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# On the Numerical Analysis of Some Variational Problems with Nonhomogeneous Boundary Conditions

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The goal of this note is to expose a new techniques to get energy estimates for nonconvex problems with nonlinear boundary conditions in term of the mesh size of a Lagrange finite elements method.

*Key words*: approximation, non-convex, calculus of variations, finite elements

## 1. Introduction

Let $\Omega$ be a bounded polyhedral domain of $\mathbf{R}^n$, $n \geq 2$, of boundary $\Gamma$. Let $w_i \in \mathbf{R}^n$, $i = 1, \ldots, p$, $p \geq 2$, and consider a function $\varphi : \mathbf{R}^n \to \mathbf{R}$ such that

$$\varphi(w_i) = 0 \quad \forall i = 1, \ldots, p, \tag{1.1}$$

$$\varphi(w) > 0 \quad \forall w \neq w_i, \quad i = 1, \ldots, p, \tag{1.2}$$

$$w_i - w_p, \quad i = 1, \ldots, p-1 \text{ are linearly independent.} \tag{1.3}$$

For instance, in a physical setting, $\varphi$ could be some stored energy density that vanishes at wells $w_i$'s. These wells stand for natural states with low or no energy. Let us denote by $\Psi$ a nonnegative continuous function such that for some $q > 0$, $c > 0$,

$$0 \leq \Psi(\xi) \leq c|\xi|^q \quad \forall \xi \in (-1, 1). \tag{1.4}$$

If $A(x)$ is a Lipschitz continuous function i.e. if

$$A \in W^{1,\infty}(\Omega) \tag{1.5}$$

we denote by $W_A^{1,\infty}(\Omega)$ the set

$$W_A^{1,\infty}(\Omega) = \{v \in W^{1,\infty}(\Omega) \,/\, v(x) = A(x) \text{ on } \Gamma\}. \tag{1.6}$$

Then, we would like to consider the problems

$$\inf_{v \in W_A^{1,\infty}(\Omega)} \int_\Omega \varphi(\nabla v(x))dx, \tag{1.7}$$

$$\inf_{v \in W_A^{1,\infty}(\Omega)} \int_\Omega \Psi(v(x) - A(x)) + \varphi(\nabla v(x))dx, \tag{1.8}$$

$$\inf_{v \in W^{1,\infty}(\Omega)} \int_\Omega \Psi(v(x) - A(x)) + \varphi(\nabla v(x))dx, \tag{1.9}$$

More precisely we would like to investigate the case when

$$\nabla A(x) \in \mathrm{Co}(w_i) \quad \text{a.e. } x \in \Omega \tag{1.10}$$

($\mathrm{Co}(w_i)$ denotes the convex hull of the $w_i$'s).

As we mentioned before, such minimization problems arise in various physical settings (see for instance the connection with continuum mechanics in [1], [2], [21], [23], [24], [25], [27], [28], [29], [30], [31]) and it is of importance to know with what accuracy a finite element method will provide an estimate of the energy level. Moreover, our theoretical analysis provides minimizing sequences for the problems at hand that describe of course the limit behavior of the system. However, we will not insist here on the fact that these problems have no minimizers and are described by the mean of these minimizing sequences. The interested reader is refered for instance to [8] or [11] for details. The relationship between (1.7), (1.9) is exposed in [3], we refer for instance to [26] for notation on Sobolev spaces.

For our purpose, let $(\mathcal{T}_h)_{h>0}$ be a family of triangulations of $\Omega$ (see [33]), that is to say satisfying

$$\forall h > 0 \begin{cases} \forall K \in \mathcal{T}_h, \ K \text{ is a } n\text{-simplex}, \\ \max\limits_{K \in \mathcal{T}_h} (h_K) = h, \\ \exists \nu > 0 \ \text{ such that } \ \forall K \in \mathcal{T}_h \dfrac{h_K}{\rho_K} \leq \nu. \end{cases} \tag{1.11}$$

where $h_K$ is the diameter of the $n$-simplex and $\rho_K$ its roundness (i.e. the largest diameter of the balls that could fit in $K$). If $P_1(K)$ is the space of polynomials of degree 1 on $K$, set

$$V_h = \{v : \Omega \to \mathbf{R} \text{ continuous}, \ v/_K \in P_1(K), \ \forall K \in \mathcal{T}_h\}. \tag{1.12}$$

Moreover, let us denote by $\widehat{A}$ the interpolate of $A$ on $\mathcal{T}_h$-i.e. $\widehat{A}$ is the unique function of $V_h$ that agrees with $A$ on the vertices of $\mathcal{T}_h$.

REMARK 1.1. If $\nabla A(x) \in \mathrm{Co}(w_i)$, note that one does not have necessarily

$$\nabla \widehat{A} \in \mathrm{Co}(w_i).$$

Then set

$$V_h^{\hat{A}} = \{v : \Omega \to \mathbf{R} \text{ continuous}, \ v/_K \in P_1(K), \forall K \in \mathcal{T}_h, \ v = \widehat{A} \text{ on } \Gamma\}. \tag{1.13}$$

Our goal is to obtain estimates for the infima

$$\inf_{v \in V_h^{\hat{A}}} \int_\Omega \varphi(\nabla v(x))dx \tag{1.14}$$

$$\inf_{v \in V_h^{\hat{A}}} \int_\Omega \Psi(v(x) - A(x)) + \varphi(\nabla v(x))dx \tag{1.15}$$

$$\inf_{v \in V_h} \int_\Omega \Psi(v(x) - A(x)) + \varphi(\nabla v(x))dx \tag{1.16}$$

in terms of the mesh size $h$. Moreover, in doing so, we will exhibit sequences providing approximations for (1.7)–(1.9).

REMARK 1.2. In the case where $A$ is affine on each face of $\Gamma$ then $\widehat{A} = A$ on $\Gamma$. Note that no assumption is needed on $\varphi$ for (1.14)–(1.16) to be defined. For (1.7)–(1.9) one might require $\varphi$ to be a Borel function.

Problems of this type have been first considered by C. Collins, D. Kinderlehrer and M. Luskin in one dimension (see [16], [17]–[18], [15]). In higher dimension estimates were also obtained in [7], [8], [10] in the case where $\nabla A$ is a constant and in [11] in the vectorial case. Sharp estimates were obtained in [14]. In the case when $A$ is nonlinear, estimates were obtained in [9] and [13]. However, in these papers the estimates are not optimal and using new test functions we are able to improve them and get in term of $h$ the same power than in the linear case i.e. when $\nabla A$ is constant (see [8], [10]).

First, remark that there is no loss of generality in assuming that

$$0 \in \mathrm{ri}(\mathrm{Co}(w_i)) \tag{1.17}$$

where $\mathrm{ri}(\mathrm{Co}(w_i))$ denotes the relative interior of $\mathrm{Co}(w_i)$-i.e. its interior in the space spanned by the $w_i - w_p$'s. Indeed, let $w \in \mathrm{ri}(\mathrm{Co}(w_i))$. Then, we can look for a $v$ of the form

$$v = u + w.x \tag{1.18}$$

where $w.x$ denotes the scalar product of $w$ and $x$. Then we are led to minimize

$$\int_\Omega \Psi(u(x) - (A(x) - w.x)) + \varphi(w + \nabla u(x))dx \tag{1.19}$$

over some space. If we set

$$\widetilde{\varphi}(\xi) = \varphi(\xi + w), \quad \widetilde{A}(x) = A(x) - w.x \tag{1.20}$$

we end up to deal with the same problem with the wells $w_i - w$ that satisfy

$$0 \in \mathrm{ri}(\mathrm{Co}(w_i - w)), \quad \nabla \widetilde{A} \in \mathrm{Co}(w_i - w) \ \text{ a.e. } \ x \in \Omega. \tag{1.21}$$

So, in what follows we will always assume that (1.17) holds. In particular, by (1.3) there exist $\alpha_i$'s, $i = 1, \ldots, p$, such that

$$0 = \sum_{i=1}^{p} \alpha_i w_i, \quad \alpha_i \in (0,1). \tag{1.22}$$

Moreover the space $W$ spanned by the $w_i - w_p$'s coincides with the space spanned by the $w_i$'s.

## 2. Energy Estimates

Our main theorem is the following:

THEOREM 2.1. *Let us assume that $\Omega$ is convex and that $\varphi$ is a function bounded on bounded subsets of $\mathbf{R}^n$ satisfying* (1.1), (1.2). *Moreover, assume that $\Psi$ is a continuous function satisfying* (1.4). *Then, if $A \in W^{1,\infty}(\Omega)$ satisfies* (1.10), *and* (1.3) *holds, there exists a constant $C$, independent of $h \in (0,1)$, such that*

$$E_h^1 = \inf_{v \in V_h^{\hat{A}}} \int_\Omega \varphi(\nabla v(x))dx \leq Ch^{1/2} \tag{2.1}$$

$$E_h^2 = \inf_{v \in V_h^{\hat{A}}} \int_\Omega \Psi(v(x) - A(x)) + \varphi(\nabla v(x))dx \leq Ch^{r/(r+1)} \tag{2.2}$$

$$E_h^3 = \inf_{v \in V_h} \int_\Omega \Psi(v(x) - A(x)) + \varphi(\nabla v(x))dx \leq Ch^{q/(q+1)} \tag{2.3}$$

*where $r = q \wedge 1$, $\wedge$ denotes the infimum of two numbers.*

REMARK 2.1. Assuming for instance that $\lim_{|\xi| \to +\infty} \varphi(\xi) = +\infty$ or $\lim_{|\xi| \to +\infty} \Psi(\xi) = +\infty$ one can show via an easy compactness argument that the infima (2.1)–(2.3) are in fact achieved (see [8] or [10] for a proof).

In order to prove Theorem 2.1 we will need some preparatory lemmas.
First:

LEMMA 2.1. *Assume $A \in W^{1,\infty}(\Omega)$ satisfies* (1.10). *Then if the segment $[x, x']$ is included in $\Omega$ one has*:

$$\bigwedge_{i=1}^{p} w_i.(x - x') \leq A(x) - A(x') \leq \bigvee_{i=1}^{p} w_i.(x - x') \tag{2.4}$$

(*we denote by $\wedge$ the infimum of numbers, by $\vee$ the supremum*).

*Proof.* It is enough to apply the mean value theorem after regularization (see [13]). ∎

REMARK 2.2. An immediate consequence of this lemma is that

$$A(x) - A(x') = 0$$

on each segment $[x, x']$ such that $x - x' \in W^{\perp}$, where $W^{\perp}$ denotes the orthogonal of $W$ that could be, of course, reduced to 0 when the $w_i$'s are spanning the whole space.

Let us denote by $v_1, \dots, v_{p-1}$ the dual basis of $w_i - w_p$ i.e. the basis such that

$$(w_i - w_p).v_j = \delta_{ij} \quad \forall i, j = 1, \dots, p-1. \tag{2.5}$$

Denote also by $x_z$ the points of the lattice of size $h^{\alpha}$, $\alpha \in (0,1)$ will be chosen later on, spanned by the $v_i$'s-i.e. for any $z = (z_1, \dots, z_{p-1})$ set

$$x_z = \sum_{i=1}^{p-1} z_i h^{\alpha} v_i. \tag{2.6}$$

Then, let us define the function $\Lambda$ by

$$\Lambda(x) = \bigvee_{z \in \mathbb{Z}^{p-1}} \Big( \bigwedge_{i=1}^{p} w_i.(x - x_z) + \widetilde{A}(x_z) \Big) \tag{2.7}$$

where $\widetilde{A}$ is the extension of $A$ (see the Appendix) that satisfies

$$\nabla \widetilde{A}(x) \in \mathrm{Co}(w_i) \quad \text{a.e.} \quad x \in \mathbf{R}^n.$$

Note that $\Lambda$ is constant in the $W^{\perp}$ direction, (see Remark 2.2), and we will consider it as a function of $x \in W$ or $x \in \mathbf{R}^n$ as well. By a unit cell of the lattice spanned by the $h^{\alpha} v_i$ we mean a set of the type

$$C_z = x_z + \left\{ \sum_{i=1}^{p-1} \beta_i h^{\alpha} v_i \,/\, \beta_i \in [0,1] \right\}$$

where $x_z$ is defined by (2.6). Then one has:

LEMMA 2.2. *Let us assume that $\Omega$ is convex. Under the above assumptions, denote by $C_{z_0}$ a unit cell spanned by $h^{\alpha} v_i$'s and by $E$ the set*

$$E = \{ z \in \mathbb{Z}^{p-1} \,/\, z_i = 0 \ \ or \ \ 1, \ \forall i = 1, \dots, p-1 \},$$

*then one has*

$$\Lambda(x) = \bigvee_{z' \in z_0 + E} \Big( \bigwedge_{i=1}^{p} w_i.(x - x_{z'}) + \widetilde{A}(x_{z'}) \Big) \quad \forall x \in C_{z_0}. \tag{2.8}$$

*i.e. in (2.7) instead of taking the supremum on a number of $z$ that could be unbounded when $h \to 0$ it is enough to take it on a fixed finite number.*

*Proof.* Assume that $x \in C_{z_0}$, and fix $z \in \mathbb{Z}^{p-1}$. Then, for some $l = 1, \ldots, p$ one has

$$\bigwedge_{i=1}^{p} w_i.(x - x_z) = w_l.(x - x_z). \tag{2.9}$$

We claim that there exists $z' \in z_0 + E$ such that

$$\bigwedge_{i=1}^{p} w_i.(x - x_z) = \bigwedge_{i=1}^{p} w_i.(x - x_{z'}) + \bigwedge_{i=1}^{p} w_i.(x_{z'} - x_z). \tag{2.10}$$

To prove that let us show that there exists a $z' \in z_0 + E$ such that

$$\begin{cases} w_i.(x - x_{z'}) \geq w_l.(x - x_{z'}) & \forall i = 1, \ldots, p, \ i \neq l, \\ w_i.(x_{z'} - x_z) \geq w_l.(x_{z'} - x_z) & \forall i = 1, \ldots, p, \ i \neq l. \end{cases} \tag{2.11}$$

Indeed, if (2.11) holds, by (2.9) one would have

$$\begin{aligned} \bigwedge_{i=1}^{p} w_i.(x - x_z) &= w_l.(x - x_z) \\ &= w_l.(x - x_{z'}) + w_l.(x_{z'} - x_z) \\ &= \bigwedge_{i=1}^{p} w_i.(x - x_{z'}) + \bigwedge_{i=1}^{p} w_i.(x_{z'} - x_z) \end{aligned} \tag{2.12}$$

and (2.10) will follow. Assume that

$$x = \sum_{k=1}^{p-1} (\beta_k + z_{0k}) h^\alpha v_k, \quad x_z = \sum_{k=1}^{p-1} z_k h^\alpha v_k, \quad x_{z'} = \sum_{k=1}^{p-1} z'_k h^\alpha v_k. \tag{2.13}$$

Then, clearly (2.9) reads

$$(w_i - w_l).x \geq (w_i - w_l).x_z \quad \forall i = 1, \ldots, p, \ i \neq l$$

or

$$\sum_{k=1}^{p-1} \{\beta_k + z_{0k}\}(w_i - w_l).v_k \geq \sum_{k=1}^{p-1} z_k (w_i - w_l).v_k \quad \forall i = 1, \ldots, p, \ i \neq l. \tag{2.14}$$

Similarly (2.11) reads

$$\begin{aligned} \sum_{k=1}^{p-1} \{\beta_k + z_{0k}\}(w_i - w_l).v_k &\geq \sum_{k=1}^{p-1} z'_k (w_i - w_l).v_k \quad \forall i = 1, \ldots, p, \ i \neq l \\ \sum_{k=1}^{p-1} z'_k \{w_i - w_l).v_k &\geq \sum_{k=1}^{p-1} z_k (w_i - w_l).v_k \quad \forall i = 1, \ldots, p, \ i \neq l. \end{aligned} \tag{2.15}$$

One has (see (2.5)), when $l \neq p$

$*$ if $i = p$,

$$(w_i - w_l).v_k = -\delta_{lk}. \tag{2.16}$$

$*$ if $i \neq p$,

$$\begin{aligned}(w_i - w_l).v_k &= [(w_i - w_p) + (w_p - w_l)].v_k \\ &= (w_i - w_p).v_k + (w_p - w_l).v_k \\ &= \delta_{ik} - \delta_{lk}.\end{aligned}$$

Hence, in this second case:

$$(w_i - w_l).v_k = \begin{cases} 0 & \text{when } k \neq i, l, \\ 1 & \text{when } k = i, \\ -1 & \text{when } k = l. \end{cases} \tag{2.17}$$

When $l = p$

$$(w_i - w_l).v_k = \delta_{ik}. \tag{2.18}$$

Let us assume first that $l \neq p$, then (2.14), and (2.15) can be written respectively

$$\begin{cases} -(\beta_l + z_{0l}) \geq -z_l \\ (\beta_i + z_{0i}) - (\beta_l + z_{0l}) \geq z_i - z_l \quad \forall i = 1, \ldots, p-1,\ i \neq l \end{cases} \tag{2.19}$$

$$\begin{cases} -(\beta_l + z_{0l}) \geq -z'_l \\ (\beta_i + z_{0i}) - (\beta_l + z_{0l}) \geq z'_i - z'_l \quad \forall i = 1, \ldots, p-1,\ i \neq l \end{cases} \tag{2.20}$$

$$\begin{cases} -z'_l \geq -z_l \\ z'_i - z'_l \geq z_i - z_l \quad \forall i = 1, \ldots, p-1,\ i \neq l. \end{cases} \tag{2.21}$$

Thus, knowing (2.19), we have to find $z' \in z_0 + E$ such that

$$\begin{cases} -(\beta_l + z_{0l}) \geq -z'_l \geq -z_l \\ (\beta_i + z_{0i}) - (\beta_l + z_{0l}) \geq z'_i - z'_l \geq z_i - z_l \quad \forall i = 1, \ldots, p-1,\ i \neq l. \end{cases} \tag{2.22}$$

If $\beta_l \in ]0, 1]$, we choose

$$z'_l = z_{0l} + 1, \quad \begin{cases} z'_i = z_{0i} + 1 & \text{if } \beta_i \geq \beta_l, \\ z'_i = z_{0i} & \text{if } \beta_i < \beta_l, \quad i = 1, \ldots, p-1,\ i \neq l. \end{cases} \tag{2.23}$$

In the case when $\beta_i \geq \beta_l$, we have $0 \leq \beta_i - \beta_l < 1$, since $z_l$, $z_i - z_l$ are integers satisfying (2.19) one has

$$\begin{cases} -(\beta_l + z_{0l}) \geq -(z_{0l} + 1) \geq -z_l \\ (\beta_i + z_{0i}) - (\beta_l + z_{0l}) \geq (z_{0i} + 1) - (z_{0l} + 1) \geq z_i - z_l \\ \qquad\qquad\qquad\qquad\qquad \forall i = 1, \ldots, p-1, \ i \neq l. \end{cases} \tag{2.24}$$

which is (2.22) in this case. When $\beta_i < \beta_l$, since $-1 \leq \beta_i - \beta_l < 0$ we have

$$\begin{cases} -(\beta_l + z_{0l}) \geq -(z_{0l} + 1) \geq -z_l \\ (\beta_i + z_{0i}) - (\beta_l + z_{0l}) \geq z_{0i} - (z_{0l} + 1) \geq z_i - z_l \\ \qquad\qquad\qquad\qquad\qquad \forall i = 1, \ldots, p-1, \ i \neq l. \end{cases} \tag{2.25}$$

which is (2.22). If $\beta_l = 0$, we choose

$$z'_l = z_{0l}, \quad \begin{cases} z'_i = z_{0i} & \text{if } \ \beta_i \in [0, 1[, \\ z'_i = z_{0i} + 1 & \text{if } \ \beta_i = 1, \quad i = 1, \ldots, p-1, \ i \neq l. \end{cases}$$

Then clearly (2.22) holds. So, it remains only to treat the case where $l = p$. In this case, the system to be solved is (see (2.14), (2.15), (2.18)):
find $z'$ such that

$$\beta_i + z_{0i} \geq z'_i \geq z_i \quad \forall i = 1, \ldots, p-1, \tag{2.26}$$

knowing that

$$\beta_i + z_{0i} \geq z_i \quad \forall i = 1, \ldots, p-1.$$

We choose

$$\begin{cases} z'_i = z_{0i} & \text{if } \ \beta_i \in [0, 1[, \\ z'_i = z_{0i} + 1 & \text{if } \ \beta_i = 1, \quad i = 1, \ldots, p-1. \end{cases}$$

Then clearly we have (2.22). This proves (2.10).

To complete the proof of Lemma 2.2, consider a point $x \in C_{z_0}$. If $x_z$ is arbitrary by (2.10), there exists a point $z' \in z_0 + E$ such that

$$\bigwedge_{i=1}^{p} w_i.(x - x_z) = \bigwedge_{i=1}^{p} w_i.(x - x_{z'}) + \bigwedge_{i=1}^{p} w_i.(x_{z'} - x_z).$$

Using Lemma 2.1 we deduce that

$$\bigwedge_{i=1}^{p} w_i.(x - x_z) \leq \bigwedge_{i=1}^{p} w_i.(x - x_{z'}) + \widetilde{A}(x_{z'}) - \widetilde{A}(x_z)$$

i.e.

$$\bigwedge_{i=1}^{p} w_i.(x - x_z) + \widetilde{A}(x_z) \le \bigwedge_{i=1}^{p} w_i.(x - x_{z'}) + \widetilde{A}(x_{z'})$$

and this completes the proof. ■

We have also:

LEMMA 2.3. *Let us assume that $\Omega$ is convex. Under the preceding assumptions one has for some positive constant $C$ independent of $h$*

$$A(x) - Ch^{\alpha} \le \Lambda(x) \le A(x) \quad \forall x \in \Omega. \tag{2.27}$$

*Proof.* Using Lemma 2.1 we get for $x \in \Omega$ :

$$\bigwedge_{i=1}^{p} w_i.(x - x_z) + \widetilde{A}(x_z) \le A(x) \quad \forall x_z.$$

Hence

$$\Lambda(x) \le A(x).$$

Denote by $P_W(\Omega)$ the orthogonal projection of $\Omega$ onto $W$. Let $x' \in P_W(\Omega)$ be the component of $x$ in $P_W(\Omega)$. There exists a $z_0$ such that $x' \in C_{z_0}$, so that

$$\begin{aligned}
\Lambda(x) &\ge \bigwedge_{i=1}^{p} w_i.(x - x_{z_0}) + \widetilde{A}(x_{z_0}) \\
&= \bigwedge_{i=1}^{p} w_i.(x' - x_{z_0}) + \widetilde{A}(x_{z_0}) \\
&\ge -Ch^{\alpha} + \widetilde{A}(x_{z_0}) - A(x') + A(x) \\
&\ge -Ch^{\alpha} + A(x)
\end{aligned}$$

we used the fact that $A(x) = A(x')$, see Remark 2.2. This completes the proof. ■

*Proof of Theorem* 2.1.

-*Proof of* (2.3). Define $\widehat{u}_h$ as the interpolate of the function $\Lambda(x)$ that has been introduced in (2.7). Clearly, $\widehat{u}_h \in V_h$. Then, first notice that

$$\nabla \Lambda(x) = w_i \ \text{ a.e. } \ x \in \Omega,$$

so that $\Lambda$ is a Lipschitz continuous function with a Lipschitz constant uniformly bounded. It then results that the same holds for $\widehat{u}_h$ (see for instance [4], [5], [6]). Then, by the mean value theorem

$$|\Lambda(x) - \widehat{u}_h(x)| \le Ch$$

for some constant $C$. Since $\alpha \in (0,1)$ we deduce from (2.27)

$$
\begin{aligned}
|A(x) - \widehat{u}_h(x)| &\leq |A(x) - \Lambda(x)| + |\Lambda(x) - \widehat{u}_h(x)| \\
&\leq Ch^\alpha + Ch \leq Ch^\alpha
\end{aligned} \tag{2.28}
$$

if one assumes $h < 1$. In particular this leads to

$$
\int_\Omega \Psi(\widehat{u}_h - A(x))dx \leq C|\Omega|h^{q\alpha} \tag{2.29}
$$

(note that (1.4) holds on any bounded subset of $\mathbf{R}$, $|\Omega|$ denotes the measure of $\Omega$.)

Next we need to estimate

$$
\int_\Omega \varphi(\nabla \widehat{u}_h(x))dx. \tag{2.30}
$$

First, notice that

$$
\nabla \widehat{u}_h = w_i
$$

except maybe on the set $I$ composed of the simplices where interpolation occured. Since on this set $\nabla \widehat{u}_h$ remains bounded one has

$$
\int_\Omega \varphi(\nabla \widehat{u}_h(x))dx = \int_I \varphi(\nabla \widehat{u}_h(x))dx \leq C|I|
$$

where $|I|$ denotes the measure of $I$. Now, interpolation occurs on a $h$-neighborhood of the ridge of $\Lambda(x)$-i.e. a $h$-neighborhood of the set where $\Lambda(x)$ has a discontinuity in its gradient. Clearly $\Lambda(x)$ has a jump in its gradient on a unit cell of the lattice spanned by $h^\alpha v_i$ when one of the function

$$
w_i.(x - x_z) + \widetilde{A}(x_z)
$$

is equal to an other. These two functions are then equal on a set of $p-2$-dimensional measure bounded by $C(h^\alpha)^{p-2}$ (this is the intersection of a hyperplane and a cell of diameter bounded by $Ch^\alpha$). Since in (2.7) the supremum is taken on a finite number of functions it is clear that

$$
|I| \leq C(h^\alpha)^{p-2}.h.N(h^\alpha) \tag{2.31}
$$

where $N(h^\alpha)$ is the number of cells of size $h^\alpha$ included in $P_W(\Omega)$ (Recall that $P_W$ denotes the projection of $\Omega$ onto $W$). Clearly,

$$
N(h^\alpha).(h^\alpha)^{p-1} \leq C.
$$

Hence (2.31) reads

$$
|I| \leq Ch^{1-\alpha}. \tag{2.32}
$$

Collecting (2.29)–(2.32) we get

$$\int_\Omega \Psi(\widehat{u}_h - A(x)) + \varphi(\nabla \widehat{u}_h(x)) \le C[h^{\alpha q} + h^{1-\alpha}].$$

The power is the best in this inequality when $1 - \alpha = \alpha q$, i.e. when $\alpha = \frac{1}{1+q}$. This completes the proof of (2.3).

REMARK 2.3. The above estimate is sharp (see for instance [14]).

*-Proof of* (2.1), (2.2). Define $\Lambda$ as before. Now, this function is not necessarily equal to $A(x)$ on the boundary of $\Omega$. So, its interpolate will not be in $V_h^{\hat{A}}$. To correct that, set

$$u_h(x) = \Lambda(x) \vee (A(x) - \operatorname{dist}(x, \Gamma))$$

where $\operatorname{dist}(x, \Gamma)$ denotes the distance from $x$ to the boundary $\Gamma$. Due to (2.27) one has

$$|A(x) - \Lambda(x)| \le Ch^\alpha.$$

So, if

$$\operatorname{dist}(x, \Gamma) \ge Ch^\alpha,$$

then,

$$u_h(x) = \Lambda(x).$$

Now, if

$$u_h(x) = A(x) - \operatorname{dist}(x, \Gamma),$$

one has necessarily

$$|u_h(x) - A(x)| = \operatorname{dist}(x, \Gamma) \le Ch^\alpha.$$

It results that in all cases one has

$$|u_h(x) - A(x)| \le Ch^\alpha.$$

Since - when $\widehat{u}_h$ denotes the interpolate of $u_h$

$$|u_h(x) - \widehat{u}_h(x)| \le Ch$$

one deduces -say for $h \le 1$

$$|\widehat{u}_h(x) - A(x)| \le Ch^\alpha. \tag{2.33}$$

So, we get (see (1.4), (2.33))

$$\int_{\Omega} \Psi(\widehat{u}_h(x) - A(x))dx \le Ch^{\alpha q}. \tag{2.34}$$

Next, as in part 1

$$\int_{\Omega} \varphi(\nabla \widehat{u}_h(x))dx \le C|I| \tag{2.35}$$

where $I$ denotes the set where we are interpolating. Recall that when

$$\operatorname{dist}(x, \Gamma) \ge Ch^{\alpha},$$

one has

$$u_h(x) = \Lambda(x).$$

So, when $\operatorname{dist}(x,\Gamma) \ge Ch^{\alpha} + h$ one has

$$\widehat{u}_h = \text{the interpolate of } \Lambda(x)$$

and, by the estimate we developed in part 1, if $I_1 = \{x/\operatorname{dist}(x,\Gamma) \ge Ch^{\alpha} + h\}$, we obtain

$$\int_{I_1} \varphi(\nabla \widehat{u}_h(x))dx \le Ch^{1-\alpha}. \tag{2.36}$$

Next we have to estimate

$$\int_{I \setminus I_1} \varphi(\nabla \widehat{u}_h(x))dx. \tag{2.37}$$

But since $I \setminus I_1$ is a $h^{\alpha}$-neighborhood of $\Gamma$. (note that $h \le h^{\alpha}$), and since the gradient of $\widehat{u}_h$ is uniformly bounded we get

$$\int_{I \setminus I_1} \varphi(\nabla \widehat{u}_h(x))dx \le Ch^{\alpha}. \tag{2.38}$$

Combining (2.34)–(2.38) we obtain

$$\begin{aligned} \int_{\Omega} \Psi(\widehat{u}_h(x) - A(x)) + \varphi(\nabla \widehat{u}_h(x))dx &\le C[h^{\alpha q} + h^{1-\alpha} + h^{\alpha}] \\ &\le [h^{\alpha r} + h^{1-\alpha}] \end{aligned} \tag{2.39}$$

and

$$\int_{\Omega} \varphi(\nabla \widehat{u}_h(x))dx \le C[h^{1-\alpha} + h^{\alpha}] \tag{2.40}$$

In (2.39) the estimate is the best when $1 - \alpha = \alpha r$, i.e. when $\alpha = \frac{1}{r+1}$. This gives (2.2). In (2.40) the estimate is optimal for $\alpha = \frac{1}{2}$ which gives (2.1). This completes the proof of Theorem (2.1). ■

REMARK 2.4. In the case where $\Omega$ is not convex, when $q \geq 1$, and under the assumptions of Theorem 2.1 one has for some constant $C$

$$E_h^i \leq Ch^{1/2} \quad \forall i = 1, 2, 3. \tag{2.41}$$

Indeed, thanks to (1.4) one has

$$E_h^1 \leq E_h^2$$

and since $V_h^{\hat{A}} \subset V_h$,

$$E_h^3 \leq E_h^2.$$

So, it is enough to prove (2.41) when $i = 2$. But clearly a polyhedral domain can be decomposed into polyhedral domains $\Omega_i$, $i = 1, \ldots, N$ that are convex. Then, if on each of these domains we construct $u_h$ as in part 2 of the proof of Theorem 2.1, the different $u_h$ are matching on the boundary of the different $\Omega_i$. Then, we call $\widehat{u}_h$ the interpolate of these functions on $\mathcal{T}_h$. The estimate of

$$\int_\Omega \Psi(\widehat{u}_h(x) - A(x)) + \varphi(\nabla \widehat{u}_h(x))dx = \sum_{i=1}^N \int_{\Omega_i} \Psi(\widehat{u}_h(x) - A(x)) + \varphi(\nabla \widehat{u}_h(x))dx$$

goes the same way than in part 2 above and the result follows.

In the case $q < 1$, one would get similarly

$$E_h^2, E_h^3 \leq Ch^{q/(q+1)}.$$

## Appendix

We would like to prove here an extension Theorem that we have used in Section 2. Let $\Omega$ be a domain of $\mathbf{R}^n$. Let $w_1, \ldots w_p \in \mathbf{R}^n$, and consider a function $A$ defined in $\Omega$. We have:

THEOREM A.1. *Assume that $\Omega$ is convex. If $A \in W^{1,\infty}(\Omega)$ satisfies*

$$\nabla A(x) \in \mathrm{Co}(w_i) \quad a.e. \quad x \in \Omega \tag{A.1}$$

*then there exists a Lipschitz continuous function $\widetilde{A}$ defined in $\mathbf{R}^n$ such that:*

$$\widetilde{A} = A \quad in \quad \Omega,$$

$$\nabla \widetilde{A}(x) \in \mathrm{Co}(w_i) \quad a.e. \quad x \in \mathbf{R}^n.$$

Before proving this theorem, remark that when $A$ satisfies (A.1) then

$$A(x) - A(x') \geq \bigwedge_{i=1}^{p} w_i.(x - x') \quad \forall x, x' \in \Omega.$$

In the following lemma we would like to establish that conversely:

LEMMA A.1. *If $\Omega$ is a domain of $\mathbf{R}^n$ and $A$ satisfies*

$$A(x) - A(x') \geq \bigwedge_{i=1}^{p} w_i.(x - x') \quad \forall x, x' \in \Omega$$

*then*

$$\nabla A(x) \in \mathrm{Co}(w_i) \quad \textit{for a.e.} \quad x \in \Omega.$$

*Proof.* We know that there exists $u_h \in W^{1,\infty}(\Omega)$ (cf. [9]) such that

$$u_h \longrightarrow A \quad \text{uniformly,}$$
$$\nabla u_h = w_i, \quad i = 1, \ldots, p.$$

By extracting eventually a subsequence we have:

$$\nabla u_h \to \nabla A \quad \text{in} \quad L^{\infty}(\Omega) \quad \text{weak} \quad *. \tag{A.2}$$

Let us now denote by $B$ an arbitrary ball included in $\Omega$. We have

$$\frac{1}{|B|} \int_B \nabla u_h(x) dx \in \mathrm{Co}(w_i).$$

On the other hand we have by (A.2)

$$\lim_{h \to 0} \frac{1}{|B|} \int_B \nabla u_h(x) dx = \frac{1}{|B|} \int_B \nabla A(x) dx.$$

Since $\mathrm{Co}(w_i)$ is closed

$$\frac{1}{|B|} \int_B \nabla A(x) dx \in \mathrm{Co}(w_i).$$

Using the Lebesgue differentiation theorem we get

$$\nabla A(x) \in \mathrm{Co}(w_i) \quad \text{a.e.} \quad x \in \Omega.$$

This completes the proof in the lemma. ■

We are now able to prove Theorem A.1. Define the function $\widetilde{A}$ on $\mathbf{R}^n$ by setting

$$\widetilde{A}(x) = \inf_{e\in\Omega}\Big\{A(e) - \bigwedge_{i=1}^{p} w_i.(e-x)\Big\} \tag{A.3}$$

First, let us show that $\widetilde{A} = A$ in $\Omega$. Let $x \in \Omega$. It is clear that $\widetilde{A}(x) \leq A(x)$. Moreover, see Lemma 2.1,

$$\forall e \in \Omega, \quad A(e) - A(x) \geq \bigwedge_{i=1}^{p} w_i.(e-x)$$

or

$$\forall e \in \Omega, \quad A(e) - \bigwedge_{i=1}^{p} w_i.(e-x) \geq A(x)$$

i.e.

$$\widetilde{A}(x) \geq A(x).$$

Hence

$$\widetilde{A}(x) = A(x) \quad \forall x \in \Omega.$$

Finally we have:

$$\forall x, y \in \mathbf{R}^n, \quad \widetilde{A}(x) - \widetilde{A}(y) \geq \bigwedge_{i=1}^{p} w_i.(x-y).$$

Indeed:

$$\begin{aligned}
\widetilde{A}(x) &= \inf_{e\in\Omega}\Big\{A(e) - \bigwedge_{i=1}^{p} w_i.(e-x)\Big\} \\
&= \inf_{e\in\Omega}\Big\{A(e) - \bigwedge_{i=1}^{p} w_i.(e-y) + \bigwedge_{i=1}^{p} w_i.(e-y) - \bigwedge_{i=1}^{p} w_i.(e-x)\Big\} \\
&\geq \inf_{e\in\Omega}\Big\{A(e) - \bigwedge_{i=1}^{p} w_i.(e-y)\Big\} + \bigwedge_{i=1}^{p} w_i.(x-y) \\
&\qquad \Big(\text{since } \bigwedge_{i=1}^{p} w_i.(e-y) - \bigwedge_{i=1}^{p} w_i.(e-x) \geq \bigwedge_{i=1}^{p} w_i.(x-y)\Big) \\
&= \widetilde{A}(y) + \bigwedge_{i=1}^{p} w_i.(x-y).
\end{aligned}$$

Taking $y \in \Omega$ this shows in particular that the infimum in (A.3) is finite. Moreover, by Lemma A.1 it follows that:

$$\nabla\widetilde{A}(x) \in \mathrm{Co}(w_i) \quad \text{for a.e. } x \in \mathbf{R}^n$$

This completes the proof of the Theorem A.1.

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