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Mixed *HP*-finite element approximations on geometric edge and boundary layer meshes in three dimensions

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Summary. In this paper, we consider the Stokes problem in a three-dimensional polyhedral domain discretized with *hp* finite elements of type $\mathbb{Q}^k$ for the velocity and $\mathbb{Q}^{k-2}$ for the pressure, defined on hexahedral meshes anisotropically and non quasi-uniformly refined towards faces, edges, and corners. The inf–sup constant of the discretized problem is independent of arbitrarily large aspect ratios. Our work generalizes a recent result for two-dimensional problems in [10,11].

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1 Introduction

It is well-known that solutions of elliptic boundary value problems in polyhedral domains have corner and edge singularities. In addition, boundary layers may also arise in laminar, viscous, incompressible flows with moderate Reynolds numbers at faces, edges, and corners. Suitably graded meshes, geometrically refined towards corners, edges, and/or faces, are required in order to achieve an exponential rate of convergence of *hp* finite element approximations; see, e.g., [2,3,8,13,14].

The Stokes and Navier–Stokes equations are mixed elliptic systems with saddle point variational form. The stability and accuracy of the corresponding finite element approximations depend on an inf–sup condition for the finite element spaces chosen for the velocity and the pressure fields. Even for stable velocity–pressure combinations, the corresponding inf–sup constants may in general be very sensitive to the aspect ratio of the mesh, thus degrading the stability if very thin elements are employed, as required for boundary-layer and singularity resolution. It has recently been shown in the two-dimensional case, for corner and boundary-layer tensor-product meshes, that the inf–sup constant of certain velocity/pressure space pairs for the Stokes problem retains the same dependence on the polynomial degree as for isotropically refined triangulations, regardless of arbitrarily large aspect ratios of the mesh; see [13, 10, 11, 1]. Analogous results in three dimensional domains appear to be lacking.

In this paper, we prove that, for the most widely used $\mathbb{Q}^k$–$\mathbb{Q}^{k-2}$ spaces on geometric boundary layer and edge meshes consisting of hexahedral elements in $\mathbb{R}^3$, the inf–sup constant decreases as $Ck^{-3/2}$, with a constant C that depends only on the mesh grading factor, but is independent of the degree k, the level of refinement, and arbitrarily large element aspect ratios. In the case of isotropically refined meshes the same constant decreases as Ck^{-1}; see [16]. Our analysis is also valid for linearly elasticity problems in nearly incompressible materials; see, e.g., [6, 7]. The same inf-sup condition is required in order to have approximations that remain stable for nearly incompressible materials.

This paper is organized as follows:
In Section 2, we introduce the continuous problem and the finite element spaces for its discretization. They are built on geometric boundary layer and edge meshes, described and constructed in Section 3. In Section 4, we describe the macro-element technique that we repeatedly employ in our proofs. In Section 5, we prove a stability result for shape-regular meshes with hanging nodes. The stability of face, edge, and corner patches for geometric boundary layer meshes is proven in Sections 6, 7, and 8, respectively. The case of geometric edge meshes is treated in Section 9.

2 Problem setting

Let $\Omega \subset \mathbb{R}^3$ be a bounded polyhedral domain. Given a vector $f \in L^2(\Omega)^3$, we consider the following problem: find a velocity $u \in H_0^1(\Omega)^3$ and a pressure $p \in L_0^2(\Omega)$, such that

$$(1) \qquad \begin{aligned} &\nu(\nabla u, \nabla v)_\Omega - (p, \nabla \cdot v)_\Omega = (f, v)_\Omega, && v \in V := H_0^1(\Omega)^3,\\ &(q, \nabla \cdot u)_\Omega = 0, && q \in M := L_0^2(\Omega). \end{aligned}$$

Here, $L_0^2(\Omega)$ denotes the subspace of $L^2(\Omega)$ of functions with vanishing mean value in Ω and, for $\mathcal{D} \subseteq \mathbb{R}^3$, $(u, v)_\mathcal{D}$ denotes the scalar product in $L^2(\Omega)$ or $L^2(\Omega)^3$.

In order to approximate (1), we replace the continuous spaces $V \times M$ by two finite element spaces $V_N \times M_N \subset V \times M$. Let $(u_N, p_N) \in V_N \times M_N$ be the solution of the corresponding discrete problem:

$$(2)\qquad \begin{aligned} \nu(\nabla u_N, \nabla v_N)_\Omega - (p_N, \nabla\cdot v_N)_\Omega &= (f, v_N)_\Omega, \qquad v_N \in V_N,\\ (q_N, \nabla\cdot u_N)_\Omega &= 0, \qquad\qquad\quad q_N \in M_N. \end{aligned}$$

A crucial role in the analysis and approximation of (1) is played by the inf–sup condition

$$(3)\qquad \inf_{0\neq p\in L_0^2(\Omega)} \sup_{0\neq v\in H_0^1(\Omega)^3} \frac{(\nabla\cdot v, p)_\Omega}{|v|_{1,\Omega}\,\|p\|_{0,\Omega}} \geq \gamma > 0,$$

which ensures its well-posedness. The corresponding discrete inf–sup condition for the finite element spaces (V_N, M_N) (also referred to as divergence stability) ensures the well-posedness and quasi-optimality of (2). Indeed, if a stability condition (3) holds for the discrete velocity and pressure spaces, with a constant γ_N, then (2) has a unique solution, and the following error estimates hold

$$\|u - u_N\|_{1,\Omega} \leq C\gamma_N^{-1} E_V(u, N) + C\nu^{-1} E_P(p, N),$$
$$\|p - p_N\|_{0,\Omega} \leq C\gamma_N^{-2} E_V(u, N) + C\gamma_N^{-1} E_P(p, N),$$

where

$$E_V(u, N) := \inf_{v\in V_N} \|u - v\|_{1,\Omega},$$
$$E_P(p, N) := \inf_{q\in M_N} \|p - q\|_{0,\Omega},$$

are the best approximation errors of the solution (u, p) of (1); see, e.g., [6].

We now specify a particular choice of finite element spaces. Given an affine hexahedral mesh $\mathcal{T}$ and a polynomial degree $k \geq 2$, in order to discretize (1), we consider the following finite element spaces:

$$(4)\qquad \begin{aligned} V_N = S_0^{k,1}(\Omega; \mathcal{T})^3 &:= \left\{ u \in H_0^1(\Omega)^3 \mid \quad u_{|K} \in \mathbb{Q}_k(K)^3 \right\},\\ M_N = S_0^{k-2,0}(\Omega; \mathcal{T}) &:= \left\{ p \in L_0^2(\Omega) \mid \quad p_{|K} \in \mathbb{Q}_{k-2}(K) \right\}. \end{aligned}$$

Here $\mathbb{Q}_k(K)$ is the space of polynomials of maximum degree k in each variable on K. The mesh $\mathcal{T}$ is said to be *regular* if it is geometrically conforming, or *irregular* if hanging nodes are present; see, e.g., [11,12]. These spaces are also known as $\mathbb{P}_k$–$\mathbb{P}_{k-2}$ in the spectral element literature. In the following, we also use the polynomial spaces $\mathbb{Q}_{r,s,m}$ of polynomials of degree r, s, and m in the first, second, and third variable, respectively.

3 Geometric meshes

In order to resolve boundary layers and/or singularities, geometrically graded meshes can be employed. They are determined by a mesh grading factor $\sigma \in (0, 1)$ and the number of layers n, the thinnest layer having width

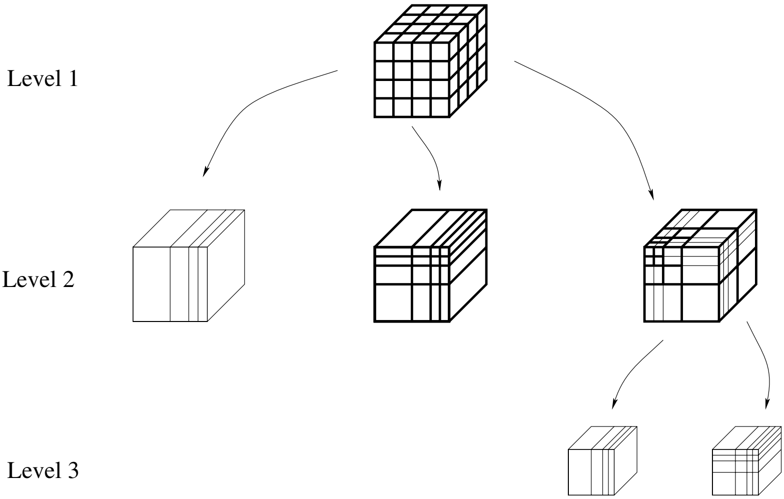

Fig. 1. Hierarchic structure of a boundary layer mesh

proportional to σ^n. Robust exponential convergence of hp finite element approximations is achieved if n is suitably chosen. For singularity resolution, n is required to be proportional to the polynomial degree k; see [2,3]. For boundary layers, the width of the thinnest layer needs to be comparable to that of the boundary layer; see [8,13,14]. In practical applications, for boundary layers of fixed width, and edge and corner singularities, n is usually chosen proportional to the polynomial degree k, with the assumption that k is sufficiently large.

3.1 Construction of geometric boundary layer meshes

A geometric boundary layer mesh $\mathcal{T}_{bl}^{n,\sigma}$ is, roughly speaking, the tensor product of meshes that are geometrically refined towards the faces. Figure 1 shows the construction of a geometric boundary layer mesh $\mathcal{T}_{bl}^{n,\sigma}$.

The mesh $\mathcal{T}_{bl}^{n,\sigma}$ is built by first considering an initial shape-regular macro-triangulation $\mathcal{T}_m$, possibly consisting of just one element, which is successively refined. This process is illustrated in Figure 1. Every macro-element can be refined isotropically (not shown) or anisotropically in order to obtain so-called face, edge, or corner patches (Figure 1, level 2). Here and in the following, we only consider patches obtained by triangulating the reference cube $\hat{Q} := I^3$, with $I := (-1, 1)$. A patch for an element $K \in \mathcal{T}_m$ is obtained by using an affine mapping $F_K : \hat{Q} \to K$. The stability properties proven for patches on the reference cube are equally valid for an arbitrary shape-regular element $K \in \mathcal{T}_m$, with a constant that is independent of the diameter of K.

A **face patch** is given by an anisotropic triangulation of the form

$$\mathcal{T}_f := \{K_x \times I \times I \mid \quad K_x \in \mathcal{T}_x\}, \tag{5}$$

where $\mathcal{T}_x$ is a mesh of $I := (-1, 1)$, geometrically refined towards, say $x = 1$, with grading factor $\sigma \in (0, 1)$ and total number of layers n; see Figure 1 (level 2, left).

An **edge patch** is given by

$$\mathcal{T}_e = \mathcal{T}_e^{bl} := \{K_{xy} \times I \mid \quad K_{xy} \in \mathcal{T}_{xy}\}, \tag{6}$$

where $\mathcal{T}_{xy}$ is a triangulation of $\hat{S} := I^2$ obtained by first considering an irregular corner mesh, geometrically refined towards a vertex of $\hat{S}$, say $(x, y) = (1, 1)$, with grading factor σ and n refinement levels (see Figure 2, level 2, left). The elements of this macro-mesh are then anisotropically refined towards the two edges $x = 1$ and $y = 1$, in order to obtain a regular mesh $\mathcal{T}_{xy}$. We refer to Figure 1 (level 2, center) for an example.

In order to build a **corner patch** $\mathcal{T}_c$, we first consider an initial, irregular, corner mesh $\mathcal{T}_{c,m}$, geometrically refined towards a vertex of $\hat{Q}$, say $(x, y, z) = (1, 1, 1)$, with grading factor σ and n refinement levels; see the mesh in bold lines in Figure 1 (level 2, right). The elements of this macro-mesh are then anisotropically refined towards the three faces $x = 1$, $y = 1$, and $z = 1$ in order to obtain a regular mesh $\mathcal{T}_c$.

The number of elements in a face, edge, and corner patch with n layers is $O(n)$, $O(n^2)$, and $O(n^3)$, respectively. Consequently, if $n = O(k)$, as is required for exponential convergence, the corresponding FE spaces have $O(k^4)$, $O(k^5)$, and $O(k^6)$ degrees of freedom, respectively.

Our main result is the following theorem; see [9–11] for the corresponding two-dimensional result.

Theorem 3.1 *Let $\mathcal{T} = \mathcal{T}_{bl}^{n,\sigma}$ be a geometric boundary layer mesh. Then, there exists a constant C that depends on the grading factor σ, but is independent of k, n, and the aspect ratio of $\mathcal{T}$, such that, for any n and $k \geq 2$,*

$$\inf_{0 \neq p \in S_0^{k-2,0}(\Omega,\mathcal{T})} \quad \sup_{0 \neq v \in S_0^{k,1}(\Omega,\mathcal{T})^3} \frac{(\nabla \cdot v, p)_\Omega}{|v|_{1,\Omega} \, \|p\|_{0,\Omega}} \geq C k^{-3/2}. \tag{7}$$

3.2 Construction of geometric edge meshes

When only singularities and no boundary layers are present (as, e.g., in Stokes flows or in nearly incompressible elasticity), it is not necessary to refine geometrically towards the faces. The corresponding geometric edge meshes $\mathcal{T}_{edge}^{n,\sigma}$ are tensor products of meshes that are geometrically refined towards the edges only. Figure 2 shows the construction of a geometric edge mesh $\mathcal{T}_{edge}^{n,\sigma}$.

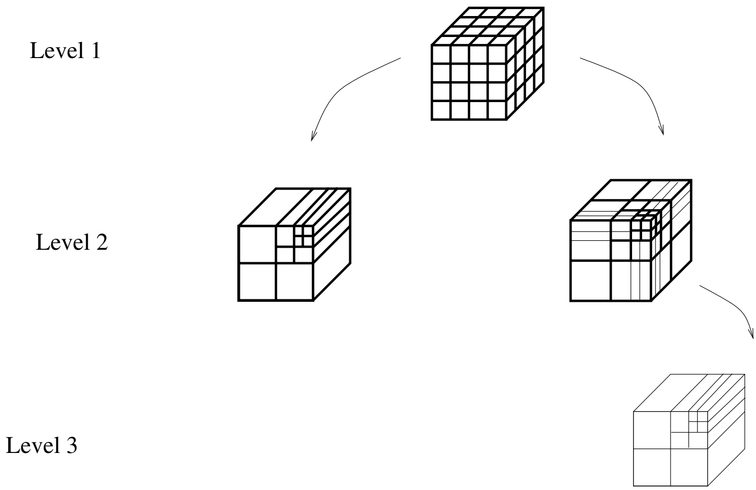

Fig. 2. Hierarchic structure of a geometric edge mesh $\mathcal{T}^{n,\sigma}_{edge}$

As in the case of a boundary layer mesh, $\mathcal{T}^{n,\sigma}_{edge}$ is built by first considering an initial shape-regular macro-triangulation $\mathcal{T}_m$, possibly consisting of just one element, which is successively refined. This process is illustrated in Figure 2. Every macro-element can be refined isotropically (not shown) or anisotropically in order to obtain so-called edge or corner patches (Figure 2, level 2).

An **edge patch** $\mathcal{T}_e$ is given by

$$\mathcal{T}_e = \mathcal{T}_e^{edge} := \{K_{xy} \times I \mid \quad K_{xy} \in \mathcal{T}_{xy}\}, \tag{8}$$

where $\mathcal{T}_{xy}$ is an irregular corner mesh, geometrically refined towards a vertex of $\hat{S}$ with grading factor σ and n refinement levels; see Figure 2 (level 2, left).

In order to build a **corner patch** $\mathcal{T}_c$, we first consider an initial, irregular, corner mesh $\mathcal{T}_{c,m}$, geometrically refined towards a vertex of $\hat{Q}$, with grading factor σ and n refinement levels; see the mesh in bold lines in Figure 2 (level 2, right). The elements of this macro-mesh are then refined towards the three edges adjacent to the vertex. We note that the macro-mesh $\mathcal{T}_{c,m}$ is the same as for a boundary layer mesh, but $\mathcal{T}_c$ is in general irregular. Figure 3 shows the difference between corner patches for boundary layer and edge meshes.

The number of elements in an edge and a corner patch with n layers is $O(n)$ and $O(n^2)$, respectively. Consequently, if $n = O(k)$, as is required for exponential convergence, the corresponding FE spaces have $O(k^4)$ and $O(k^5)$ degrees of freedom, respectively; see [3].

In section 9, we show that Theorem 3.1 also holds for an edge mesh $\mathcal{T} = \mathcal{T}^{n,\sigma}_{edge}$.

Fig. 3. Geometrically refined corner patches for boundary layer (left) and edge (right) meshes

4 Macro-element technique

In order to prove Theorem 3.1, we repeatedly use a macro-element technique; see [15,16,9,11]. Given a mesh $\mathcal{T}$, it is enough to prove the divergence-stability for a couple of low dimensional spaces, typically $S_0^{2,1}(\Omega,\mathcal{T})^3$ and $S_0^{0,0}(\Omega,\mathcal{T})$, on a macro-mesh contained in or coinciding with $\mathcal{T}$, and the stability of local higher order spaces defined on the single elements K of the macro-mesh, $S_0^{k,1}(K)^3$ and $S_0^{k-2,0}(K)$ in this case.

The following theorem holds.

Theorem 4.1 *Let $\mathcal{F}$ be a family of irregular or regular affine meshes on the reference element $\hat{Q}$, also containing the trivial triangulation $\hat{\mathcal{T}} = \hat{Q}$. On a bounded polyhedral domain $\Omega \subset \mathbb{R}^3$, let $\mathcal{T}$ be an affine mesh which is obtained from a (coarser) affine shape-regular macro-element mesh $\mathcal{T}_m$ in the following way: Some elements of $\mathcal{T}_m$ are further partitioned into $F_K(\hat{\mathcal{T}})$ where $\hat{\mathcal{T}} \in \mathcal{F}$ and F_K is the affine mapping between $\hat{Q}$ and K. Let $k \geq 2$ be a polynomial degree. Assume that there exists a space $X_N \subseteq S_0^{k,1}(\Omega,\mathcal{T})^3 \subset H_0^1(\Omega)^3$ such that*

$$\inf_{0\neq p\in S_0^{0,0}(\Omega,\mathcal{T}_m)} \sup_{0\neq v\in X_N} \frac{(\nabla\cdot v,\, p)_\Omega}{|v|_{1,\Omega}\,\|p\|_{0,\Omega}} \geq C_1, \tag{9}$$

with a constant $C_1 > 0$ independent of k. Assume further that the family $\mathcal{F}$ is uniformly stable in the sense that there holds

$$\inf_{0\neq p\in S_0^{k-2,0}(\hat{Q},\hat{\mathcal{T}})} \sup_{0\neq v\in S_0^{k,1}(\hat{Q},\hat{\mathcal{T}})^3} \frac{(\nabla\cdot v,\, p)_{\hat{Q}}}{|v|_{1,\hat{Q}}\|p\|_{0,\hat{Q}}} \geq C_2\, k^{-1}, \tag{10}$$

for all $\hat{\mathcal{T}} \in \mathcal{F}$ and all k.

Then the spaces $S_0^{k,1}(\Omega, \mathcal{T})^3$ *and* $S_0^{k-2,0}(\Omega, \mathcal{T})$ *satisfy (7) with a constant* C *that can be bounded by*

$$C \geq \tilde{C}\,\frac{C_2}{\kappa^3}\,\min\{1, C_1^2\},$$

where κ *is the aspect ratio of the elements of* $\mathcal{T}_m$ *and* $\tilde{C}$ *is independent of* k, κ, C_1, *and* C_2.

We refer to [15, 16, 9, 11] for a proof. In particular we note that a shape-regular macro-mesh $\mathcal{T}_m$ is required since locally refined meshes $F_K(\hat{\mathcal{T}})$ are employed for $K \in \mathcal{T}_m$.

We apply the macro-element technique recursively in our analysis. This is illustrated in Figure 1. At the top level, we have the shape-regular macro-mesh $\mathcal{T}_m$, which is successively refined. Every macro-element can be refined isotropically (not shown), or anisotropically towards a face (second level, left), or an edge (second level, center), or a corner (second level, right). The divergence stability for the shape-regular macro-mesh at the top level and the isotropically refined patches is well-known; see [16]. We then prove the stability of the single patches for the higher order spaces:

- **Face patch**. We build a Fortin operator, generalizing the analysis in [10, Sect. 3].
- **Edge patch**. We use a macro-element technique applied to a refined auxiliary edge mesh and next construct a Fortin operator that maps the refined velocity space into that on the original edge patch.
- **corner patch**. For the corner patch, we generalize the two-dimensional analysis in [11, Sect. 4]. We prove the stability for low order spaces on the corner mesh in bold lines in Figure 1 (second level, right) and then use the stability of the other patches. We note that the refined elements of this macro-mesh are face and edge patches.

5 Shape-regular meshes with hanging nodes

In this section we prove a lower bound for the inf–sup constant of conforming velocity and pressure pairs on isotropic meshes with hanging nodes that decreases as k^{-1}, thus generalizing the result for regular meshes in [16]. This result, which also generalizes the two-dimensional one in [11] and is of interest in itself, will then be employed for the analysis of edge and corner patches. For simplicity we only consider a particular mesh in full detail, but note that our construction can be employed for more general irregular meshes in a straightforward way. We consider the mesh $\mathcal{T}_{c,m}$, which is a macro-mesh employed for the construction of an edge patch; see Section 3. We note that $\mathcal{T}_{c,m}$ is not quasi-uniform.

We first need to introduce a low-order velocity space $\mathcal{L}_0^{1,1}(\hat{Q}, \mathcal{T}_{c,m})$ on the corner macro-mesh $\mathcal{T}_{c,m}$.

$$S_0^{1,1}(\hat{Q}, \mathcal{T}_{c,m})^3 \subset \mathcal{L}_0^{1,1}(\hat{Q}, \mathcal{T}_{c,m}) \subset S_0^{2,1}(\hat{Q}, \mathcal{T}_{c,m})^3.$$

Given an element $K \in \mathcal{T}_{c,m}$ such that $K = F_K(\hat{Q})$, we introduce some notations associated to its faces. Let the faces of $\hat{Q}$ perpendicular to the $\hat{x}$-axis be

$$\begin{aligned} \hat{\Gamma}_1^x &:= \{\hat{x} = -1\} \times (-1, 1)^2, \\ \hat{\Gamma}_2^x &:= \{\hat{x} = +1\} \times (-1, 1)^2. \end{aligned}$$

The two other sets of faces $\hat{\Gamma}_i^j$, for $j = y, z$ and $i = 1, 2$, are defined in a similar way. Let Γ_i^j, for $j = x, y, z$ and $i = 1, 2$, be the corresponding faces of K and $n_{j,i}$ the unit vectors which are perpendicular to them, pointing outward.

To the two x-faces of $\hat{Q}$, $\hat{\Gamma}_i^x$, $i = 1, 2$, we can associate the two functions

$$\begin{aligned} \hat{q}_{x,1} &:= (1 - \hat{x}) \cdot (1 + \hat{y})(1 - \hat{y}) \cdot (1 + \hat{z})(1 - \hat{z}), \\ \hat{q}_{x,2} &:= (1 + \hat{x}) \cdot (1 + \hat{y})(1 - \hat{y}) \cdot (1 + \hat{z})(1 - \hat{z}), \end{aligned}$$

respectively. We note that $\hat{q}_{x,1}$, for instance, vanishes on all the faces except on $\hat{\Gamma}_1^x$ and, when restricted to this face, is a polynomial in $\mathbb{Q}_2$. The functions $\hat{q}_{j,i}$, for $j = y, z$ and $i = 1, 2$, associated to the other faces can be defined in a similar way by suitable permutations of the indices.

We now define, for $j = x, y, z$ and $i = 1, 2$, the vector functions

$$w_{j,i} := n_{j,i} \left(\hat{q}_{j,i} \circ F_K^{-1} \right) \in \mathbb{Q}_2(K)^3,$$

and the local space

$$\mathcal{L}^1(K) := \mathbb{Q}_1(K)^3 \oplus span\{w_{j,i}; \quad j = x, y, z; \ i = 1, 2\}.$$

The corresponding global space is

$$\mathcal{L}_0^{1,1}(\hat{Q}, \mathcal{T}_{c,m}) = \mathcal{L}_0^{1,1}(\hat{Q}) := \left\{ v \in H_0^1(\hat{Q})^3 | \quad v_{|K} \in \mathcal{L}^1(K), \ K \in \mathcal{T}_{c,m} \right\}.$$

Before proving the divergence stability for the low-order spaces $\mathcal{L}_0^{1,1}(\hat{Q}, \mathcal{T}_{c,m})$ and $S_0^{0,0}(\hat{Q}, \mathcal{T}_{c,m})$, we need to introduce a Clément-type interpolation operator for the three-dimensional irregular mesh $\mathcal{T}_{c,m}$. We begin by introducing some notations for the corner macro-mesh $\mathcal{T}_{c,m} = \mathcal{T}_{c,m}^{n,\sigma}$, refined towards a vertex, e.g., $V = (1, 1, 1)$.

The corner macro-mesh $\mathcal{T}_{c,m}^{n,\sigma}$ can be constructed recursively. This is illustrated in Figure 4. Let $\mathcal{T}_{c,m}^{0,\sigma} = \hat{Q}$. Then, $\mathcal{T}_{c,m}^{1,\sigma}$ is obtained by partitioning $\mathcal{T}_{c,m}^{0,\sigma}$ into eight elements by dividing its sides in a $\sigma{:}(1 - \sigma)$ ratio. Let

$$\mathcal{T}_{c,m}^{1,\sigma} = \{\Omega_{i,1}, \ 1 \leq i \leq 8\},$$

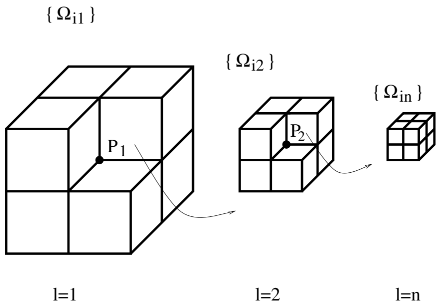

Fig. 4. Recursive construction of an irregular corner mesh $\mathcal{T}_{c,m}$

with $\Omega_{8,1}$ denoting the element that contains V. At the next refinement level $l = 2$, we partition $\Omega_{8,1}$ into eight parallelepipeds in a similar way. The final mesh $\mathcal{T}^{n,\sigma}_{c,m}$ is obtained after $l = n$ refinement steps. At an intermediate refinement level $1 \le l \le n-1$, there are seven new parallelepipeds introduced at level l that do not touch V:

$$\{\Omega_{i,l},\ 1 \le i \le 7\}, \quad 1 \le l \le n-1.$$

For $l = n$, let

$$\{\Omega_{i,n},\ 1 \le i \le 8\},$$

be the new eight parallelepipeds obtained after the last refinement. We remark that the $\{\Omega_{i,l},\ 1 \le i \le 7,\ l \le n\}$ are disjoint and that

$$\overline{\Omega} = \left(\bigcup_{l=1}^{n}\bigcup_{i=1}^{7}\overline{\Omega}_{i,l}\right) \cup \overline{\Omega}_{8,n}.$$

We next consider the linear space $S^{1,1}_0(\hat{Q}, \mathcal{T}^{n,\sigma}_{c,m})$. It is spanned by the n nodal basis functions $\{\phi_l\}$ associated to the regular nodes $\{P_l,\ 1 \le l \le n\}$ of $\mathcal{T}^{n,\sigma}_{c,m}$. We note that P_l is the node that is common to the elements $\{\Omega_{l,i},\ 1 \le i \le 7\}$ at level l; (see Figure 4) and that

$$\mathcal{O}_l := supp\,\{\phi_l\} = \left(\bigcup_{i=1}^{7}\overline{\Omega}_{i,l}\right) \cup \left(\bigcup_{i=1}^{7}\overline{\Omega}_{i,l+1}\right), \quad 1 \le l \le n-1,$$
$$\mathcal{O}_n := supp\,\{\phi_n\} = \bigcup_{i=1}^{8}\overline{\Omega}_{i,n}.$$

Let finally $\mathcal{E}(\mathcal{T}^{n,\sigma}_{c,m})$ be the set of all faces e of the elements in $\mathcal{T}^{n,\sigma}_{c,m}$ and, for $e \in \mathcal{E}(\mathcal{T}^{n,\sigma}_{c,m})$, let h_e be the diameter of e.

Our Clément type operator is then defined in the following way.

Definition 5.1 *Given $u \in H_0^1(\hat{Q})$, let*

$$Iu := \sum_{l=1}^{n} a_l \, \phi_l \in S_0^{1,1}(\hat{Q}, \mathcal{T}_{c,m}^{n,\sigma}),$$

where

$$a_l := \frac{\int_{\mathcal{O}_l} u \, dx}{|\mathcal{O}_l|}, \quad 1 \le l \le n,$$

and $|\mathcal{O}_l|$ is the volume of $\mathcal{O}_l$.

The following error estimate holds; see [11, Prop. 4.5] for the corresponding two-dimensional result.

Lemma 5.2 *There exists a constant C that depends on σ but is otherwise independent of $\mathcal{T}_{c,m}^{n,\sigma}$ such that*

$$\sum_{K \in \mathcal{T}_{c,m}^{n,\sigma}} h_K^{-2} \|u - Iu\|_{0,K}^2 + \sum_{K \in \mathcal{T}_{c,m}^{n,\sigma}} |u - Iu|_{1,K}^2 + \sum_{e \in \mathcal{E}(\mathcal{T}_{c,m}^{n,\sigma})} h_e^{-1} \|u - Iu\|_{0,e}^2 \le C |u|_{1,\hat{Q}}^2.$$

Proof. Given an element $K = \Omega_{i,l}$, of diameter h_K, let ω_K be the union of the supports of the basis functions associated to its nodes. We note in particular that for $2 \le l \le n$ and $1 \le i \le 7$ these functions are ϕ_{l-1} and ϕ_l, for $l = 1$ and $1 \le i \le 7$ we only have ϕ_1, and for $l = n$ and $i = 8$ we only have ϕ_n. We can then write

$$\begin{aligned} \|Iu\|_{0,K}^2 &\le 2 \sum_{\mathcal{O}_l \supset K} |a_l|^2 \, \|\phi_l\|_{0,K}^2 \\ &\le 2 \sum_{\mathcal{O}_l \supset K} \frac{|K|}{|\mathcal{O}_l|} \|u\|_{0,\mathcal{O}_l}^2 \le 2 \sum_{\mathcal{O}_l \supset K} \|u\|_{0,\mathcal{O}_l}^2 \le 4 \|u\|_{0,\omega_K}^2. \end{aligned} \tag{11}$$

We now define

$$\tilde{u} := u - |\omega_K|^{-1} \int_{\omega_K} u \, dx.$$

Using (11), we can write

$$\|u - Iu\|_{0,K} = \|\tilde{u} - I\tilde{u}\|_{0,K} \le 3 \|\tilde{u}\|_{0,\omega_K}.$$

Since the diameter of ω_K is comparable to h_K, using the Poincaré inequality, we obtain

$$\|u - Iu\|_{0,K} \le C h_K |\tilde{u}|_{1,\omega_K} = C h_K |u|_{1,\omega_K}, \tag{12}$$

with a constant C that only depends on the shape of ω_K, and thus on σ but not on h_K.

Using an inverse estimate on K, we can write

$$|Iu|_{1,K} = |I\tilde{u}|_{1,K} \le Ch_K^{-1}\|I\tilde{u}\|_{0,K} \le Ch_K^{-1}(\|\tilde{u}\|_{0,\omega_K} + \|\tilde{u} - I\tilde{u}\|_{0,K}).$$

By applying the Poincaré inequality on ω_K and (12), we obtain

$$(13) \qquad |Iu|_{1,K} \le C|u|_{1,\omega_K},$$

with a constant that only depends on σ.

We are now left with the bounds for the face contributions. Given a face $e \subset \partial K$, we can use a trace estimate and obtain

$$h_e^{-1}\|u - Iu\|_{0,e}^2 \le C(h_K^{-2}\|u - Iu\|_{0,K}^2 + |u - Iu|_{1,K}^2).$$

We note that the constant C depends on the aspect ratio of K, and thus on σ, but not on h_K. Using (12) and (13), we find

$$(14) \qquad h_e^{-1}\|u - Iu\|_{0,e}^2 \le C|u|_{1,\omega_K}^2.$$

The proof is concluded by using (12), (13), and (14) and by summing over the elements. □

For the macro-mesh $\mathcal{T}_{c,m}^{n,\sigma}$, we are now ready to prove the following result.

Lemma 5.3 *There exists a constant C, depending on σ, but otherwise independent of $\mathcal{T}_{c,m}^{n,\sigma}$, such that*

$$(15) \qquad \inf_{0 \ne p \in S_0^{0,0}(\hat{Q},\mathcal{T}_{c,m}^{n,\sigma})} \; \sup_{0 \ne v \in \mathcal{L}_0^{1,1}(\hat{Q},\mathcal{T}_{c,m}^{n,\sigma})} \frac{(\nabla \cdot v, p)_{\hat{Q}}}{|v|_{1,\hat{Q}}\|p\|_{0,\hat{Q}}} \ge C.$$

Proof. The proof is similar to [11, Th. 4.9] and presented here for completeness.

We first need to define some local spaces associated to the patches $\{\mathcal{O}_l\}$. For $1 \le l \le n$, we set

$$\begin{aligned} S^{0,0}(\mathcal{O}_l) &:= \{p \in L^2(\mathcal{O}_l) |\; p_{|K} \in \mathbb{Q}_0,\; K \subset \mathcal{O}_l\} \\ \mathcal{L}_0^{1,1}(\mathcal{O}_l) &:= \{v \in H_0^1(\mathcal{O}_l)^3 |\; v_{|K} \in \mathcal{L}^1(K),\; K \subset \mathcal{O}_l\} \\ N_l &:= \mathbb{Q}_0(\mathcal{O}_l), \end{aligned}$$

and consider the orthogonal decomposition

$$S^{0,0}(\mathcal{O}_l) = N_l \oplus W_l. \tag{16}$$

We then define $\mathcal{E}(\mathcal{O}_l)$ as the set of all interelement faces in $\mathcal{O}_l$ and $\mathcal{E}_0(\mathcal{O}_l)$ as the subset of $\mathcal{E}(\mathcal{O}_l)$ of faces that do not have hanging nodes in their mid-point. Analogous definitions hold for the global sets $\mathcal{E}(\mathcal{T}_{c,m}^{n,\sigma})$ and $\mathcal{E}_0(\mathcal{T}_{c,m}^{n,\sigma})$.

On each patch $\mathcal{O}_l$, we define a mesh-dependent seminorm by

$$|p|_{\mathcal{O}_l}^2 := \sum_{e\in\mathcal{E}_0(\mathcal{O}_l)} h_e \int_e |[p]_e|^2 ds, \quad p \in S^{0,0}(\mathcal{O}_l),$$

where $[p]_e$ is the jump of p across a face e. The global seminorm is defined by

$$|p|_h^2 := \sum_{e\in\mathcal{E}_0(\mathcal{T}_{c,m}^{n,\sigma})} h_e \int_e |[p]_e|^2 ds,$$

A scaling argument gives, for $p \in W_j$,

$$\sup_{0\neq v\in\mathcal{L}_0^{1,1}(\mathcal{O}_l)} \frac{(\nabla\cdot v,\, p)_{\mathcal{O}_j}}{|v|_{1,\mathcal{O}_j}} \geq \tilde{\gamma}\ |p|_{\mathcal{O}_j}, \tag{17}$$

with $\tilde{\gamma}$ depending on the shape of $\mathcal{O}_j$, and thus depending on σ but not on h or l.

Given $p \in S_0^{0,0}(\hat{Q}, \mathcal{T}_{c,m}^{n,\sigma})$, we set $p_l := p_{|\mathcal{O}_l}$. According to (16), we have the decomposition

$$p_l = c_l + q_l,$$

where $c_l \in N_l$ is constant and $q_l \in W_l$. The stability condition (17) implies that, for each q_l there exists $v_l \in \mathcal{L}_0^{1,1}(\mathcal{O}_l)$, such that

$$(\nabla\cdot v_l, q_l)_{\mathcal{O}_l} \geq \tilde{\gamma}|q_l|_{\mathcal{O}_l}^2, \quad |v_l|_{1,\mathcal{O}_l} \leq |q_l|_{\mathcal{O}_l},$$

and therefore

$$(\nabla\cdot v_l,\, p_l)_{\mathcal{O}_l} \geq \tilde{\gamma}|p_l|_{\mathcal{O}_l}^2, \quad |v_l|_{1,\mathcal{O}_l} \leq |p_l|_{\mathcal{O}_l}.$$

If we define $v := \sum_{l=1}^n v_l$, we have

$$(\nabla\cdot v,\, p)_{\hat{Q}} = \sum_{l=1}^n (\nabla\cdot v_l,\, p)_{\hat{Q}} = \sum_{l=1}^n (\nabla\cdot v_l,\, p_l)_{\mathcal{O}_l} \geq \tilde{\gamma} \sum_{l=1}^n |p_l|_{\mathcal{O}_l}^2 \geq C|p_l|_h^2,$$

and

$$|v|_{1,\hat{Q}}^2 \leq \sum_{l=1}^n |v_l|_{1,\hat{Q}}^2 \leq C|p_l|_h^2,$$

which are equivalent to

$$\sup_{0\neq v\in\mathcal{L}_0^{1,1}(\hat{Q},\mathcal{T}_{c,m}^{n,\sigma})} \frac{(\nabla\cdot v,\, p)_{\hat{Q}}}{|v|_{1,\hat{Q}}} \geq C_1|p|_h. \tag{18}$$

We now show that we can replace the seminorm with a norm in (18). The continuous stability condition (3) ensures that for $p \in S_0^{0,0}(\hat{Q}, \mathcal{T}_{c,m}^{n,\sigma})$ there exists $v \in H_0^1(\hat{Q})$, such that

$$(\nabla\cdot v,\, p)_{\hat{Q}} \geq \gamma\|p\|_{0,\hat{Q}}^2, \quad |v|_{1,\hat{Q}}^2 \leq \|p\|_{0,\hat{Q}}^2.$$

We define $u \in S_0^{1,1}(\hat{Q}, \mathcal{T}_{c,m}^{n,\sigma})^3$ by

$$u_i := I v_i, \quad i = x, y, z.$$

Using integration by parts over the elements, Cauchy–Schwarz, and Lemma 5.2, we find

$$\begin{aligned}
(\nabla\cdot u,\, p)_{\hat{Q}} &= (\nabla\cdot(u-v),\, p)_{\hat{Q}} + (\nabla\cdot v,\, p)_{\hat{Q}} \\
&\geq \sum_{e\in\mathcal{E}_0(\mathcal{T}_{c,m}^{n,\sigma})} \int_e ((u-v)\cdot n)\,[p]_e ds + \gamma\|p\|_{0,\hat{Q}}^2 \\
&\geq -\left(\sum_{e\in\mathcal{E}(\mathcal{T}_{c,m}^{n,\sigma})} h_e^{-1}\|u-v\|_{0,e}^2\right)^{1/2} |p|_h + \gamma\|p\|_{0,\hat{Q}}^2 \\
&\geq \|p\|_{0,\hat{Q}}^2\left(C_3 - C_2\frac{|p|_h}{\|p\|_{0,\hat{Q}}}\right).
\end{aligned}$$

Since $|u|_{1,\hat{Q}} \leq C\|p\|_{0,\hat{Q}}$, we have

$$\sup_{0\neq u\in\mathcal{L}_0^{1,1}(\hat{Q},\mathcal{T}_{c,m}^{n,\sigma})} \frac{(\nabla\cdot u,\, p)_{\hat{Q}}}{|u|_{1,\hat{Q}}} \geq \|p\|_{0,\hat{Q}}\left(C_4 - C_5\frac{|p|_h}{\|p\|_{0,\hat{Q}}}\right). \tag{19}$$

Combining (18) and (19), we then have

$$\sup_{0\neq v\in\mathcal{L}_0^{1,1}(\hat{Q},\mathcal{T}_{c,m}^{n,\sigma})} \frac{(\nabla\cdot v,\, p)_{\hat{Q}}}{|v|_{1,\hat{Q}}} \geq \|p\|_{0,\hat{Q}}\ \min_{t\geq 0} f(t),$$

with $t := |p|_h/\|p\|_{0,\hat{Q}}$ and $f(t) := \max\{C_4 - C_5 t, C_1 t\}$. The proof is concluded by noticing that $\min_{t\geq 0} f(t) = (C_1C_4)/(C_1+C_5) > 0$. □

We then obtain the following stability result for the irregular mesh $\mathcal{T}_{c,m}$, by using Lemma 5.3 and the stability result for single elements $K \in \mathcal{T}_{c,m}$.

Theorem 5.4 *Let $\mathcal{T}_{c,m}$ be a corner macro-mesh with grading factor σ and n layers. Then, there exists a constant C, that depends on σ, but is independent of k and n, such that*

$$\inf_{0\neq p\in S_0^{k-2,0}(\hat{Q},\mathcal{T}_{c,m})} \sup_{0\neq v\in S_0^{k,1}(\hat{Q},\mathcal{T}_{c,m})^3} \frac{(\nabla\cdot v, p)_{\hat{Q}}}{|v|_{1,\hat{Q}}\|p\|_{0,\hat{Q}}} \geq Ck^{-1}. \tag{20}$$

Proof. It is enough to use Theorem 4.1 with $\mathcal{T} = \mathcal{T}_{c,m}$ and $\mathcal{T}_m = \mathcal{T}_{c,m}$. □

6 Face patches

A face patch is given by a mesh $\mathcal{T}_f$ of the form (5). For this patch, we prove that the inf–sup constant γ_N is independent of $\mathcal{T}_x$ and, consequently, of σ and n. In addition it decreases as Ck^{-1}, as in the case of shape-regular meshes.

In this section, we generalize the analysis in [10, Sect. 3] for boundary layer patches in two-dimensions, by building a Fortin operator $\Pi_k : H_0^1(\hat{Q})^3 \longrightarrow S_0^{k,1}(\hat{Q}, \mathcal{T}_f)$, that satisfies the following property.

Theorem 6.1 *There exists a constant C, independent of k and the diameter and the aspect ratio of $\mathcal{T}_f$, such that, for all $v \in H_0^1(\hat{Q})^3$,*

$$|\Pi_k v|_{1,\hat{Q}} \leq Ck|v|_{1,\hat{Q}}, \tag{21}$$

$$(\nabla\cdot v, p)_{\hat{Q}} = (\nabla\cdot\Pi_k v, p)_{\hat{Q}}, \quad p \in S_0^{k-2,0}(\hat{Q}, \mathcal{T}_f). \tag{22}$$

It is then immediate to see that if the inf–sup condition (3) for the continuous spaces $H_0^1(\hat{Q})^3$–$L_0^2(\hat{Q})$ holds and a Fortin operator Π_k that satisfies Theorem 6.1 can be found, the following inf–sup condition for the discrete spaces holds

$$\inf_{0\neq p\in S_0^{k-2,0}(\hat{Q},\mathcal{T}_f)} \sup_{0\neq v\in S_0^{k,1}(\hat{Q},\mathcal{T}_f)^3} \frac{(\nabla\cdot v, p)_{\hat{Q}}}{|v|_{1,\hat{Q}}\|p\|_{0,\hat{Q}}} \geq Ck^{-1}, \tag{23}$$

with a constant C that is independent of k and of the diameter and the aspect ratio of $\mathcal{T}_f$.

6.1 The Fortin operator for the face patch

We begin by defining an operator on the reference cube $\hat{Q}$. We first need to define some of the geometric objects of $\hat{Q}$:
Let the faces of $\hat{Q}$ perpendicular to the x-axis be

$$\Gamma_{\pm}^x := \{x = \pm 1\} \times (-1, 1)^2.$$

The two other sets of faces $\Gamma^i_\pm$, $i = y, z$, are defined in a similar way. The edges of $\hat{Q}$ parallel to the x, y, and z-axis are denoted by E^x_j, E^y_j, and E^z_j, $j = 1, \ldots, 4$, respectively. Finally, let $\{P_i, \quad i = 1, \cdots, 8\}$ be the set of vertices of $\hat{Q}$. Similar definitions hold for an element $K \in \mathcal{T}_f$.

Definition 6.2 *Let $r, s, m \geq 2$ and $v \in H^{\epsilon+3/2}(\hat{Q})$, $\epsilon > 0$. We define $u = I_{r,s,m}\, v$ as the unique polynomial in $\mathbb{Q}_{r,s,m}(\hat{Q})$ satisfying the following $(r+1)(s+1)(m+1)$ conditions:*

$$u(P_i) = v(P_i), \quad i = 1, \ldots, 8, \tag{24}$$

$$\int_{E^x_i} (u - v)p\, dx = 0, \quad p \in \mathbb{Q}_{r-2}, \quad i = 1, \ldots, 4, \tag{25}$$

$$\int_{E^y_i} (u - v)p\, dy = 0, \quad p \in \mathbb{Q}_{s-2}, \quad i = 1, \ldots, 4, \tag{26}$$

$$\int_{E^z_i} (u - v)p\, dz = 0, \quad p \in \mathbb{Q}_{m-2}, \quad i = 1, \ldots, 4, \tag{27}$$

$$\int_{\Gamma^x_\pm} (u - v)p\, dydz = 0, \quad p \in \mathbb{Q}_{s-2,m-2}, \tag{28}$$

$$\int_{\Gamma^y_\pm} (u - v)p\, dxdz = 0, \quad p \in \mathbb{Q}_{r-2,m-2}, \tag{29}$$

$$\int_{\Gamma^z_\pm} (u - v)p\, dxdy = 0, \quad p \in \mathbb{Q}_{r-2,s-2}, \tag{30}$$

$$\int_{\hat{Q}} (u - v)p\, dxdydz = 0, \quad p \in \mathbb{Q}_{r-2,s-2,m-2}. \tag{31}$$

We note that $I_{r,s,m}$ cannot be defined on the whole space $H^1(\hat{Q})$ since values at the edges or vertices of $\partial\hat{Q}$ are not defined in general. However, it can be defined on the space

$$H(\hat{Q}) := \{v \in H^1(\hat{Q}) \mid \quad v = 0 \text{ on } \Gamma^y_\pm \text{ and } \Gamma^z_\pm\}.$$

In this case, v can be assumed to be zero on $\partial\hat{Q} \setminus (\Gamma^x_- \cup \Gamma^x_+)$ in Definition 6.2.

An interpolation operator $I^K_{r,s,m}$ can also be defined on an affinely mapped element $K = F_K(\hat{Q}) \in \mathcal{T}_f$, for functions in $H(K)$. Here, $H(K)$ is defined in a similar way as $H(\hat{Q})$.

Our Fortin operator is then defined locally using the operators $\{I^K_{r,s,m}\}$.

Definition 6.3 *Let $v = (v_x, v_y, v_z) \in H^1_0(\hat{Q})^3$. We define*

$$u = (u_x, u_y, u_z) := \Pi_k v$$

as the unique vector in $S^{k,1}_0(\hat{Q}, \mathcal{T}_f)^3$ that satisfies, for $i = x, y, z$ and $K \in \mathcal{T}_f$,

$$u_i := I^K_{k,k,k} v_i, \quad on\ K.$$

Note that Π_k is well defined, since, for $v \in H^1_0(\hat{Q})^3$, the restrictions of v_i to $K \in \mathcal{T}_f$, for $i = x, y, z$, belong to $H(K)$.

6.2 Proof of Theorem 6.1

Let $\mathcal{I}(\hat{Q})$ be the set of polynomials on $\hat{Q}$ and $\mathcal{I}_0(\hat{Q})$ its subspace of polynomials that vanish on $\Gamma^y_\pm$ and $\Gamma^z_\pm$.

We will first consider the operator $I_{r,s,m} : \mathcal{I}(\hat{Q}) \longrightarrow \mathbb{Q}_{r,s,m}(\hat{Q})$ and introduce a suitable basis for the two polynomial spaces, which allows a convenient representation of $I_{r,s,m}$.

Let $\{L_i(x),\ i \in \mathbb{N}_0\}$ be the set of Legendre polynomials of degree i on I. We also set $L_{-1} = L_{-2} = 0$. We consider the one-dimensional basis $\{U_i(x),\ i \in \mathbb{N}_0\}$ defined by

$$(32) \qquad U_0(x) = 1, \quad U_1(x) = x, \qquad U_i(x) = \int_{-1}^{x} L_{i-1}(t)dt \quad i \geq 2.$$

The set $\{U_i(x)U_j(y)U_l(z);\ i, j, l \in \mathbb{N}_0\}$ is thus a basis for $\mathcal{I}(\hat{Q})$. Indeed, each $v \in \mathcal{I}(\hat{Q})$ can be uniquely written as

$$(33) \qquad v(x, y, z) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{l=0}^{\infty} a_{ijl} U_i(x)U_j(y)U_l(z),$$

where only finitely many terms are non-vanishing. For a polynomial $v \in \mathbb{Q}_{r,s,m}(\hat{Q})$ the sum is taken for $i \leq r,\ j \leq s,\ l \leq m$.

We recall that, if $\gamma_i := \frac{1}{2i+1}$, $i \in \mathbb{N}_0$, with $\gamma_{-1} = 1$ and $\gamma_{-2} = 0$, we have

$$(34) \qquad \int_I L_i(x)L_j(x)dx = 2\gamma_i\delta_{ij},$$

and

$$(35) \qquad U_i(x) = \gamma_{i-1}\left(L_i(x) - L_{i-1}(x)\right), \quad i \in \mathbb{N}_0.$$

In addition, using (32), (34), and (35), we can show the identities

$$(36) \qquad \int_I U_i(x)U_j(x)dx = \begin{cases} 2\gamma_{i-1}^2\gamma_i, & j = i = 0, 1, \\ 2\gamma_{i-1}^2(\gamma_i + \gamma_{i-2}), & j = i \geq 2, \\ -2\gamma_{i-1}\gamma_i\gamma_{i+1}, & j = i+2,\ i \geq 0, \\ -2\gamma_{i-3}\gamma_{i-2}\gamma_{i-1}, & j = i-2,\ i \geq 2, \\ 0, & \text{otherwise.} \end{cases}$$

Lemma 6.4 *Let $v \in \mathcal{I}(\hat{Q})$ be written in the form (33). Then*

$$\|v_x\|^2_{0,\hat{Q}} = \sum_{i=1}^{\infty}\sum_{j,l=0}^{\infty} 8\gamma_{i-1}\gamma_j\gamma_l \left((\gamma_{j-1}\gamma_{l-1}a_{i,j,l} - \gamma_{j-1}\gamma_{l+1}a_{i,j,l+2}) - (\gamma_{j+1}\gamma_{l-1}a_{i,j+2,l} - \gamma_{j+1}\gamma_{l+1}a_{i,j+2,l+2})\right)^2, \tag{37}$$

$$\|v_y\|^2_{0,\Gamma^x_\pm} := \|v_y\|^2_{0,\Gamma^x_-} + \|v_y\|^2_{0,\Gamma^x_+} = \sum_{i=0}^{1}\sum_{j=1}^{\infty}\sum_{l=0}^{\infty} 8\,\gamma_{j-1}\gamma_l(\gamma_{l-1}a_{i,j,l} - \gamma_{l+1}a_{i,j,l+2})^2, \tag{38}$$

$$\|v\|^2_{0,\Gamma^x_\pm} := \|v\|^2_{0,\Gamma^x_-} + \|v\|^2_{0,\Gamma^x_+} = \sum_{i=0}^{1}\sum_{j,l=0}^{\infty} 8\,\gamma_j\gamma_l \left((\gamma_{j-1}\gamma_{l-1}a_{i,j,l} - \gamma_{j-1}\gamma_{l+1}a_{i,j,l+2}) - (\gamma_{j+1}\gamma_{l-1}a_{i,j+2,l} - \gamma_{j+1}\gamma_{l+1}a_{i,j+2,l+2})\right)^2. \tag{39}$$

The corresponding expressions for $\|v_y\|^2_{0,\hat{Q}}$, $\|v_z\|^2_{0,\hat{Q}}$, $\|v_z\|^2_{0,\Gamma^x_\pm}$, and for the norms on the other faces are obtained by permutations of the indices.

Proof. The proof of (37) can be carried out in a similar way as in the two-dimensional case (see [10]) and as in the simpler case where $v \in \mathcal{I}(\hat{Q}) \cap H^1_0(\hat{Q})$ (see [16]).

For (38), it is enough to realize that the restriction of v to a face, e.g., $x = 1$,

$$v(1, y, z) = \sum_{i=0}^{1}\sum_{j=1}^{\infty}\sum_{l=0}^{\infty} a_{ijl}U_i(1)U_j(y)U_l(z),$$

is a polynomial in y and z. The expressions for the L^2-norm of the derivatives of a polynomial in two variables proven in [10, Lem. 3.14], can be thus employed. The proof of (39) can be carried out in a similar way as for (37). □

Lemma 6.5 *Let $v \in \mathcal{I}(\hat{Q})$ be written in the form (33). Then*

$$u(x, y, z) = (I_{r,s,m}v)(x, y, z) = \sum_{i=0}^{r}\sum_{j=0}^{s}\sum_{l=0}^{m} a_{ijl}U_i(x)U_j(y)U_l(z).$$

Proof. Let

$$u(x, y, z) = \sum_{i=0}^{r}\sum_{j=0}^{s}\sum_{l=0}^{m} b_{ijl}U_i(x)U_j(y)U_l(z).$$

We first note that the only contributions that do not vanish on the boundary in the expansions of v and u are for $0 \le i \le 1$ or $0 \le j \le 1$ or $0 \le l \le 1$.

Condition (24) ensures that

$$b_{ijl} = a_{ijl}, \quad 0 \le i, j, l \le 1, \tag{40}$$

since $u - v$ vanishes at the vertices of $\hat{Q}$.

We next consider condition (25), with $p(x) = L'_{n-1}(x)$, $n = 2, \ldots, r$, and the edge

$$E_1^x = \{(x, y, z), \quad x \in I, \; y = -1, \; z = -1\}.$$

We have

$$\sum_{i=0}^{r} (b_{i00} - b_{i01} - b_{i10} + b_{i11}) \int_I U_i L'_{n-1} dx = \sum_{i=0}^{\infty} (a_{i00} - a_{i01} - a_{i10} + a_{i11}) \int_I U_i L'_{n-1} dx.$$

Integrating by parts and using (40), we obtain

$$b_{n00} - b_{n01} - b_{n10} + b_{n11} = a_{n00} - a_{n01} - a_{n10} + a_{n11}, \quad n = 2, \ldots, r.$$

Using (25) for the remaining edges, we obtain the four conditions, for $n = 2, \ldots, r$,

$$(b_{n00} - a_{n00}) \pm (b_{n01} - a_{n01}) \pm (b_{n10} - a_{n10}) + (b_{n11} - a_{n11}) = 0,$$

and finally

$$b_{nij} = a_{nij}, \quad 2 \le n \le r, \quad 0 \le i, j \le 1. \tag{41}$$

Using (26) and (27), we find, in a similar way,

$$b_{inj} = a_{inj}, \quad 2 \le n \le s, \quad 0 \le i, j \le 1, \tag{42}$$

$$b_{ijn} = a_{ijn}, \quad 2 \le n \le m, \quad 0 \le i, j \le 1. \tag{43}$$

We next consider condition (28), with

$$p(y, z) = L'_{n-1}(y) L'_{q-1}(z), \quad n = 2, \ldots, s, \quad q = 2, \ldots, m,$$

and the face Γ_-^x. We have

$$\sum_{j=0}^{s} \sum_{l=0}^{m} (b_{0jl} - b_{1jl}) \int_I U_j L'_{n-1} dy \int_I U_l L'_{q-1} dz = \sum_{j=0}^{\infty} \sum_{l=0}^{\infty} \left(a_{0jl} - a_{1jl}\right) \int_I U_j L'_{n-1} dy \int_I U_l L'_{q-1} dz.$$

Integrating by parts and using (40), (41), (42), and (43), we obtain

$$(b_{0nq} - b_{1nq}) - (a_{0nq} - a_{1nq}) = 0, \quad n = 2, \ldots, s, \quad q = 2, \ldots, m.$$

Using then (25) for Γ_+^x, we obtain the two conditions, for $n = 2, \ldots, s$ and $q = 2, \ldots, m$,

$$(b_{0nq} - a_{0nq}) \pm (b_{1nq} - a_{1nq}) = 0,$$

and finally

$$b_{inq} = a_{inq}, \quad 2 \le n \le s, \quad 2 \le q \le m, \quad 0 \le i \le 1. \tag{44}$$

Using (29) and (30), we find, in a similar way,

$$b_{niq} = a_{niq}, \quad 2 \le n \le r, \quad 2 \le q \le m, \quad 0 \le i \le 1, \tag{45}$$

$$b_{nqi} = a_{nqi}, \quad 2 \le n \le r, \quad 2 \le q \le s, \quad 0 \le i \le 1. \tag{46}$$

We finally consider condition (31), with

$$p(x, y, z) = L'_{n-1}(x) L'_{q-1}(y) L'_{t-1}(z), \;\; 2 \le n \le r, \; 2 \le q \le s, \; 2 \le t \le m.$$

We have

$$\begin{aligned}
&\sum_{i=0}^{r} \sum_{j=0}^{s} \sum_{l=0}^{m} b_{ijl} \int_I U_i L'_{n-1} dx \int_I U_j L'_{q-1} dy \int_I U_l L'_{t-1} dz \\
&\quad = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{l=0}^{\infty} a_{ijl} \int_I U_i L'_{n-1} dx \int_I U_j L'_{q-1} dy \int_I U_l L'_{t-1} dz.
\end{aligned}$$

Integrating by parts and using the previously proven conditions, we obtain

$$b_{nqt} = a_{nqt}, \quad 2 \le n \le r, \; 2 \le q \le s, \; 2 \le t \le m,$$

which concludes the proof. □

Lemma 6.6 *Let $v \in \mathcal{I}_0(\hat{Q})$ be written in the form (33) and $u := I_{r,s,m} v$. Then*

$$\|u_x\|^2_{0,\hat{Q}} \le C\, sm\, \|v_x\|^2_{0,\hat{Q}}, \tag{47}$$

$$\|u_y\|^2_{0,\hat{Q}} \le C\, rm\, \|v_y\|^2_{0,\hat{Q}} + C\, m\, S_y, \tag{48}$$

$$\|u_z\|^2_{0,\hat{Q}} \le C\, rs\, \|v_z\|^2_{0,\hat{Q}} + C\, s\, S_z, \tag{49}$$

where

$$\begin{aligned}
S_y &:= \sum_{i=0}^{1} \sum_{j=1}^{s} \sum_{l=0}^{m-2} 4 \gamma_{r-1} \gamma_{j-1} \gamma_l (\gamma_{l-1} a_{ijl} - \gamma_{l+1} a_{i,j,l+2})^2, \\
S_z &:= \sum_{i=0}^{1} \sum_{j=0}^{s-1} \sum_{l=1}^{m} 4 \gamma_{r-1} \gamma_j \gamma_{l-1} (\gamma_{j-1} a_{ijl} - \gamma_{j+1} a_{i,j+2,l})^2.
\end{aligned}$$

Proof. We will first prove a bound for u_x in case $v \in \mathcal{I}(\hat{Q})$. Using Lemma 6.5 and (37), we see that the sums in (37) can be decomposed into four parts

$$\begin{aligned}\|u_x\|^2_{0,\hat{Q}} &= \sum_{i=1}^{r}\sum_{j=0}^{s-2}\sum_{l=0}^{m-2} + \sum_{i=1}^{r}\sum_{j=0}^{s-2}\sum_{l=m-1}^{m} + \sum_{i=1}^{r}\sum_{j=s-1}^{s}\sum_{l=0}^{m-2} + \sum_{i=1}^{r}\sum_{j=s-1}^{s}\sum_{l=m-1}^{m}\\ &= A + B_1 + B_2 + D.\end{aligned}$$

Using (37), we immediately have

$$A \le \|v_x\|^2_{0,\hat{Q}}.$$

We next consider B_1 and note that B_1 consists of just two terms in l, for $l = m-1$ and $l = m$. We first consider the term for $l = m$ and suppose that m is odd. We can write, for i and j fixed,

$$\begin{aligned}&(\gamma_{j-1}\gamma_{m-1}a_{ijm} - \gamma_{j+1}\gamma_{m-1}a_{i,j+2,m})\\ &\quad= \sum_{l=0}^{\frac{m-3}{2}}[-(\gamma_{j-1}\gamma_{2l}a_{i,j,2l+1} - \gamma_{j+1}\gamma_{2l}a_{i,j+2,2l+1})\\ &\qquad+(\gamma_{j-1}\gamma_{2l+2}a_{i,j,2l+3} - \gamma_{j+1}\gamma_{2l+2}a_{i,j+2,2l+3})]\\ &\qquad+(\gamma_{j-1}a_{i,j,1} - \gamma_{j+1}a_{i,j+2,1}).\end{aligned}$$

Taking the square of both sides, we obtain

$$\begin{aligned}&(\gamma_{j-1}\gamma_{m-1}a_{ijm} - \gamma_{j+1}\gamma_{m-1}a_{i,j+2,m})^2\\ &\quad\le (m-1)\sum_{l=0}^{\frac{m-3}{2}}[-(\gamma_{j-1}\gamma_{2l}a_{i,j,2l+1} - \gamma_{j+1}\gamma_{2l}a_{i,j+2,2l+1})\\ &\qquad+(\gamma_{j-1}\gamma_{2l+2}a_{i,j,2l+3} - \gamma_{j+1}\gamma_{2l+2}a_{i,j+2,2l+3})]^2\\ &\qquad+2(\gamma_{j-1}a_{i,j,1} - \gamma_{j+1}a_{i,j+2,1})^2.\end{aligned}$$

The term for $l = m-1$ can be bounded in a similar way: for odd m, we obtain

$$\begin{aligned}&(\gamma_{j-1}\gamma_{m-2}a_{i,j,m-1} - \gamma_{j+1}\gamma_{m-2}a_{i,j+2,m-1})^2\\ &\quad\le (m-1)\sum_{l=0}^{\frac{m-3}{2}}[-(\gamma_{j-1}\gamma_{2l-1}a_{i,j,2l} - \gamma_{j+1}\gamma_{2l-1}a_{i,j+2,2l})\\ &\qquad+(\gamma_{j-1}\gamma_{2l+1}a_{i,j,2l+2} - \gamma_{j+1}\gamma_{2l+1}a_{i,j+2,2l+2})]^2\\ &\qquad+2(\gamma_{j-1}a_{i,j,0} - \gamma_{j+1}a_{i,j+2,0})^2.\end{aligned}$$

Analogous expressions can be found for even m. Using (37), we obtain

$$B_1 \le Cm\|v_x\|^2_{0,\hat{Q}} + C\sum_{i=1}^{r}\sum_{j=0}^{s-2}\sum_{l=0}^{1} 4\gamma_{i-1}\gamma_j\gamma_{m-1}(\gamma_{j-1}a_{i,j,l} - \gamma_{j+1}a_{i,j+2,l})^2.$$

In a similar way, we also find

$$B_2 \leq Cs\|v_x\|^2_{0,\hat{Q}} + C\sum_{i=1}^{r}\sum_{j=0}^{1}\sum_{l=0}^{m-2} 4\gamma_{i-1}\gamma_{s-1}\gamma_l(\gamma_{l-1}a_{i,j,l} - \gamma_{l+1}a_{i,j,l+2})^2.$$

We finally consider the last term D and note that D consists of four terms in j and l, for $j = s-1, s$ and $l = m-1, m$, which can be bounded as before, by employing one telescoping series for j and one for l. We obtain

$$\begin{aligned} D \leq{} & Csm\|v_x\|^2_{0,\hat{Q}} \\ & + Cm\sum_{i=1}^{r}\sum_{j=0}^{1}\sum_{l=0}^{m-2} 4\gamma_{i-1}\gamma_{s-1}\gamma_l(\gamma_{l-1}a_{i,j,l} - \gamma_{l+1}a_{i,j,l+2})^2 \\ & + Cs\sum_{i=1}^{r}\sum_{j=0}^{s-2}\sum_{l=0}^{1} 4\gamma_{i-1}\gamma_j\gamma_{m-1}(\gamma_{j-1}a_{i,j,l} - \gamma_{j+1}a_{i,j+2,l})^2 \\ & + C\sum_{i=1}^{r}\sum_{j=0}^{1}\sum_{l=0}^{1} 4\gamma_{i-1}\gamma_{m-1}\gamma_{s-1}a^2_{i,j,l}. \end{aligned}$$

We note that the corresponding bounds for u_y and u_z can be found by permutations of the indices. Inequality (47) can be found by noticing that, if $v \in \mathcal{I}_0(\hat{Q})$, we have

$$a_{ijl} = 0, \quad 0 \leq j \leq 1 \quad \text{or} \quad 0 \leq l \leq 1.$$

Inequalities (48) and (49) can be found in a similar way. □

Lemma 6.7 *Let $v \in \mathcal{I}_0(\hat{Q})$ and $u := I_{r,s,m}v$. Then*

(50) $$\|u_y\|^2_{0,\hat{Q}} \leq C\,rm\,\|v_y\|^2_{0,\hat{Q}} + C\,\frac{m}{r}\,\|v_y\|^2_{0,\Gamma^x_\pm},$$

(51) $$\|u_z\|^2_{0,\hat{Q}} \leq C\,rs\,\|v_z\|^2_{0,\hat{Q}} + C\,\frac{s}{r}\,\|v_z\|^2_{0,\Gamma^x_\pm},$$

(52) $$\|u_y\|^2_{0,\hat{Q}} \leq C\,rm\,\|v_y\|^2_{0,\hat{Q}} + C\,\frac{ms^4}{r}\,\|v\|^2_{0,\Gamma^x_\pm},$$

(53) $$\|u_z\|^2_{0,\hat{Q}} \leq C\,rs\,\|v_z\|^2_{0,\hat{Q}} + C\,\frac{sm^4}{r}\,\|v\|^2_{0,\Gamma^x_\pm},$$

(54) $$\|u_y\|^2_{0,\hat{Q}} \leq C\,rm\,\|v_y\|^2_{0,\hat{Q}} + C\,\frac{ms^2}{r}\,\|v\|^2_{1/2,00,\Gamma^x_\pm},$$

(55) $$\|u_z\|^2_{0,\hat{Q}} \leq C\,rs\,\|v_z\|^2_{0,\hat{Q}} + C\,\frac{sm^2}{r}\,\|v\|^2_{1/2,00,\Gamma^x_\pm},$$

where

$$\|v\|^2_{1/2,00,\Gamma^x_\pm} := \|v\|^2_{H^{1/2}_{00}(\Gamma^x_-)} + \|v\|^2_{H^{1/2}_{00}(\Gamma^x_+)}.$$

Proof. We only consider the terms in u_y in detail. Those in u_z can be treated in a similar way.

We immediately find (50) by using (38) and noting that $\gamma_{r-1} \le C/r$.

In order to prove (52), we need to bound

$$\begin{aligned} S_y &= \sum_{i=0}^{1}\sum_{j=1}^{s}\sum_{l=0}^{m-2} 4\gamma_{r-1}\gamma_{j-1}\gamma_l(\gamma_{l-1}a_{ijl} - \gamma_{l+1}a_{i,j,l+2})^2 \\ &= \sum_{j=1}^{s}\sum_{l=0}^{m-2} 4\gamma_{r-1}\gamma_{j-1}\gamma_l(\gamma_{l-1}a_{0jl} - \gamma_{l+1}a_{0,j,l+2})^2 \\ &\quad + \sum_{j=1}^{s}\sum_{l=0}^{m-2} 4\gamma_{r-1}\gamma_{j-1}\gamma_l(\gamma_{l-1}a_{1jl} - \gamma_{l+1}a_{1,j,l+2})^2 \\ &=: S_0 + S_1. \end{aligned}$$

We first consider S_0. Recalling that $\gamma_{s-1} \le \gamma_{j-1}$, for $j \le s$, we can write

$$S_0 \le 4\frac{\gamma_{r-1}}{\gamma_{s-1}} \sum_{l=0}^{m-2} \gamma_l \sum_{j=1}^{s} \gamma_{j-1}^2(\gamma_{l-1}a_{0jl} - \gamma_{l+1}a_{0,j,l+2})^2. \tag{56}$$

We next set, for a fixed l,

$$A_j := \gamma_{j-1}(\gamma_{l-1}a_{0jl} - \gamma_{l+1}a_{0,j,l+2}),$$

and denote $J = J(l) \le s$ the index j such that

$$A_J^2 = \max_{0\le j\le s}\{A_j^2\}.$$

We first assume that J is even. If $A_J^2 > 0$, then $J \ge 2$ since $a_{ijl} = 0$ for $0 \le j \le 1$. Noting that $A_0 = 0$, we can then write

$$A_J = -\sum_{j=0}^{\frac{J-2}{2}}(A_{2j} - A_{2j+2}),$$

and bound A_J^2 by

$$\begin{aligned} A_J^2 \le \frac{J}{2}\sum_{j=0}^{\frac{J-2}{2}}&[(\gamma_{2j-1}\gamma_{l-1}a_{0,2j,l} - \gamma_{2j-1}\gamma_{l+1}a_{0,2j,l+2}) \\ &- (\gamma_{2j+1}\gamma_{l-1}a_{0,2j+2,l} - \gamma_{2j+1}\gamma_{l+1}a_{0,2j+2,l+2})]^2. \end{aligned}$$

Using this bound and (56), we find

$$\begin{aligned}
S_0 &\le 4\frac{\gamma_{r-1}}{\gamma_{s-1}} \sum_{l=0}^{m-2} \gamma_l \, s \, A^2_{J(l)} \\
&\le 4\frac{\gamma_{r-1}}{\gamma_{s-1}} \sum_{l=0}^{m-2} \gamma_l \, s \frac{J}{2} \sum_{j=0}^{\frac{J-2}{2}} [(\gamma_{2j-1}\gamma_{l-1}a_{0,2j,l} - \gamma_{2j-1}\gamma_{l+1}a_{0,2j,l+2}) \\
&\quad -(\gamma_{2j+1}\gamma_{l-1}a_{0,2j+2,l} - \gamma_{2j+1}\gamma_{l+1}a_{0,2j+2,l+2})]^2 \\
&\le 2\frac{\gamma_{r-1}}{\gamma_s^2} \sum_{l=0}^{m-2} \gamma_l \, s^2 \sum_{j=0}^{\infty} \gamma_j [(\gamma_{j-1}\gamma_{l-1}a_{0,j,l} - \gamma_{j-1}\gamma_{l+1}a_{0,j,l+2}) \\
&\quad -(\gamma_{j+1}\gamma_{l-1}a_{0,j+2,l} - \gamma_{j+1}\gamma_{l+1}a_{0,j+2,l+2})]^2,
\end{aligned}$$

and, using (39),

$$S_0 \le C\frac{s^4}{r} \, \|v\|^2_{0,\Gamma^x_\pm}.$$

We note that, in case $A^2_J = 0$, this bound trivially holds. The case of J odd can be treated in a similar way.

Using a similar bound for S_1, we find (52).

Using now (50), (52), and an interpolation argument between the spaces $L^2(\Gamma^x_-) \times L^2(\Gamma^x_+)$ and $H^1_0(\Gamma^x_-) \times H^1_0(\Gamma^x_+)$, we find (55); see the proof of [10, Th. 3.5] for more details. □

The following corollary is a straightforward consequence of (54), (55), Lemma 6.7, the trace theorem, and the Poincaré inequality.

Corollary 6.8 *Let $v \in \mathcal{I}_0(\hat{Q})$ and $u := I_{r,s,m}v$. Then*

$$\|u_y\|^2_{0,\hat{Q}} \le C\, rm \, \|v_y\|^2_{0,\hat{Q}} + C\,\frac{ms^2}{r}\,|v|^2_{1,\hat{Q}}, \tag{57}$$

$$\|u_z\|^2_{0,\hat{Q}} \le C\, rs \, \|v_z\|^2_{0,\hat{Q}} + C\,\frac{sm^2}{r}\,|v|^2_{1,\hat{Q}}. \tag{58}$$

We are now ready to give a bound for the case of a general element in $\mathcal{T}_f$.

Lemma 6.9 *Let $v \in H^1(K)$, with $K = (x_1, x_2) \times (-1, 1)^2$. Suppose in addition that v vanishes on all of ∂K except on Γ^x_- and Γ^x_+. Then there exists a constant $C > 0$, independent of v, r, s, m, and K such that*

$$|I^K_{r,s,m}v|^2_{1,K} \le C\left(\max\{sm, rm, rs\} + \frac{ms}{r}\max\{s, m\}\right)|v|^2_{1,K}.$$

If, in addition, $r = s = m = k \ge 2$, then

$$|I^K_{k,k,k}v|_{1,K} \le Ck|v|_{1,K}.$$

Proof. We first note that, since $\mathcal{I}_0(\hat{Q})$ is dense in $H(\hat{Q})$, (47), (57), and (58) also hold for $v \in H(\hat{Q})$.

Let now $h := (x_2 - x_1)/2$. Then, $F_K : \hat{Q} \to K$ is given by

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} h\,\hat{x} + \dfrac{x_1 + x_2}{2} \\ \hat{y} \\ \hat{z} \end{bmatrix}.$$

If $\hat{v} := v \circ F_K$, we have

$$\begin{aligned} \|\hat{v}_{\hat{x}}\|^2_{0,\hat{Q}} &= h\|v_x\|^2_{0,K}, \quad \|(I_{r,s,m}\hat{v})_{\hat{x}}\|^2_{0,\hat{Q}} = h\|(I^K_{r,s,m}v)_x\|^2_{0,K} \\ \|\hat{v}_{\hat{y}}\|^2_{0,\hat{Q}} &= \frac{1}{h}\|v_y\|^2_{0,K}, \quad \|(I_{r,s,m}\hat{v})_{\hat{y}}\|^2_{0,\hat{Q}} = \frac{1}{h}\|(I^K_{r,s,m}v)_y\|^2_{0,K}, \\ \|\hat{v}_{\hat{z}}\|^2_{0,\hat{Q}} &= \frac{1}{h}\|v_z\|^2_{0,K}, \quad \|(I_{r,s,m}\hat{v})_{\hat{z}}\|^2_{0,\hat{Q}} = \frac{1}{h}\|(I^K_{r,s,m}v)_z\|^2_{0,K}. \end{aligned}$$

In addition, we have

$$|\hat{v}|^2_{1,\hat{Q}} \le \frac{1}{h}\,|v|^2_{1,K}.$$

Inequalities (47), (57), and (58) then give

$$\begin{aligned} h\,\|(I^K_{r,s,m}v)_x\|^2_{0,K} &\le C\,sm\,h\,\|v_x\|^2_{0,K}, \\ h^{-1}\,\|(I^K_{r,s,m}v)_y\|^2_{0,K} &\le C\,rm\,h^{-1}\,\|v_y\|^2_{0,K} + C\,\frac{ms^2}{r}\,h^{-1}\,|v|^2_{1,K}, \\ h^{-1}\,\|(I^K_{r,s,m}v)_z\|^2_{0,K} &\le C\,rs\,h^{-1}\,\|v_z\|^2_{0,K} + C\,\frac{sm^2}{r}\,h^{-1}\,|v|^2_{1,K}, \end{aligned}$$

which concludes the proof. □

We are now ready to prove Theorem 6.1:
Inequality (21) is a direct consequence of Lemma 6.9. We then note that, since the pressure space $S_0^{k-2,0}(\hat{Q}, \mathcal{T}_f)$ consists of discontinuous functions, it is enough to prove (22) on a single element $K \in \mathcal{T}_f$. Let $p \in S_0^{k-2,0}(\hat{Q}, \mathcal{T}_f)$ and $v \in H_0^1(\hat{Q})^3$. We have

$$(\nabla \cdot v, p)_K = -(v, \nabla p)_K + (v \cdot n, p)_{\partial K}.$$

Since ∇p and p are polynomials of degree $k-2$ on K, using the definition of Π_k, we find

$$(\nabla \cdot v, p)_K = -(\Pi_k v, \nabla p)_K + (\Pi_k v \cdot n, p)_{\partial K} = (\nabla \cdot (\Pi_k v), p)_K,$$

which concludes the proof.

Fig. 5. Auxiliary shape-regular edge mesh $\tilde{\mathcal{T}}_e$ (left) obtained from a macro-mesh $\tilde{\mathcal{T}}_{e,m}$ (right)

7 Edge patches

An edge patch is given by a mesh $\mathcal{T}_e$ of the form (6). For this patch, we prove that the inf–sup constant γ_N depends on σ, but is independent of n and the aspect ratio of the elements of $\mathcal{T}_e$. In addition we show a lower bound that decreases as $Ck^{-3/2}$. As a tool for the proof, we first introduce a refined auxiliary mesh $\tilde{\mathcal{T}}_e \subset \mathcal{T}_e$ for which the inf–sup constant decreases as k^{-1}. We then construct a Fortin operator that maps the velocity space on $\tilde{\mathcal{T}}_e$ into that on $\mathcal{T}_e$.

The auxiliary edge mesh $\tilde{\mathcal{T}}_e$ is shown in Figure 5 and is obtained by refining $\mathcal{T}_e$ in the z-direction in a minimal way, in order to obtain a shape-regular mesh. We note that $\tilde{\mathcal{T}}_e$ can also be obtained by first considering an initial shape-regular, irregular macro-mesh $\tilde{\mathcal{T}}_{e,m}$, geometrically refined towards an edge, say $(x, y) = (1, 1)$, with grading factor σ and n layers. The elements of $\tilde{\mathcal{T}}_{e,m}$ are then geometrically refined towards the two faces $x = 1$ and $y = 1$. Since $\tilde{\mathcal{T}}_e \subset \mathcal{T}_e$, we have the inclusions

$$S_0^{k,1}(\hat{Q}, \mathcal{T}_e)^3 \subset S_0^{k,1}(\hat{Q}, \tilde{\mathcal{T}}_e)^3, \qquad S_0^{k-2,0}(\hat{Q}, \mathcal{T}_e) \subset S_0^{k-2,0}(\hat{Q}, \tilde{\mathcal{T}}_e).$$

We have the following stability result

Lemma 7.1 *Let $\tilde{\mathcal{T}}_e$ be an auxiliary edge triangulation with grading factor σ and n layers. Then, there exists a constant C, that depends on σ, but is independent of k, n, and the aspect ratio of $\tilde{\mathcal{T}}_e$, such that*

$$\text{(60)} \qquad \inf_{0\neq p\in S_0^{k-2,0}(\hat{Q},\tilde{\mathcal{T}}_e)} \; \sup_{0\neq v\in S_0^{k,1}(\hat{Q},\tilde{\mathcal{T}}_e)^3} \frac{(\nabla\cdot v, p)_{\hat{Q}}}{|v|_{1,\hat{Q}}\|p\|_{0,\hat{Q}}} \geq Ck^{-1}.$$

Proof. We use Theorem 4.1 with $\Omega = \hat{Q}$, $\mathcal{T} = \tilde{\mathcal{T}}_e$, and $\mathcal{T}_m = \tilde{\mathcal{T}}_{e,m}$. The irregular mesh $\tilde{\mathcal{T}}_{e,m}$ is shape-regular and a stability result for low-order spaces can be proven in the same way as for Lemma 5.3: there exists a constant C, only depending on σ, such that

$$(61)\qquad \inf_{0\neq p\in S_0^{0,0}(\hat{Q},\tilde{\mathcal{T}}_{e,m})}\ \sup_{0\neq v\in S_0^{2,1}(\hat{Q},\tilde{\mathcal{T}}_{e,m})^3} \frac{(\nabla\cdot v, p)_{\hat{Q}}}{|v|_{1,\hat{Q}}\|p\|_{0,\hat{Q}}} \geq C,$$

which proves (9). We finally note that the anisotropically refined elements in $\tilde{\mathcal{T}}_{e,m}$ are particular face patches and that (10) then holds. □

Despite the fact that a better bound for the inf–sup condition for the auxiliary mesh is obtained here (see Theorem 7.4), $\tilde{\mathcal{T}}_e$ is not computationally efficient since it has a number of elements that grows exponentially with the number of layers n. Since the polynomial degree k needs to be of the same order as n in order to achieve exponential convergence, this would impose a number of degrees of freedom that grows exponentially with k.

We next define a one-dimensional projection. Let $\chi : [0, +\infty) \to \mathbb{R}$ be a Lipschitz continuous cut-off function that is equal to one in $[0, 1]$, decreases to zero in $[1, 1+\mu]$, $\mu > 0$, and is equal to zero in $[1+\mu, \infty)$. If $k' = (1+\mu)k - 1$, we define $\pi^{\chi,k} : H_0^1(I) \to \mathbb{Q}_{k'}(I)$ as

$$\pi^{\chi,k}\left(\sum_{i=2}^{\infty} a_i U_i\right) = \sum_{i=2}^{\infty} \chi\left(\frac{i}{k}\right) a_i U_i,$$

where the polynomials $\{U_i\}$ have been introduced in Section 6.2. The following stability properties hold. We refer to Lemma 3.2, Lemma 3.3, and Remark 3.4 in [4] for a proof.

Lemma 7.2 *There is a constant $C > 0$ independent of k and χ, such that*

$$|\pi^{\chi,k} v|_{1,I}^2 \leq |v|_{1,I}^2,$$
$$\|\pi^{\chi,k} v\|_{0,I}^2 \leq C\mu\|\chi'\|_\infty^2\, \|v\|_{0,I}^2.$$

For our analysis we make a particular choice of χ. We impose that χ decreases linearly from one to zero in $[1, 1+\mu]$ and set $\mu = 1/k$. We note that $k' = k$. Let $\pi^k : H_0^1(I) \to \mathbb{Q}_k(I)$ be the operator corresponding to this choice of χ. We have

$$\pi^k\left(\sum_{i=2}^{\infty} a_i U_i\right) = \sum_{i=2}^{\infty} \chi\left(\frac{i}{k}\right) a_i U_i = \sum_{i=2}^{k} a_i U_i.$$

Lemma 7.3 *There is a constant $C > 0$ independent of k such that*

$$|\pi_z v|_{1,I} \leq |v|_{1,I},$$
$$\|\pi_z v\|_{0,I} \leq C\sqrt{k}\, \|v\|_{0,I}.$$

Proof. It is enough to apply Lemma 7.2 with $\mu = 1/k$ and $\|\chi'\|_\infty = 1/\mu = k$. □

We next define the operator $\Pi_e : S_0^{k,1}(\hat{Q}, \tilde{\mathcal{T}}_e)^3 \to S_0^{k,1}(\hat{Q}, \mathcal{T}_e)^3$

$$\Pi_e = I_{xy} \circ \pi_z^k = \pi_z^k,$$

where I_{xy} is the identity operator for functions in x and y and π_z^k is the operator π^k applied to functions in z. We note that Π_e is well-defined since for a fixed z the restrictions of velocities in $S_0^{k,1}(\hat{Q}, \tilde{\mathcal{T}}_e)^3$ and $S_0^{k,1}(\hat{Q}, \mathcal{T}_e)^3$ coincide. The operator Π_e provides a Fortin operator and allows to prove the following result.

Theorem 7.4 *Let $\mathcal{T}_e$ be an edge triangulation with grading factor σ and n layers. Then, there exists a constant C, that depends on σ, but is independent of k, n, and the aspect ratio of $\mathcal{T}_e$, such that*

$$\inf_{0 \neq p \in S_0^{k-2,0}(\hat{Q}, \mathcal{T}_e)} \sup_{0 \neq v \in S_0^{k,1}(\hat{Q}, \mathcal{T}_e)^3} \frac{(\nabla \cdot v, p)_{\hat{Q}}}{|v|_{1,\hat{Q}} \|p\|_{0,\hat{Q}}} \geq C k^{-3/2}. \tag{62}$$

Proof. Using the definition of π_z and Lemma 7.3 we can easily prove

$$(\nabla \cdot (\Pi_e \tilde{v}), p)_{\hat{Q}} = (\nabla \cdot \tilde{v}, p)_{\hat{Q}}, \quad \tilde{v} \in S_0^{k,1}(\hat{Q}, \tilde{\mathcal{T}}_e)^3, \; p \in S_0^{k-2,0}(\hat{Q}, \mathcal{T}_e),$$
$$|\Pi_e \tilde{v}|^2_{1,\hat{Q}} \leq Ck |\tilde{v}|^2_{1,\hat{Q}}, \quad \tilde{v} \in S_0^{k,1}(\hat{Q}, \tilde{\mathcal{T}}_e)^3.$$

Let now $p \in S_0^{k-2,0}(\hat{Q}, \mathcal{T}_e)$. Thanks to Lemma 7.1, we can find a velocity $\tilde{v} \in S_0^{k,1}(\hat{Q}, \tilde{\mathcal{T}}_e)^3$ such that

$$(\nabla \cdot \tilde{v}, p)_{\hat{Q}} = \|p\|^2_{0,\hat{Q}}, \qquad |\tilde{v}|_{1,\hat{Q}} \leq Ck\|p\|_{0,\hat{Q}}.$$

If $v = \Pi_e \tilde{v}$, we then have

$$(\nabla \cdot v, p)_{\hat{Q}} = (\nabla \cdot \tilde{v}, p)_{\hat{Q}} = \|p\|^2_{0,\hat{Q}}$$

and

$$|v|^2_{1,\hat{Q}} \leq Ck|\tilde{v}|^2_{1,\hat{Q}} \leq Ck^3\|p\|^2_{0,\hat{Q}},$$

which concludes the proof. □

8 Corner patches

A corner patch is given by a geometric mesh $\mathcal{T}_c$, with grading factor σ and n layers. For this patch, we prove that the inf–sup constant γ_N depends on σ, but is independent of n and the aspect ratio of $\mathcal{T}_c$. In our analysis, we generalize the result in [11, Sect. 4] for two-dimensional corner patches.

Our stability result is given in the next theorem and is obtained by using Lemma 5.3 and noticing that the anisotropically refined elements in the macro-mesh $\mathcal{T}_{c,m}^{n,\sigma}$ are particular face and edge patches; see Section 3.1 and Figure 1.

Theorem 8.1 *Let $\mathcal{T}_c$ be a corner patch with grading factor σ and n layers. Then, there exists a constant C, that depends on σ, but is independent of k, n, and the aspect ratio of $\mathcal{T}_c$, such that*

$$\inf_{0\neq p\in S_0^{k-2,0}(\hat{Q},\mathcal{T}_c)} \sup_{0\neq v\in S_0^{k,1}(\hat{Q},\mathcal{T}_c)^3} \frac{(\nabla\cdot v, p)_{\hat{Q}}}{|v|_{1,\hat{Q}}\|p\|_{0,\hat{Q}}} \geq Ck^{-3/2}. \tag{63}$$

Proof. It is enough to use Theorem 4.1 with $\Omega = \hat{Q}$, $\mathcal{T} = \mathcal{T}_c$, and $\mathcal{T}_m = \mathcal{T}_{c,m}$. □

9 Stability on geometric edge meshes

We now consider the case of geometric edge meshes $\mathcal{T} = \mathcal{T}_{edge}^{n,\sigma}$, introduced in Section 3.2. As before, we employ a macro-element technique, described in Figure 2.

At the top level, we have the shape-regular macro-mesh $\mathcal{T}_m$, which is successively refined, either isotropically, or anisotropically towards an edge (second level, left) or a corner (second level, right). The divergence stability for the shape-regular macro-mesh at the top level and the isotropically refined patches is proven in [16]. We then need to prove the stability of the single patches for the higher order spaces.

9.1 Edge patches

For an edge patch, the same analysis for the case of a boundary layer mesh in Section 7 can be carried out here. Indeed, an edge patch is given by a mesh $\mathcal{T}_e$ of the form (8), where the two-dimensional triangulation $\mathcal{T}_{xy}$ is an irregular corner mesh, with grading factor σ and n layers. The following theorem holds.

Theorem 9.1 *Let $\mathcal{T}_e$ be an edge triangulation with grading factor σ and n layers. Then, there exists a constant C, that depends on σ, but is independent of k, n, and the aspect ratio of $\mathcal{T}_e$, such that*

$$\inf_{0\neq p\in S_0^{k-2,0}(\hat{Q},\mathcal{T}_e)} \sup_{0\neq v\in S_0^{k,1}(\hat{Q},\mathcal{T}_e)^3} \frac{(\nabla\cdot v, p)_{\hat{Q}}}{|v|_{1,\hat{Q}}\|p\|_{0,\hat{Q}}} \geq Ck^{-3/2}. \tag{64}$$

Proof. We first consider an auxiliary shape-regular edge mesh $\tilde{\mathcal{T}}_{e,m} \subset \mathcal{T}_e$; see Figure 5. In the same way as for Lemma 7.1, we can prove that there exists a constant C only depending on σ such that

$$\inf_{0\neq p\in S_0^{k-2,0}(\hat{Q},\tilde{\mathcal{T}}_{e,m})} \sup_{0\neq v\in S_0^{k,1}(\hat{Q},\tilde{\mathcal{T}}_{e,m})^3} \frac{(\nabla\cdot v, p)_{\hat{Q}}}{|v|_{1,\hat{Q}}\|p\|_{0,\hat{Q}}} \geq Ck^{-1}.$$

The Fortin operator

$$\Pi_e = I_{xy}\circ\pi_z^k = \pi_z^k : S_0^{k,1}(\hat{Q},\tilde{\mathcal{T}}_{e,m})^3 \longrightarrow S_0^{k,1}(\hat{Q},\mathcal{T}_e)^3$$

then enables to prove the result, as for the proof of Theorem 7.4. □

9.2 Corner patches

For a corner patch, the analysis is similar to that in Section 8. A corner patch is given by a mesh $\mathcal{T}_c$ obtained by refining an initial irregular corner mesh $\mathcal{T}_{c,m}$ towards the edges only. The following theorem holds.

Theorem 9.2 *Let $\mathcal{T}_c$ be a corner patch with grading factor σ and n layers. Then, there exists a constant C, that depends on σ, but is independent of k, n, and the aspect ratio of $\mathcal{T}_c$, such that*

$$\inf_{0\neq p\in S_0^{k-2,0}(\hat{Q},\mathcal{T}_c)} \sup_{0\neq v\in S_0^{k,1}(\hat{Q},\mathcal{T}_c)^3} \frac{(\nabla\cdot v, p)_{\hat{Q}}}{|v|_{1,\hat{Q}}\|p\|_{0,\hat{Q}}} \geq Ck^{-3/2}. \tag{65}$$

Proof. We use a macro-element technique with $\mathcal{T}_{c,m}$ as macro-mesh. The macro-mesh $\mathcal{T}_{c,m}$ is the same as in the case of boundary layer meshes and, consequently, Lemma 5.3 holds. The proof is concluded by noticing that the anisotropically refined elements in $\mathcal{T}_{c,m}$ are particular edge patches and by using Theorem 4.1 with $\mathcal{T} = \mathcal{T}_c$ and $\mathcal{T}_m = \mathcal{T}_{c,m}$. □

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