

Qual Quant (2012) 46:1947–1952  
DOI 10.1007/s11135-011-9459-4

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RESEARCH NOTE

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# Moments structure of $\ell_1$ -stochastic volatility models

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Published online: 3 March 2011  
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**Abstract** We consider Taylor's stochastic volatility model (SVM) when the innovations of the hidden log-volatility process have a Laplace distribution ( $\ell_1$  exponential density), rather than the standard Gaussian distribution ( $\ell_2$ ) usually employed. Recently many investigations have employed  $\ell_1$  metric to allow better modeling of the abrupt changes of regime observed in financial time series. However, the estimation of SVM is known to be difficult because it is a non-linear with an hidden markov process. Moreover, an additional difficulty yielded by the use of  $\ell_1$  metric is the not differentiability of the likelihood function. An alternative consists in using a generalized or efficient method-of-moments (GMM/EMM) estimation. For this purpose, we derive here the moments and autocovariance function of such  $\ell_1$ -based stochastic volatility models.

**Keywords** Stochastic volatility model · Laplace innovations · Autocovariance function · Variance gamma model

**JEL Classification** C22

## 1 Introduction

Stochastic volatility (SV) model introduced by [Taylor \(1986\)](#) is the empirical discrete-time version of continuous-time models of finance theory, in particular of the models used in option pricing problem ([Hull and White 1987](#)). It can also be regarded as an Euler discretisation of a diffusion. SV models are a good alternative to ARCH models, with theoretical properties that fit in a more appropriate way the stylized features observed in finance. In particular, the

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log-normal SV model seems to capture more leptokurticity of the marginal distributions of the financial data than the conditional GARCH model (Shephard 1996). Taylor's SV model can be defined as

$$y_t = \sigma_t \epsilon_t \quad (1a)$$

$$\sigma_t = \exp(h_t) \quad (1b) \quad (1)$$

$$h_t = \mu + \phi(h_{t-1} - \mu) + \eta_t \quad (1c),$$

for  $t \leq T$ , where  $y_t$  are the log-returns over a unit time period,  $\sigma_t$  are the volatilities,  $\phi > 0$  and  $\mu$  are the two parameters respectively representing the autoregressive coefficient and the drift of the log-volatilities  $h_t$ ,  $\epsilon_t$  and  $\eta_t$  are two mutually and serially independent innovations centered around zero with respective variance one and  $\sigma_\eta$ . In general,  $\eta_t$  and  $\epsilon_t$  are assumed to be Gaussian. The properties of model (1) are given by Taylor (1994) and Shephard (1996). In particular, Andersen (1994) shows the condition  $\log(\phi) < 0$  guarantees the strictly stationarity and the ergodicity of the volatility process (Andersen 1994, Theorem 2.1, p. 82).

A stylized feature of financial time series is volatility clustering and abrupt changes of volatility regimes. In the recent years, modeling abrupt changes in the volatility has been one of the main challenge in the financial literature because they provide a good explanation of the (global) non-stationary feature of the volatility (See Starica and Granger 2005, among others). Abrupt changes can be obtained by SV models if the stochastic innovations in (1c) can take large (either positive or negative) values with a greater probability than with Gaussian innovations. To that aim we propose to employ Laplace( $\theta, \lambda$ ) innovations with density

$$f(\eta_t; \beta, \lambda) = \frac{\lambda}{2} \exp(-\lambda |\eta_t - \theta|) \quad (2)$$

centered around  $\theta = 0$ . The use of  $\ell_1$ -based innovations has been successful to recover process with abrupt changes with wavelet-based estimators (Donoho and Johnstone 1994) and Markov random field-based estimators (Sardy and Tseng 2004). This idea bears similarity with the investigation of Madan and Seneta (1990), Madan et al. (1998) which considers a Variance Gamma (VG) model for modeling the non-Gaussian nature of stock market returns and future Index prices.<sup>1</sup> The VG model can be expressed by Eq. 1a, where  $\sigma_t$  follows a Gamma process and the continuous-time stochastic process which is consistent with this VG model is given by

$$S(t) = S(0) \exp(L(t)),$$

where  $S(t)$  are the stock prices and  $L(t)$  is a Laplace motion (Kotz et al. 2001). Hence, the Laplace motion is defined as a Brownian motion, denoted  $B(\tau)$ , evaluated at random time distributed as a Gamma process, denoted  $\gamma(t)$ . This Brownian is said subordinated to the process  $\gamma(t)$ . Therefore, the Laplace motion is defined as  $L(t) \stackrel{d}{=} B(\gamma(t))$ . Another appeal of the Laplace motion is it can be written as a compound Poisson process with independent and random jumps. In this sense it is a pure jumps process able then to capture abrupt changes. The Laplace SV model we study here also extends from VG model in that the additional parameter  $\phi$  allows control of the level of persistence of the time series.

However, two pitfalls are combined in the estimation of such model: in addition to the issue of the hidden Markov process, the likelihood function of this model is not differentiable due to the use of the  $\ell_1$  exponential density, which makes the likelihood-based estimation methods trickier to apply here than in the Gaussian version of (1) (Chib et al. 2002).

<sup>1</sup> Indeed, the estimates of the risk neutral processes are known to not by Gaussian.

We explore the moment properties of such models. Our results could be exploited for instance in a method of moment estimation procedure (Taylor 1986; Melino and Turnbull 1990; Andersen and Sørensen 1996; Andersen et al. 1999).

### 2 Moment properties

Andersen (1994) shows that for  $\phi > 0$ , the  $r$ th moment of  $h_t$  exists if and only if  $\phi^r < 1$  and  $\mathbb{E}(\eta_t^r) < \infty$ . Therefore, for (1c) with  $\eta_t$  iid Laplace(0,  $\lambda$ ), first-order and second-order (unconditional) moments of  $h_t$  exist and are given by  $\mathbb{E}(h_t) = \mu$  and  $\mathbb{V}(h_t) = \frac{2}{\lambda^2(1-\phi^2)}$ . Moreover, the probability distribution function of the marginal distribution of  $(h_t - \mu)$  is given by the following proposition.

**Proposition 1** *The marginal probability distribution function of the process  $\{h_t - \mu\}_t$  in the  $\ell_1$ -SVM defined by (1) – (2) is given by*

$$f(h_t - \mu) = \frac{\lambda}{2} \sum_{i=0}^{\infty} |\phi|^{-i} g_i(\phi) \exp(-\lambda |h_t - \mu| \phi^{-i}), \tag{3}$$

where the function  $g_i(\phi)$  is:

$$g_i(\phi) = (-1)^i \prod_{k=1}^i \left( \frac{\phi^{2k}}{1 - \phi^{2k}} \right) \left( \prod_{k=1}^{\infty} (1 - \phi^{2k}) \right)^{-1}, \text{ for } i \in \mathbb{N},$$

with the convention that  $\prod_{i=1}^0 (\cdot) \equiv 1$ .

The moments of  $\{h_t - \mu\}_t$  are given by  $\mathbb{E}((h_t - \mu)^r) = \frac{r!}{\lambda^r} \sum_{i=0}^{\infty} \phi^{ir} g_i(\phi)$ , for  $r$  even, and zero otherwise.

*Proof* Let  $\{X_t\}_t$  be a stationary process generated by an ARMA(p,q) process such that  $\Phi(L)X_t = \Theta(L)\eta_t$ , where  $\eta_t$  is independent, identically Laplace (double-exponential) distributed,  $\Phi(L) = 1 - \sum_{j=1}^p \phi_j L^j$ ,  $\Theta(L) = 1 - \sum_{j=1}^q \theta_j L^j$  with all roots of  $\Phi(L) = 0$  and  $\Theta(L) = 0$  lying outside the unit circle. The MA( $\infty$ ) representation is given by  $X_t = \Psi(L)\eta_t$ , where  $\Psi(L) = \Phi^{-1}(L)\Theta(L) = \sum_{j=0}^{\infty} \psi_j L^j$ . Damsleth and El-Shaarawi (1989, pp. 62–63) give the marginal probability distribution function of  $X_t$ . This pdf is given by:

$$f(x) = \frac{\lambda}{2} \sum_{j=0}^{\infty} |\psi_j|^{-1} g_j(\psi_j) \exp(-\lambda |x/\psi_j|),$$

where

$$g_j(\psi_j) = \prod_{i=0, i \neq j}^{\infty} (1 - |\psi_i/\psi_j|^2)^{-1}.$$

In the  $\ell_1$ -SVM,  $\{h_t - \mu\}_t$  is generated by an AR(1), therefore its marginal pdf is obtained using  $\psi_j = \phi^j$ , where  $\phi \in (0, 1)$ . □

Since the marginal distribution of  $h_t$  is symmetric about zero, the odd unconditional (centered) moments of  $y_t$  are zeros. The following theorem gives the unconditional moments function of  $y_t$ .

**Theorem 1** For the  $\ell_1$ -SVM defined by (1) – (2) which satisfies  $\phi \in (0, 1)$ , the  $r$ th unconditional moment of  $y_t$  exists if  $\lambda > r$  and is given by:

$$m_R = \frac{r! \lambda^2 \exp(r\mu)}{2^{\frac{r}{2}} (r/2)!} \sum_{i=0}^{\infty} \frac{\phi^{-2i} g_i(\phi)}{(\lambda \phi^{-i} + r)(\lambda \phi^{-i} - r)},$$

for  $r$  even and  $m_r = 0$  otherwise.

*Proof* Using  $h_t$  independent of  $\epsilon_t$ , which is *iid* standard Gaussian, and denoting  $\mathbb{E}(y_t^r) = m_r$ , we have  $m_r = \mathbb{E}(\exp(rh_t)) \mathbb{E}(\epsilon_t^r) = \mathbb{E}(\exp(r(h_t - \mu))) \exp(r\mu) \mathbb{E}(\epsilon_t^r)$ .

We use the following Corollary of the dominated convergence theorem to derive  $\mathbb{E}(\exp(r(h_t - \mu)))$ .

**Corollary** Let  $\{f_n\}_{n \in \mathbb{N}}$  be a sequence of integrable functions on a measure space  $(\Omega, F, \mathbb{P})$  such that we have  $\sum_{n=1}^{\infty} \int_{\Omega} |f_n| d\mathbb{P} < \infty$ , then  $\sum_{n=1}^{\infty} f_n(x)$  converges almost everywhere to an integrable function  $f$  and  $\int_{\Omega} \sum_{n=1}^{\infty} f_n(x) d\mathbb{P} = \sum_{n=1}^{\infty} \int_{\Omega} f_n(x) d\mathbb{P}$ .

Let us set  $f_i(h) = (\lambda/2) \phi^{-i} g_i(\phi) \exp(rh - \lambda|h|\phi^{-i})$ , we have

$$\int_{\mathbb{R}} |f_i(h)| dh = -\frac{|g_i(\phi)| \lambda^2}{r^2 \phi^{2i} - \lambda^2},$$

for  $\lambda > r$  and  $\phi \in (0, 1)$ . Denoting  $G_i = -\frac{g_i(\phi) \lambda^2}{r^2 \phi^{2i} - \lambda^2}$ , we verify that  $\sum_{i=1}^{\infty} G_i < \infty$  for  $|\phi| < 1$  through

$$\lim_{i \rightarrow \infty} |G_{i+1}/G_i| = \lim_{i \rightarrow \infty} \frac{\phi^{2(i+1)} (r^2 \phi^{2i} - \lambda^2)}{(1 - \phi^{2(i+1)}) (r^2 \phi^{2(i+1)} - \lambda^2)} = 0.$$

Here  $\lim_{i \rightarrow \infty} |G_{i+1}/G_i| < 1$ , so  $\sum_{i=1}^{\infty} |G_i|$  and  $\sum_{i=1}^{\infty} G_i$  converge.

Therefore, we can write:

$$\begin{aligned} \mathbb{E}(\exp(r(h_t - \mu))) &= \int_{\mathbb{R}} \exp(rh) (\lambda/2) \sum_{i=0}^{\infty} \phi^{-i} g_i(\phi) \exp(-\lambda|h|\phi^{-i}) dh \\ &= (\lambda/2) \sum_{i=0}^{\infty} \phi^{-i} g_i(\phi) \int_{\mathbb{R}} \exp(rh - \lambda|h|\phi^{-i}) dh. \end{aligned}$$

Straightforward algebra leads to the moments function. □

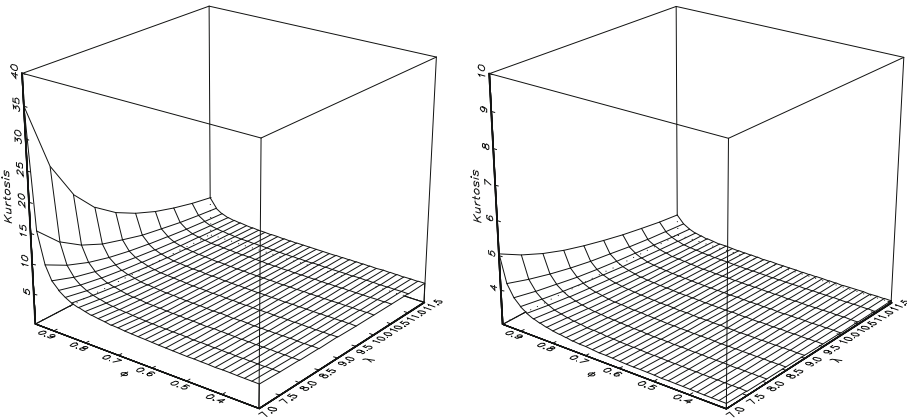
In particular from Theorem 1 we obtain the kurtosis of the marginal distribution of  $y_t$ , defined as  $\kappa_y = m_4/m_2^2$ :

$$\kappa_y = 3 \left( \lambda^2 \sum_{i=0}^{\infty} \frac{g_i(\phi)}{(\lambda^2 - 4\phi^{2i})} \right)^{-2} \sum_{i=0}^{\infty} \frac{g_i(\phi)}{(\lambda^2 - 16\phi^{2i})}, \quad \text{for } \lambda > 4.$$

Figure 1 compares the kurtosis functions between the Laplace and log-normal SV models. Hence, for  $\phi$  fixed, the unconditional distribution of the process  $y_t$  gets more leptokurtic as  $\lambda$  decreases. Therefore, the parameter  $\lambda$  may be regarded as a measure of the long tailedness.

Kurtosis for marginal distribution of  $L_1$ -SVM process

Kurtosis for marginal distribution of log-Gaussian-SVM process



**Fig. 1** Kurtosis functions when  $y_t$  follows: (left) a Laplace-SVM; (right) the corresponding log-normal SVM, where  $\kappa_y = 3 \exp\left(\frac{\sigma_\eta^2}{1-\phi^2}\right)$  and  $\sigma_\eta^2 = 2/\lambda^2$

**Theorem 2** For  $\lambda > 4$ , the autocovariance function of the squared values of  $y_t$ , denoted  $\gamma_s = \text{cov}(y_t^2 y_{t-s}^2)$ , for  $s \in \mathbb{N}^*$ , is finite and given by:

$$\gamma_s = \exp(4\mu) \left( A_1(s, \phi, \lambda) A_2(s, \phi, \lambda) - \lambda^4 A_3(\phi, \lambda)^2 \right),$$

where  $A_1(s, \phi, \lambda) = \prod_{i=0}^{s-1} \frac{\lambda^2}{\lambda^2 - 4\phi^{2i}}$ ,  $A_2(s, \phi, \lambda) = \prod_{i=s}^{\infty} \frac{\lambda^2}{\lambda^2 - 4(1+\phi^s)^2 \phi^{2(i-s)}}$ , and  $A_3(\phi, \lambda) = \sum_{i=0}^{\infty} \frac{\phi^{-2i} g_i(\phi)}{(\lambda|\phi|^{-i} + 2)(\lambda|\phi|^{-i} - 2)}$ , with the convention  $\prod_{i=0}^{-1} (\cdot) \equiv 1$ .

The autocorrelation function  $\rho_s = \gamma_s/m_4$  is given by:

$$\rho_s = \frac{A_1(s, \phi, \lambda) A_2(s, \phi, \lambda) - \lambda^4 A_3(\phi, \lambda)^2}{3\lambda^2 A_0(\phi, \lambda) - \lambda^4 A_3(\phi, \lambda)^2},$$

with  $A_0(\phi, \lambda) = \sum_{i=0}^{\infty} \frac{g_i(\phi)}{(\lambda^2 - 16\phi^{2i})}$ .

*Proof* Using  $h_t$  independent of  $\epsilon_t$ , we write  $\gamma_s = \mathbb{E}(y_t^2 y_{t-s}^2) - \mathbb{E}(y_t^2)^2 = \mathbb{E}(\exp(2h_t + 2h_{t-s})) - \mathbb{E}(y_t^2)^2$ , where the second term is given by Theorem 1:  $\mathbb{E}(y_t^2)^2 = \lambda^2 \exp(4\mu) A_3(\phi, \lambda)$  where  $A_3(\phi, \lambda)$  is given in the statement of the Theorem.

For the first term  $\mathbb{E}(\exp(2h_t + 2h_{t-s}))$ , we note that  $h_t - \mu = \sum_{i=0}^{\infty} \phi^i \eta_{t-i}$  and  $\sum_{i=0}^{\infty} \phi^i \eta_{t-i} + \sum_{i=0}^{\infty} \phi^i \eta_{t-s-i} = \sum_{i=0}^{s-1} \phi^i \eta_{t-i} + (1 + \phi^s) \sum_{i=0}^{\infty} \phi^i \eta_{t-s-i}$ .

We get  $\exp(2h_t + 2h_{t-s}) = \exp(4\mu) \prod_{i=0}^{s-1} \exp(2\phi^i \eta_{t-i}) \prod_{i=s}^{\infty} \exp(2(1 + \phi^s) \phi^{i-s} \eta_{t-i})$ .

Using  $\mathbb{E}(\exp(c\eta_t)) = \frac{\lambda^2}{(\lambda+c)(\lambda-c)}$ , with  $c$  a constant, and taking the expectation of the expression of  $\exp(2h_t + 2h_{t-s})$  we have:  $\mathbb{E}(\exp(2h_t + 2h_{t-s})) = \exp(4\mu) A_1(s, \phi, \lambda) A_2(s, \phi, \lambda)$  where the functions  $A_1(s, \phi, \lambda)$  and  $A_2(s, \phi, \lambda)$  are defined in the Theorem's statement.

The expression of the autocorrelation function is then immediate using:

$$\rho_s = \frac{A_1(s, \phi, \lambda) A_2(s, \phi, \lambda) - \lambda^4 A_3(\phi, \lambda)^2}{\mathbb{V}(y_t^2)}.$$

□

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