alg Mod}^* \mathbf{H}$  satisfies an identity of the form  $\varphi \approx \mathbf{1}$  for each axiom  $\varphi$  of the presentation of  $\mathbf{H}$  given above. Denote any identity so obtained by  $\varphi[\approx \mathbf{1}]$ . By algebraisability and Conditions (AX1)–(AX3),  $\text{Alg Mod}^* \mathbf{H}$  is axiomatised by the identities (A1)[ $\approx \mathbf{1}$ ]–(A13)[ $\approx \mathbf{1}$ ] together with the quasi-identities:

$$x \approx \mathbf{1} \text{ and } x \Rightarrow y \approx \mathbf{1} \text{ implies } y \approx \mathbf{1} \tag{5.1}$$

$$x \Rightarrow y \approx \mathbf{1} \text{ and } y \Rightarrow x \approx \mathbf{1} \text{ implies } x \approx y. \tag{5.2}$$

LEMMA 5.3. *Alg Mod\*  $\mathbf{H}$  is the variety of all  $\mathcal{FL}_{ew}$ -algebras.*

PROOF. Let  $\mathbf{A} \in \text{Alg Mod}^* \mathbf{H}$ . From the proof of [33, Proposition 2], we have that the  $\langle *, \Rightarrow, 1 \rangle$ -reducts of members of  $\text{Alg Mod}^* \mathbf{H}$  are pocrim. In particular, therefore,  $\langle A; *, \Rightarrow, 1 \rangle$  is a pocrim. Further, the identities (A6)[ $\approx \mathbf{1}$ ]–(A8)[ $\approx \mathbf{1}$ ] and (A9)[ $\approx \mathbf{1}$ ]–(A11)[ $\approx \mathbf{1}$ ] guarantee that for all  $a, b \in A$ ,  $a \wedge b$  and  $a \vee b$  are the greatest lower bound and least upper bound of  $\{a, b\}$  respectively with regards to the pocrim partial order  $\sqsubseteq$ .<sup>13</sup> Hence  $\langle A; \wedge, \vee \rangle$  is a lattice whose lattice order  $\leq$  is  $\sqsubseteq$ . By [41, Lemma 3.11],  $\mathbf{A}$  is a commutative, integral, residuated lattice. The identity (A13)[ $\approx \mathbf{1}$ ] can now be seen to assert that  $0 \leq a$  for all  $a \in A$ , whence  $\mathbf{A} \in \mathcal{FL}_{ew}$ . Hence  $\text{Alg Mod}^* \mathbf{H} \subseteq \mathcal{FL}_{ew}$ .

Conversely, from the well-developed arithmetic of  $\mathcal{FL}_{ew}$ -algebras [8, 19] it readily follows that  $\mathcal{FL}_{ew}$  satisfies the identities (A1)[ $\approx \mathbf{1}$ ]–(A13)[ $\approx \mathbf{1}$ ] together with the quasi-identities (5.1)–(5.2). Hence  $\mathcal{FL}_{ew} \subseteq \text{Alg Mod}^* \mathbf{H}$ . ■

LEMMA 5.4.  *$\mathbf{H}$  is  $\mathbf{FL}_{ew}$ .*

PROOF. From Lemmas 5.2 and 5.3 we have that  $\mathbf{H}$  is regularly algebraisable with equivalent variety semantics  $\mathcal{FL}_{ew}$ , while from Galatos and Ono [18, Theorems 3.3 and 3.4] we have that  $\mathbf{FL}_{ew}$  is regularly algebraisable, also with equivalent variety semantics  $\mathcal{FL}_{ew}$ .<sup>14</sup> From Theorem 2.2(1) we conclude that  $\mathbf{H} = \mathbf{S}^{\text{ASL}} \mathcal{FL}_{ew} = \mathbf{FL}_{ew}$  as desired. ■

<sup>13</sup>For the definition of the pocrim partial order, see Part I of this series [41, Section 3].

<sup>14</sup>The results of [18, Theorem 3.3, Theorem 3.4] show only that  $\mathbf{FL}_{ew}$  is algebraisable with equivalent variety semantics  $\mathcal{FL}_{ew}$ . However, it is easy to verify Condition (ALG6) holds for  $\mathbf{FL}_{ew}$ .

Let  $\mathbf{NFL}_{ew}$  denote the axiomatic extension of  $\mathbf{H}$  by the four axioms labelled (Double Negation), (Distributivity), (3-Potency), and (Nelson) of Section 1. Since any extension of a regularly algebraisable deductive system  $\mathbf{S}$  is itself regularly algebraisable, from Lemma 5.2 we have that  $\mathbf{NFL}_{ew}$  is regularly algebraisable. Moreover, from Lemma 5.4 and Condition (EQV1) we have that  $\text{Alg Mod}^* \mathbf{NFL}_{ew}$  is the subvariety of  $\mathcal{FL}_{ew}$  determined by the identities

$$\sim \sim x \Rightarrow x \approx \mathbf{1} \tag{5.3}$$

$$(x \wedge (y \vee z)) \Rightarrow ((x \wedge y) \vee (x \wedge z)) \approx \mathbf{1} \tag{5.4}$$

$$(x \Rightarrow (x \Rightarrow (x \Rightarrow y))) \Rightarrow (x \Rightarrow (x \Rightarrow y)) \approx \mathbf{1} \tag{5.5}$$

$$((x \Rightarrow (x \Rightarrow y)) \wedge (\sim y \Rightarrow (\sim y \Rightarrow \sim x))) \Rightarrow (x \Rightarrow y) \approx \mathbf{1}. \tag{5.6}$$

In [23, p. 18] Kowalski and Ono essentially observe that a variety of  $\mathcal{FL}_{ew}$ -algebras satisfies (5.3) if and only if it satisfies (DN). By (5.3), therefore,  $\text{Alg Mod}^* \mathbf{NFL}_{ew} \models$  (DN). Further, it is part of the folklore of lattice theory that a variety of lattices is distributive if and only if it satisfies the lattice inequality  $x \wedge (y \vee z) \leq (x \wedge y) \vee (x \wedge z)$ . From (5.4) it follows that  $\text{Alg Mod}^* \mathbf{NFL}_{ew} \models$  (D7)–(D8). Additionally, it is well known from the theory of BCK-algebras that any class of BCK-algebras satisfying the BCK-identity  $(x \Rightarrow^{n+1} y) \Rightarrow (x \Rightarrow^n y) \approx \mathbf{1}$  is  $n + 1$ -potent.<sup>15</sup> From (5.5) we thus have that  $\text{Alg Mod}^* \mathbf{NFL}_{ew} \models$  (E<sub>2</sub>). Summarising in the terminology of Part I [41]:  $\text{Alg Mod}^* \mathbf{NFL}_{ew}$  is a variety of 3-potent, distributive, classical  $\mathcal{FL}_{ew}$ -algebras.

LEMMA 5.5. *The variety  $\text{Alg Mod}^* \mathbf{NFL}_{ew}$  satisfies the identity:*

$$(x \Rightarrow (x \Rightarrow y)) \wedge (\sim y \Rightarrow (\sim y \Rightarrow \sim x)) \approx x \Rightarrow y. \tag{N}$$

PROOF. See Appendix A. ■

From Lemma 5.5 and the remarks directly preceding the lemma we have

COROLLARY 5.6.  *$\text{Alg Mod}^* \mathbf{NFL}_{ew}$  is the variety of Nelson  $\mathcal{FL}_{ew}$ -algebras.*

The main result of this paper, Theorem 1.1, now follows from directly from Corollary 5.6, Theorem 2.1, and Theorem 4.6.

By [41, Corollary 3.8], a Nelson algebra satisfies the identity  $x \Rightarrow y \approx x \Rightarrow (x \Rightarrow y)$ , where  $\Rightarrow$  is defined as in  $(\Rightarrow_{\text{def}})$ , if and only if it is term equivalent to a Boolean algebra. Thus  $\mathcal{NFL}_{ew} \not\models x \Rightarrow y \approx x \Rightarrow (x \Rightarrow y)$ . It follows that the deductive system  $\mathbf{NFL}_{ew}$  is not contractive, *i.e.*, (c) is not a rule of  $\mathbf{NFL}_{ew}$ . From Theorem 3.1 we thus have

---

<sup>15</sup>For the definitions of the terms  $x \Rightarrow^{n+1} y$  and  $n + 1$ -potent, see [41, Section 3].

LEMMA 5.7.<sup>16</sup>  $\mathbf{NFL}_{ew}$  is a substructural logic over  $\mathbf{FL}_{ew}$ .

The main result of this series of papers, Theorem 1.2, now follows directly from Theorem 1.1 and Lemma 5.7.

**Added in proof.** The results of this paper, together with results obtained recently by Busaniche and Cignoli in [11], imply  $\mathbf{N}$  is definitionally equivalent to the extension  $\mathbf{NFL}'_{ew}$  of the deductive system  $\mathbf{H}$  by the axioms of (Double Negation), (3-potency), and the rule of inference

$$(p * p) \Rightarrow (q * q), (\sim p * \sim p) \Rightarrow (\sim q * \sim q) \vdash p \Rightarrow q.$$

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## A. Appendix

In the following (machine-oriented) proof of Lemma 5.1,  $A, B, C$  and  $D$  denote arbitrary constants for which the hypothesis of the lemma holds and for which the corresponding conclusions necessarily follow. The justification  $[i, j]$  indicates an application of modus ponens with major premise  $i$  and minor premise  $j$ . Steps 1–7 are axioms of  $\mathbf{H}$ ; Steps 8 and 9 are the hypotheses of the lemma; and Steps 20 and 21 give the desired conclusions. Steps 20 and 21 of the proof are flagged with “\*” for easy identification.

LEMMA 5.1. *The following rules of inference are derived rules of  $\mathbf{H}$ :*

$$\begin{aligned} p \Rightarrow q, q \Rightarrow p, r \Rightarrow s, s \Rightarrow r &\vdash_{\mathbf{H}} (p \wedge r) \Rightarrow (q \wedge s) \\ p \Rightarrow q, q \Rightarrow p, r \Rightarrow s, s \Rightarrow r &\vdash_{\mathbf{H}} (p \vee r) \Rightarrow (q \vee s). \end{aligned}$$

---

<sup>16</sup>Lemma 5.7 continues to hold with respect to Galatos and Ono’s conception of substructural logic over  $\mathbf{FL}$ . Hence the main result of this series of papers, Theorem 1.2, remains valid when formulated in the framework of [18, 19].

PROOF.

- |   |              |
|---|--------------|
| 1. $(p \Rightarrow q) \Rightarrow ((q \Rightarrow r) \Rightarrow (p \Rightarrow r))$            | [(A1)]       |
| 2. $(p \wedge q) \Rightarrow p$   | [(A6)]       |
| 3. $(p \wedge q) \Rightarrow q$   | [(A7)]       |
| 4. $(p \Rightarrow q) \Rightarrow ((p \Rightarrow r) \Rightarrow (p \Rightarrow (q \wedge r)))$ | [(A8)]       |
| 5. $p \Rightarrow (p \vee q)$   | [(A9)]       |
| 6. $p \Rightarrow (q \vee p)$   | [(A10)]      |
| 7. $(p \Rightarrow q) \Rightarrow ((r \Rightarrow q) \Rightarrow ((p \vee r) \Rightarrow q))$   | [(A11)]      |
| 8. $A \Rightarrow B$  | [Assumption] |
| 9. $C \Rightarrow D$  | [Assumption] |
| 10. $(p \Rightarrow q) \Rightarrow ((p \wedge r) \Rightarrow q)$                                | [1, 2]       |
| 11. $(p \Rightarrow q) \Rightarrow ((r \wedge p) \Rightarrow q)$                                | [1, 3]       |
| 12. $(B \Rightarrow p) \Rightarrow (A \Rightarrow p)$   | [1, 8]       |
| 13. $(D \Rightarrow p) \Rightarrow (C \Rightarrow p)$   | [1, 9]       |
| 14. $A \Rightarrow (B \vee p)$  | [12, 5]      |
| 15. $(p \Rightarrow (B \vee q)) \Rightarrow ((A \vee p) \Rightarrow (B \vee q))$                | [7, 14]      |
| 16. $C \Rightarrow (p \vee D)$  | [13, 6]      |
| 17. $(A \wedge p) \Rightarrow B$  | [10, 8]      |
| 18. $((A \wedge p) \Rightarrow q) \Rightarrow ((A \wedge p) \Rightarrow (B \wedge q))$          | [4, 17]      |
| 19. $(p \wedge C) \Rightarrow D$  | [11, 9]      |
| *20. $(A \vee C) \Rightarrow (B \vee D)$  | [15, 16]     |
| *21. $(A \wedge C) \Rightarrow (B \wedge D)$  | [18, 19]     |

■

In the (machine-oriented) proof of Lemma 5.5 below, the justification  $[i \rightarrow j]$  indicates paramodulation from  $i$  into  $j$ ; that is, unifying the left-hand side of  $i$  with a subterm of  $j$ , instantiating  $j$  with the corresponding substitution, and replacing the subterm with the corresponding instance of the right-hand side of  $i$ . The labels (D3), (M1), etc., in Steps 1–2, 4, and 6–10 indicate identities established in Part I [41].

LEMMA 5.5. *The variety  $\text{Alg Mod}^* \mathbf{NFL}_{ew}$  satisfies the identity:*

$$(x \Rightarrow (x \Rightarrow y)) \wedge (\sim y \Rightarrow (\sim y \Rightarrow \sim x)) \approx x \Rightarrow y. \quad (\text{N})$$

PROOF.

- |   |                 |
|---|-----------------|
| 1. $x \vee y \approx y \vee x$  | [(D3)]          |
| 2. $x * \mathbf{1} \approx x$   | [(M1)]          |
| 3. $\sim x := x \Rightarrow \mathbf{0}$   | [( $\sim$ def)] |
| 4. $\sim \sim x \approx x$  | [(DN)]          |
| 5. $((x \Rightarrow (x \Rightarrow y)) \wedge (\sim y \Rightarrow (\sim y \Rightarrow \sim x))) \Rightarrow (x \Rightarrow y) \approx \mathbf{1}$ | [(5.6)]         |

- 6.  $x \Rightarrow (y \Rightarrow x) \approx \mathbf{1}$  [(3.17)]
- 7.  $x \Rightarrow (y \Rightarrow z) \approx y \Rightarrow (x \Rightarrow z)$  [(3.18)]
- 8.  $(x \Rightarrow y) \wedge (x \Rightarrow z) \approx x \Rightarrow (y \wedge z)$  [(4.2)]
- 9.  $(x * (x \Rightarrow y)) \vee y \approx y$  [(4.3)]
- 10.  $(x \Rightarrow y) \wedge (z \Rightarrow y) \approx (x \vee z) \Rightarrow y$  [(4.4)]
- 11.  $x \Rightarrow (y \Rightarrow \mathbf{0}) \approx y \Rightarrow \sim x$  [3  $\rightarrow$  7]
- 12.  $x \Rightarrow (((x \Rightarrow (x \Rightarrow y)) \wedge (\sim y \Rightarrow (\sim y \Rightarrow \sim x))) \Rightarrow y) \approx \mathbf{1}$  [7  $\rightarrow$  5]
- 13.  $x \vee (y * (y \Rightarrow x)) \approx x$  [9  $\rightarrow$  1]
- 14.  $(x \Rightarrow y) \vee (y * \mathbf{1}) \approx x \Rightarrow y$  [6  $\rightarrow$  13]
- 15.  $(x \Rightarrow y) \vee (z * (x \Rightarrow (z \Rightarrow y))) \approx x \Rightarrow y$  [7  $\rightarrow$  13]
- 16.  $x \Rightarrow \sim y \approx y \Rightarrow \sim x$  [3  $\rightarrow$  11]
- 17.  $\sim x \Rightarrow \sim y \approx y \Rightarrow x$  [4  $\rightarrow$  16]
- 18.  $(x \Rightarrow y) \vee y \approx x \Rightarrow y$  [2  $\rightarrow$  14]
- 19.  $x \vee (y \Rightarrow x) \approx y \Rightarrow x$  [18  $\rightarrow$  1]
- 20.  $(x \Rightarrow y) \vee (x \Rightarrow (z \Rightarrow y)) \approx z \Rightarrow (x \Rightarrow y)$  [7  $\rightarrow$  19]
- 21.  $x \Rightarrow (((x \Rightarrow (x \Rightarrow y)) \wedge (\sim y \Rightarrow (x \Rightarrow y))) \Rightarrow y) \approx \mathbf{1}$  [17  $\rightarrow$  12]
- 22.  $x \Rightarrow (((x \Rightarrow (x \Rightarrow y)) \wedge (x \Rightarrow (\sim y \Rightarrow y))) \Rightarrow y) \approx \mathbf{1}$  [7  $\rightarrow$  21]
- 23.  $x \Rightarrow ((x \Rightarrow ((x \Rightarrow y) \wedge (\sim y \Rightarrow y))) \Rightarrow y) \approx \mathbf{1}$  [8  $\rightarrow$  22]
- 24.  $x \Rightarrow ((x \Rightarrow ((x \vee \sim y) \Rightarrow y)) \Rightarrow y) \approx \mathbf{1}$  [10  $\rightarrow$  23]
- 25.  $(x \Rightarrow y) \vee ((x \Rightarrow ((x \vee \sim y) \Rightarrow y)) * \mathbf{1}) \approx x \Rightarrow y$  [24  $\rightarrow$  15]
- 26.  $(x \Rightarrow y) \vee (x \Rightarrow ((x \vee \sim y) \Rightarrow y)) \approx x \Rightarrow y$  [2  $\rightarrow$  25]
- 27.  $(x \vee \sim y) \Rightarrow (x \Rightarrow y) \approx x \Rightarrow y$  [26  $\rightarrow$  20]
- 28.  $(x \Rightarrow (x \Rightarrow y)) \wedge (\sim y \Rightarrow (x \Rightarrow y)) \approx x \Rightarrow y$  [27  $\rightarrow$  10]
- 29.  $(x \Rightarrow (x \Rightarrow y)) \wedge (\sim y \Rightarrow (\sim y \Rightarrow \sim x)) \approx x \Rightarrow y$  [17  $\rightarrow$  28]

■

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