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New distinct curves having the same complement in the projective plane

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Abstract In 1984, Yoshihara conjectured that if two plane irreducible curves have isomorphic complements, they are projectively equivalent, and proved the conjecture for a special family of unicuspidal curves. Recently, Blanc gave counterexamples of degree 39 to this conjecture, but none of these is unicuspidal. In this text, we give a new family of counterexamples to the conjecture, all of them being unicuspidal, of degree $4m + 1$ for any $m \geq 2$. In particular, we have counterexamples of degree 9, which seems to be the lowest possible degree.

1 The conjecture

In the sequel, we will work with algebraic varieties over a fixed ground field $\mathbb{K}$, which can be arbitrary.

Conjecture 1.1 ([2]) *Suppose that the ground field is algebraically closed of characteristic zero. Let $C \subset \mathbb{P}^2$ be an irreducible curve. Suppose that $\mathbb{P}^2 \backslash C$ is isomorphic to $\mathbb{P}^2 \backslash D$ for some curve D. Then C and D are projectively equivalent, i.e. there is an automorphism $\alpha : \mathbb{P}^2 \to \mathbb{P}^2$ such that $\alpha(C) = D$.*

This conjecture leads to several alternatives. Let $\psi : \mathbb{P}^2 \backslash C \to \mathbb{P}^2 \backslash D$ be an isomorphism. If the conjecture holds, then:

- either ψ extends to an automorphism of $\mathbb{P}^2$ and we can choose $\alpha := \psi$.
- or ψ extends to a strict birational map $\psi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$. In this case, there is an automorphism $\alpha : \mathbb{P}^2 \to \mathbb{P}^2$ such that $\alpha(C) = D$.

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Otherwise, if ψ gives a counterexample to the conjecture, then:

- either C and D are not isomorphic.
- or C and D are isomorphic, but not by an automorphism of $\mathbb{P}^2$.

In this text, we are going to study the conjecture in the case of curves of type I.

Definition 1.2 We say that a curve $C \subset \mathbb{P}^2$ is of **type I** if there is a point $p \in C$ such that $C\backslash p$ is isomorphic to $\mathbb{A}^1$.
We say that a curve $C \subset \mathbb{P}^2$ is of **type II** if there is a line $L \subset \mathbb{P}^2$ such that $C\backslash L$ is isomorphic to $\mathbb{A}^1$.

All curves of type II are of type I, but the converse is false in general. Moreover, a curve of type I is a line, a conic, or a unicuspidal curve (a curve with one singularity of cuspidal type).

In the case of curves of type II, Yoshihara [2] showed that the conjecture is true, but in general the conjecture does not hold. Some counterexamples are given in [1], but these curves are not of type I.

In this article, we give a new family of counterexamples, of degree $4m + 1$ for any $m \geq 2$. These are all of type I, and some of them have degree 9, which seems to be the lowest possible degree (see the end of the article for more details). In Sect. 2 we give a general way to construct examples, that we precise in Sect. 3. The last section is the conclusion.

2 The construction

We begin with giving a general construction, which provides isomorphisms of the form $\mathbb{P}^2\backslash C \to \mathbb{P}^2\backslash D$ where C, D are curves in $\mathbb{P}^2$. We start with the following definition:

Definition 2.1 We say that a morphism $\pi : S \to \mathbb{P}^2$ is a (-1)**—tower resolution** of a curve C if:

(1) $\pi = \pi_1 \circ \cdots \circ \pi_m$ where π_i is the blow-up of a point p_i,
(2) $\pi_i(p_{i+1}) = p_i$ for $i = 1, \ldots, m-1$,
(3) the strict transform of C in S is a smooth curve, isomorphic to $\mathbb{P}^1$, and has self-intersection -1.

The isomorphisms of the form $\mathbb{P}^2\backslash C \to \mathbb{P}^2\backslash D$ are closely related to $(-1)-$tower resolutions of C and D because of the following Lemma:

Lemma 2.2 ([1]) *Let $C \subset \mathbb{P}^2$ be an irreducible algebraic curve and $\psi : \mathbb{P}^2\backslash C \to \mathbb{P}^2\backslash D$ an isomorphism. Then, either ψ extends to an automorphism of $\mathbb{P}^2$, or it extends to a strict birational map $\phi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$.*

Consider the second case. Let $\chi : X \to \mathbb{P}^2$ a minimal resolution of the indeterminacies of ϕ, call $\tilde{E}_1, \ldots, \tilde{E}_m$ and $\tilde{C}$ the strict transforms of its exceptional curves and C in X and set $\epsilon := \phi \circ \chi$. Then:

(1) *χ is a $(-1)-$tower resolution of C*
(2) *ϵ collapses $\tilde{C}, \tilde{E}_1, \ldots, \tilde{E}_{m-1}$ and $\epsilon(\tilde{E}_m) = D$,*
(3) *ϵ is a $(-1)-$tower resolution of D.*

Remark 2.3 This lemma shows that if C does not admit a $(-1)-$tower resolution, then every isomorphism $\mathbb{P}^2\backslash C \to \mathbb{P}^2\backslash D$ extends to an automorphism of $\mathbb{P}^2$. So counterexamples will be given by rational curves with only one singularity.

We start with a smooth conic $Q \subset \mathbb{P}^2$ and $\phi \in \mathbf{Aut}(\mathbb{P}^2 \backslash Q)$ which extends to a strict birational map $\phi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$. Call $p_1, \ldots, p_m$ the indeterminacies points of ϕ; according to Lemma 2.2, we can order the points so that p_1 is a point of $\mathbb{P}^2$ and p_i is infinitely near to p_{i-1} for $i = 2, \ldots, n$. Consider $\chi : X \to \mathbb{P}^2$, a minimal resolution of the indeterminacies of ϕ and set $\epsilon := \phi \circ \chi$. Lemma 2.2 says that:

(1) χ is a $(-1)-$tower resolution of Q,
(2) ϵ collapses $\tilde{Q}, \tilde{E}_1, \ldots, \tilde{E}_{m-1}$ and $\epsilon(\tilde{E}_m) = Q$,
(3) ϵ is a $(-1)-$tower resolution of Q.

Now, consider a line $L \subset \mathbb{P}^2$, which is tangent to Q at $p \neq p_1$. Since ϕ contracts Q, then $C := \phi(L)$ is a curve with a unique singular point which is $\phi(Q)$. Since $L \cap (\mathbb{P}^2 \backslash Q) \simeq \mathbb{A}^1$, we have $C \cap (\mathbb{P}^2 \backslash Q) \simeq \mathbb{A}^1$, which means that C is of type I.

Consider now a birational map $f \in \mathbf{Aut}(\mathbb{P}^2 \backslash L)$ which extends to a strict birational map $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ and satisfies:

(1) $f(Q) = Q$,
(2) $f(p_1) = p_1$.

Now, we are going to get a new birational map $\phi' : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ which restricts to an automorphism of $\mathbb{P}^2 \backslash Q$ using the p_i's and f. Set:

$$p_i' := f(p_i).$$

Note that p_i' is a well-defined point infinitely near to p_{i-1}' for $i > 1$.
Let's call $\chi' : X' \to \mathbb{P}^2$ the blow-up of the p_i''s and $\tilde{E}_1', \ldots, \tilde{E}_m'$ and $\tilde{Q}'$ the strict transforms of the exceptional curves of χ' and of Q in X'.
Since $f(Q) = Q$ and f is an isomorphism in a neighbourhood of p_1, the intersections between $\tilde{E}_1, \ldots, \tilde{E}_m$ and $\tilde{Q}'$ are the same as those between $\tilde{E}_1, \ldots, \tilde{E}_m$ and $\tilde{Q}$. Then there is a morphism $\epsilon' : X' \to \mathbb{P}^2$ which contracts $\tilde{E}_1', \ldots, \tilde{E}_{m-1}'$ and $\tilde{Q}'$. Moreover, $\epsilon'(\tilde{E}_m')$ is a conic, and up to composing by an automorphism of $\mathbb{P}^2$, we can suppose that $\epsilon'(\tilde{E}_m') = Q$.
By construction, the birational map ϕ' restricts to an automorphism of $\mathbb{P}^2 \backslash Q$. In fact, none of the p_i''s belongs to L (as proper or infinitely near point), so $\phi'(L)$ is well defined. Moreover, ϕ' collapses Q, so $D := \phi'(L)$ is a curve with a unique singular point which is $\phi'(Q)$.
Set then $\psi := \phi' \circ f \circ \phi^{-1}$. We have the following commutative diagram:

$$\begin{tikzcd}
& X' \arrow[dl, "\chi'"'] \arrow[dr, "\epsilon'"] & \\
\mathbb{P}^2 \arrow[rr, dashed, "\phi'"] & & \mathbb{P}^2 \\
& X \arrow[dl, "\chi"'] \arrow[dr, "\epsilon"] & \\
\mathbb{P}^2 \arrow[rr, dashed, "\phi"] \arrow[uu, dashed, "f"] & & \mathbb{P}^2 \arrow[uu, dashed, "\psi"']
\end{tikzcd}$$

Lemma 2.4 *The map $\psi : \mathbb{P}^2 \backslash C \to \mathbb{P}^2 \backslash D$ induced by the birational map defined above is an isomorphism.*

Proof Since $\phi, \phi' \in \mathbf{Aut}(\mathbb{P}^2 \backslash Q)$ and $f \in \mathbf{Aut}(\mathbb{P}^2 \backslash L)$, we only have to check that $\psi(Q) = Q$.
Let $\chi : X \to \mathbb{P}^2$ (resp. $\chi' : X' \to \mathbb{P}^2$) be a minimal resolution of the indeterminacies of ϕ (resp. ϕ') and write $\epsilon := \phi \circ \chi$ (resp. $\epsilon' := \phi' \circ \chi'$). Call $\tilde{E}_1, \ldots, \tilde{E}_m$ (resp. $\tilde{E}_1', \ldots, \tilde{E}_m'$) the strict transforms of the exceptional curves of χ (resp. χ') in X (resp. X').

It follows from Lemma 2.2 that $\epsilon(\tilde{E}_m) = Q$ (resp. $\epsilon'(\tilde{E}'_m) = Q$). Then factorising ψ we get $\psi(Q) = Q$. □

Now we study the automorphisms $\alpha \in \mathbf{Aut}(\mathbb{P}^2)$ such that $\alpha(C) = D$.

Lemma 2.5 *If $\alpha \in \mathbf{Aut}(\mathbb{P}^2)$ sends C onto D, then $a := (\phi')^{-1} \circ \alpha \circ \phi$ is an automorphism of $\mathbb{P}^2$ and satisfies:*

(1) $a(L) = L$,
(2) $a(Q) = Q$,
(3) $a(p_i) = p'_i$ *for* $i = 1, \ldots, m$.

$$\begin{tikzcd}
 & X' \arrow[dl, "\chi'"'] \arrow[dr, "\epsilon'"] & \\
\mathbb{P}^2 \arrow[rr, dashed, "\phi'"] & & \mathbb{P}^2 \\
 & X \arrow[dl, "\chi"'] \arrow[dr, "\epsilon"] & \\
\mathbb{P}^2 \arrow[uu, dashed, "a"] \arrow[rr, dashed, "\phi"] & & \mathbb{P}^2 \arrow[uu, dashed, "\alpha"']
\end{tikzcd}$$

Proof Call $q_1, \ldots, q_m$ (resp. $q'_1, \ldots, q'_m$) the points blown-up by ϵ (resp. ϵ'). Then these points are the singular points of C (resp. D). Since α is an automorphism such that $\alpha(C) = D$, then α sends q_i on q'_i for $i = 1, \ldots, m$, and lifts to an isomorphism $X \to X'$ which sends $\tilde{E}_i$ on $\tilde{E}'_i$ for $i = 1, \ldots, m - 1$ and $\tilde{Q}$ on $\tilde{Q}'$.
Since Q is the conic through $q_1, \ldots, q_5$, then $\alpha(Q) = Q$, and the isomorphism $X \to X'$ sends $\tilde{E}_m$ on $\tilde{E}'_m$. So χ and χ' contract the curves in X and X' which correspond by mean of this isomorphism, and we deduce that $a \in \mathbf{Aut}(\mathbb{P}^2)$.
It follows then that a sends p_i on p'_i, $a(Q) = Q$ and that $a(L) = L$. □

3 The counterexample

In this section, we describe more explicitly the construction given in the previous section, by giving more concrete examples.
We choose $n \geq 1$ and will define $\Delta \colon X \to \mathbb{P}^2$ which is the blow-up of some points $p_1, \ldots, p_{4+2n}$, such that $p_1 \in \mathbb{P}^2$, and for $i \geq 2$ the point p_i is infinitely near to p_{i-1}. We call E_i the exceptional curve associated to p_i and $\tilde{E}_i$ its strict transform in X. The points will be chosen so that:

- p_i belongs to Q (as proper or infinitely near points) if and only if $i \in \{1, \ldots, 4\}$,
- p_i belongs (as a proper or infinitely near point) to E_4 if and only if $i \in \{5, \ldots, 4 + n\}$,
- $p_i \in E_{i-1} \backslash E_{i-2}$ if $i \in \{5 + n, \ldots, 4 + 2n\}$.

Note that $p_1, \ldots, p_{4+n}$ are fixed by these conditions, and that $p_{5+n}, \ldots, p_{4+2n}$ depend on parameters. On the surface X, we obtain the following dual graph of curves (see Fig. 1).

The symmetry of the graph implies the existence of a birational morphism $\epsilon \colon X \to \mathbb{P}^2$ which contracts the curves $\tilde{E}_1, \ldots, \tilde{E}_{3+2n}, \tilde{Q}$, and which sends E_{4+2n} on a conic. We may choose that this conic is Q, so that $\phi = \epsilon \circ \Delta^{-1}$ restricts to an automorphism of $\mathbb{P}^2 \backslash Q$.
Calculating auto-intersection, the image by ϕ of a line of the plane which does not pass through p_1 has degree $4n + 1$.

3.1 Choosing the points

Now we are going to choose the birational maps f and the points which define ϕ in order to get two curves which give a counterexample to the conjecture of Yoshihara.

We choose that L is the line of equation $z = 0$, Q is the conic of equation $xz = y^2$ and $p_1 = (0 : 0 : 1)$.

We define the birational map $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ by:

$$f(x : y : z) = \left(\mu^2(\lambda xz + (1-\lambda)y^2) : \mu yz : z^2\right) \quad \text{with } \lambda, \mu \in \mathbb{K}^* \quad \text{and} \quad \lambda \neq 1.$$

The map f preserves Q, and is an isomorphism at a local neighbourhood of p_1. In consequence, f sends respectively $p_1, \ldots, p_{4+2n}$ on some points $p'_1, \ldots, p'_{4+2n}$ which will define $\Delta' \colon X \to \mathbb{P}^2$, $\epsilon' \colon X' \to \mathbb{P}^2$ and $\phi' = \epsilon' \circ (\Delta')^{-1}$ in the same way as ϕ was constructed.

We describe now the points p_i and p'_i in local coordinates.

Since f preserves Q and fixes p_1, we have $p'_i = p_i$ for $i = 1, \ldots, 4$. Locally, the blow-up of $p_1, \ldots, p_4$ corresponds to:

$$\phi_4 \colon \mathbb{A}^2 \to \mathbb{P}^2, \quad \phi_4(x, y) = (xy^4 + y^2 : y : 1).$$

The curve E_4 corresponds to $y = 0$, and the conic $\tilde{Q}$ to $x = 0$. The lift of f in these coordinates is:

$$(x, y) \mapsto (\lambda\mu^2 x, \mu y).$$

The blow-up of the points $p_5, \ldots, p_{4+n}$ (which are equal to $p'_5, \ldots, p'_{4+n}$) now corresponds to:

$$\phi_{4+n} \colon \mathbb{A}^2 \to \mathbb{A}^2, \quad \phi_{4+n}(x, y) = (x, x^n y).$$

So the lift of f corresponds to:

$$(x, y) \mapsto \left(\lambda\mu^2 x, \frac{y}{\lambda^n \mu^{2n-1}}\right).$$

We set $p_{4+n+i} = (0, a_i)$ for $i \in \{1, \ldots, n\}$ with $a_n \neq 0$. The blow-up of $p_{5+n}, \ldots, p_{4+n+i}$ now corresponds to:

$$\phi_{4+n+i} : \mathbb{A}^2 \to \mathbb{A}^2, \quad \phi_{4+n+i}(x, y) = \left(x, x^i y + P_i(x)\right) \quad \text{where } P_i(x) = a_1 x^{i-1} + \cdots + a_i.$$

Since f sends p_i on p'_i, we can set $p'_{4+n+i} = (0, b_i)$ for $i \in \{1, \ldots, n\}$ with $b_n \neq 0$. The blow-up of $p'_{5+n}, \ldots, p'_{4+n+i}$ then corresponds to:

$$\phi'_{4+n+i} : \mathbb{A}^2 \to \mathbb{A}^2, \quad \phi'_{4+n+i}(x, y) = \left(x, x^i y + Q_i(x)\right) \quad \text{where } Q_i(x) = b_1 x^{i-1} + \cdots + b_i.$$

So the lift of f corresponds to:

$$(x, y) \mapsto \left(\lambda\mu^2 x, \frac{x^i y + P_i(x) - \lambda^i \mu^{2i-1} Q_i(\lambda\mu^2 x)}{\lambda^i \mu^{2i-1} x^i}\right).$$

The curves E_{4+n+i} and E'_{4+n+i} correspond to $x = 0$ in both local charts. Since f is a local isomorphism which sends p_i on p'_i for each i, it has to be defined on the line $x = 0$. Because P_i and Q_i have both degree $i - 1$, this implies that:

$$P_i(x) = \lambda^i \mu^{2i-1} Q_i(\lambda\mu^2 x) \quad \text{for } i = 1, \ldots, n.$$

In particular, the coefficients satisfy:

$$a_i = \lambda^i \mu^{2i-1} b_i \quad \text{for } i = 1, \ldots, n.$$

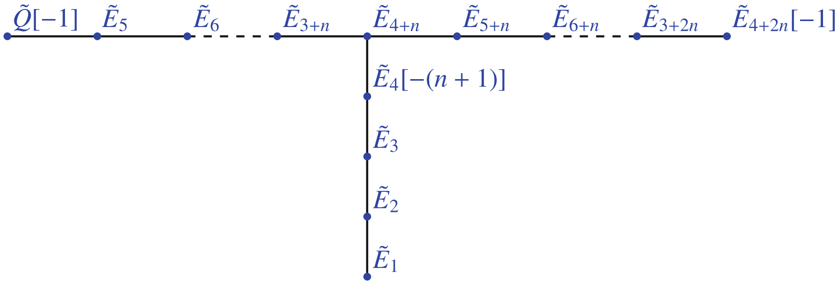

Fig. 1 The dual graph of the curves $\tilde{E}_1, \ldots, \tilde{E}_{3+2n}, E_{4+2n}, \tilde{Q}$. Two curves have an edge between them if and only they intersect, and their self-intersection is written in brackets, if and only if it is not -2

3.2 The counterexample

Now to get a counter example, we must show that any automorphism $a : \mathbb{P}^2 \to \mathbb{P}^2$ such that $a(L) = L, a(Q) = Q$ and $a(p_1) = p_1$ does not send p_i on p_i' for at least one $i \in \{5+n, \ldots, 4+2n\}$. Let's start with the following Lemma:

Lemma 3.1 *Let $a : \mathbb{P}^2 \to \mathbb{P}^2$ be an automorphism such that $a(L) = L, a(Q) = Q$ and $a(p_1) = p_1$. Then a is of the form:*

$$a(x : y : z) = \left(k^2 x : ky : z\right) \quad \textit{where } k \in \mathbb{K}^*.$$

Proof Follows from a direct calculation. □

Theorem 3.2 *If $n \geq 2$, the curves C and D obtained from the construction of the previous section give a counter example to the conjecture.*

Proof Choose $a_n = a_{n-1} = 1$.
Since a is an automorphism, it lifts to an automorphism which sends E_{4+n+i} on E'_{4+n+i}. Put $\lambda = 1$ and $\mu = k$ in the formula for f. Then this lift corresponds to:

$$(x, y) \mapsto \left(k^2 x, \frac{x^i y + P_i(x) - k^{2i-1} Q_i(k^2 x)}{k^{2i-1} x^i}\right)$$

where P_i and Q_i are the polynomials defined above.
Since E_{4+n+i} and E'_{4+n+i} both correspond to $x = 0$ in local charts, this lift has to be well defined on $x = 0$. So since P_i and Q_i both have degree $i - 1$, we get:

$$P_i(x) = k^{2i-1} Q_i(k^2 x) \quad \text{for } i = 1, \ldots, n$$

and the constant terms satisfy $a_i = k^{2i-1} b_i$ for $i = 1, \ldots, n$.
Since $a_n, a_{n-1} \neq 0$, then $b_n, b_{n-1} \neq 0$. As explained in the previous section, a sends p_i on p_i', so we get:

$$\lambda^i \mu^{2i-1} b_i = k^{2i-1} b_i \quad \text{for } i = 1, \ldots, n.$$

This formula for $i = n$ and $i = n - 1$ gives $\lambda = 1$ or $\mu = 0$, which leads to a contradiction. □

4 Conclusion

We conclude by observing that the curves C and D of the previous construction have degree $4n + 1$ (using Fig. 1) and are of type I. In particular, we get a counterexample with a curve of degree 9 when $n = 2$. One can check by direct computation that the conjecture holds for irreducible curves of type I up to degree 5, because there is only one curve of degree 5 which is of type I and not of type II, up to automorphism of $\mathbb{P}^2$. One can also check that all irreducible curves of type I of degree 6, 7 and 8 are of type II. So the curves of degree 9 given by this construction leads to a counterexample of minimal degree among the curves of type I.

If we consider the conjecture for all rational curves, the counterexamples in [1] are of degree 39 (and not of type I). So we have new counterexamples with curves of lower degree. It seems that the curves of degree 9 give counterexamples of minimal degree among the rational curves, but it hasn't been shown yet.

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