

On the asymptotic expansion of the colored Jones polynomial for torus knots

Jérôme Dubois · Rinat Kashaev

Received: 25 November 2005 / Revised: 16 October 2006 / Published online: 10 August 2007
© Springer-Verlag 2007

Abstract In the asymptotic expansion of the hyperbolic specification of the colored Jones polynomial of torus knots, we identify different geometric contributions, in particular Chern–Simons invariant and twisted Reidemeister torsion with coefficients in the adjoint representation.

Mathematics Subject Classification (2000) 57M27 · 57Q10 · 58J28

1 Introduction

In this paper we consider the “quantum hyperbolic invariant” of a knot K defined by the formula

$$\langle K \rangle_N = \lim_{h \rightarrow 2\pi i/N} J'_N(K; h), \quad \forall N \in \mathbb{Z}_{>1}.$$

Here

$$J'_N(K; h) = \frac{J_N(K; h)}{J_N(\bigcirc; h)}$$

This work is supported in part by the Swiss National Science Foundation, the first author (J. Dubois) is also supported by the European Community with Marie Curie Intra–European Fellowship (MEIF–CT–2006–025316). While writing the paper, J. Dubois visited the CRM. He thanks the CRM for its hospitality.

J. Dubois · R. Kashaev

Section de Mathématiques, Université de Genève CP 64, 2–4 Rue du Lièvre, 1211 Genève 4, Switzerland
e-mail: Rinat.Kashaev@math.unige.ch

J. Dubois (✉)

Centre de Recerca Matemàtica, Apartat 50, 08193 Bellaterra, Spain
e-mail: JDubois@crm.cat

where $J_N(K; h)$ is the N th colored Jones polynomial and \bigcirc stands for the unknot. In the standard normalization one has

$$J_N(\bigcirc; h) = \frac{\sinh(Nh/2)}{\sinh(h/2)}$$

and it is known that in this normalization the N th colored Jones polynomial vanishes at the point $h = 2\pi i/N$ for any knot or link. Thus, the quantity $\langle K \rangle_N$ is a well-defined invariant.

The aim of the present paper is to give a geometrical interpretation for the terms in the asymptotic expansion of $\langle K \rangle_N$ at large N in the case of torus knots. This work is motivated by the “*volume conjecture*” of [9, 16] for the invariant $\langle K \rangle_N$ which states that

$$2\pi \lim_{N \rightarrow \infty} \frac{\log |\langle K \rangle_N|}{N} = v_3 \|S^3 \setminus K\| \tag{1}$$

where v_3 denotes the hyperbolic volume of a regular ideal tetrahedron in \mathbb{H}^3 and $\|S^3 \setminus K\|$ is the simplicial or Gromov norm of the three-manifold $S^3 \setminus K$.

For any torus knot, the asymptotic expansion at large N is derived in [10] to all orders, and one has $|\langle K \rangle_N| = O(N^{3/2})$. This is in fact the optimal estimation in the sense that there exists a sequence of integers of the form $N_j = 2pq(1 + 2j)$ such that $\lim_{j \rightarrow \infty} |\langle K \rangle_{N_j}|/N_j^{3/2}$ exists and is not zero (see also [26], we thank H. Zheng for posing this question to us). In particular, the volume conjecture appears to be trivially true in this case as all torus knots are known to have vanishing simplicial norm.

In this paper, we prove that the non-abelian Reidemeister torsion and the Chern–Simons invariant appear in the asymptotic expansion of $\langle K \rangle_N$. The precise form of this result is as follows.

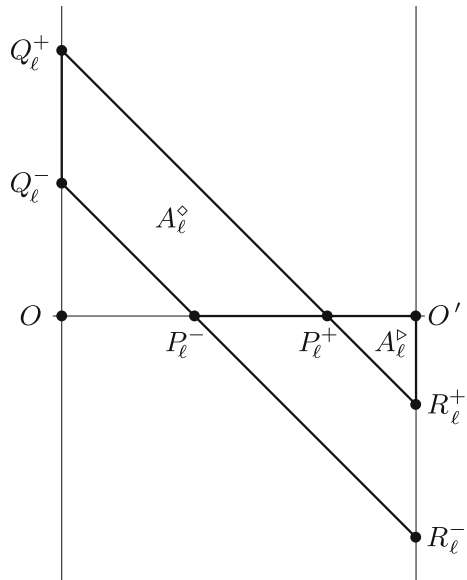
For a knot $K \subset S^3$, let $N(K)$ be a tubular neighborhood of K , $M_K = S^3 \setminus N(K)$ its exterior. Let $\mathbb{T}_\lambda^K(\rho) \in \mathbb{C}$ denote the sign-determined $SL_2(\mathbb{C})$ -twisted Reidemeister torsion at an irreducible representation $\rho : \pi_1(M_K) \rightarrow SL_2(\mathbb{C})$ (with respect to the longitude λ of the knot K) associated to K defined by the first author in [5], see Subsect. 3.4 of the present paper. Let $\Delta_K(t)$ denote the Alexander polynomial of K normalized so that $\Delta_K(t) = \Delta_K(t^{-1})$. According to the works of Milnor and Turaev [14, 22], the analytic map

$$\tau_K(z) = \frac{2 \sinh(z)}{\Delta_K(e^{2z})}, \quad z \in \mathbb{C} \setminus \{z \mid \Delta_K(e^{2z}) = 0\}, \tag{2}$$

is essentially equal to the abelian Reidemeister torsion associated to K (see Sect. 3.3 and especially Proposition 2). The poles of $\tau_K(z)$ describe the so-called *bifurcation points* of the $SL_2(\mathbb{C})$ -character variety of the knot exterior, corresponding to those abelian characters which are limits of non-abelian ones (see Burde [1]).

Let K be the torus knot of type (p, q) . The non-abelian part of its character variety, denoted by $X^{\text{non-abelian}}(M_K)$, consists of $N_{p,q} = (p - 1)(q - 1)/2$ connected components, and each component intersects the abelian part in exactly two bifurcation points. One can characterize the ℓ th non-abelian component by a unique pair of distinct positive

Fig. 1 Euclidian picture associated to the ℓ th component of $X^{\text{ab}}(M_K)$



integers $0 < k_\ell^- < k_\ell^+ < pq$ satisfying certain conditions (see Subsect. 2.2 for details and in particular Theorem 2). In this case, the non-abelian Reidemeister torsion \mathbb{T}_λ^K is locally constant on the $SL_2(\mathbb{C})$ -character variety (Proposition 3). This is due to the structure of torus knot exteriors which are Seifert fibered spaces. We consider the euclidean picture in Fig. 1 where $O = (0, 0)$, $O' = (\frac{1}{2}, 0)$, $P_\ell^\pm = (\frac{k_\ell^\pm}{2pq}, 0)$, and two parallel segments passing through the points P_ℓ^\pm with slope $-pq$. In fact, this picture essentially describes the real slice of ℓ th component of the character variety. Let A_ℓ^\diamond and A_ℓ° be the double areas of the trapezoid $P_\ell^- P_\ell^+ Q_\ell^- Q_\ell^+$ and the triangle $P_\ell^+ R_\ell^+ O'$ respectively (a priori these are positive real numbers, but A_ℓ^\diamond is in fact an integer as we will see). We note that the set $\{A_\ell^\circ \pmod{\mathbb{Z}} \mid 1 \leq \ell \leq (p-1)(q-1)/2\}$ is essentially the Chern–Simons invariant of the knot K (see Subsect. 4.2 for details). We define the quantity

$$Z_N(K) = \sum_{\ell=1}^{N_{p,q}} \varepsilon_\ell \sqrt{|\mathbb{T}_\lambda^K(\chi_\ell)|} A_\ell^\diamond e^{-2\pi i N A_\ell^\circ}, \quad \varepsilon_\ell = (-1)^{[k_\ell^+/p] + [k_\ell^+/q]} \tag{3}$$

where $N_{p,q} = (p-1)(q-1)/2$, $[x]$ is the integer part of the real number x , χ_ℓ is any character in the ℓ th component (as the torsion is locally constant), and the positive value of the square root is assumed.

Theorem 1 For the (p, q) torus knot K the following asymptotic equality holds:

$$2i^{pqN} e^{\frac{i\pi}{2N}(\frac{p+q}{q+p})} \cdot \langle K \rangle_N = C_{p,q} N^{3/2} Z_N(K) + (-1)^{pqN} \sum_{n=1}^{\infty} \frac{a_n(K)}{2^n n!} \left(\frac{i\pi}{pqN}\right)^{n-1} \tag{4}$$

where $C_{p,q} = \sqrt{2pq}e^{i\pi/4}$, and $a_n(K)$ are finite type invariants.

Remark 1 The sequence $(a_n(K))_n$ is not only defined for torus knots but for all knots by the following general formula:

$$a_n(K) = \frac{\partial^{2n}(z\tau_K(z))}{\partial z^{2n}} \Big|_{z=0}.$$

Each $a_n(K)$ is in this sense a finite type invariant (see Sect. 6).

Equation (4) gives a geometrical interpretation for each contribution in the asymptotic expansion of $\langle K \rangle_N$ in terms of classical invariants of knots. We can note consistency with the *asymptotic expansion conjecture* (see [18, Sect. 7.2]), namely appearances of the Chern–Simons invariant and the Reidemeister torsion, but we can also see some peculiarity in our formula, for example, the presence of the multiplication factors A_ℓ° and ε_ℓ .

We remark that, Murakami [17] studied the asymptotic behavior of the colored Jones polynomial of torus knots $\log(J'_N(K; 2\pi ir/N))/N$ in the complementary case $r \neq 1$. It would be interesting to understand the geometrical contributions in that case also. Besides, Hikami [7] interprets the asymptotic expansion of the invariant $\langle K \rangle_N$ for $(2, q)$ torus knots from a different point of view, namely in terms of q -series identities.

Organization

In Sect. 2 we recall some well known facts about the character varieties and fix the notation. In Sect. 3, we review the abelian and the non-abelian Reidemeister torsions for knots as presented in [4], state the result of the computation for torus knots (Proposition 3) and Yamaguchi’s Theorem 3 on the behavior of the non-abelian Reidemeister torsion near a bifurcation point. Section 4 deals with the Chern–Simons invariant for knot exteriors. We compute it for torus knots using the technique of [11]. In Sect. 5, we review the quantum hyperbolic invariant for torus knots and, in particular, the integral representation of [10]. Section 6 contains the proof of Theorem 1; the proof combines all results given in the preceding sections, especially the computation of the residues of the abelian Reidemeister torsion at bifurcation points and the Chern–Simons invariant for torus knots.

2 Review on $SL_2(\mathbb{C})$ -character varieties

2.1 Review on character variety

Given a finitely generated group π we let

$$R(\pi; SL_2(\mathbb{C})) = \text{Hom}(\pi; SL_2(\mathbb{C}))$$

denote the space of $\mathrm{SL}_2(\mathbb{C})$ -representations of π . This space is endowed with the compact-open topology. Here π is assumed to have the discrete topology and the Lie group $\mathrm{SL}_2(\mathbb{C})$ is endowed with the usual one. A representation $\rho: \pi \rightarrow \mathrm{SL}_2(\mathbb{C})$ is called *abelian* if $\rho(\pi)$ is an abelian subgroup of $\mathrm{SL}_2(\mathbb{C})$. A representation ρ is called *reducible* if there exists a proper subspace $U \subset \mathbb{C}^2$ such that $\rho(g)U \subset U$, for all $g \in \pi$. Of course, any abelian representation is reducible. A non reducible representation is called *irreducible*.

The group $\mathrm{SL}_2(\mathbb{C})$ acts on the representation space $R(\pi; \mathrm{SL}_2(\mathbb{C}))$ by conjugation, but the naive quotient $R(\pi; \mathrm{SL}_2(\mathbb{C}))/\mathrm{SL}_2(\mathbb{C})$ is not Hausdorff in general. Following [2], we will focus on the *character variety* $X(\pi) = X(\pi; \mathrm{SL}_2(\mathbb{C}))$ which is the set of *characters* of π . Associated to a representation $\rho \in R(\pi, \mathrm{SL}_2(\mathbb{C}))$ is its character $\chi_\rho: \pi \rightarrow \mathbb{C}$, defined by $\chi_\rho(g) = \mathrm{tr}(\rho(g))$. Here $\mathrm{tr}(\rho(g))$ denotes the trace of the matrix $\rho(g)$. Let $g \in \pi$, following [2] we let $I_g: X(\pi) \rightarrow \mathbb{C}$ denote the function defined by $I_g: \rho \mapsto \mathrm{tr}(\rho(g))$.

In some sense $X(\pi)$ is the “algebraic quotient” of $R(\pi; \mathrm{SL}_2(\mathbb{C}))$ by the action by conjugation of $\mathrm{PSL}_2(\mathbb{C})$. It is well known that $R(\pi, \mathrm{SL}_2(\mathbb{C}))$ and $X(\pi)$ have the structure of complex algebraic affine sets (see [2]).

Let $R^{\mathrm{irr}}(\pi; \mathrm{SL}_2(\mathbb{C}))$ denote the subset of irreducible representations of π in $\mathrm{SL}_2(\mathbb{C})$, and let $X^{\mathrm{irr}}(\pi)$ denote its image under the map $R(\pi; \mathrm{SL}_2(\mathbb{C})) \rightarrow X(\pi)$. Note that two irreducible representations of π in $\mathrm{SL}_2(\mathbb{C})$ with the same character are conjugate by an element of $\mathrm{SL}_2(\mathbb{C})$, see [2, Proposition 1.5.2].

For a knot K in S^3 , let $\Pi(K)$ denote its group. Let $\Pi(K)'$ be the subgroup generated by the commutators of $\Pi(K)$. It is well known that

$$\Pi(K)/\Pi(K)' \cong H_1(S^3 \setminus K; \mathbb{Z}) \cong \mathbb{Z}$$

is generated by the meridian μ of K . As a consequence, each abelian representation of $\Pi(K)$ in $\mathrm{SL}_2(\mathbb{C})$ is conjugate either to $\varphi_z: \Pi(K) \rightarrow \mathrm{SL}_2(\mathbb{C})$ defined by $\varphi_z(\mu) = \begin{pmatrix} e^z & 0 \\ 0 & e^{-z} \end{pmatrix}$, with $z \in \mathbb{C}$, or to a representation ρ with $\rho(\mu) = \pm \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

Let $R^{\mathrm{ab}}(\Pi(K))$ denote the space of abelian representations, its image under the map $R(\Pi(K); \mathrm{SL}_2(\mathbb{C})) \rightarrow X(\Pi(K))$ is denoted $X^{\mathrm{ab}}(\Pi(K))$. Similarly we write $X^{\mathrm{nab}}(\Pi(K))$ for the image of $R(\Pi(K); \mathrm{SL}_2(\mathbb{C})) \setminus R^{\mathrm{ab}}(\Pi(K))$. Observe that $X^{\mathrm{irr}}(\Pi(K)) \subset X^{\mathrm{nab}}(\Pi(K))$ and that the converse is false.

If M is a three-dimensional manifold, then we write $X^{\mathrm{ab}}(M) = X^{\mathrm{ab}}(\pi_1(M))$, $X^{\mathrm{nab}}(M) = X^{\mathrm{nab}}(\pi_1(M))$, $X^{\mathrm{irr}}(M) = X^{\mathrm{irr}}(\pi_1(M))$ etc.

2.2 Character variety of torus knot groups

Let $\mathcal{T}(p, q)$ denote the torus knot of type (p, q) where $p, q > 1$ are coprime integers and set $\mathcal{M}(p, q) = M_{\mathcal{T}(p, q)}$. The group of $\mathcal{T}(p, q)$ admits the following well known presentation

$$\Pi(p, q) = \langle a, b \mid a^p = b^q \rangle.$$

Let $r, s \in \mathbb{Z}$ be such that $ps - qr = 1$. The meridian of $\mathcal{T}(p, q)$ is represented by the word $\mu = a^{-r}b^s$ and the longitude by $\lambda = a^p\mu^{-pq} = b^q\mu^{-pq}$.

Theorem 2 *The non-abelian part $X^{\text{non-abelian}}(\mathcal{M}(p, q))$ of the character variety $X(\mathcal{M}(p, q))$ is the disjoint union of $N_{p,q} = (p - 1)(q - 1)/2$ copies of \mathbb{C} indexed by the pair (α, β) which satisfies the following conditions:*

$$1 \leq \alpha \leq p - 1, \quad 1 \leq \beta \leq q - 1, \quad \alpha \equiv \beta \pmod{2}. \tag{5}$$

For $\chi \in X^{\text{non-abelian}}(\mathcal{M}(p, q))$ on the component indexed by (α, β) , we have

$$\chi(a) = 2 \cos\left(\frac{\alpha\pi}{p}\right), \quad \chi(b) = 2 \cos\left(\frac{\beta\pi}{q}\right).$$

Moreover, we have $X^{\text{irr}}(\mathcal{M}(p, q)) = X^{\text{non-abelian}}(\mathcal{M}(p, q)) \setminus X^{\text{abelian}}(\mathcal{M}(p, q))$ and the component of $X^{\text{non-abelian}}(\mathcal{M}(p, q))$ indexed by (α, β) is attached to $X^{\text{abelian}}(\mathcal{M}(p, q))$ at the two abelian representations $\varphi_{i\pi \frac{k^-}{pq}}$ and $\varphi_{i\pi \frac{k^+}{pq}}$ where the integers k^- and k^+ satisfy the following conditions:

$$\begin{aligned} 0 < k^- < k^+ < pq, & \tag{6} \\ k^- = \varepsilon^-(\beta ps - \varepsilon\alpha qr) \pmod{pq}, & \tag{7} \\ k^+ = \varepsilon^+(\beta ps + \varepsilon\alpha qr) \pmod{pq}. & \tag{8} \end{aligned}$$

Here $\varepsilon, \varepsilon^-, \varepsilon^+ \in \{\pm 1\}$.

We will use this description in the proof of Theorem 1. The first part of Theorem 2 is due to Klassen [12] and Le [13]. Further notice the following remarks.

Remark 2 The integers k^- and k^+ satisfying conditions (7) and (8) verify the following properties:

1. k^\pm is not divisible neither by p nor by q ;
2. $k^+ \pm k^- \in 2\mathbb{Z}$;
3. either p divides $k^+ + k^-$, and q divides $k^+ - k^-$; or q divides $k^+ + k^-$, and p divides $k^+ - k^-$.

Remark 3 The quantities $m = \frac{k^+ - k^-}{2}$ and $l = \frac{(k^+)^2 - (k^-)^2}{4pq}$ have the following properties:

1. $m, l \in \mathbb{Z}_{>0}$,
2. $m < \frac{pq}{2}$,
3. either p divides m or q divides m ,
4. m divides pql ,
5. $\frac{m^2}{pq} < l < m - \frac{m^2}{pq}$.

Remark 4 In the case of $(2, q)$ torus knots one has a more explicit description of the character variety $X(\mathcal{M}(2, q))$, see [13] for details. We use the fact that $\mathcal{T}(2, q)$ is a

two-bridge knot and its group admits the following Wirtinger presentation:

$$\Pi(2, q) = \langle a, b \mid wa = bw \rangle \quad \text{where } w = (ab)^{(q-1)/2}.$$

We write $x = I_a: \rho \mapsto \text{tr } \rho(a)$, $y = I_{ab}: \rho \mapsto \text{tr } \rho(ab)$. The character variety is parametrized by x and y (see Riley [20]). Specifically, its abelian part $X^{\text{ab}}(\mathcal{M}(p, q))$ is described by the equation $y - x^2 + 2 = 0$. The non-abelian part $X^{\text{nab}}(\mathcal{M}(2, q))$ is described by the polynomial equation $\Phi(2, q) = 0$, where $\Phi(2, q) \in \mathbb{Z}[x, y]$ satisfies the linear recurrence relation

$$\Phi(2, q) = y\Phi(2, q-2) - \Phi(2, q-4) \quad \text{and} \quad \Phi(2, 1) = 1, \Phi(2, 3) = y - 1. \quad (9)$$

The non-abelian part $X^{\text{nab}}(\mathcal{M}(2, q))$ of the character variety is the disjoint union of $(q-1)/2$ copies of \mathbb{C} parametrized by x , indexed by $\ell \in \{1, \dots, (q-1)/2\}$ and attached to $X^{\text{ab}}(\mathcal{M}(2, q))$ at the $(q-1)$ abelian representations φ_z for

$$z \in \left\{ i\pi - i\pi \frac{2\ell - 1}{2q}, i\pi \frac{2\ell - 1}{2q} \right\}.$$

As a consequence, the $(q-1)/2$ pairs of integers (k^-, k^+) are given by the following formulas:

$$k_\ell^- = 2\ell - 1 \quad \text{and} \quad k_\ell^+ = 2q - 2\ell + 1,$$

where $\ell \in \{1, \dots, (q-1)/2\}$. Besides we have

$$m_\ell = q - 2\ell + 1, \quad A_\ell^\diamond = \frac{q - 2\ell + 1}{2}, \quad A_\ell^\triangleright = \frac{(2\ell - 1)^2}{8q}.$$

Remark 5 It is more complicated to compute the pairs (k^-, k^+) for arbitrary torus knots, and in fact we do not know any “closed” formula for them. In Table 1 we give some concrete examples.

3 Review on $\text{SL}_2(\mathbb{C})$ -twisted Reidemeister torsion

3.1 Preliminaries: sign-determined torsion of a CW-complex

We review the basic notions and results about the sign-determined Reidemeister torsion introduced by Turaev which are needed in this paper. Details can be found in Milnor’s survey [15] and in Turaev’s monograph [24].

Table 1 Numerical invariants of the non-abelian components of the character variety for some torus knots

(p, q)	$N_{p,q}$	(k_ℓ^-, k_ℓ^+)	$m_\ell = \frac{k_\ell^+ - k_\ell^-}{2}$	$A_\ell^\diamond = \frac{k_\ell^{+2} - k_\ell^{-2}}{4pq}$	$A_\ell^\triangleright = \frac{(pq - k_\ell^+)^2}{4pq}$
(3,4)	3	(1,7)	3	1	$\frac{25}{48}$
		(2,10)	4	2	$\frac{1}{12}$
		(5,11)	3	2	$\frac{1}{48}$
(3,5)	4	(1,11)	5	2	$\frac{4}{15}$
		(2,8)	3	1	$\frac{49}{60}$
		(4,14)	5	3	$\frac{1}{60}$
		(7,13)	3	2	$\frac{1}{15}$
(4,5)	6	(1,9)	4	1	$\frac{121}{80}$
		(2,18)	8	4	$\frac{1}{20}$
		(3,13)	5	2	$\frac{49}{80}$
		(6,14)	4	2	$\frac{9}{20}$
		(7,17)	5	3	$\frac{9}{80}$
		(11,19)	4	3	$\frac{1}{80}$
		(4,7)	9	(1,15)	7
(2,26)	12			6	$\frac{1}{28}$
(3,11)	4			1	$\frac{289}{112}$
(5,19)	7			3	$\frac{81}{112}$
(6,22)	8			4	$\frac{9}{28}$
(9,23)	7			4	$\frac{25}{112}$
(10,18)	4			2	$\frac{25}{28}$
(13,27)	7			5	$\frac{1}{112}$
(17,25)	4			3	$\frac{9}{112}$

Torsion of a chain complex

Let $C_* = (0 \rightarrow C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} \dots \xrightarrow{d_1} C_0 \rightarrow 0)$ be a chain complex of finite dimensional vector spaces over \mathbb{C} . Choose a basis \mathbf{c}^i for C_i and a basis \mathbf{h}^i for the i th homology group $H_i = H_i(C_*)$. The torsion of C_* with respect to this choice of bases is defined as follows.

Let \mathbf{b}^i be a sequence of vectors in C_i such that $d_i(\mathbf{b}^i)$ is a basis of $B_{i-1} = \text{im}(d_i : C_i \rightarrow C_{i-1})$ and let $\tilde{\mathbf{h}}^i$ denote a lift of \mathbf{h}^i in $Z_i = \ker(d_i : C_i \rightarrow C_{i-1})$. The set of vectors $d_{i+1}(\mathbf{b}^{i+1})\tilde{\mathbf{h}}^i \mathbf{b}^i$ is a basis of C_i . Let $[d_{i+1}(\mathbf{b}^{i+1})\tilde{\mathbf{h}}^i \mathbf{b}^i / \mathbf{c}^i] \in \mathbb{C}^*$ denote the determinant of the transition matrix between those bases (the entries of this matrix are coordinates of vectors in $d_{i+1}(\mathbf{b}^{i+1})\tilde{\mathbf{h}}^i \mathbf{b}^i$ with respect to \mathbf{c}^i). The *sign-determined Reidemeister torsion* of C_* (with respect to the bases \mathbf{c}^* and \mathbf{h}^*) is the following alternating product (see [23, Definition 3.1]):

$$\text{Tor}(C_*, \mathbf{c}^*, \mathbf{h}^*) = (-1)^{|C_*|} \cdot \prod_{i=0}^n [d_{i+1}(\mathbf{b}^{i+1})\tilde{\mathbf{h}}^i \mathbf{b}^i / \mathbf{c}^i]^{(-1)^{i+1}} \in \mathbb{C}^*. \tag{10}$$

Here $|C_*| = \sum_{k \geq 0} \alpha_k(C_*)\beta_k(C_*)$, where $\alpha_i(C_*) = \sum_{k=0}^i \dim C_k$, $\beta_i(C_*) = \sum_{k=0}^i \dim H_k$.

The torsion $\text{Tor}(C_*, \mathbf{c}^*, \mathbf{h}^*)$ does not depend on the choices of \mathbf{b}^i and $\tilde{\mathbf{h}}^i$. Further observe that if C_* is acyclic (i.e. if $H_i = 0$ for all i), then $|C_*| = 0$.

Torsion of a CW-complex

Let W be a finite CW-complex and $\rho \in \text{Hom}(\pi_1(W); \text{SL}_2(\mathbb{C}))$ be a representation. We define the $\mathfrak{sl}_2(\mathbb{C})_\rho$ -twisted cochain complex of W to be

$$C^*(W; \mathfrak{sl}_2(\mathbb{C})_\rho) = \text{Hom}_{\pi_1(X)}(C_*(\tilde{W}; \mathbb{Z}); \mathfrak{sl}_2(\mathbb{C})_\rho).$$

Here $C_*(\tilde{W}; \mathbb{Z})$ is the complex of the universal covering with integer coefficients which is in fact a $\mathbb{Z}[\pi_1(W)]$ -module (via the action of $\pi_1(W)$ on \tilde{W} as the covering group), and $\mathfrak{sl}_2(\mathbb{C})_\rho$ denotes the $\mathbb{Z}[\pi_1(W)]$ -module via the composition $Ad \circ \rho$, where $Ad: \text{SL}_2(\mathbb{C}) \rightarrow \text{Aut}(\mathfrak{sl}_2(\mathbb{C}))$, $A \mapsto Ad_A$ is the adjoint representation. This cochain complex $C^*(W; \mathfrak{sl}_2(\mathbb{C})_\rho)$ computes the $\mathfrak{sl}_2(\mathbb{C})_\rho$ -twisted cohomology of W which we denote as $H_\rho^*(W)$.

Let $\{e_1^{(i)}, \dots, e_{n_i}^{(i)}\}$ be the set of i -dimensional cells of W . We lift them to the universal covering and we choose an arbitrary order and an arbitrary orientation for the cells $\{\tilde{e}_1^{(i)}, \dots, \tilde{e}_{n_i}^{(i)}\}$. If $\mathcal{B} = \{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ is an orthonormal basis of $\mathfrak{sl}_2(\mathbb{C})$, then we consider the corresponding ‘‘dual’’ basis over \mathbb{C}

$$\mathbf{c}_{\mathcal{B}}^i = \left\{ \tilde{e}_{1,\mathbf{a}}^{(i)}, \tilde{e}_{1,\mathbf{b}}^{(i)}, \tilde{e}_{1,\mathbf{c}}^{(i)}, \dots, \tilde{e}_{n_i,\mathbf{a}}^{(i)}, \tilde{e}_{n_i,\mathbf{b}}^{(i)}, \tilde{e}_{n_i,\mathbf{c}}^{(i)} \right\}$$

of $C^i(W; \mathfrak{sl}_2(\mathbb{C})_\rho) = \text{Hom}_{\pi_1(X)}(C_*(\tilde{W}; \mathbb{Z}); \mathfrak{sl}_2(\mathbb{C})_\rho)$. Now choosing for each i a basis \mathbf{h}^i for the twisted cohomology $H_\rho^i(W)$, we can compute

$$\text{Tor}(C^*(W; \mathfrak{sl}_2(\mathbb{C})_\rho), \mathbf{c}_{\mathcal{B}}^*, \mathbf{h}^*).$$

The cells $\{\tilde{e}_j^{(i)}\}_{0 \leq i \leq \dim W, 1 \leq j \leq n_i}$ are in one-to-one correspondence with the cells of W , their order and orientation induce an order and an orientation for the cells $\{e_j^{(i)}\}_{0 \leq i \leq \dim W, 1 \leq j \leq n_i}$. Again, corresponding to these choices, we get a basis c^i over \mathbb{R} for $C^i(W; \mathbb{R})$.

Choose a *cohomology orientation* of W , which is an orientation of the real vector space $H^*(W; \mathbb{R}) = \bigoplus_{i \geq 0} H^i(W; \mathbb{R})$. Let \mathfrak{o} denote this chosen orientation. Provide each vector space $H^i(W; \mathbb{R})$ with a reference basis h^i such that the basis $\{h^0, \dots, h^{\dim W}\}$ of $H^*(W; \mathbb{R})$ is positively oriented with respect to \mathfrak{o} . Compute the sign-determined Reidemeister torsion $\text{Tor}(C^*(W; \mathbb{R}), c^*, h^*)$ of the resulting based

and cohomology based chain complex and consider its sign

$$\tau_0 = \text{sgn} \left(\text{Tor}(C^*(W; \mathbb{R}), c^*, h^*) \right) \in \{\pm 1\}.$$

We define the sign-determined $Ad \circ \rho$ -twisted Reidemeister torsion of W to be

$$\text{TOR}(W; Ad \circ \rho, \mathbf{h}^*, \sigma) = \tau_0 \cdot \text{Tor}(C^*(W; \mathfrak{sl}_2(\mathbb{C})_\rho), \mathbf{c}_B^*, \mathbf{h}^*) \in \mathbb{C}^*. \tag{11}$$

This definition only depends on the combinatorial class of W , the conjugacy class of ρ , the choice of \mathbf{h}^* and the cohomology orientation σ . It is independent of the orthonormal basis \mathcal{B} of $\mathfrak{sl}_2(\mathbb{C})$, of the choice of the lifts $\tilde{e}_j^{(i)}$, and of the choice of the positively oriented basis of $H^*(W; \mathbb{R})$. Moreover, it is independent of the order and the orientation of the cells (because they appear twice).

One can prove that TOR is invariant under cellular subdivision, homeomorphism and simple homotopy equivalences. In fact, it is precisely the sign $(-1)^{|C_*|}$ in (10) which ensures all these important invariance properties to hold.

Canonical orientation of knot exteriors

In the case of knot exteriors in which we are interested, there exists a canonical cohomology orientation essentially defined by the meridian of the knot. The aim of this paragraph is to review it in details.

The exterior M_K of K is a three-dimensional CW-complex which has the same simple homotopy type as a two-dimensional CW-complex. We equip M_K with its *canonical cohomology orientation* defined as follows (see [24, Sect. V.3]). We have

$$H^*(M_K; \mathbb{R}) = H^0(M_K; \mathbb{R}) \oplus H^1(M_K; \mathbb{R})$$

and we base this \mathbb{R} -vector space with $\{[\![pt]\!], \mu^*\}$. Here $[\![pt]\!]$ is the cohomology class of a point, and $\mu^*: \mu \mapsto 1$ is the dual of the meridian μ of K . This reference basis of $H^*(M_K; \mathbb{R})$ induces the so-called canonical cohomology orientation of M_K . In the sequel, we let σ denote the canonical cohomology orientation of M_K .

3.2 Regularity for representations

In this subsection we briefly review two notions of regularity (see [5, 19]). In the sequel $K \subset S^3$ denotes an oriented knot. We let $\Pi(K) = \pi_1(M_K)$ denote its group. The meridian μ of K is supposed to be oriented according to the rule $\ell k(K, \mu) = +1$, while the longitude λ is oriented according to the condition $\text{int}(\mu, \lambda) = +1$. Here $\text{int}(\cdot, \cdot)$ denotes the intersection form on ∂M_K .

We say that $\rho \in R^{\text{irr}}(\Pi(K); \text{SL}_2(\mathbb{C}))$ is *regular* if $\dim H_\rho^1(M_K) = 1$. This notion is invariant by conjugation and thus it is well-defined for irreducible characters.

Example 1 For the torus knot $\mathcal{T}(p, q)$, one can prove that each irreducible representation of $\Pi(p, q)$ in $\text{SL}_2(\mathbb{C})$ is regular.

In the case of the figure-eight knot, one can also prove that each irreducible representation of its group in $SL_2(\mathbb{C})$ is regular.

Observe that for a regular representation ρ , we have

$$\dim H_\rho^1(M_K) = 1, \dim H_\rho^2(M_K) = 1 \text{ and } H_\rho^j(M_K) = 0 \text{ for all } j \neq 1, 2.$$

Let γ be a simple closed unoriented curve in ∂M_K . Among irreducible representations we focus on the γ -regular ones. We say that an irreducible representation $\rho: \Pi(K) \rightarrow SL_2(\mathbb{C})$ is γ -regular, if (see [19, Definition 3.21]):

1. the inclusion $\alpha: \gamma \hookrightarrow M_K$ induces an *injective* map

$$\alpha^*: H^1(M_K; \mathfrak{sl}_2(\mathbb{C})_\rho) \rightarrow H^1(\gamma; \mathfrak{sl}_2(\mathbb{C})_\rho),$$

2. if $\text{tr}(\rho(\pi_1(\partial M_K))) \subset \{\pm 2\}$, then $\rho(\gamma) \neq \pm \mathbf{1}$.

It is easy to see that this notion is invariant by conjugation and that γ -regularity implies regularity (the converse is false). Thus, for $\chi \in X^{\text{irr}}(M_K)$ the notion of γ -regularity is well-defined.

Example 2 For the torus knot $\mathcal{T}(p, q)$, one can prove that each irreducible representation of $\Pi(p, q)$ in $SL_2(\mathbb{C})$ is μ -regular and also λ -regular.

Here is an alternative formulation, see [5, Proposition 3]. Fix a generator P^ρ of $H_\rho^0(\partial M_K)$. We recall that

$$H_\rho^0(\partial M_K) = \mathfrak{sl}_2(\mathbb{C})^{\text{Ad} \circ \rho(\pi_1(\partial M_K))} = \{v \in \mathfrak{sl}_2(\mathbb{C}) \mid \forall g \in \pi_1(\partial M_K) \text{ Ad}_{\rho(g)}v = v\}.$$

The inclusion $\alpha: \gamma \hookrightarrow M_K$ and the cup product induce the linear form $f_\gamma^\rho: H_\rho^1(M_K) \rightarrow \mathbb{C}$. We explicitly have

$$f_\gamma^\rho(v) = B_{\mathfrak{sl}_2(\mathbb{C})}(P^\rho, v(\gamma)), \text{ for all } v \in H_\rho^1(M_K).$$

Here $B_{\mathfrak{sl}_2(\mathbb{C})}$ denotes the killing form of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$.

Proposition 1 [5, Proposition 3] *An irreducible representation ρ of $\Pi(K)$ in $SL_2(\mathbb{C})$ is γ -regular if and only if the linear form $f_\gamma^\rho: H_\rho^1(M_K) \rightarrow \mathbb{C}$ is an isomorphism.*

Remark 6 In fact, a regular representation ρ is γ -regular if and only if the linear form f_γ^ρ is non degenerated (i.e. $f_\gamma^\rho \neq 0$).

3.3 Review on the abelian Reidemeister torsion for knot exteriors

The aim of this subsection is to compute the Reidemeister torsion of the exterior of K twisted by the adjoint representation associated to an abelian representation of $\Pi(K)$ in terms of the Alexander polynomial of K .

Let $\varphi_z : \Pi(K) \rightarrow \text{SL}_2(\mathbb{C})$ be the abelian representation such that

$$\varphi_z(\mu) = \begin{pmatrix} e^z & 0 \\ 0 & e^{-z} \end{pmatrix}$$

and suppose that φ_z is not boundary-central: $\varphi_z(\partial M_K) \not\subset \{\pm \mathbf{1}\}$ (i.e. $z \neq 2\pi ki$ and $z \neq (\pi + 2\pi k)i$). When e^{2z} is not a zero of the Alexander polynomial Δ_K of K we say that φ_z is regular. In this case, one can prove following Klassen’s arguments [12, Theorem 19] that $H_{\varphi_z}^i(M_K) \cong H^i(M_K; \mathbb{Z}) \otimes \mathbb{C}$, for all i .

Let $h^{(0)} = P^\rho \in \mathfrak{sl}_2(\mathbb{C})$ be a fixed generator of $H_\rho^0(\partial M_K)$; then $H_{\varphi_z}^0(M_K)$ is generated by $h^{(0)}$ and $H_{\varphi_z}^1(M_K)$ is generated by $h^{(1)} = h_1^{(0)} + \dots + h_{2n}^{(0)}$, where $h_k^{(0)}$ is the vector in $\mathfrak{sl}_2(\mathbb{C})^{2n}$ of which all entries are zero except the one of index k which is equal to $h^{(0)}$.

With above notation and choices, we have (see [14, Theorem 4], [22, Subsect. 1.2] and [4, Proposition 4.4])

Proposition 2 *Let φ_z be a regular abelian representation which is not boundary-central. The $(\text{Ad} \circ \varphi_z)$ -twisted Reidemeister torsion of M_K calculated in the basis $\{h^{(0)}, h^{(1)}\}$ of $H_{\varphi_z}^*(M_K)$ and with respect to the canonical cohomology orientation of M_K has the following form*

$$\text{TOR} \left(M_K; \text{Ad} \circ \varphi_z, \{h^{(0)}, h^{(1)}\}, \mathfrak{o} \right) = -\tau_K(z)\tau_K(-z). \tag{12}$$

Here $\tau_K(z)$ is the analytic map defined in Eq. (2).

Remark 7 If K is the trivial knot, then

$$\text{TOR} \left(M_K; \text{Ad} \circ \varphi_z, \{h^{(0)}, h^{(1)}\}, \mathfrak{o} \right) = 4 \sinh^2(z)$$

is the twisted Reidemeister torsion of the solid torus M_K .

3.4 Review on the non-abelian Reidemeister torsion for knot exteriors

This subsection gives a review of the constructions made in [4, Sect. 6]. In particular, we shall explain how to construct distinguished bases for the twisted cohomology of knot exteriors.

How to construct natural bases for the twisted cohomology

Let ρ be a regular representation of $\Pi(K)$. One has a distinguished isomorphism induced by the cup product and the Killing form (which explicitly depends on the invariant vector P^ρ), see [4, Lemmas 5.1, 5.2]:

$$\phi_{P^\rho} : H_\rho^2(M_K) \rightarrow H^2(M_K; \mathbb{Z}) \otimes \mathbb{C}.$$

Let c be the generator of $H^2(\partial M_K; \mathbb{Z}) = \text{Hom}(H_2(\partial M_K; \mathbb{Z}), \mathbb{Z})$ corresponding to the fundamental class $[[\partial M_K]] \in H_2(\partial M_K; \mathbb{Z})$ induced by the orientation of ∂M_K . The reference generator of $H^2_\rho(M_K)$ is defined by

$$h_\rho^{(2)} = \phi_{P_\rho}^{-1}(c). \tag{13}$$

Let ρ be a λ -regular representation of $\Pi(K)$. The reference generator of $H^1_\rho(M_K)$ is defined by

$$h_\rho^{(1)}(\lambda) = (f_\lambda^\rho)^{-1}(1). \tag{14}$$

The Reidemeister torsion for knot exteriors

Let $\rho: \Pi(K) \rightarrow \text{SL}_2(\mathbb{C})$ be a λ -regular representation. The Reidemeister torsion \mathbb{T}_λ^K at ρ is defined to be

$$\mathbb{T}_\lambda^K(\rho) = \text{TOR} \left(M_K; \text{Ad} \circ \rho, \{h_\rho^{(1)}(\lambda), h_\rho^{(2)}\}, \mathfrak{o} \right) \in \mathbb{C}^*.$$

It is an invariant of knots. Moreover, if ρ_1 and ρ_2 are two λ -regular representations which have the same character then $\mathbb{T}_\lambda^K(\rho_1) = \mathbb{T}_\lambda^K(\rho_2)$. Thus \mathbb{T}_λ^K defines a smooth function on the set $X_\lambda^{\text{irr}}(M_K) = \{\chi \in X^{\text{irr}}(M_K) \mid \chi \text{ is } \lambda\text{-regular}\} \subset \mathcal{R}\text{eg}(K)$.

3.5 The Reidemeister torsion for torus knots

The aim of this subsection is to state the result of the computation of the $\text{SL}_2(\mathbb{C})$ -twisted Reidemeister torsion for torus knots, see [5, Sect. 6.2]. This result will be used to prove Theorem 3.

Proposition 3 ([5]) *If $\chi \in X^{\text{irr}}(\mathcal{M}(p, q))$ lies in the component of the character variety indexed by the pair (α, β) which satisfies conditions (5), then*

$$\mathbb{T}_\lambda^{\mathcal{J}(p,q)}(\chi) = \frac{16}{p^2q^2} \sin^2 \left(\frac{\pi\alpha}{p} \right) \sin^2 \left(\frac{\pi\beta}{q} \right). \tag{15}$$

Remark 8 Example 2 gives $X_\lambda^{\text{irr}}(\mathcal{M}(p, q)) = X^{\text{irr}}(\mathcal{M}(p, q))$, thus $\mathbb{T}_\lambda^{\mathcal{J}(p,q)}$ is defined on the whole character variety of $\Pi(p, q)$ and not a priori only on $X_\lambda^{\text{irr}}(\mathcal{M}(p, q))$.

Moreover, $\mathbb{T}_\lambda^{\mathcal{J}(p,q)}$ is locally constant on the character variety.

Remark 9 The fact that \mathbb{T}_λ^K is locally constant on the character variety in the case of torus knots is a particular property of such knots due to the fact that the exterior of a torus knot is a Seifert fibered manifold with two exceptional fibers (and a regular one).

In general, \mathbb{T}_λ^K is, of course, not locally constant. An example is given by the figure-eight knot 4_1 , following [5], we have

$$(\mathbb{T}_\lambda^{4_1}(\rho))^2 = \frac{1}{17 + 4 \text{tr}(\rho(\lambda))}. \tag{16}$$

3.6 Behavior of the non-abelian Reidemeister torsion near a bifurcation point

Let $z \in \mathbb{C}$. Consider the abelian representation $\varphi_z: \Pi(K) \rightarrow \text{SL}_2(\mathbb{C})$ defined by

$$\varphi_z: \mu \mapsto \begin{pmatrix} e^z & 0 \\ 0 & e^{-z} \end{pmatrix}$$

and suppose that φ_z is not boundary-central (i.e. $\varphi_z(\partial M_K) \not\subset \{\pm 1\}$).

A result of Burde [1] and de Rham [3] states that there exists a reducible non-abelian representation $\rho_z: \pi_1(M_K) \rightarrow \text{SL}_2(\mathbb{C})$ which has the same character as φ_z if and only if $\Delta_K(e^{2z}) = 0$. Furthermore, a recent result of Heusener et al. [6] states specifically that if e^{2z} is a simple zero of Δ_K (i.e. $\Delta_K(e^{2z}) = 0$ and $\Delta'_K(e^{2z}) \neq 0$), then the corresponding representation ρ_z is a *bifurcation point*, i.e. ρ_z is limit of irreducible representations and is a smooth point of the $\text{SL}_2(\mathbb{C})$ -representation variety (contained in a unique irreducible four-dimensional component of the $\text{SL}_2(\mathbb{C})$ -representation variety).

In the initial version of this paper, based on explicit calculations in the cases of torus knots and the figure-eight knot, we conjectured a relation between the residues of the analytic map τ_K and the limit values of the non-abelian Reidemeister torsion at the corresponding bifurcation points. Shortly after, this conjecture has been proved by Yamaguchi [25]. The result is as follows.

Theorem 3 (see [25, Theorem 1]) *Let $z_0 \in \mathbb{C}$ be such that e^{2z_0} is a simple zero of the Alexander polynomial of K . Let χ_{z_0} denote the character corresponding to φ_{z_0} . The residue of τ_K at z_0 satisfies:*

$$\left(2 \text{Res}_{z=z_0} \tau_K(z) \right)^2 = \pm \lim_{\chi \rightarrow \chi_{z_0}} \mathbb{T}_\lambda^K(\chi).$$

Here the limit is taken for irreducible $\text{SL}_2(\mathbb{C})$ -characters χ which converge to χ_{z_0} .

In the case of torus knots, which we are interested in that paper, Theorem 3 is a trivial application of Formula (15) and the proof reduces in this case to a direct computation of the residue at each bifurcation points.

Proof (Sketch of the proof of Theorem 3) The proof is divided in two main parts.

1. *Existence part.* The first point is to prove the existence of the limit:

$$\lim_{\substack{\chi \rightarrow \chi_{z_0} \\ \chi \in \mathcal{X}_\lambda^{\text{irr}}(M_K)}} \mathbb{T}_\lambda^K(\chi). \tag{17}$$

The existence of limit (17) is guaranteed by the fact that the reference bases of the twisted cohomology groups, defined in Eqs. (13) and (14), depend smoothly on the λ -regular character and can be smoothly extended at χ_{z_0} . Thus $\mathbb{T}_\lambda^K(\chi_{z_0})$ makes sense and is equal to limit (17).

2. *Computations part.* With the preceding fact in mind, Theorem 3 directly follows from the next lemma in which we compute the Reidemeister torsion at a reducible non-abelian representation.

Lemma 1 *Let φ be a reducible and non-abelian representation of $\Pi(K)$ in $\mathrm{SL}_2(\mathbb{C})$ such that*

$$\varphi: \mu \mapsto \begin{pmatrix} t & * \\ 0 & 1/t \end{pmatrix}, \quad t \in \mathbb{C}^*.$$

If we suppose that t is a simple zero of Δ_K (i.e. $\Delta_K(t) = 0$ and $\Delta'_K(t) \neq 0$), then

$$\mathbb{T}_\lambda^K(\varphi) = \frac{(t-1)(1/t-1)}{\Delta'_K(t)\Delta'_K(1/t)}. \quad (18)$$

Essentially, Formula (18) can be considered as an analogue of Milnor–Turaev’s formula for Reidemeister torsion at an abelian representation replacing abelian representations by appropriate reducible non-abelian ones and the Alexander polynomial by its derivative, see [14, Theorem 4], [22, Subsect. 1.2].

4 $\mathrm{SL}_2(\mathbb{C})$ -Chern–Simons invariant for torus knots

In the late 1980s, E. Witten considered a quantum field theory whose Lagrangian is the Chern–Simons functional. He argued that the Chern–Simons path integral on a three-manifold with an embedded link gives a (formal) three-dimensional interpretation of the Jones polynomial of links. Below we follow Kirk–Klassen’s approach for calculating the Chern–Simons invariant for manifolds with boundary.

Let M be a three-dimensional manifold. Let \mathcal{A} denote the space of $\mathfrak{sl}_2(\mathbb{C})$ -valued one-forms on M . The Chern–Simons functional $CS: \mathcal{A} \rightarrow \mathbb{C}$ is defined by

$$CS(A) = \frac{1}{8\pi^2} \int_M \mathrm{tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A).$$

In the case of a manifold with boundary this integral is not gauge invariant. Here we recall the definition of the Chern–Simons invariant for knot exteriors (and more generally for three-dimensional manifolds whose boundary consists of a two-dimensional torus) and explicitly compute it in the case of torus knots.

4.1 Chern–Simons invariant for knot exteriors

In this subsection we review the work of Kirk and Klassen [11] which allows us to compute the Chern–Simons invariant for torus knots.

Let M denote a three-dimensional manifold whose boundary is non-empty and consists of a single two-dimensional torus $T = \partial M$. Following some ideas coming

from Physics [21], the Chern–Simons invariant of M is considered in [11] as a lift of the restriction map induced by the inclusion $T \hookrightarrow M$:

$$\begin{array}{ccc}
 & & \mathcal{B}_T \\
 & \nearrow C_M & \downarrow \\
 X(M) & \longrightarrow & X(T)
 \end{array}$$

Here \mathcal{B}_T is a \mathbb{C}^* -bundle over $X(T)$ defined as follows. Let (μ, λ) be an oriented basis for $\pi_1(T)$. The map $t: X(T) \rightarrow \mathbb{C}^3, \rho \mapsto (\text{tr } \rho(\mu), \text{tr } \rho(\lambda), \text{tr } \rho(\mu\lambda))$ is an algebraic embedding of the character variety $X(T)$. Let $V(T)$ be the two-dimensional vector space (over \mathbb{C}):

$$V(T) = \text{Hom}(\pi_1(T); \mathbb{C}).$$

The map $V(T) \rightarrow X(T)$ defined by $v \mapsto (x \mapsto 2 \cos(2\pi v(x)))$ is a branched covering. The covering group G is isomorphic to a semi-direct product of $\mathbb{Z} \oplus \mathbb{Z}$ and $\mathbb{Z}/2$ with the presentation

$$G = \langle x, y, b \mid xyx^{-1}y^{-1} = bxbx = byby = b^2 = 1 \rangle.$$

Via the isomorphism $V(T) \rightarrow \mathbb{C}^2$ defined by

$$v \mapsto (v(\mu), v(\lambda))$$

the action of G on $V(T) \cong \mathbb{C}^2$ is as follows

$$x(\alpha, \beta) = (\alpha + 1, \beta), \quad y(\alpha, \beta) = (\alpha, \beta + 1), \quad b(\alpha, \beta) = (-\alpha, -\beta).$$

Now we extend the action of G to the the product $V(T) \times \mathbb{C}^*$ by the formulas:

$$x(\alpha, \beta; z) = (\alpha + 1, \beta; ze^{2\pi i\beta}) \tag{19}$$

$$y(\alpha, \beta; z) = (\alpha, \beta + 1; ze^{-2\pi i\alpha}) \tag{20}$$

$$b(\alpha, \beta; z) = (-\alpha, -\beta; z). \tag{21}$$

Thus, the quotient \mathbb{C}^* -bundle \mathcal{B}_T over $X(T)$ is defined by the formula

$$\mathcal{B}_T = (V(T) \times \mathbb{C}^*)/G.$$

Since \mathcal{B}_T is a quotient of $\mathbb{C}^2 \times \mathbb{C}^*$ we write $[\alpha, \beta; z]$ for equivalence classes, so for example

$$[\alpha, \beta; z] = [\alpha + m, \beta + n; ze^{2\pi i(m\beta - n\alpha)}].$$

The map $C_M : \rho \mapsto C_M(\rho) = [\gamma_\mu, \gamma_\lambda, e^{2\pi i C_{SM}(\rho)}]$ defines a lift of the restriction map induced by $T \hookrightarrow M$, see [11, Theorems 2.1, 3.2]. Here $(\gamma_\mu, \gamma_\lambda)$ denotes a lift in \mathbb{C}^2 of the restriction $\rho|_{\pi_1(T)}$.

One can compute C_M using the following Kirk–Klassen formula.

Theorem 4 ([11, Theorems 2.7, 3.2 and Corollary 2.6]) *Let M denote an oriented three-dimensional manifold whose boundary $\partial M = T$ consists of a two-dimensional torus. Let (μ, λ) denote an oriented basis for $\pi_1(T)$.*

1. *Let $\rho(t) : \pi_1(M) \rightarrow \text{SL}_2(\mathbb{C})$, $t \in [0, 1]$, be a path of representations. Let $(\gamma_\mu(t), \gamma_\lambda(t))$ denote a lift of $\rho(t)|_{\pi_1(T)}$ to \mathbb{C}^2 . Suppose*

$$C_M(\rho(t)) = [\gamma_\mu(t), \gamma_\lambda(t); z(t)]$$

for all t . Then

$$z(1) \cdot z(0)^{-1} = \exp \left(2\pi i \int_0^1 \gamma_\mu(t) \gamma'_\lambda(t) - \gamma'_\mu(t) \gamma_\lambda(t) dt \right). \tag{22}$$

Furthermore, if $\rho(1)$ is the trivial representation, then $z(1) = 1$.

2. *There is an inner product*

$$\langle \cdot, \cdot \rangle : \mathcal{B}_T \times \mathcal{B}_{-T} \rightarrow \mathbb{C}^* \tag{23}$$

given by taking the pair $([\gamma_\mu, \gamma_\lambda, z], [\gamma_\mu, \gamma_\lambda, w])$ to $z/w \in \mathbb{C}^*$.

Remark 10 One can observe that Theorem 4 is true not only for three-dimensional manifolds but also for some three-dimensional orbifolds. For example, for the orbifold $W = D^2/\mathbb{Z}_2 \times S^1$ where the generator of the group \mathbb{Z}_2 acts on the two-dimensional disk D^2 by Euclidian rotation thru the angle π , the boundary ∂W is the two-dimensional torus T . One has $\pi_1(W) = \mathbb{Z}_2 \times \mathbb{Z}$. If we identify the meridian μ and the longitude λ of the boundary with the generators of \mathbb{Z}_2 and \mathbb{Z} , respectively, then $X(W)$ is given by the equation $\chi(\mu) = \pm 2$. An easy application of Theorem 4 gives

$$C_W(\chi) = \left[n/2, \gamma_\lambda; e^{i\pi n \gamma_\lambda} \right]$$

for some integer n and a character $\chi \in X(W)$, where for odd n we have fixed the normalization by the condition $C_W(\chi_0) = [1/2, 0, 1]$ at the representation χ_0 with trivial longitude. One can easily check that this is a well-defined lift of $X(W) \rightarrow X(T)$ to the bundle \mathcal{B}_T .

4.2 The areas $A_\ell^\triangleright, A_\ell^\diamond$ and the Chern–Simons invariant for knot exteriors

The space $X(\partial M_K)$ has the canonical holomorphic symplectic form ω , and by using this additional structure we define a knot invariant as follows. One has a branched

covering mapping $\pi : \mathbb{C}^2 \rightarrow X(\partial M_K)$ given by the canonical projection to the orbit space of the group action $(x, y) \mapsto (\pm x + m, \pm y + n), (m, n) \in \mathbb{Z}^2$. One can choose a fundamental domain for this action given by $D = D_1 \cap D_2$, where $D_1 = \{(x, y) \in \mathbb{C}^2 \mid 0 < \Re x < 1/2\}$ and $D_2 = \{(x, y) \in \mathbb{C}^2 \mid |\Re y| < 1/2\}$. Thus, topologically the set $X(\partial M_K)$ is given by the quotient space \overline{D}/\sim with respect to the equivalence relation generated by $(is, y) \sim (-is, -y), (1/2 + is, y) \sim (1/2 - is, -y)$, and $(x, 1/2 + it) \sim (x, -1/2 + it)$, where $x, y \in \mathbb{C}, s, t \in \mathbb{R}$. The pullback of the holomorphic symplectic form is given by the explicit formula $\pi^*\omega = 2dx \wedge dy$.

Let $X_{\pm 2}(M_K)$ denote the finite sets of characters $\chi \in X^{\text{irr}}(M_K)$ such that $\chi(\mu) = \pm 2$, where μ is the meridian of K . Each of them is mapped to the set of branching points of $X(\partial M_K)$. One can observe that $\sharp(X_{\pm 2}(\mathcal{T}(p, q))) = (p - 1)(q - 1)/2$ and if K is a hyperbolic knot then the characters corresponding to the lift (from the group $\text{PSL}_2(\mathbb{C})$ to $\text{SL}_2(\mathbb{C})$) of the holonomy associated to the hyperbolic structure is in $X_{\pm 2}(M_K)$. Thus in all these cases, the sets $X_{\pm 2}(M_K)$ are non-empty.

Let \mathcal{A}_K be the set of paths in $X^{\text{nab}}(M_k)$ such that each $\gamma \in \mathcal{A}_k$ runs between a bifurcation point and a character in $X_{\pm 2}(M_k)$ and whose image in $X(\partial M_k)$ admits a (unique) lift to a path $\tilde{\gamma}$ in $\overline{D_1}$. For any $\gamma \in \mathcal{A}_K$ let $\chi_\gamma \in X_{\pm 2}(M_K)$ be the end point of γ . For each $\gamma \in \mathcal{A}_K$ we associate a triangle t_γ in $\overline{D_1}$ as follows. If $\chi_\gamma \in X_2(M_K)$, then t_γ is the triangle with sides $\tilde{\gamma}$ and two straight segments connecting the end points of $\tilde{\gamma}$ with the point $(0, 0)$. On the other hand, if $\chi_\gamma \in X_{-2}(M_K)$, then t_γ is the triangle with sides $\tilde{\gamma}$ and two straight segments connecting the end points of $\tilde{\gamma}$ with the point $(1/2, 0)$. See Fig. 1 for the case of ℓ th component of $X(\mathcal{M}(p, q))$. We define a function $cs_K : \mathcal{A}_K \rightarrow \mathbb{C}$ which associates to $\gamma \in \mathcal{A}_K$ the complex symplectic area of the projected triangle $\pi(t_\gamma)$. This function generalizes the Chern–Simons invariant defined in [11]. Suppose that K is the (p, q) torus knot. Let γ be such that $\chi_\gamma \in X_{-2}(M_K)$ and χ_γ is on the ℓ th component of $X(M_K)$, then $|cs_K(\gamma)| = A_\ell^\triangleright$.

The exterior of the knot M_K and the orbifold $W(K) = N(K)/\mathbb{Z}_2$, where \mathbb{Z}_2 acts by rotation on the first component of $N(K) = D^2 \times S^1$, have the same boundary. In $W(K)$ the meridian μ bounds a two-dimensional orbifold D^2/\mathbb{Z}_2 , hence each character $\chi' \in X(W(K))$ satisfies $\chi'(\mu) = \pm 2$. Let $\chi \in X(M_K)$ be such that χ and χ' coincide on the peripheral system of K , then $\chi \in X_{\pm 2}(M_K)$. We define on $X_{\pm 2}(M_K)$ a complex valued discrete function by

$$CS_K : X_{\pm 2}(M_K) \rightarrow \mathbb{C}^*, \quad CS_K(\chi) = \langle C_{M_K}(\chi), C_{W(K)}(\chi') \rangle, \tag{24}$$

where χ and χ' coincide on the peripheral system of K .

By construction CS_K is a knot invariant. In this paper we call this discrete function the *Chern–Simons invariant of the knot K* . Let $\gamma \in \mathcal{A}_K$ and let χ_γ be the end point of γ . We have

$$CS_K(\chi_\gamma) = e^{2\pi i cs_K(\gamma)}.$$

Suppose that $\gamma_1, \gamma_2 \in \mathcal{A}_K$ have coinciding end points, but start from two bifurcation points on one and the same connected components of $X(M_K)$. Then the two corresponding triangles have areas differing by an integer (since both of them give the

same $CS_K(\chi_\gamma)$. In the case of torus knots the difference of those areas is precisely the symplectic area A_ℓ^\diamond of the trapezoid in Fig. 1.

4.3 Computation of the Chern–Simons invariant for torus knots

Proposition 4 *Let $\rho : \Pi(p, q) \rightarrow \text{SL}_2(\mathbb{C})$ be a non-abelian representation whose character lies in the component of $X(\mathcal{M}(p, q))$ parametrized by (α, β) (see Theorem 2). If the matrix $\rho(\mu)$ is conjugate to*

$$\rho(\mu) \simeq \begin{pmatrix} e^{2\pi i \gamma_\mu} & * \\ 0 & e^{-2\pi i \gamma_\mu} \end{pmatrix}$$

then

$$C_{\mathcal{M}(p,q)}(\rho) = \left[\gamma_\mu, \frac{1}{2} - pq\gamma_\mu; \exp \left(2\pi i \left(\frac{(\beta ps + \varepsilon \alpha qr)^2}{4pq} - \frac{\gamma_\mu}{2} \right) \right) \right]. \tag{25}$$

Here $ps - qr = 1$ and the result does not depend on the choice of $\varepsilon \in \{\pm 1\}$.

The proof is based on Kirk–Klassen’s Theorem 4.

Proof (Proof of Proposition 4) Our computation is based on the fact that a non-abelian representation ρ of $\Pi(p, q)$ which lies in the component parametrized by (α, β) is connected to the trivial representation ϑ by a path of representations which contains the bifurcation point φ_{z_k} , where $z_k = i\pi \frac{k}{pq}$ with k satisfying conditions of Theorem (2). This path is divided into two distinct parts connected by φ_{z_k} :

1. the path $(\varphi_t)_{0 \leq t \leq \theta}$ of abelian representations which connects ϑ to φ_{z_k} ,
2. the path $(\rho_t)_{0 \leq t \leq 1}$ of non-abelian representations which connects $\rho = \rho_0$ to $\varphi_{z_k} = \rho_1$.

The computation is done in two steps.

1. First, applying the second part of Kirk–Klassen’s Theorem 4 to the abelian representation φ_{z_k} where $z_k = i\pi \frac{k}{pq}$, we get:

$$C_{\mathcal{M}(p,q)}(\varphi_{z_k}) = \left[\frac{k}{2pq}, 0; 1 \right]. \tag{26}$$

Observe that for any abelian representation φ one has $\varphi(\lambda) = \mathbf{1}$.

2. Now, we apply Kirk–Klassen’s Theorem 4 to the path $(\rho_t)_{0 \leq t \leq 1}$ of non-abelian representations. Suppose that $C_{\mathcal{M}(p,q)}(\rho_t) = [\gamma_\mu(t), \gamma_\lambda(t); z(t)]$. As $\lambda = a^p \mu^{-pq}$, we have $\gamma_\lambda(t) = \frac{1}{2} - pq\gamma_\mu(t)$ (because a^p is central in $\Pi(p, q)$ and thus $\rho_t(a^p) = -\mathbf{1}$) and thus

$$\gamma'_\lambda(t) = -pq\gamma'_\mu(t).$$

Moreover we have $\gamma_\lambda(1) = \frac{1-k}{2}$. Choosing appropriate representative for Formula (26), one has

$$C_{\mathcal{M}(p,q)}(\varphi_{z_k}) = \left[\frac{k}{2pq}, 0; 1 \right] = \left[\frac{k}{2pq}, \frac{1-k}{2}; \exp \left(-2\pi i \frac{k(1-k)}{2pq} \right) \right].$$

With this, Formula (22) gives us

$$z(1) \cdot z(0)^{-1} = \exp \left(-i\pi \int_0^1 \gamma'_\mu(t) dt \right) = \exp \left(2\pi i \left(\frac{\gamma_\mu}{2} - \frac{k}{4pq} \right) \right).$$

Thus,

$$z(0) = \exp \left(2\pi i \left(\frac{k^2}{4pq} - \frac{\gamma_\mu}{2} \right) \right)$$

which achieves the proof.

We are now ready to compute the discrete function

$$CS_{\mathcal{T}(p,q)} : X_{\pm 2}(\mathcal{M}(p, q)) \rightarrow \mathbb{C}^*$$

defined in Eq. (24). For each $1 \leq \ell \leq (p-1)(q-1)/2$, let χ_ℓ^\pm be the unique irreducible character of $\Pi(p, q)$ with $\chi_\ell^\pm(\mu) = \pm 2$ and which lies in the ℓ th connected component of $X^{\text{irr}}(\mathcal{M}(p, q))$. We have $X_{\pm 2}(\mathcal{M}(p, q)) = \{\chi_\ell^\pm \mid 1 \leq \ell \leq (p-1)(q-1)/2\}$.

Corollary 1 *The map $CS_{\mathcal{T}(p,q)}$ defined in Eq. (24) satisfies*

$$CS_{\mathcal{T}(p,q)}(\chi_\ell^+) = e^{i\pi \frac{k_\ell^- - 2}{2pq}} \quad \text{and} \quad CS_{\mathcal{T}(p,q)}(\chi_\ell^-) = e^{-i\pi \frac{(pq - k_\ell^+)^2}{2pq}}. \tag{27}$$

Here k_ℓ^\pm are the two integers which satisfy conditions of Subsect. 2.2.

Remark 11 We have

$$CS_{\mathcal{T}(p,q)}(\chi_\ell^+) = e^{i\pi \frac{k_\ell^{+2}}{2pq}}.$$

This is just the consequence of the fact that $A_\ell^\diamond = \frac{k_\ell^{+2} - k_\ell^{-2}}{4pq}$ is an integer (see Remark 3).

Remark 12 The Chern–Simons invariant for torus knots has also been discussed in papers [7, 8].

Proof (Proof of Corollary 1) The computation of $C_{W(\mathcal{T}(p,q))}(\rho)$ for any character of $\Pi(p, q)$ in $SL_2(\mathbb{C})$ is done in Remark 10. Now, applying the inner product (23) we get

$$CS_{\mathcal{T}(p,q)}(\chi_\ell^+) = \left\langle C_{\mathcal{M}(p,q)}(\chi_\ell^+), C_{W(\mathcal{T}(p,q))}(\chi_\ell^{+'}) \right\rangle = e^{i\pi \frac{k_\ell^- - 2}{2pq}}$$

and

$$CS_{\mathcal{T}(p,q)}(\chi_\ell^-) = \left\langle C_{\mathcal{M}(p,q)}(\chi_\ell^-), C_{W(\mathcal{T}(p,q))}(\chi_\ell^{-'}) \right\rangle = e^{-i\pi \frac{(pq - k_\ell^+)^2}{2pq}}.$$

Here χ_ℓ^\pm and $\chi_\ell^{\pm'}$ coincide on the peripheral system of the knot.

5 Review on the quantum hyperbolic invariant for torus knots

Using the relationship between the colored Jones polynomial and the quantum hyperbolic invariant, one has the following result.

Proposition 5 ([10, Lemma 2]) *Let p and q be two coprime integers. For the torus knot $\mathcal{T}(p, q)$ of type (p, q) , the quantum hyperbolic invariant has the following complex integral representation:*

$$2 \cdot \langle \mathcal{T}(p, q) \rangle_N = \left(\frac{pqN}{2} \right)^{3/2} \cdot e^{-\frac{i\pi}{2N} \left(\frac{p}{q} + \frac{q}{p} + \frac{N}{2} \right)} \cdot \int_{\mathcal{C}} e^{\pi pqN(z + \frac{i}{2}z^2)} z^2 \tau_{\mathcal{T}(p,q)}(z) dz. \tag{28}$$

Here the path of integration \mathcal{C} is the image of the real line under the mapping $\mathbb{R} \ni x \mapsto xe^{i\phi} \in \mathbb{C}$ where ϕ is to be chosen by the convergence condition.

Proposition 5 is the main ingredient to prove:

Proposition 6 ([10, Theorem]) *Let p and q be two coprime integers. For the torus knot $\mathcal{T}(p, q)$ of type (p, q) , the quantum hyperbolic invariant has the following asymptotic expansion at large N :*

$$e^{\frac{i\pi}{2N} \left(\frac{p}{q} + \frac{q}{q} \right)} \cdot \langle \mathcal{T}(p, q) \rangle_N = \sum_{k=1}^{pq-1} \langle \mathcal{T}(p, q) \rangle_N^{(k)} + \langle \mathcal{T}(p, q) \rangle_N^{(\infty)}. \tag{29}$$

Here

$$\langle \mathcal{T}(p, q) \rangle_N^{(k)} = \left(\frac{pqN}{2} \right)^{3/2} \cdot \frac{e^{-\frac{i\pi}{4}}}{2} \cdot \operatorname{Res}_{z=i\pi \frac{k}{pq}} \left(e^{\pi pqN(z + \frac{i}{2}z^2)} z^2 \tau_{\mathcal{T}(p,q)}(z) \right), \tag{30}$$

and

$$\langle \mathcal{T}(p, q) \rangle_N^{(\infty)} = \frac{i^{pqN}}{4} \sum_{n=0}^{\infty} \frac{a_n(K)}{n!} \left(\frac{i\pi}{2pqN} \right)^{n-1} \tag{31}$$

where $a_n(K) = \frac{\partial^{2n}(z\tau_K(z))}{\partial z^{2n}} \Big|_{z=0}$.

Formula (29) is our starting point in the proof of Theorem 1. Moreover a direct computation gives:

$$\begin{aligned} \langle \mathcal{T}(p, q) \rangle_N^{(k)} &= 2 \left(\frac{N}{2pq} \right)^{3/2} \cdot e^{\frac{i\pi}{4}} \cdot (-1)^{(N-1)k} \cdot e^{-\frac{i\pi k^2}{2pq}N} \cdot k^2 \\ &\cdot \sin(\pi k/p) \sin(\pi k/q). \end{aligned} \tag{32}$$

6 Proof of the Theorem 1

The proof is a direct computation of each part of equality (4) and combines Propositions 6 and 4.

Proof (Proof of the Theorem 1) For each $\ell \in \{1, \dots, (p-1)(q-1)/2\}$, the ℓ th connected component of $X^{\text{irr}}(\mathcal{M}(p, q))$ intersects the abelian one at the two bifurcation points $\varphi_{i\pi \frac{k_\ell^+}{pq}}$ and $\varphi_{i\pi \frac{k_\ell^-}{pq}}$. Here k_ℓ^\pm are the two integers which satisfy conditions of

Subsect. 2.2. We rewrite the sum $\sum_{k=1}^{pq-1} \langle \mathcal{T}(p, q) \rangle_N^{(k)}$, where $\langle \mathcal{T}(p, q) \rangle_N^{(k)}$ is defined in Eq. (32), as follows.

First, observe that $\langle \mathcal{T}(p, q) \rangle_N^{(k)} = 0$ for all integers k which are divisible by p or q . Write

$$\langle \mathcal{T}(p, q) \rangle_N^{(j)} = \Gamma_j \cdot \sin(\pi j/p) \sin(\pi j/q),$$

where

$$\Gamma_j = 2 \left(\frac{N}{2pq} \right)^{3/2} e^{\frac{i\pi}{4}} e^{-i\pi N \frac{j^2}{2pq}} (-1)^{(N-1)j} j^2.$$

Next, observe that the integers k_ℓ^\pm indexed by $\ell \in \{1, \dots, (p-1)(q-1)/2\}$ exhaust all the integers between 1 and $pq - 1$ which are mutually prime with pq . Thus, we have

$$\sum_{j=1}^{pq-1} \langle \mathcal{T}(p, q) \rangle_N^{(j)} = \sum_{\ell=1}^{(p-1)(q-1)/2} \langle \mathcal{T}(p, q) \rangle_N^{(k_\ell^-)} + \langle \mathcal{T}(p, q) \rangle_N^{(k_\ell^+)}.$$

It is easy to observe that (see Remarks 2, 3)

$$\sin(\pi k_\ell^+/p) \sin(\pi k_\ell^+/q) = -\sin(\pi k_\ell^-/p) \sin(\pi k_\ell^-/q).$$

As a consequence, we obtain

$$\sum_{j=1}^{pq-1} \langle \mathcal{J}(p, q) \rangle_N^{(j)} = \sum_{\ell=1}^{(p-1)(q-1)/2} (\Gamma_{k_\ell^+} - \Gamma_{k_\ell^-}) \cdot \sin(\pi k_\ell^+ / p) \sin(\pi k_\ell^+ / q).$$

Using the properties in Remark 3 (1), we have

$$\Gamma_{k_\ell^+} - \Gamma_{k_\ell^-} = 2 \left(\frac{N}{2pq} \right)^{3/2} e^{\frac{i\pi}{4}} e^{-i\pi N \frac{k_\ell^{+2}}{2pq}} (-1)^{(N-1)k_\ell^+} (k_\ell^{+2} - k_\ell^{-2}).$$

Thus,

$$\sum_{j=1}^{pq-1} \langle \mathcal{J}(p, q) \rangle_N^{(j)} = N^{3/2} e^{\frac{i\pi}{4}} \sqrt{\frac{pq}{2}} \cdot \Sigma$$

where

$$\begin{aligned} \Sigma = & \sum_{\ell=1}^{(p-1)(q-1)/2} \frac{(-1)^{(N-1)k_\ell^+}}{4pq} (k_\ell^{+2} - k_\ell^{-2}) \cdot e^{-i\pi N \frac{k_\ell^{+2}}{2pq}} \\ & \cdot \frac{4}{pq} \sin(\pi k_\ell^+ / p) \sin(\pi k_\ell^+ / q) \end{aligned} \tag{33}$$

Here is a geometric explanation of the quantities which appear in Eq. (33).

Using Theorem 3, one has

$$\sqrt{\left| \mathbb{T}_\lambda^{\mathcal{J}(p,q)}(\chi_\ell) \right|} = \frac{4}{pq} \sin(\pi k_\ell^+ / p) \sin(\pi k_\ell^+ / q) = (-1)^{k_\ell^+ - 1} 2 \operatorname{Res}_{z=i\pi \frac{k_\ell^+}{pq}} \tau_{\mathcal{J}(p,q)}(z).$$

Corollary 1 gives

$$e^{i\pi \frac{k_\ell^{+2}}{2pq}} = CS_{\mathcal{J}(p,q)}(\chi_\ell^+)$$

where χ_ℓ^+ lies in the ℓ th component of $X^{\text{irr}}(\mathcal{M}(p, q))$ and satisfies $\chi_\ell^+(\mu) = 2$ (see Corollary 1). Writing CS_ℓ instead of $CS_{\mathcal{J}(p,q)}(\chi_\ell^+)$ we arrive at the formula

$$\sum_{j=1}^{pq-1} \langle \mathcal{J}(p, q) \rangle_N^{(j)} = N^{3/2} e^{\frac{i\pi}{4}} \sqrt{\frac{pq}{2}} \cdot \Sigma'$$

where

$$\Sigma' = \sum_{\ell=1}^{(p-1)(q-1)/2} \frac{(-1)^{(N-1)k_\ell^+}}{4pq} (k_\ell^{+2} - k_\ell^{-2}) \cdot CS_\ell^{-N} \cdot \sqrt{\left| \mathbb{T}_\lambda^{\mathcal{J}(p,q)}(\chi_\ell) \right|}.$$

Now we compute the symplectic areas A_ℓ^\diamond and A_ℓ^\triangleright . Recall that A_ℓ^\diamond and A_ℓ^\triangleright are respectively the double areas of the trapezoid $P_\ell^- P_\ell^+ Q_\ell^+ Q_\ell^-$ and the triangle $P_\ell^+ R_\ell^+ O'$, see Fig. 1. So, one has

$$A_\ell^\diamond = \frac{k_\ell^{+2} - k_\ell^{-2}}{4pq} \quad \text{and} \quad A_\ell^\triangleright = \frac{(pq - k_\ell^+)^2}{4pq}.$$

Combining the two last formulas, we finally obtain

$$\sum_{j=1}^{pq-1} \langle \mathcal{J}(p, q) \rangle_N^{(j)} = N^{3/2} \sqrt{\frac{pq}{2}} \cdot \frac{e^{i\pi/4}}{i^{pqN}} \sum_{\ell=1}^{(p-1)(q-1)/2} \varepsilon_\ell \sqrt{\left| \mathbb{T}_\lambda^{\mathcal{J}(p,q)}(\chi_\ell) \right|} A_\ell^\diamond e^{-2\pi i N A_\ell^\triangleright},$$

where $\varepsilon_\ell = (-1)^{\lfloor k_\ell^+/p \rfloor + \lfloor k_\ell^+/q \rfloor}$, thus

$$\sum_{j=1}^{pq-1} \langle \mathcal{J}(p, q) \rangle_N^{(j)} = \sqrt{\frac{pq}{2}} \cdot \frac{e^{i\pi/4}}{i^{pqN}} \cdot N^{3/2} Z_N(\mathcal{J}(p, q))$$

which achieves the proof of formula (4).

We now prove that $a_n(K)$ is a finite type invariant for each n . One has

$$a_n(K) = \left. \frac{\partial^{2n}(z\tau_K(z))}{\partial z^{2n}} \right|_{z=0}, \quad \text{where } \tau_K(z) = \frac{2 \sinh(z)}{\Delta_K(e^{2z})}$$

As a consequence, $a_n(K)$ is a linear combination of products of coefficients of the Alexander polynomial $\Delta_K(t)$. Moreover, each coefficient of $\Delta_K(t)$ is a finite type invariant, a linear combination and a product of finite type invariants is also a finite type invariant. Thus $a_n(K)$ is of finite type.

We finish by a remark.

Remark 13 Let K be the $(2, q)$ torus knot. Each component of the non-abelian part of the character variety is parametrized by $\rho \mapsto \text{tr} \rho(\mu)$. The invariant $Z_N(K)$ can be written as a single integral on the $SU(2)$ -moduli space of the knot group as follows.

Let $\widehat{R}(\mathcal{M}(2, q)) = \text{Hom}^{\text{irr}}(\Pi(2, q); SU(2))/SO(3)$ be the $SU(2)$ -moduli space and let

$$CS_{\mathcal{J}(2,q)} : \widehat{R}(\mathcal{M}(2, q)) \rightarrow S^1$$

denote the map defined by $CS_{\mathcal{J}(2,q)}(\rho) = CS_\ell$ if ρ lies in the ℓ th component of $\widehat{R}(\mathcal{M}(2,q))$. Recall that $\widehat{R}(\mathcal{M}(2,q))$ is a one-dimensional smooth manifold (see [4]). One can prove the following formula:

$$Z_N(\mathcal{J}(2,q)) = \frac{1}{\pi} \int_{\widehat{R}(\mathcal{M}(2,q))} \frac{CS_{\mathcal{J}(2,q)}^{-N}}{\sqrt{|\mathbb{T}_\lambda^{\mathcal{J}(2,q)}|}} \omega^{\mathcal{J}(2,q)}. \quad (34)$$

Here $\omega^{\mathcal{J}(2,q)}$ denotes the one-volume form on $\widehat{R}(\mathcal{M}(2,q))$ defined in [4, Sect. 6] using the Reidemeister torsion. Formula (34) can be considered as an analogue for knots of the Witten formal integral for closed three-dimensional manifolds.

Acknowledgements The authors thank J.E. Andersen, S. Baseilhac, K. Hikami, V. Huynh Quang, T. Le, V. Turaev, Y. Yamaguchi and C. Weber for helpful discussions. They also thank the referee for his comments.

References

- Burde, G.: Darstellungen von Knotengruppen (German). *Math. Ann.* **173**, 24–33 (1967)
- Culler, M., Shalen, P.: Varieties of group representations and splittings of 3-manifolds. *Ann. Math.* **117**, 109–146 (1983)
- Rham, G. de : Introduction aux polynomes d'un nœud (French). *Enseign. Math.* **13**(2), 187–194 (1967)
- Dubois, J.: Non abelian Reidemeister torsion and volume form on the $SU(2)$ -representation space of knot groups. *Ann. Inst Fourier* **55**, 1685–1734 (2005)
- Dubois, J.: Non abelian twisted Reidemeister torsion for fibered knots. *Can. Math. Bull.* **49**, 55–71 (2006)
- Heusener, M., Porti, J., Suárez, E.: Deformations of reducible representations of 3-manifold groups into $SL_2(\mathbb{C})$. *J. Reine Angew. Math.* **530**, 191–227 (2001)
- Hikami, K.: Volume Conjecture and asymptotic expansion of q -series. *Exp. Math.* **12**, 319–337 (2003)
- Hikami, K., Kirillov, A.N.: Torus knot and minimal model. *Phys. Lett. B* **575**, 343–348 (2003)
- Kashaev, R.: The hyperbolic volume of knots from quantum dilogarithm. *Lett. Math. Phys.* **39**, 269–275 (1997)
- Kashaev, R., Tirkkonen, O.: Proof of the volume conjecture for torus knots. *J. Math. Sci. N. Y.* **115**, 2033–2036 (2003)
- Kirk, P., Klassen, E.: Chern–Simons invariants of 3-manifolds decomposed along tori and the circle bundle over the representation space of T^2 . *Commun. Math. Phys.* **153**, 521–557 (1993)
- Klassen, E.: Representations of knot groups in $SU(2)$. *Trans. Am. Math. Soc.* **326**, 795–828 (1991)
- Le, T.: Varieties of representations and their subvarieties of cohomology jumps for certain knot groups. *Russ. Acad. Sci. Sb. Math.* **78**, 187–209 (1994)
- Milnor, J.: A duality theorem for Reidemeister Torsion. *Ann. Math.* **76**, 134–147 (1962)
- Milnor, J.: Whitehead torsion. *Bull. Am. Math. Soc.* **72**, 358–426 (1966)
- Murakami, H., Murakami, J.: The colored Jones polynomials and the simplicial volume of a knot. *Acta Math.* **186**, 85–104 (2001)
- Murakami, H.: Asymptotic behaviors of the colored Jones polynomials of a torus knot. *Int. J. Math.* **15**, 547–555 (2004)
- Ohtsuki, T.: Problems on Invariants of Knots and 3-Manifolds. *Geom. Topol. Monogr.* **4**, 377–572 (2002)
- Porti, J.: Torsion de Reidemeister pour les variétés hyperboliques. *Mem. Am. Math. Soc.* **128**(612), $x + 139$ pp (1997)
- Riley, R.: Nonabelian representations of 2-bridge knot groups. *Quart. J. Math. Oxf.* **35**, 191–208 (1984)
- Ramadas, T., Singer, I., Weitsman, J.: Some comments on Chern–Simons gauge theory. *Comm. Math. Phys.* **126**, 409–420 (1989)
- Turaev, V.: Reidemeister torsion in knot theory, English version. *Russ. Math. Surv.* **41**, 119–182 (1986)
- Turaev, V.: Introduction to combinatorial torsions. Birkhäuser, Basel (2001)

24. Turaev, V.: Torsions of 3-dimensional manifolds. Birkhäuser, Basel (2002)
25. Yamaguchi, Y.: The limit values of the non-abelian twisted Reidemeister torsion associated to knots. Preprint arXiv:math.GT/0512277 (2005)
26. Zheng, H.: Proof of the volume conjecture for Whitehead double of tours knots. Preprint arXiv:math.GT/0508138 (2005)