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Boundary Non-crossings of Brownian Pillow

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Abstract Let $B_0(s,t)$ be a Brownian pillow with continuous sample paths, and let $h,u:[0,1]^2\to\mathbb{R}$ be two measurable functions. In this paper we derive upper and lower bounds for the boundary non-crossing probability

$$\psi(u;h):=\mathbf{P}\big\{B_0(s,t)+h(s,t)\le u(s,t),\forall s,t\in[0,1]\big\}.$$

Further we investigate the asymptotic behaviour of $\psi(u;\gamma h)$ with γ tending to ∞ and solve a related minimisation problem.

Keywords Boundary non-crossing probability · Brownian pillow with trend · Large deviations · Smallest concave majorant · Reproducing kernel Hilbert space · Small ball probabilities

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1 Introduction

Let $B_0(s,t), s,t\in[0,1]$ be a Brownian pillow with continuous sample paths. Its covariance function K is the product of two covariance functions defined by

$$K\big((s_1,t_1),(s_2,t_2)\big)=K_1(s_1,t_1)K_2(s_2,t_2),\quad s_i,t_i\in[0,1],\ i=1,2,$$

with $K_i(s,t)=\min(s,t)-ts,\ \ i=1,2,$ the covariance function of a Brownian bridge.

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Our concern in this article is the boundary non-crossing probability

$$\psi(u;h) := \mathbf{P}\big\{B_0(s,t) + h(s,t) \le u(s,t), \forall s,t \in [0,1]\big\} \tag{1.1}$$

with a trend function h and a measurable boundary function u.

When considering a Brownian bridge and a Brownian motion, the corresponding non-crossing probability can be explicitly calculated if h and u are polygonal lines, see e.g. [5, 11, 14, 26, 29] and the references therein. Such explicit formulae are not available in our setup of the multi-parameter processes.

Our novel results presented below are:

(a) upper and lower bounds for $\psi(u;h)$,
(b) a large deviation type result for the boundary non-crossing probability $\psi(u;\gamma h)$ with $\gamma \to \infty$, and
(c) we solve a related minimisation problem.

We comment briefly the result mentioned in (b). Given a function $g : [0,\infty)^2 \to \mathbb{R}$, we denote by g'' its partial derivative obtained by differentiating both components, provided that it exists. From the large deviation theory (see e.g. [24] or [21]) for any positive constant c and any trend function $h : [0,1]^2 \to \mathbb{R}$ with a square-integrable partial derivative h'' (i.e. $\int_{[0,1]^2} (h''(s,t))^2\, ds\, dt < \infty$), we obtain

$$\begin{aligned}\lim_{\gamma\to\infty} 2\gamma^{-2} \ln \boldsymbol{P}\Big\{\sup_{s,t\in[0,1]} \big(B_0(s,t) + \gamma h(s,t)\big) \le c\Big\}\\ = -\int_{[0,1]^2} \big(\underline{h}''(s,t)\big)^2\, ds\, dt \in (-\infty, 0]\end{aligned} \tag{1.2}$$

with $\underline{h}$ the solution of the minimisation problem

$$\inf_{g\ge h} \int_{[0,1]^2} \big(g''(s,t)\big)^2\, ds\, dt, \tag{1.3}$$

where the functions $g : [0,1]^2 \to \mathbb{R}$ in the minimisation problem are assumed to possess a square-integrable partial derivative g'', and g, h vanish on the boundary of $[0,1]^2$.

Compared to (1.2), our new result is a sharper asymptotic estimate of the boundary non-crossing probability of interest. In the special case h being a product of two concave functions $h_1, h_2 : [0,1] \to [0,\infty)$ with $h_i(0) = h_i(1) = 0$, $i = 1,2$, we show (see below (4.5))

$$\begin{aligned}&\boldsymbol{P}\Big\{\sup_{s,t\in[0,1]} \big(B_0(s,t) + \gamma h_1(s) h_2(t)\big) \le c\Big\}\\ &= \exp\bigg(-\frac{\gamma^2}{2} \prod_{i=1,2} \int_{[0,1]} \big(h_i'(x)\big)^2 \lambda(dx) + c\gamma \prod_{i=1,2} \big[h_i'(1) - h_i'(0)\big] + z(\gamma)\bigg),\end{aligned} \tag{1.4}$$

where

$$-A\gamma^{2/3} \ln^3 \gamma \le z(\gamma) \le \ln \boldsymbol{P}\Big\{\sup_{s,t\in[0,1]} B_0(s,t) \le c\Big\}$$

holds for all large γ with positive constant A not depending on γ. Here h_i' is a right-continuous version of the derivative of h_i, $i = 1, 2$, and λ is the Lebesgue measure on $[0, 1]$.

We derive (1.4) utilising a known small ball result for a Brownian pillow. Indeed the small ball problem for both a Brownian pillow and a Brownian sheet is investigated by several authors, see [6–8, 10, 16–18, 20, 23, 28] among many other references.

A consequence of the Gaussian shift inequality (see [22]) and (1.4) is the following bound (set D for the set of all concave functions $f : [0, 1] \to [0, \infty)$):

$$\boldsymbol{P}\Big\{\sup_{s,t\in[0,1]} B_0(s,t) \le c\Big\} \le \inf_{h\in D} \Phi\left(c^2\left(\frac{h'(1) - h'(0)}{\int_0^1 (h'(x))^2\,\lambda(dx)}\right)^2\right) \tag{1.5}$$

with Φ the distribution function of a Gaussian random variable with mean 0 and variance 1. Since the upper bound in (1.5) is not smaller than $1/2$, the above inequality is of some interest, provided that $\psi(0; c) \in (1/2, 1)$.

Organisation of the paper: In the next section we present some notation and preliminary results. The main results are discussed in Sect. 3. Section 4 explains the simple situation where the trend function h is a product of two trend functions. Proofs of all the results are relegated to Sect. 5 followed by a short Appendix with two results on the Riemann–Stieltjes integral.

2 Preliminaries

We first introduce a Hilbert space related to the covariance function of a Brownian pillow, which can also be seen as tensor product of Hilbert spaces related to the covariance function of a Brownian bridge. Then we provide a result utilised in solving the minimisation problem (1.3).

The reproducing kernel Hilbert space (RKHS) related to the covariance function of a Brownian pillow, denoted by $\mathcal{H}_2^0$, is given by

$$\begin{aligned}\mathcal{H}_2^0 := \Big\{ & h : [0,1]^2 \to \mathbb{R} \mid \exists h'' \in L_2\big([0,1]^2, \lambda^2\big), \text{ with} \\ & h(s,t) = \int_{[0,s]\times[0,t]} h''(x,y)\,\lambda^2(dx,dy), \\ & h(0,s) = h(1,s) = h(t,0) = h(t,1) = 0,\ \forall s,t \in [0,1]\Big\},\end{aligned}$$

where $L_2([0,1]^2, \lambda^2)$ is the set of all real functions on $[0,1]^2$ square integrable with respect to the Lebesgue measure λ^2 on $[0,1]^2$. The inner product is

$$\langle h_1, h_2\rangle = \int_{[0,1]^2} h_1''(x,y) h_2''(x,y)\,\lambda^2(dx,dy), \quad h_1, h_2 \in \mathcal{H}_2^0,$$

and the corresponding norm of $h \in \mathcal{H}_2^0$ is $\|h\| := \langle h, h\rangle^{1/2}$.

As shown in [17], another approach to deal with $\mathcal{H}_2^0$ is to construct this Hilbert space as the tensor product of two RKHS, i.e. $\mathcal{H}_2^0 = \mathcal{H}_1^0 \otimes \mathcal{H}_1^0$ with the RKHS $\mathcal{H}_1^0$ of the covariance function of a Brownian bridge defined by

$$\mathcal{H}_1^0 := \Big\{ h : [0,1] \to \mathbb{R} \mid \exists h' \in L_2\big([0,1],\lambda\big) \text{ with}$$
$$h(s) = \int_{[0,s]} h'(x)\,\lambda(dx),\ h(0) = h(1) = 0 \Big\},$$

where $L_2([0,1],\lambda)$ is the set of all real functions on $[0,1]$ square integrable with respect to λ. The inner product of $\mathcal{H}_1^0$ is

$$\langle h_1, h_2 \rangle = \int_{[0,1]} h_1'(x) h_2'(x)\,\lambda(dx), \quad h_1, h_2 \in \mathcal{H}_1^0,$$

and the corresponding norm is denoted again by $\|\cdot\|$. Any element $h \in \mathcal{H}_2^0$ can be identified by $h_1, h_2 \in \mathcal{H}_1^0$ so that $h = h_1 \otimes h_2$ (see [17]).

In the following, for any trend function $h \in \mathcal{H}_2^0$, we denote by h'' its right-continuous derivative.

Lemma 2 in [15] is crucial for our next result. Define the closed convex sets

$$V := \big\{ h \in \mathcal{H}_2^0 : h(s,t) \le 0, \forall s,t \in [0,1] \big\},$$
$$W := \big\{ h \in \mathcal{H}_2^0 : h(s,t) \ge 0, \forall s,t \in [0,1] \big\},$$

and let $\widetilde{V}, \widetilde{W}$ be the polar cones of V and W, respectively, defined by

$$\widetilde{V} := \big\{ h \in \mathcal{H}_2^0 : \langle h, v \rangle \le 0, \forall v \in V \big\}, \qquad \widetilde{W} := \big\{ h \in \mathcal{H}_2^0 : \langle h, v \rangle \ge 0, \forall v \in W \big\}.$$

Further denote by $BV_H(T), T \subset \mathbb{R}^2$ the class of functions $f : T \to \mathbb{R}$ which have bounded variation in the sense of Hardy (see e.g. [1, 25]).

Lemma 2.1 *Let $h \in \mathcal{H}_2^0$ be a given function, and let $V_{p,h}, \widetilde{V}_{p,h}$ be the unique projections of h into V and the polar cone $\widetilde{V}$, respectively.*

(a) *If $\widetilde{V}''_{p,h}$ is a right-continuous partial derivative of $\widetilde{V}_{p,h}$ such that $\widetilde{V}''_{p,h} \in BV_H([0,1]^2)$, then for any function $g : [0,1]^2 \to [0,\infty)$ Riemann–Stieltjes integrable with respect to $\widetilde{V}''_{p,h}$, the Riemann–Stieltjes integral $I(g) := \int_{[0,1]^2} g(s,t)\, d\widetilde{V}''_{p,h}(s,t)$ satisfies $I(g) \ge 0$.*

(b) *We have*

$$h = V_{p,h} + \widetilde{V}_{p,h}, \quad \langle V_{p,h}, \widetilde{V}_{p,h} \rangle = 0. \tag{2.1}$$

(c) *If $h = h_1 + h_2$ with $h_1 \in V, h_2 \in \widetilde{V}$ such that $\langle h_1, h_2 \rangle = 0$, then $h_1 = V_{p,h}$ and $h_2 = \widetilde{V}_{p,h}$.*

(d) *The unique solution $\underline{h}$ of the minimisation problem*

$$\min_{g \ge h, g \in \mathcal{H}_2^0} \|g\| \tag{2.2}$$

is $\underline{h} = \widetilde{V}_{p,h}$ satisfying further $\|\underline{h}\| = \min\{\|g\| : g \in \widetilde{V}, g \ge h\}$.

We note in passing that a similar decomposition to (2.1) can be stated for $h \in \mathcal{H}_2^0$ in terms of the unique projections $W_{p,h}, \widetilde{W}_{p,h}$ of h into W and the polar cone $\widetilde{W}$, respectively. Furthermore, (b) and (c) hold for some general Hilbert space.

We write alternatively $\underline{h}, \overline{h}$ instead of $\widetilde{V}_{p,h}, \widetilde{W}_{p,h}$. The above lemma immediately implies

$$\begin{aligned} \overline{h}(s,t) \le h(s,t) \le \underline{h}(s,t), \quad \forall s,t \in [0,1], \quad \text{and} \\ \|h\| \ge \max\big(\|\underline{h}\|, \|\overline{h}\|\big), \quad \forall h \in \mathcal{H}_2^0. \end{aligned} \tag{2.3}$$

Furthermore, for any two functions $h, q \in \mathcal{H}_2^0$ such that $q \ge h$, (1.3) and Lemma 2.1 yield

$$\|q\| \ge \|h\|, \tag{2.4}$$

provided that $\underline{h} = h,\ \underline{q} = q$.

3 Main Results

Let $B_0(s,t), s,t \in [0,1]$ be a Brownian pillow with continuous sample paths, and let $h \in \mathcal{H}_2^0$ be a given trend function. For some measurable boundary function $u : [0,1]^2 \to \mathbb{R}$, we define the boundary non-crossing probability $\psi(u;h)$ as in (1.1). Throughout the rest of the paper we assume that $\psi(u;0) \in (0,1)$. Since $h \in \mathcal{H}_2^0$, the Cameron–Martin formula (see e.g. [19, 22, 24] or [23]) implies

$$\begin{aligned} \psi(u;h) = \exp\left(-\frac{1}{2}\|h\|^2\right) \\ \times \mathbf{E}\left\{\exp\left(\int_{[0,1]^2} h''(s,t)\, dB_0(s,t)\right)\mathbf{1}\big(B_0(s,t) \le u(s,t), \forall s,t \in [0,1]\big)\right\}, \end{aligned} \tag{3.1}$$

where $\mathbf{1}(\cdot)$ is the indicator function.

Li and Kuelbs [22] show that the Cameron–Martin translation implies important shift inequalities for some general Gaussian processes. Applying their Theorem 1′, we have

$$\Phi\big(\theta - \|h\|\big) \le \psi(u;h) \le \Phi\big(\theta + \|h\|\big), \tag{3.2}$$

where Φ is the Gaussian distribution function on $\mathbb{R}$ with mean 0 and variance 1, and θ is such that $\Phi(\theta) = \psi(u;0)$. When $\|h\|$ is small, the lower and upper bounds in (3.2) are close to the non-crossing probability of interest, since $\lim_{\gamma\to 0}\psi(u;\gamma h) = \psi(u;0) = \Phi(\theta)$. As $\gamma \to \infty$, the upper bound in (3.2) tends to 1, whereas the lower bound and $\psi(u;\gamma h)$ tend to 0. Note in passing that as in [27] we obtain

$$\big|\psi(u;\gamma h) - \psi(u;0)\big| \le 2\Phi\big(\gamma\|h\|/2\big) - 1 \le \frac{\gamma\|h\|}{\sqrt{2\pi}}, \quad \forall \gamma \in (0,\infty). \tag{3.3}$$

One important criteria which we will look at when discussing bounds for the non-crossing probability of interest is their performance for both small or large trend functions. In our first result below we provide upper and lower bounds for the boundary non-crossing probability $\psi(u;h)$. If we consider further the trend function γh, then the bounds perform well as $\gamma \to 0$.

Proposition 3.1 *Let $h, u : [0,1]^2 \to \mathbb{R}$ be two measurable functions such that $\psi(u;0) \in (0,1)$. If $h \in \mathcal{H}_2^0$, then we have*

$$\Phi\big(\theta - \|\underline{h}\|\big) \le \psi(u;h) \le \Phi\big(\theta + \|\overline{h}\|\big), \quad \theta := \Phi^{-1}\big(\psi(u;0)\big), \tag{3.4}$$

with $\underline{h}, \overline{h}$ as defined in Sect. 2 and Φ^{-1} the inverse of Φ. Furthermore

$$-\frac{\|\underline{h}\|}{\sqrt{2\pi}} \le \psi(u;h) - \psi(u;0) \le \frac{\|\overline{h}\|}{\sqrt{2\pi}}. \tag{3.5}$$

When $h \ne \underline{h}$ or $h \ne \overline{h}$, in view of (2.3), we see that (3.5) yields better bounds than (3.3). By (3.5) we obtain

$$-\gamma\frac{\|\underline{h}\|}{\sqrt{2\pi}} \le \psi(u;\gamma h) - \psi(u;0) \le \gamma\frac{\|\overline{h}\|}{\sqrt{2\pi}}, \quad \forall\gamma > 0, \tag{3.6}$$

which is of some interest as γ tends to 0, since both the lower and upper bounds converge to 0.

As mentioned in the Introduction, if γ tends to infinity, then we have the logarithmic asymptotic behaviour

$$\lim_{\gamma\to\infty} 2\gamma^{-2} \ln \psi(u;\gamma h) = -\|\underline{h}\|^2, \quad \forall h \in \mathcal{H}_2^0, \tag{3.7}$$

with $\underline{h}$ the unique solution of the minimisation problem (2.2).

Next, we derive explicit upper and lower bounds for $\psi(u;h)$, which perform asymptotically better (for trend function becoming large) than those implied by (3.4).

Proposition 3.2 *Let $h \in \mathcal{H}_2^0$ be a given trend function, and let $u, l : [0,1]^2 \to \mathbb{R}$ be two measurable functions. If the partial derivative $\underline{h}''$ of the projection of h into its polar cone satisfies $\underline{h}'' \in BV_H([0,1]^2)$ and is right continuous, then*

$$\underline{h} := \inf_{g \ge h,\, g \in \widetilde{V},\, g \in BV_H([0,1]^2)} g, \tag{3.8}$$

and further $\underline{h}$ is the smallest majorant of h such that its right-continuous partial derivative belongs to $BV_H([0,1]^2)$ and generates a finite positive measure.

Moreover, if the Riemann–Stieltjes integral $\int_{[0,1]^2} v(s,t)\,d\underline{h}''(s,t)$ is finite for both $v = l$ and $v = u$ and $\psi(u;0) \in (0,1)$, then

$$\psi(u;h) \le \psi(u;h-\underline{h}) \exp\left(-\frac{1}{2}\|\underline{h}\|^2 + \int_{[0,1]^2} u(s,t)\,d\underline{h}''(s,t)\right) \tag{3.9}$$

and

$$\psi(u;h) \ge \mathbf{P}\big\{l(s,t) \le B_0(s,t) \le u(s,t), \forall s,t \in [0,1]\big\}$$
$$\times \exp\left(-\frac{1}{2}\|\underline{h}\|^2 + \int_{[0,1]^2} l(s,t)\, d\underline{h}''(s,t)\right). \tag{3.10}$$

Remarks

(a) If $u(s,t) := c \in (0,\infty), \forall s,t \in [0,1]$, then (3.9) implies

$$\psi(c;h) \le \psi(c;h-\underline{h})$$
$$\times \exp\left(-\frac{1}{2}\|\underline{h}\|^2 + c\big[\underline{h}''(1,1) - \underline{h}''(1,0) - \underline{h}''(0,1) + \underline{h}''(0,0)\big]\right). \tag{3.11}$$

A lower bound for $\psi(c;h)$ is derived using (3.10) with $l(s,t) := -c$, $\forall s,t \in [0,1]$.

(b) As in the proof of Proposition 3.2, it can be shown that if the trend function $h \in \mathcal{H}_2^0$ is such that its right-continuous partial derivative h'' satisfies $h'' \in BV_H([0,1]^2)$ and furthermore h'' generates a positive measure on $[0,1]^2$, then the unique solution of the minimisation problem (2.2) is $\underline{h} = h$.

(c) An upper bound for $\psi(u;h)$ is the discrete boundary non-crossing probability

$$\psi_n(u;h) := \mathbf{P}\big\{B_0(s_i,t_i) + h(s_i,t_i) \le u(s_i,t_i), \forall (s_i,t_i) \in T_n\big\}$$

with $T_n := \{(s_i,t_i), i = 1,\dots,n\} \subset [0,1]^2$. Hashorva [13] shows the asymptotic behaviour (considering a Brownian bridge) of the corresponding discrete boundary non-crossing probability.

Next, we discuss the asymptotic behaviour of $\psi(u;\gamma h)$ as $\gamma \to \infty$. Exact asymptotics of the non-crossing probabilities of the Brownian motion with trend is derived in [12], which was motivated by a large deviation type result obtained in [3]. As in [4], we expect that our novel asymptotic result will have some implications for statistical applications.

Proposition 3.3 *Let* $h, \underline{h}, u$ *be as in Proposition* 3.2. *Suppose that there exist functions* $u_\varepsilon \in \mathcal{H}_2^0, \varepsilon > 0$, *such that* $\|u_\varepsilon\| = O(1/\varepsilon)$ *and*

$$\lim_{\varepsilon\to 0} u_\varepsilon(s,t) = u(s,t), \quad u_\varepsilon(s,t) \le u(s,t) - \varepsilon,\ \forall s,t \in [0,1]. \tag{3.12}$$

If the Riemann–Stieltjes integral $I_\epsilon := \int_{[0,1]^2} u_\epsilon(s,t)\, d\underline{h}''(s,t)$ *exists and* $|I_\epsilon| \le M \in (0,\infty), \forall \epsilon > 0$, *then*

$$\lim_{\epsilon\to 0} I_\epsilon = I := \int_{[0,1]^2} u(s,t)\, d\underline{h}''(s,t), \quad |I| \le M, \tag{3.13}$$

and

$$\psi(u;\gamma h)=\exp\left(-\frac{\gamma^2}{2}\|\underline{h}\|^2+\gamma\int_{[0,1]^2}u(s,t)\,d\underline{h}''(s,t)+z(\gamma)\right), \tag{3.14}$$

where for all large γ,

$$\begin{aligned}-A\gamma^{2/3}\ln^3\gamma &\le z(\gamma)\\ &\le \ln \mathbf{P}\big\{B_0(s,t)\le u(s,t),\forall s,t\in[0,1]:\underline{h}(s,t)=h(s,t)\big\}\end{aligned} \tag{3.15}$$

with positive constant A *not depending on* γ.

In view of the above asymptotics and (3.4), we obtain a simple upper bound for $\psi(u;0)$.

Corollary 3.4 *Let* $u:[0,1]^2\to\mathbb{R}$ *be a measurable function satisfying the assumptions of Proposition* 3.3. *Then we have*

$$\psi(u;0)\le \inf_{h\in\mathcal{H}_2^0,h''\in BV_H([0,1]^2):\|\underline{h}\|>0}\Phi\left(\|\underline{h}\|^{-1}\int_{[0,1]^2}u(s,t)\,d\underline{h}''(s,t)\right). \tag{3.16}$$

Remarks

(a) If the function u in Proposition 3.3 satisfies $u(s,t)>\mu\in(0,\infty),\forall s,t\in[0,1]$, where (s,t) belongs to the boundary of $[0,1]^2$, and there exist functions $w_\varepsilon:[0,1]^2\to\mathbb{R},\varepsilon>0$ such that $uw_\varepsilon\in\mathcal{H}_2^0,\varepsilon>0$, then we may define u_ε in Proposition 3.3 by $u_\varepsilon:=uw_\varepsilon-\epsilon,\epsilon>0$. When u is a positive constant, then functions $u_\varepsilon,\varepsilon>0$, satisfying the assumption of Proposition 3.3 can be easily constructed. If u_ε is continuous, then the Riemann–Stieltjes integral $I_\varepsilon:=\int_{[0,1]^2}u_\varepsilon(s,t)\,d\underline{h}''(s,t)$ in Proposition 3.3 is finite.

(b) When $\underline{h}''$ is almost surely continuous with respect to the Lebesgue measure λ^2, then instead of assuming that $\underline{h}$ has a bounded variation in the sense of Hardy (Lemma 2.1, Propositions 3.2 and 3.3) we may impose the weaker assumption that $\underline{h}$ has a bounded variation in the sense of Vitali (see Appendix below and Lemma 6.2).

(c) Our results can be easily extended to the d-dimensional setup by considering a Brownian pillow $B_0(s_1,\dots,s_d),s_i\in[0,1],i\le d$, with continuous sample paths. The term $\ln^3\gamma$ in (3.15) should then be replaced by $\ln^{2d-1}\gamma$.

(d) Similar results can be stated for considering instead of B_0 a Brownian sheet $B(s,t),s,t\in[0,\infty)$, with continuous sample paths. For instance Proposition 3.2 holds with $\underline{h}$ the solution of the minimisation problem (1.3), where g,h have square-integrable partial derivatives satisfying further $g(0,s)=h(0,s)=h(t,0)=g(t,0)=0,s,t\in[0,\infty)$.

4 Product Trend Functions

As demonstrated in the previous section, the non-crossing probability $\psi(u;h)$ can be bounded by some functions which depend on the solution of the minimisation problem (2.2). We discuss below an instance where the solution of (2.2) can be easily determined. Let therefore $h_1, h_2 \in \mathcal{H}_1^0$, and let $B_0(s)$, $s \in [0,1]$, denote a Brownian bridge with continuous sample paths. If $u_1, u_2 : [0,1] \to \mathbb{R}$ are two measurable functions with $u_i(0), u_i(1) > 0$, $i = 1, 2$, then we have (see [2])

$$
\begin{aligned}
&\mathbf{P}\big\{B_0(s) + h_i(s) \le u_i(s), \forall s \in [0,1]\big\} \\
&\quad \le \mathbf{P}\big\{B_0(s) \le u_i(s) + \widetilde{h}_i(s) - h_i(s), \forall s \in [0,1]\big\} \\
&\qquad \times \exp\left(-\frac{1}{2}\|\widetilde{h}_i\|^2 + \int_{[0,1]} u_i(s)\, d\big(-\widetilde{h}_i'(s)\big)\right),
\end{aligned}
$$

where $\widetilde{h}_i$, $i = 1, 2$, is the smallest concave majorant of h_i, and $\widetilde{h}_i'$ is a right-continuous derivative of $\widetilde{h}_i$. Furthermore, $\widetilde{h}_i$ is the unique solution of the minimisation problem

$$
\min_{g \in \mathcal{H}_1^0, g \ge h_i} \|g\|, \quad i = 1, 2. \tag{4.1}
$$

Set in the following $h(s,t) := h_1(s)h_2(t), \widetilde{h}(s,t) := \widetilde{h}_1(s)\widetilde{h}_2(t), s, t \in [0,1]$, and write $h = h_1 \times h_2, \widetilde{h} = \widetilde{h}_1 \times \widetilde{h}_2$. In the next lemma we show that for special trend functions, the unique solution of (2.2) with $h = h_1 \times h_2 \in \mathcal{H}_2^0$ is simply $\widetilde{h}$.

Lemma 4.1 *Let $h := h_1 \times h_2, h_1, h_2 \in \mathcal{H}_1^0$, and denote by $\widetilde{h}_i, i = 1, 2$, the smallest concave majorant of $h_i, i = 1, 2$. If*

$$
\widetilde{h}(s,t) \ge h(s,t), \quad \forall s, t \in [0,1], \tag{4.2}
$$

then the unique solution $\underline{h}$ of (1.3) is $\underline{h} := \widetilde{h}$.

Clearly, (4.2) holds if h_1, h_2 are both nonnegative functions. In the special case that also u is a product function we have the following immediate result.

Corollary 4.2 *Let $h_i, \widetilde{h}_i, i = 1, 2$, satisfy the assumption of Lemma 4.1, and let $u_i, l_i : [0,1] \to \mathbb{R}, i = 1, 2$, be measurable functions. If the Riemann–Stieltjes integral $\int_{[0,1]} v_i(s)\, d(-\widetilde{h}_i'(s))$ is a finite constant for $i = 1, 2$ and $v_i = l_i$ or $v_i = u_i$, then we have*

$$
\psi(u; h) \le \psi\big(u; h - \widetilde{h}\big) \exp\left(-\frac{1}{2}\|\widetilde{h}_1\|^2 \|\widetilde{h}_2\|^2 \prod_{i=1,2} \int_{[0,1]} u_i(s)\, d\big(-\widetilde{h}_i'(s)\big)\right) \tag{4.3}
$$

with $h := h_1 \times h_2, \widetilde{h} := \widetilde{h}_1 \times \widetilde{h}_2, u := u_1 \times u_2$, and further

$$
\begin{aligned}
\psi(u; h) &\ge \mathbf{P}\big\{l_1(s)l_2(t) \le B(s,t) \le u_1(s)u_2(t), \forall s, t \in [0,1]\big\} \\
&\quad \times \exp\left(-\frac{1}{2}\|\widetilde{h}_1\|^2 \|\widetilde{h}_2\|^2 + \prod_{i=1,2} \int_{[0,1]} l_i(s)\, d\big(-\widetilde{h}_i'(s)\big)\right).
\end{aligned} \tag{4.4}
$$

Corollary 4.3 *Under the assumptions and the notation of Corollary 4.2, if further* $\min_{s\in[0,1]} u_i(s) > C \in (0,\infty)$, $i=1,2$, *and* u_i, $i=1,2$, *are absolutely continuous with* u_i' *satisfying* $\int_{[0,1]}(u_i'(s))^2\,\lambda(ds)<\infty$, *then we have*

$$\begin{aligned}\psi(u_1\times u_2;\gamma h_1\times h_2)\\ =\exp\left(-\frac{\gamma^2}{2}\|\widetilde{h}_1\|^2\|\widetilde{h}_2\|^2+\gamma\prod_{i=1}^{2}\int_{[0,1]}u_i(s)\,d\big(-\widetilde{h}_i'(s)\big)+z(\gamma)\right)\end{aligned}\tag{4.5}$$

with $z(\gamma)$ *satisfying*

$$\begin{aligned}-A\gamma^{2/3}\ln^3\gamma &\le z(\gamma)\\ &\le \ln \mathbf{P}\big\{B_0(s,t)\le u_1(s)u_2(t),\forall s,t\in[0,1]:\widetilde{h}_1(s)\widetilde{h}_2(t)=h_1(s)h_2(t)\big\}\end{aligned}$$

for all large γ, *where* A *is a positive constant not depending on* γ. *Furthermore*

$$\psi(u;0)\le \inf_{h_1,h_2\in\mathcal{H}_1^0:\|\widetilde{h}_1\|\|\widetilde{h}_2\|>0}\Phi\left(\big(\|\widetilde{h}_1\|\,\|\widetilde{h}_2\|\big)^{-1}\prod_{i=1,2}\int_{[0,1]}u_i(s)\,d\big(-\widetilde{h}_i'(s)\big)\right).\tag{4.6}$$

5 Proofs

Proof of Lemma 2.1 Let $g,h\in\mathcal{H}_2^0$ be two given functions. If $h''\in BV_H([0,1]^2)$ with h'' a right-continuous partial derivative of h, then we have by (6.2) and the integration by parts formula (see Lemmas 2 and 3 in [25] and (6.1))

$$\begin{aligned}\langle g,h\rangle &= \int_{[0,1]^2} g''(s,t)h''(s,t)\,\lambda^2(ds,dt)\\ &= \int_{[0,1]^2} h''(s,t)\,dg(s,t)\\ &= \int_{[0,1]^2} g(s,t)\,dh''(s,t).\end{aligned}\tag{5.1}$$

Consequently, for any $g\in V$, by the assumption on $\widetilde{V}_{p,h}''$ we have $\langle g,\widetilde{V}_{p,h}\rangle\le 0$. Hence for any function $g:[0,1]^2\to[0,\infty)$ which is Riemann–Stieltjes integrable with respect to $\widetilde{V}_{p,h}''$ on $[0,1]^2$, for the corresponding Riemann–Stieltjes integral, we have

$$\int_{[0,1]^2} g(s,t)\,d\widetilde{V}_{p,h}''\ge 0.\tag{5.2}$$

The proof of statements (b) and (c) follows immediately by Lemma 2 in [15].

We show next statement (d). Let $\widetilde{h} \in \mathcal{H}_2^0$ be a given function such that $\widetilde{h} := g + h$ with $g(s,t) \geq 0, \forall s,t \in [0,1]$. By the properties of $\widetilde{V}_{p,h}$ we have $\langle \widetilde{V}_{p,h}, g\rangle \geq 0$, hence we may write

$$\begin{aligned}
\|\widetilde{h}\|^2 &= \|g+h\|^2 \\
&= \|\widetilde{V}_{p,h} + g + h - \widetilde{V}_{p,h}\|^2 \\
&= \|\widetilde{V}_{p,h}\|^2 + 2\langle \widetilde{V}_{p,h}, g + h - \widetilde{V}_{p,h}\rangle + \|g + h - \widetilde{V}_{p,h}\|^2 \\
&= \|\widetilde{V}_{p,h}\|^2 + 2\langle \widetilde{V}_{p,h}, g\rangle + 2\langle V_{p,h}, \widetilde{V}_{p,h}\rangle + \|g + h - \widetilde{V}_{p,h}\|^2 \\
&= \|\widetilde{V}_{p,h}\|^2 + 2\langle \widetilde{V}_{p,h}, g\rangle + \|g + h - \widetilde{V}_{p,h}\|^2 \\
&\geq \|\widetilde{V}_{p,h}\|^2 + 2\langle \widetilde{V}_{p,h}, g\rangle \\
&\geq \|\widetilde{V}_{p,h}\|^2.
\end{aligned}$$

Since further $\widetilde{V}_{p,h}(s,t) \geq h(s,t), \forall s,t \in [0,1]$, it follows that the solution of the minimisation problem (2.2) is $\widetilde{V}_{p,h}$. Clearly, its solution is unique, and thus the result follows. □

Proof of Proposition 3.1 By (2.3) and (3.2) we see that (3.4) follows easily. The proof of (3.5) can be established along the lines of the proof of Lemma 5 in [15], thus the result. □

Proof of Proposition 3.2 Let $V, \widetilde{V}$ be as in Section 2, and let $\widetilde{V}_{p,h}$ be the projection of h into the polar cone $\widetilde{V}$. In view of statement (b) of Lemma 2.1,

$$h = V_{p,h} + \widetilde{V}_{p,h}, \qquad \|h\|^2 = \|\widetilde{V}_{p,h}\|^2 + \|V_{p,h}\|^2.$$

Furthermore, $\psi(u;h) \geq \psi(u;\widetilde{V}_{p,h})$. Next, applying the Cameron–Martin formula, we obtain (set $\mathbf{1}_u(B_0(s,t)) := \mathbf{1}(B_0(s,t) \leq u(s,t), \forall s,t \in [0,1])$

$$\begin{aligned}
\psi(u;h) &= \exp\Big(-\frac{1}{2}\|h\|^2\Big)\mathbf{E}\Big\{\exp\Big(\int_{[0,1]^2} h''(s,t)\,dB_0(s,t)\Big)\mathbf{1}_u\big(B_0(s,t)\big)\Big\} \\
&= \exp\Big(-\frac{1}{2}\|\widetilde{V}_{p,h}\|^2\Big)\mathbf{E}\Big\{\exp\Big(-\frac{1}{2}\|V_{p,h}\|^2 + \int_{[0,1]^2} V''_{p,h}(s,t)\,dB_0(s,t) \\
&\quad + \int_{[0,1]^2} \widetilde{V}''_{p,h}(s,t)\,dB_0(s,t)\Big)\mathbf{1}_u\big(B_0(s,t)\big)\Big\}.
\end{aligned}$$

Since $\widetilde{V}''_{p,h} \in BV_H([0,1]^2)$ is right continuous and $B_0(s,t)$ has continuous sample paths, by the integration by parts formula (6.1) for the Riemann–Stieltjes integral we have almost surely

$$\int_{[0,1]^2} B_0(s,t)\,d\widetilde{V}''_{p,h}(s,t) = \int_{[0,1]^2} \widetilde{V}''_{p,h}(s,t)\,dB_0(s,t).$$

Consequently, we may further write (recall (5.2))

$$\begin{aligned}\psi(u;h) &= \mathbf{E}\Big\{\exp\Big(-\frac{1}{2}\|V_{p,h}\|^2 + \int_{[0,1]^2} V''_{p,h}(s,t)\,dB_0(s,t) \\ &\quad + \int_{[0,1]^2} B_0(s,t)\,d\widetilde{V}''_{p,h}(s,t)\Big)\mathbf{1}_u\big(B_0(s,t)\big)\Big\} \\ &\le \exp\Big(-\frac{1}{2}\big\|\widetilde{V}_{p,h}\big\|^2 + \int_{[0,1]^2} u(s,t)\,d\widetilde{V}''_{p,h}(s,t)\Big) \\ &\quad \times \mathbf{E}\Big\{\exp\Big(-\frac{1}{2}\|V_{p,h}\|^2 + \int_{[0,1]^2} V''_{p,h}(s,t)\,dB_0(s,t)\Big)\mathbf{1}_u\big(B_0(s,t)\big)\Big\} \\ &= \exp\Big(-\frac{1}{2}\big\|\widetilde{V}_{p,h}\big\|^2 + \int_{[0,1]^2} u(s,t)\,d\widetilde{V}''_{p,h}(s,t)\Big)\psi(u;V_{p,h}).\end{aligned}$$

Clearly, by the definition $\psi(u;h) \ge \psi(u;\widetilde{V}_{p,h})$. Applying (3.7) to $\psi(u;\gamma\widetilde{V}_{p,h})$, $\gamma > 0$, we find

$$\ln\psi(u;\gamma h) = -\big(1+o(1)\big)\frac{\gamma^2}{2}\big\|\widetilde{V}_{p,h}\big\|^2, \quad \gamma\to\infty,$$

hence by (3.7) the unique solution of (2.2) equals $\widetilde{V}_{p,h}$. Since $\widetilde{V}_{p,h} \ge h$ and $\widetilde{V}_{p,h} \in \widetilde{V}$, we have $\underline{h} = \widetilde{V}_{p,h}$, and (3.8) follows.

We next show the last claim (3.10). Using again the Cameron–Martin formula, we have

$$\begin{aligned}\psi(u;h) &\ge \psi(u;\underline{h}) \\ &\ge \mathbf{P}\big\{l(s,t) \le B_0(s,t) + \underline{h}(s,t) \le u(s,t), \forall s,t\in[0,1]\big\} \\ &= \exp\Big(-\frac{1}{2}\|\underline{h}\|^2\Big)\mathbf{E}\Big\{\exp\Big(\int_{[0,1]^2} \underline{h}''(s,t)\,dB_0(s,t)\Big) \\ &\quad \times \mathbf{1}\big(l(s,t) \le B_0(s,t) \le u(s,t), \forall s,t\in[0,1]\big)\Big\} \\ &= \mathbf{P}\big\{l(s,t) \le B_0(s,t) \le u(s,t), \forall s,t\in[0,1]\big\} \\ &\quad \times \exp\Big(-\frac{1}{2}\|\underline{h}\|^2 + \int_{[0,1]^2} l(s,t)\,d\underline{h}''(s,t)\Big),\end{aligned}$$

hence the proof is established. □

Proof of Proposition 3.3 Set next

$$\underline{h}_\epsilon(s,t) := \underline{h}(s,t) - u_\epsilon(s,t), \quad \forall s,t\in[0,1].$$

Applying the Cameron–Martin formula, we obtain

$$\begin{aligned}
\psi(u;h) &\ge \psi(u;\underline{h}) \\
&= \mathbf{P}\big\{B_0(s,t)+\underline{h}(s,t)\le u(s,t), \forall s,t\in[0,1]\big\} \\
&\ge \mathbf{P}\big\{B_0(s,t)+\underline{h}(s,t)\le u_\epsilon(s,t)+\epsilon, \forall s,t\in[0,1]\big\} \\
&> \exp\Big(-\frac{1}{2}\|\underline{h}_\epsilon\|^2\Big)\mathbf{E}\Big\{\exp\Big(\int_{[0,1]^2}\underline{h}''_\epsilon(s,t)\,dB_0(s,t)\Big) \\
&\qquad \times \mathbf{1}\big(-\epsilon\le B_0(s,t)\le\epsilon, \forall s,t\in[0,1]\big)\Big\}.
\end{aligned}$$

Define the Gaussian random variable

$$Z := \int_{[0,1]^2}\underline{h}''_\epsilon(s,t)\,dB_0(s,t).$$

Clearly, Z has mean 0 and variance $\|\underline{h}_\epsilon\|^2$. For $\varepsilon>0$ small enough, we have $\|\underline{h}_\epsilon\|\in(0,\infty)$. For any constant $C\in\mathbb{R}$ and ε small enough, we may write

$$\begin{aligned}
&\mathbf{E}\big\{\exp(Z)\mathbf{1}\big(-\epsilon\le B_0(s,t)\le\epsilon, \forall s,t\in[0,1]\big)\big\} \\
&\quad = \mathbf{E}\big\{\exp(Z)\mathbf{1}\big(-\epsilon\le B_0(s,t)\le\epsilon, \forall s,t\in[0,1]\big)\big[\mathbf{1}(Z<C)+\mathbf{1}(Z\ge C)\big]\big\} \\
&\quad \ge \mathbf{E}\big\{\exp(Z)\mathbf{1}\big(-\epsilon\le B_0(s,t)\le\epsilon, \forall s,t\in[0,1]\big)\mathbf{1}(Z\ge C)\big\} \\
&\quad \ge \exp(C)\mathbf{P}\big\{-\epsilon\le B_0(s,t)\le\epsilon, \forall s,t\in[0,1], Z\ge C\big\} \\
&\quad = \exp(C)\Big[\mathbf{P}\Big\{\sup_{s,t\in[0,1]}\big|B_0(s,t)\big|<\epsilon\Big\} \\
&\qquad - \mathbf{P}\big\{-\epsilon\le B_0(s,t)\le\epsilon, \forall s,t\in[0,1], Z<C\big\}\Big] \\
&\quad \ge \exp(C)\Big[\mathbf{P}\Big\{\sup_{s,t\in[0,1]}\big|B_0(s,t)\big|<\epsilon\Big\} - \mathbf{P}\{Z\le C\}\Big] \\
&\quad = \exp(C)\Big[\mathbf{P}\Big\{\sup_{s,t\in[0,1]}\big|B_0(s,t)\big|<\epsilon\Big\} - \Phi\big(C/\|\underline{h}_\epsilon\|\big)\Big].
\end{aligned}$$

By the small ball asymptotic result (see [7–9, 16]) we have

$$\boldsymbol{P}\Big\{\sup_{s,t\in[0,1]}\big|B_0(s,t)\big|<\epsilon\Big\} \ge \exp\Big(-K\frac{\ln^3(1/\epsilon)}{\epsilon^2}\Big)$$

for some positive constant K and all $\epsilon>0$ small enough. Since

$$\|\underline{h}_\epsilon\|^2 = \|\underline{h}\|^2 - 2\int_{[0,1]^2}u_\varepsilon(s,t)\,d\underline{h}''(s,t) + \|u_\varepsilon\|^2 = O\big(1/\varepsilon^2\big),$$

choosing $C := -K_*\|\underline{h}_\epsilon\|\ln^{3/2}(1/\epsilon)/\epsilon$, $K_*\in(0,\infty)$, $K_*^2>K$ and using the Mills-ratio asymptotics for Gaussian random variables for all $\epsilon>0$ small enough and some positive constants c_1, c_2, we have

$$\mathbf{E}\big\{\exp(Z)\mathbf{1}\big(-\epsilon \le B_0(s,t) \le \epsilon, \forall s,t \in [0,1]\big)\big\} \ge \exp\left(-\frac{c_1}{\epsilon} - \frac{c_2 \ln^3(1/\epsilon)}{\epsilon^2}\right),$$

implying thus

$$\psi(u;h) \ge \exp\left(-\frac{1}{2}\|\underline{h}\|^2 + \int_{[0,1]^2} u_\epsilon(s,t)\, d\underline{h}''(s,t) - \frac{c_1}{\epsilon} - \frac{c_2 \ln^3(1/\epsilon)}{\epsilon^2}\right).$$

Recalling that $\lim_{\varepsilon\to 0} u_\epsilon(s,t) = u(s,t), \forall s,t \in [0,1]$ and $\|u_\epsilon\|^2 = O(1/\varepsilon^2)$, we obtain using the result of Proposition 3.2 (set next $\epsilon := \gamma^{-1/3}, \gamma > 0$)

$$\psi(u;\gamma h) = \exp\left(-\frac{\gamma^2}{2}\|\underline{h}\|^2 + \gamma I + z(\gamma)\right), \quad \gamma \to \infty,$$

where $|I| \le M$ with $I := \int_{[0,1]^2} u(s,t)\, d\underline{h}''(s,t)$ and

$$-A\gamma^{2/3}\ln^3\gamma \le z(\gamma) \le \ln \boldsymbol{P}\big\{B_0(s,t) \le u(s,t), \forall s,t \in [0,1] : \underline{h}(s,t) = h(s,t)\big\}$$

is satisfied for all γ large and a positive constant A not depending on γ. Hence the result follows. □

Proof of Lemma 4.1 Set $V := \{h \in \mathcal{H}_2^0 : h(s,t) \le 0, \forall s,t \in [0,1]\}$ and $\underline{h} := \widetilde{h}_1 \times \widetilde{h}_2$. By the assumptions the function $g := \underline{h} - h_1 \times h_2$ belongs to V. Furthermore, for any $v \in V$, we have

$$\langle v, \underline{h}\rangle = \int_{[0,1]^2} v(s,t)\, d\big(\widetilde{h}_1'(s)\widetilde{h}_2'(t)\big) \le 0.$$

Consequently $\underline{h}$ belongs to the polar cone $\widetilde{V}$ of V. In view of statement (c) in Lemma 2.1, the proof follows if we show that g is orthogonal to $\underline{h}$. Since $\widetilde{h}_i - h_i$ is orthogonal to $\widetilde{h}_i, i = 1,2$ (see [2]), we have

$$\begin{aligned}\langle g, \underline{h}\rangle &= \big\langle \widetilde{h}_1 \times \widetilde{h}_2 - h_1 \times h_2, \widetilde{h}_1 \times \widetilde{h}_2\big\rangle \\ &= \big\langle \widetilde{h}_1 \times \big(\widetilde{h}_2 - h_2\big), \widetilde{h}_1 \times \widetilde{h}_2\big\rangle - \big\langle\big(\widetilde{h}_1 - h_1\big) \times h_2, \widetilde{h}_1 \times \widetilde{h}_2\big\rangle \\ &= 0,\end{aligned}$$

hence the result follows. □

Proof of Corollary 4.3 The proof follows easily by the assumptions on $u_i, i = 1,2$. □

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Appendix

In this short section we provide two results for the Riemann–Stieltjes integral.

Let $f : [0, 1]^2 \to \mathbb{R}$ be a given function. If $f(s,t) = g(s,t) + g_1(s) + g_2(t)$ with $g \in BV_H([0, 1]^2)$ and g_1, g_2 two other functions, then h has bounded variation in the sense of Vitali (write $f \in BV_V([0, 1]^2)$). In fact f can be expressed as the difference of two real functions defined on $[0, 1]^2$ which generate a positive measure on $[0, 1]^2$. Thus the class of functions with bounded variation in the sense of Vitali consists of all real functions defined on $[0, 1]^2$ generating a finite signed measure.

If $g : [0, 1]^2 \to \mathbb{R}$ is continuous, then it is well known that the Riemann–Stieltjes integral $\int_{[0,1]^2} g(x, y)\, df(x, y)$ exists, provided that $f \in BV_V([0, 1]^2)$. In the next lemma we present an integration by parts formula; the case $f \in BV_H([0, 1]^2)$ is discussed in Lemma 1 in [25].

Lemma 6.1 *Let $f, g : [0, 1]^2 \to \mathbb{R}$ be two given functions. If g is continuous such that $g(s,t) = 0$ for all (s,t) in the boundary of $[0, 1]^2$ and $f \in BV_V([0, 1]^2)$, then the integration by parts formula for the Riemann–Stieltjes integral reads*

$$\int_{[0,1]^2} g(x, y)\, df(x, y) = \int_{[0,1]^2} f(x, y)\, dg(x, y). \tag{6.1}$$

Proof The proof follows with similar arguments as in Lemma 2 in [25], since the four single sums in expression (3.8) therein are equal to 0 due to the fact that g vanishes on the boundary of $[0, 1]^2$. □

Lemma 6.2 *Let $f, g : [0, 1]^2 \to \mathbb{R}$ be two given functions. Assume that g is absolutely continuous with $g(s,t) = \int_{[0,s]\times[0,t]} h(x, y)\, \lambda^2(dx, dy), s, t \in [0, 1]$. If $f \in BV_V([0, 1]^2)$ and f is almost surely continuous with respect to λ^2, then we have*

$$\int_{[0,1]^2} g(x, y)\, df(x, y) = \int_{[0,1]^2} f(x, y)h(x, y)\, d\lambda^2(dx, dy). \tag{6.2}$$

Proof The proof follows with similar arguments as in Lemma 3 in [25]. □

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