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Integral Equations and Operator Theory

Anisotropic Operator Symbols Arising From Multivariate Jump Processes

Nils Reich

Abstract. It is shown that infinitesimal generators $\mathcal{A}$ of certain multivariate pure jump Lévy copula processes give rise to a class of anisotropic symbols that extends the well-known classes of pseudo differential operators of Hörmander-type. In addition, we provide minimal regularity convergence analysis for a sparse tensor product finite element approximation to solutions of the corresponding stationary Kolmogorov equations $\mathcal{A}u = f$. The computational complexity of the presented approximation scheme is essentially independent of the underlying state space dimension.

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1. Introduction

On $\mathbb{R}^n$, $n \geq 2$, consider the integrodifferential equation

$$\mathcal{A}u = f, \tag{1.1}$$

where $\mathcal{A}$ denotes an integrodifferential operator of anisotropic order $\underline{\alpha} \in \mathbb{R}^n$, i.e. $\mathcal{A} : H^{\underline{\alpha}}(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is continuous. Here $H^{\underline{s}}$, $\underline{s} \in \mathbb{R}^n$, denotes the anisotropic Sobolev space

$$H^{\underline{s}}(\mathbb{R}^n) = \left\{ f \in \mathcal{S}'(\mathbb{R}^n) \, : \, \left\| \sum_{i=1}^{n} (1 + \xi_i^2)^{s_i/2} \widehat{f} \right\|_{L^2(\mathbb{R}^n)} < \infty \right\}.$$

We assume that the operator $\mathcal{A}$ is a pseudo differential operator with symbol $p : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, i.e.

$$\mathcal{A}u(x) = \mathcal{A}_p u(x) := - \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} p(x,\xi) \widehat{u}(\xi) d\xi, \quad u \in \mathcal{S}(\mathbb{R}^n). \tag{1.2}$$

In [21, 39], it was shown that such integral operators occur as infinitesimal generators of certain Lévy copula processes X. In this case (1.1) can be regarded as the stationary part of the Kolmogorov equation of X. Such equations occur, for instance, in the field of asset pricing in multidimensional Lévy models as introduced in [21, 34, 39, 47].

In terms of Bessel potential spaces corresponding to a continuous negative definite reference function $\psi(\cdot)$, symbols arising from rather general stochastic processes have been studied in [19, 20, 26, 27, 30, 45]. For an overview, we refer to the monographs [31, 32, 33]. However, classical numerical analysis of (1.1) is based on a Sobolev space characterization of the operator $\mathcal{A}$. To this end, we shall see below that the infinitesimal generators of Lévy copula and certain Feller processes give rise to a new class of pseudo differential operators with symbols that extend the classes $S^m_{1,0}$, $m \in \mathbb{R}$, of Hörmander (cf. e.g. [28, 48]). The operators in this class act continuously on *anisotropic* Sobolev spaces and their symbols admit a more complex singularity structure than classical pseudo differential operators.

The structure of anisotropic symbols and their corresponding distributional integral kernels has been analyzed by many authors since the 1960s: Extending the fundamental results of [8, 9] that were obtained for homogeneous singular operators, in [17, 18] a symbolic calculus is constructed for certain anisotropic operators with kernels of mixed homogeneity, spectral asymptotics are considered in [3, 5, 43] and the references therein. Furthermore, for the closely related analysis of hypo- and multi-quasi-elliptic operators we refer to [1, 3, 4, 6, 22, 25, 42, 41]. Even though the focus of this work lies on classical Sobolev- and hence L^2-based results, note that a great number of $\mathcal{L}^p$-boundedness results for (different classes of) anisotropic integral operators can be found in [10, 16, 29, 40, 46] and the references there.

Finally, in order to obtain numerical solutions of (1.1) we shall also extend the numerical analysis of [7, 21, 24] to obtain a minimal regularity finite element discretization of (1.1) with essentially dimension independent convergence rates for the class of anisotropic operators under consideration. For related numerical analysis we also refer to [23, 49] and the references therein. In addition, the symbol estimates provide the basis for further numerical analysis such as wavelet compression techniques, see [37, 38].

The outline of this work is as follows:

In Section 2 we recall the fundamentals of Lévy copula processes and their characteristic exponents.

Section 3 provides the new classes of anisotropic symbols and some examples.

In Section 4 it is shown that symbols of infinitesimal generators of certain Lévy copula processes are indeed contained in these new symbol classes. These symbols are in general not contained in the classes of Hörmander-type.

Finally, in Section 5 we show that the (stationary) Kolmogorov equations for operators with such anisotropic symbols can be discretized very efficiently using a wavelet finite element scheme. Based on the symbol estimates of the previous sections, a priori convergence analysis is provided.

2. Motivation: Infinitesimal generators of Lévy copula processes

Based on [21, 34, 47], in this Section we briefly introduce Lévy copula processes and characterize their infinitesimal generators. Recall that a stochastic process $L = (L_t)_{t\geq 0}$ with state space $\mathbb{R}^n$ and $L_0 = 0$ *a.s.* is a Lévy process if it has independent increments, is temporally homogeneous and stochastically continuous.

The characteristic function Φ_L and the characteristic exponent ψ^L of L are defined by

$$\Phi_L(\xi) = \exp(-t\psi^L(\xi)) = \mathbb{E}(\exp(i\langle \xi, L_t\rangle)), \qquad \xi \in \mathbb{R}^n, \quad t > 0.$$

The characteristic exponent $\psi^L(\xi)$ is also called Lévy *symbol*. The infinitesimal generator $\mathcal{A}$ of L and the associated bilinear form $\mathcal{E}(\cdot,\cdot)$ are given by

$$\mathcal{A}u(x) = -\int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle}\psi^L(\xi)\widehat{u}(\xi)d\xi, \quad u \in C_0^\infty(\mathbb{R}^n), \tag{2.1}$$

$$\mathcal{E}(u,v) = \langle \mathcal{A}u, v\rangle = -(2\pi)^n \int_{\mathbb{R}^n} \psi^L(\xi)\widehat{u}(\xi)\overline{\widehat{v}(\xi)}d\xi, \quad u,v \in \mathcal{S}(\mathbb{R}^n). \tag{2.2}$$

Furthermore, the characteristic exponent ψ^L admits the *Lévy-Khinchin representation*

$$\psi^L(\xi) = i\langle \gamma, \xi\rangle + Q(\xi) + \int_{\mathbb{R}^n\setminus\{0\}} (1 - e^{i\langle \xi, x\rangle} + \frac{i\langle \xi, x\rangle}{1+|z|^2})\nu(dx), \tag{2.3}$$

where $Q(\xi)$ denotes the quadratic form $\frac{1}{2}\xi^\top \mathbf{Q}\xi$ with a symmetric, nonnegative definite matrix $\mathbf{Q}$, a drift vector $\gamma \in \mathbb{R}^n$ and the Lévy measure $\nu(dx)$ which satisfies

$$\int_{\mathbb{R}^n} (1 \wedge |x|^2)\nu(dx) < \infty. \tag{2.4}$$

Any Lévy process L is completely determined by its *characteristic triple* $(\mathbf{Q}, \gamma, \nu)$ in (2.3). We speak of a *pure jump* Lévy process if $\mathbf{Q} = 0$ and $\gamma = 0$.

We shall now define a pure jump Lévy copula process. It is denoted by X: For each $i = 1,\dots,n$ the i-th marginal Lévy measure of X is given by $\nu_i(dx_i) = k_i^{\beta_i}(x_i)\,dx_i$ with densities $k_i^{\beta_i} : \mathbb{R}\setminus\{0\} \to \mathbb{R}$. These densities are defined by

$$k_i^{\beta_i}(x_i) = c_i \frac{e^{-\beta_i|x_i|}}{|x_i|^{1+\alpha_i}}, \tag{2.5}$$

where $0 < \alpha_1,\dots,\alpha_n < 2$ and $\beta_1,\dots,\beta_n \in \mathbb{R}_{\geq 0}$ are governing the Lévy densities' tail behavior and $c_i > 0$ are constants. The strongest singularity of all marginal Lévy measures is given by

$$\overline{\alpha} := |\underline{\alpha}|_\infty = \max\{\alpha_i : i = 1,\dots,n\} < 2. \tag{2.6}$$

To characterize the dependence among the margins, let $F : \overline{\mathbb{R}^n} \to \overline{\mathbb{R}}$ be a Lévy copula as defined in [21, 34] that is homogeneous of order 1, i.e. $F(t\xi_1,\dots,t\xi_n) = tF(\xi_1,\dots,\xi_n)$ for all $t > 0$ and $\xi \in \mathbb{R}^n$.

By Sklar's Theorem, [34, Theorem 3.6], we know that if the partial derivatives $\partial_1 \dots \partial_n F$ exist in a distributional sense, then one can compute the Lévy density of the multivariate Lévy copula process by differentiation as follows:

$$\nu(dx_1, \dots, dx_n) = [\partial_1 \dots \partial_n F]\,(U_1(x_1), \dots, U_n(x_n))\nu_1(dx_1) \dots \nu_n(dx_n), \qquad (2.7)$$

where $\nu_1(dx_1), \dots, \nu_n(dx_n)$ are the marginal Lévy measures defined above and U_i, $i = 1, \dots, n$, denote the corresponding marginal tail integral

$$U_i(x_i) = \begin{cases} \nu_i([x_i, \infty)), & \text{if } x_i > 0, \\ -\nu_i((-\infty, x_i]), & \text{if } x_i < 0. \end{cases}$$

Herewith, one obtains

$$\nu(dx_1, \dots, dx_n) = [\partial_1 \dots \partial_n F]\,(U_1(x_1), \dots, U_n(x_n))k_1^{\beta_1}(x_1) \dots k_n^{\beta_n}(x_n)\,dx_1 \dots dx_n, \qquad (2.8)$$

and this can be written as

$$\nu(dx_1, \dots, dx_n) = k^{\underline{\beta}}(x_1, \dots, x_n)\,dx_1 \dots dx_n, \qquad (2.9)$$

with $\underline{\beta} = (\beta_1, \dots, \beta_n)$. To define the copula process X we specify its characteristic exponent using the Lévy-Khinchin representation (2.3). Since we are interested in pure jump processes, the characteristic exponent ψ^X of X is given by

$$\begin{aligned} \psi^X(\xi) &= \int_{\mathbb{R}^n} (1 - e^{i\langle \xi, y\rangle} + \frac{i\langle \xi, y\rangle}{1 + |y|^2})\nu(dy) \\ &= \int_{\mathbb{R}^n} (1 - e^{i\langle \xi, y\rangle} + \frac{i\langle \xi, y\rangle}{1 + |y|^2})k^{\underline{\beta}}(y)dy, \end{aligned} \qquad (2.10)$$

with $k^{\underline{\beta}}$ as in (2.8) and (2.9). Herewith the Lévy copula process X is completely determined (see e.g. [44, Section 2.11]).

Definition 2.1. The Lévy copula process X is said to have *$\underline{\alpha}$-stable margins* if its marginal Lévy densities in (2.5) are of the form

$$k_i^{\beta_i}(x_i) = c_i \frac{1}{|x_i|^{1+\alpha_i}}, \qquad \text{for all } i = 1, \dots, n,$$

i.e. $\beta_1 = \dots = \beta_n = 0$ in (2.5). If $\beta_i > 0$ for all $i = 1, \dots, n$ then the Lévy copula process X is said to have *tempered stable margins.*

Lemma 2.2. *For any Lévy copula process X with marginal Lévy densities as in* (2.5) *there holds*

$$\psi^X(\xi) = \int_{\mathbb{R}^n} (1 - \cos\langle \xi, y\rangle)k^{\underline{\beta}}(y)dy. \qquad (2.11)$$

Proof. The symmetry of (2.5) implies that the density $k^{\underline{\beta}}$ is symmetric with respect to each coordinate axis. A simple change of coordinates in (2.10) implies that $\psi^X = \overline{\psi^X}$, i.e. ψ^X is real-valued. Thus, the result follows from [31, Corollary 3.7.9]. □

Since, by (2.11), the characteristic exponent ψ^X is real-valued it obviously satisfies the so-called sector condition (cf. e.g. [31]). From [2, Theorem 3.7] one therefore infers that $\mathcal{E}(\cdot,\cdot)$ defined in (2.2) is in fact a (translation invariant) Dirichlet form. In the important case that X has $\underline{\alpha}$-stable margins, i.e. $\beta_i = 0$ for all $i = 1, \dots, n$ in (2.5), the domain $D(\mathcal{E})$ of the Dirichlet form $\mathcal{E}(\cdot,\cdot)$ is well known:

Proposition 2.3. *The domain $\mathcal{D}(\mathcal{E})$ of the Dirichlet form associated to the generator of a Lévy copula process with $\underline{\alpha}$-stable margins can be identified with the anisotropic space $H^{\underline{\alpha}/2}(\mathbb{R}^n)$ with $\underline{\alpha} = (\alpha_1, \dots, \alpha_n)$ as in* (2.5).

Proof. [21, Theorem 3.7]. □

From Proposition 2.3 one infers

Corollary 2.4. *The domain $\mathcal{D}(\mathcal{A})$ of the infinitesimal generator of a Lévy copula process X with $\underline{\alpha}$-stable margins can be identified with $H^{\underline{\alpha}}(\mathbb{R}^n)$.*

We conclude this section by an example of a Lévy copula that shall be of reference throughout this work:

Example. The cardinal example for our purposes is the Clayton family of Lévy copulas taken from [34, Example 5.2]: Let $n \geq 2$. For $\theta > 0$, the function F_θ defined as

$$F_\theta(u_1, \dots, u_n) = 2^{2-n} \left(\sum_{i=1}^{n} |u_i|^{-\theta} \right)^{-1/\theta} \left(\eta 1_{\{u_1 \cdots u_n \geq 0\}} - (1-\eta) 1_{\{u_1 \cdots u_n < 0\}} \right), \tag{2.12}$$

defines a two parameter family of Lévy copulas which resembles the Clayton family of ordinary copulas. It is a Lévy copula homogeneous of order 1, for any $\theta > 0$ and any $\eta \in [0, 1]$.

We shall frequently write $a \lesssim b$ to express that a is bounded by a constant multiple of b, uniformly with respect to all parameters on which a and b may depend. Then $a \sim b$ means $a \lesssim b$ and $b \lesssim a$.

3. Anisotropic operators and their symbol classes

Recall that for any symbol $p : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, the corresponding operator $\mathcal{A}_p$ is defined by

$$\mathcal{A}_p u(x) = -\int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} p(x,\xi) \widehat{u}(\xi) d\xi, \quad u \in \mathcal{S}(\mathbb{R}^n). \tag{3.1}$$

Furthermore, denote the axes in $\mathbb{R}^n$ by $\Lambda := \{x \in \mathbb{R}^n : x_i = 0 \text{ for some } i \in \{1, \dots, n\}\}$. Herewith we can define a suitable class of anisotropic symbols and corresponding operators.

Definition 3.1. A function $p : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is called a symbol in class $\Gamma^{\underline{\alpha}}(\mathbb{R}^n)$, $\underline{\alpha} \in \mathbb{R}^n$, if $p(\cdot, \xi) \in C^\infty(\mathbb{R}^n)$ for all $\xi \in \mathbb{R}^n$, $p(x, \cdot) \in C^\infty(\mathbb{R}^n \backslash \Lambda) \cap C(\mathbb{R}^n)$ for all $x \in \mathbb{R}^n$, and for any $\underline{\tau}, \underline{\tau}' \in \mathbb{N}_0^n$ there holds

$$\left|\partial_x^{\underline{\tau}'} \partial_\xi^{\underline{\tau}} p(x,\xi)\right| \lesssim \prod_{i \in \mathcal{I}_{\underline{\tau}}} |\xi_i|^{\alpha_i - \tau_i} \cdot \sum_{k \notin \mathcal{I}_{\underline{\tau}}} (1 + |\xi_k|^2)^{\frac{\alpha_k}{2}}, \quad \text{for all } \ x, \xi \in \mathbb{R}^n, \tag{3.2}$$

where we set $\mathcal{I}_{\underline{\tau}} := \{i : \tau_i > 0\}$. The multiindex $\underline{\alpha}$ is called the (anisotropic) order of the symbol p and the operator $\mathcal{A}_p$.

Some possible realizations of operators $\mathcal{A}$ with symbols $p \in \Gamma^{\underline{\alpha}}(\mathbb{R}^n)$ are:

Example. If for any $\underline{\tau} \in \mathbb{N}_0^n$ the function $p \in C^\infty(\mathbb{R}^n \backslash \Lambda) \cap C(\mathbb{R}^n)$ satisfies

$$\left|\partial_\xi^{\underline{\tau}} p(\xi)\right| \lesssim \sum_{i=1}^n (1 + |\xi_i|^2)^{\frac{\alpha_i - \tau_i}{2}}, \quad \text{for all } \xi \in \mathbb{R}^n,$$

then $p \in \Gamma^{\underline{\alpha}}(\mathbb{R}^n)$ and $\mathcal{A}_p$ is admissible in this setting.

Example. Consider a symbol $p : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ in the Hörmander class $S_{1,0}^\alpha$ with non-negative order α, i.e. there exists some $\alpha \in \mathbb{R}_{\geq 0}$ such that for all $\underline{\tau} \in \mathbb{N}_0^n$ there holds

$$\left|\partial_\xi^{\underline{\tau}} p(\xi)\right| \lesssim (1 + |\xi|^2)^{\frac{\alpha - |\underline{\tau}|}{2}}, \quad \text{for all } \xi \in \mathbb{R}^n. \tag{3.3}$$

Then $p \in \Gamma^{\underline{\alpha}}(\mathbb{R}^n)$ with $\alpha_1 = \ldots = \alpha_n = \alpha$. To see this, one may use that for $\underline{\tau} \in \mathbb{N}_0^n$ there holds

$$\prod_{i=1}^n \left(1 + |\xi_i|^2\right)^{\frac{\tau_i}{2}} \leq \prod_{i=1}^n \left(1 + \sum_{j=1}^n |\xi_j|^2\right)^{\frac{\tau_i}{2}} = \left(1 + |\xi|^2\right)^{\frac{|\underline{\tau}|}{2}},$$

and thus

$$\left(1 + |\xi|^2\right)^{-\frac{|\underline{\tau}|}{2}} \leq \prod_{i=1}^n \left(1 + |\xi_i|^2\right)^{-\frac{\tau_i}{2}}. \tag{3.4}$$

Furthermore,

$$\left(1 + |\xi|^2\right)^{\frac{\alpha}{2}} \lesssim \left(\sum_{i=1}^n \left(1 + |\xi_i|^2\right)\right)^{\frac{\alpha}{2}} \lesssim \sum_{i=1}^n \left(1 + |\xi_i|^2\right)^{\frac{\alpha}{2}}, \tag{3.5}$$

since $\alpha \geq 0$. Clearly, (3.4) and (3.5) imply that (3.2) holds for any symbol $p \in C^\infty(\mathbb{R}^n)$ that satisfies (3.3). Note that this statement does not remain true if $\alpha < 0$ in (3.3).

Example. Also, symbols of the following structure belong to $\Gamma^{\underline{\alpha}}(\mathbb{R}^n)$ with suitable $\underline{\alpha} \in \mathbb{R}^n$:

$$p(x,\xi) = \sum_{j=1}^M b_j(x) \psi_j(\xi),$$

for some $M \in \mathbb{N}$. Here it is assumed that each $\psi_j : \mathbb{R}^n \to \mathbb{R}$ satisfies (3.2). The functions $b_j : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ are assumed to be C^∞-functions with bounded derivatives. Note that similar symbols have already been studied in terms of the symbol classes $S_{\varrho}^{m,\psi}$ of [27], see e.g. [26, 30].

It is straightforward to see that if a symbol $p : \mathbb{R}^n \to \mathbb{R}^n$ is independent of the state variable x, then the order $\underline{\alpha} \in \mathbb{R}^n$ of $p \in \Gamma^{\underline{\alpha}}(\mathbb{R}^n)$ has a natural interpretation in terms of mapping properties of the corresponding bilinear form $\mathcal{E}(u,v) := \langle \mathcal{A}_p u, v\rangle$:

Lemma 3.2. *Let $p \in \Gamma^{\underline{\alpha}}(\mathbb{R}^n)$ be independent of x and let $\mathcal{A}_p$ be the corresponding pseudo differential operator. Then the bilinear form $\mathcal{E}(\cdot,\cdot) = \langle \mathcal{A}_p\cdot,\cdot\rangle$ corresponding to $\mathcal{A}_p$ acts continuously on the anisotropic space $H^{\underline{\alpha}/2}(\mathbb{R}^n)$, i.e. there exists some constant $c > 0$ such that*

$$|\mathcal{E}(u,v)| \leq c\|u\|_{H^{\underline{\alpha}/2}(\mathbb{R}^n)}\|v\|_{H^{\underline{\alpha}/2}(\mathbb{R}^n)}, \qquad \textit{for all } u,v \in H^{\underline{\alpha}/2}(\mathbb{R}^n). \tag{3.6}$$

Proof. For $u, v \in H^{\underline{\alpha}/2}(\mathbb{R}^n)$ there holds

$$\mathcal{E}(u,v) = (2\pi)^n \int_{\mathbb{R}^n} p(\xi)\widehat{u}(\xi)\overline{\widehat{v}(x)}d\xi.$$

Thus, by (3.2), the Cauchy-Schwarz inequality yields

$$\begin{aligned}
&(2\pi)^{-n}\,|\mathcal{E}(u,v)| \\
&= \int_{\mathbb{R}^n} \sum_{k=1}^n (1+|\xi_k|^2)^{\alpha_k/2}\left|\widehat{u}(\xi)\overline{\widehat{v}(x)}\right| d\xi \\
&\leq \left(\int_{\mathbb{R}^n} \sum_{k=1}^n (1+|\xi_k|^2)^{\alpha_k/2}\,|\widehat{u}(\xi)|^2\,d\xi\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^n} \sum_{k=1}^n (1+|\xi_k|^2)^{\alpha_k/2}\,|\widehat{v}(\xi)|^2\,d\xi\right)^{\frac{1}{2}} \\
&= \|u\|_{H^{\underline{\alpha}/2}(\mathbb{R}^n)}\|v\|_{H^{\underline{\alpha}/2}(\mathbb{R}^n)}.
\end{aligned}$$

□

From Lemma 3.2 one immediately infers

Corollary 3.3. *Let $p \in \Gamma^{\underline{\alpha}}(\mathbb{R}^n)$ be independent of x and let $\mathcal{A}_p$ be the corresponding pseudo differential operator. Then $\mathcal{A}_p$ maps the anisotropic space $H^{\underline{\alpha}}(\mathbb{R}^n)$ continuously into $L^2(\mathbb{R}^n)$, i.e. there exists some constant $c' > 0$ such that*

$$\|\mathcal{A}_p u\|_{L^2(\mathbb{R}^n)} \leq c'\|u\|_{H^{\underline{\alpha}}(\mathbb{R}^n)}, \qquad \textit{for all } u \in H^{\underline{\alpha}}(\mathbb{R}^n).$$

Remark 3.4. In order to prove the continuity of general operators $\mathcal{A}_p$, with x-dependent symbol $p \in \Gamma^{\underline{\alpha}}(\mathbb{R}^n)$, further smoothness assumptions on p are required. For instance, the Calderón-Vaillancourt Theorem can be employed to obtain the desired estimates if the partial derivatives $\partial_x^{\underline{\tau}'}\partial_{\xi}^{\underline{\tau}}p$, $|\underline{\tau}'|, |\underline{\tau}| \leq 3$, exist and are continuous on the whole $\mathbb{R}^n \times \mathbb{R}^n$, see e.g. [32, Theorem 2.5.3]. However, since in this work we are mainly interested in symbols arising from Lévy processes (which are stationary) we omit such considerations here.

4. Anisotropic symbol estimates

In this section, we prove anisotropic symbol estimates for the characteristic exponent $\psi^X : \mathbb{R}^n \to \mathbb{R}$ of a Lévy copula process defined by (2.10). We will see that indeed $\psi^X \in \Gamma^{\underline{\alpha}}(\mathbb{R}^n)$ with α_i, $i = 1, \ldots, n$, given by (2.5).

4.1. Symbol estimates for processes with stable margins

At first, we consider the generator $\mathcal{A}$ of a Lévy copula process X^0 with $\underline{\alpha}$-stable margins. Its symbol is denoted by ψ^{X^0}. The following two lemmas provide the necessary estimates:

Lemma 4.1. *There holds,*

$$\psi^{X^0}(\xi_1, \ldots, \xi_n) \lesssim \sum_{i=1}^{n} (1 + |\xi_i|^2)^{\frac{\alpha_i}{2}}, \quad \text{for all } (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n.$$

Proof. By [21, Theorem 3.3], $\psi^{X^0} : \mathbb{R}^n \to \mathbb{R}$ is an anisotropic distance function such that for any $t > 0$,

$$\psi^{X^0}(t^{\frac{1}{\alpha_1}}\xi_1, \ldots, t^{\frac{1}{\alpha_n}}\xi_n) = t \cdot \psi^{X^0}(\xi_1, \ldots, \xi_n), \quad \text{for all } \xi \in \mathbb{R}^n. \tag{4.1}$$

Since all anisotropic distance functions of the same homogeneity are equivalent,

$$\psi^{X^0}(\xi_1, \ldots, \xi_n) \sim |\xi_1|^{\alpha_1} + \ldots + |\xi_n|^{\alpha_n},$$

and the result follows. □

To state the following lemma, recall that for $\underline{\tau} \in \mathbb{N}_0^n$ we denote

$$\mathcal{I}_{\underline{\tau}} := \{i \in \{1, \ldots, n\} : \tau_i > 0\}, \tag{4.2}$$

and let S^{n-1} be the unit sphere in $\mathbb{R}^n$.

Lemma 4.2. *Let $\underline{\tau} \in \mathbb{N}_0^n$. Suppose there exists some constant $c > 0$ such that*

$$\left|\partial_\xi^{\underline{\tau}} \psi^{X^0}(\xi)\right| \leq c \cdot \prod_{i \in \mathcal{I}_{\underline{\tau}}} |\xi_i|^{\alpha_i - \tau_i} \cdot \sum_{k \notin \mathcal{I}_{\underline{\tau}}} (1 + |\xi_k|^2)^{\frac{\alpha_k}{2}}, \quad \text{for all } \xi \in S^{n-1}. \tag{4.3}$$

Then there holds,

$$\left|\partial_\xi^{\underline{\tau}} \psi^{X^0}(\xi)\right| \lesssim \prod_{i \in \mathcal{I}_{\underline{\tau}}} |\xi_i|^{\alpha_i - \tau_i} \cdot \sum_{k \notin \mathcal{I}_{\underline{\tau}}} (1 + |\xi_k|^2)^{\frac{\alpha_k}{2}}, \tag{4.4}$$

for all $\xi \in \mathbb{R}^n$ such that $|\xi_i| \geq 1$ if $i \in \mathcal{I}_{\underline{\tau}}$.

Proof. Without loss of generality one may assume that $\tau_i \geq 1$ for at least one $i \in \{1, \ldots, n\}$. Otherwise, the claim in (4.4) coincides with Lemma 4.1. By differentiation of (4.1) one obtains,

$$\left|\partial_\xi^{\underline{\tau}} \psi^{X^0}(\xi)\right| = t^{\frac{t_1}{\alpha_1} + \ldots + \frac{t_n}{\alpha_n} - 1} \left|\partial_\xi^{\underline{\tau}} \psi^{X^0}(t^{\frac{1}{\alpha_1}}\xi_1, \ldots, t^{\frac{1}{\alpha_n}}\xi_n)\right|, \quad t > 0, \xi \in \mathbb{R}^n.$$

By [15, Lemma 2.1, (iv)], the mapping $t \to |(t^{\frac{1}{\alpha_1}}\xi_1, \ldots, t^{\frac{1}{\alpha_n}}\xi_n)|$, $\xi \neq 0$, maps $(0, \infty)$ onto itself. Thus, one can choose $t = t(\xi)$, such that

$$|(t^{\frac{1}{\alpha_1}}\xi_1, \ldots, t^{\frac{1}{\alpha_n}}\xi_n)| = 1.$$

By (4.3) one obtains

$$\begin{aligned}
&\left|\partial_\xi^{\underline{\tau}}\psi^{X^0}(\xi)\right| \\
&\leq c \cdot t^{\frac{t_1}{\alpha_1}+\ldots+\frac{t_n}{\alpha_n}-1} \cdot \prod_{i \in \mathcal{I}_{\underline{\tau}}} |t^{\frac{1}{\alpha_i}}\xi_i|^{\alpha_i - \tau_i} \cdot \sum_{k \notin \mathcal{I}_{\underline{\tau}}} (1 + |t^{\frac{1}{\alpha_k}}\xi_k|^2)^{\frac{\alpha_k}{2}} \\
&\leq c \cdot t^{\frac{t_1}{\alpha_1}+\ldots+\frac{t_n}{\alpha_n}} t^{-1} \cdot \prod_{i \in \mathcal{I}_{\underline{\tau}}} \left(t\,|\xi_i|^{\alpha_i - \tau_i} t^{-\frac{\tau_i}{\alpha_i}}\right) \cdot \sum_{k \notin \mathcal{I}_{\underline{\tau}}} (1 + |t^{\frac{1}{\alpha_k}}\xi_k|^2)^{\frac{\alpha_k}{2}} \\
&= c \cdot t^{|\mathcal{I}_{\underline{\tau}}|-1} \cdot \prod_{i \in \mathcal{I}_{\underline{\tau}}} |\xi_i|^{\alpha_i - \tau_i} \cdot \sum_{k \notin \mathcal{I}_{\underline{\tau}}} (1 + |t^{\frac{1}{\alpha_k}}\xi_k|^2)^{\frac{\alpha_k}{2}}.
\end{aligned}$$

Since there exists some $i \in \{1, \ldots, n\}$ with $|\xi_i| \geq 1$, $t^{\frac{2}{\alpha_1}}\xi_1^2 + \ldots + t^{\frac{2}{\alpha_n}}\xi_n^2 = 1$ implies $t^{\frac{1}{\alpha_i}} \leq \frac{1}{|\xi_i|} \leq 1$. Thus, $t \leq 1$ and the result follows. □

Remark 4.3. The technical assumption (4.3) is satisfied by all common examples of anisotropic distance functions (cf. e.g. [15]). Furthermore, using the Lévy-Khinchin representation (2.11) it can be shown that (4.3) is satisfied if the underlying Lévy copula is of Clayton-type as in (2.12). Nonetheless, to prove the validity of (4.3) in general, one requires further analytical properties of the Lévy copula.

The combination of Lemmas 4.1 and 4.2 implies $\psi^{X^0} \in \Gamma^{\underline{\alpha}}(\mathbb{R}^n)$ with α_i, $i = 1, \ldots, n$, given by (2.5). In the following section, we extend this result to the case of tempered stable margins.

4.2. Symbol estimates for processes with tempered stable margins

Let X be a Lévy copula process as defined in Section 2. Suppose that the marginal densities of X are given by (2.5) with $\beta_1, \ldots, \beta_n > 0$. The structure of the density $k^{\underline{\beta}}$ of X is illustrated in Figure 1. Throughout, we denote by $\psi^{X^0} : \mathbb{R}^n \to \mathbb{R}$ the symbol of a Lévy copula process X^0 with $\underline{\alpha}$-stable margins corresponding to X. In particular, X and X^0 share the same $\alpha_1, \ldots, \alpha_n$ in (2.5). The Lévy density of X^0 is denoted by $k^0 : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$.

Denote by $k^{\underline{\beta}} : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ the Lévy density of X defined in Section 2. Since for any $\underline{\tau} \in \mathbb{N}_0^n$ there holds $(1 - \cos\langle x, \xi\rangle)\partial_x^{\underline{\tau}}(x_1^{\tau_1} \ldots x_n^{\tau_n} k^{\underline{\beta}}(x)) \in L^1(\mathbb{R}^n)$ for all $\xi \in \mathbb{R}^n$, one may apply integration by parts to obtain,

$$\begin{aligned}
\left|\xi_1^{\tau_1} \ldots \xi_n^{\tau_n} \partial_\xi^{\underline{\tau}} \psi^X(\xi)\right| &= \left|\xi_1^{\tau_1} \ldots \xi_n^{\tau_n} \int_{\mathbb{R}^n} f(\langle x, \xi\rangle) x_1^{\tau_1} \ldots x_n^{\tau_n} k^{\underline{\beta}}(x) dx\right| \\
&= \left|\int_{\mathbb{R}^n} (1 - \cos\langle x, \xi\rangle)\partial_x^{\underline{\tau}}\left(x_1^{\tau_1} \ldots x_n^{\tau_n} k^{\underline{\beta}}(x)\right) dx\right|,
\end{aligned} \tag{4.5}$$

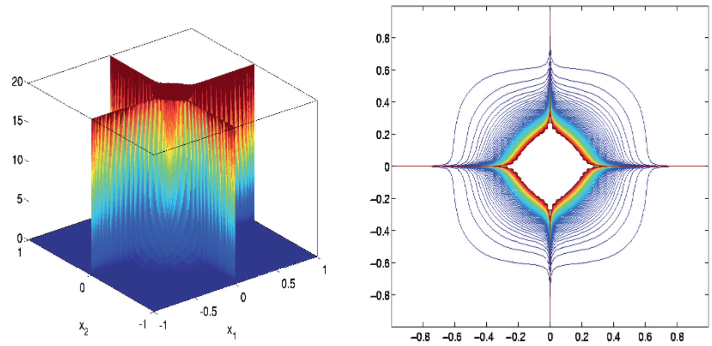

FIGURE 1. Illustration of a two-dimensional density $k^{\underline{\beta}}$ under a Clayton-type Lévy copula with marginal densities defined by (2.5) with $\alpha_1 = 1$, $\alpha_2 = 1$, $\beta_1 = 2$, $\beta_2 = 2$.

where f is either cos or sin depending on whether $|\underline{\tau}|$ is even or odd. By the Riemann-Lebesgue Lemma, the singularity structure (and strength) of

$$k^{\underline{\beta}}_{\underline{\tau}}(x) := \partial^{\underline{\tau}}_{x}(x_1^{\tau_1} \dots x_n^{\tau_n} k^{\underline{\beta}}(x))$$

governs the behavior of $|\xi_1^{\tau_1} \dots \xi_n^{\tau_n} \partial^{\underline{\tau}}_{\xi_1} \psi^X(\xi)|$ as $|\xi| \to \infty$. To study this structure, from now on, we make the following technical assumption on the underlying copula F.

Assumption 4.4. *Assume for any $\underline{\tau} \in \mathbb{N}_0^n$ the underlying Lévy copula F satisfies*

$$\partial^{\underline{\tau}}_{x}\big(\partial_1 \dots \partial_n F(x)\big) = \partial_1 \dots \partial_n F(x) \cdot \prod_{i=1}^{n} \frac{1}{|x_i|^{\tau_i}} \cdot b_\tau(x), \qquad \text{for all } x \in \mathbb{R}^n, \tag{4.6}$$

where $b_\tau : \mathbb{R}^n \to \mathbb{R}$ is uniformly bounded.

Herewith, one obtains the following crucial result:

Proposition 4.5. *Under Assumption 4.4, for any $\underline{\tau} \in \mathbb{N}_0^n$ and $x \in \mathbb{R}^n$, $|x| \le 1$, there holds*

$$\big|\partial^{\underline{\tau}}_{x}\big(x_1^{\tau_1} \dots x_n^{\tau_n} k^{\underline{\beta}}(x_1, \dots, x_n)\big)\big| \lesssim k^0(x_1, \dots, x_n). \tag{4.7}$$

The proof of Proposition 4.5 is long and technical. It is detailed in Appendix A.

Remark 4.6. Assumption 4.4 is often satisfied in practice. For instance, in dimension $n = 2$, the Clayton-type Lévy copulas F_θ given by (2.12) satisfy (4.6) for any $\theta > 0$ with bounded function $b_{\underline{\tau}}(x_1, x_2)$ of the form

$$\sum_{k_1=0}^{\tau_1-1} \sum_{k_2=0}^{\tau_2-1} \left[a_{\underline{k}} \frac{|x_1|^{k_1\theta} |x_2|^{k_2\theta} (b_{k_2}|x_2|^\theta - b_{k_1}|x_1|^\theta)}{(|x_1|^\theta + |x_2|^\theta)^{k_1+k_2+1}} + c_{\underline{k}} \frac{d_{k_1}|x_1|^{k_1\theta} d_{k_2}|x_2|^{k_2\theta}}{(|x_1|^\theta + |x_2|^\theta)^{k_1+k_2}} \right],$$

where $\underline{k} = (k_1, k_2)$ and $a_{\underline{k}}, b_{k_i}, c_{\underline{k}}, d_{k_i} \neq 0$ for $k_i = 0, \ldots, \tau_i - 1$, $i = 1, 2$, are some suitable coefficients depending only on θ and k_i.

With Proposition 4.5 one obtains the desired symbol estimates.

Theorem 4.7. *If the Lévy copula F satisfies Assumption 4.4 then there holds*

$$\left|\psi^X(\xi)\right| \lesssim \sum_{i=1}^{n} (1 + |\xi_i|^2)^{\frac{\alpha_i}{2}}, \quad \textit{for all } \ \xi \in \mathbb{R}^n. \tag{4.8}$$

Furthermore, for $\underline{\tau} \in \mathbb{N}_0^n$ there holds,

$$\left|\partial_\xi^{\underline{\tau}} \psi^X(\xi)\right| \lesssim \prod_{i \in \mathcal{I}_{\underline{\tau}}} |\xi_i|^{\alpha_i - \tau_i} \cdot \sum_{k \notin \mathcal{I}_{\underline{\tau}}} (1 + |\xi_k|^2)^{\frac{\alpha_k}{2}}, \tag{4.9}$$

for all $\xi \in \mathbb{R}^n$ such that $|\xi_i| > 1$ if $i \in \mathcal{I}_{\underline{\tau}}$. Here, as above, $\mathcal{I}_{\underline{\tau}} = \{i : \tau_i > 0\}$.

Proof. Let ψ^{X^0} be the characteristic exponent of the $\underline{\alpha}$-stable copula process X^0 corresponding to X, i.e. the margins of both processes share the same $\alpha_1, \ldots, \alpha_n$ in (2.5). We split the integral

$$\begin{aligned}\left|\xi_1^{\tau_1} \ldots \xi_n^{\tau_n} \cdot \partial_\xi^{\underline{\tau}} \psi^X(\xi)\right| \leq &\left|\int_{B_1(0)} (1 - \cos\langle \xi, x\rangle) k_{\underline{\tau}}^{\beta}(x) dx\right| \\ &+ \left|\int_{\mathbb{R}^n \setminus B_1(0)} (1 - \cos\langle \xi, x\rangle) k_{\underline{\tau}}^{\beta}(x) dx\right|,\end{aligned}$$

where $B_1(0)$ denotes the unit ball in $\mathbb{R}^n$. Since $k_{\underline{\tau}}^{\beta} \in L^1(\mathbb{R}^n \setminus B_1(0))$, by the Riemann-Lebesgue Lemma, for each $\underline{\tau} \in \mathbb{N}_0^n$ there exists some constant $D > 0$ such that

$$\left|\int_{\mathbb{R}^n \setminus B_1(0)} (1 - \cos\langle \xi, x\rangle) k_{\underline{\tau}}^{\beta}(x) dx\right| \leq D, \quad \text{for all } \xi \in \mathbb{R}^n. \tag{4.10}$$

Thus, using Proposition 4.5, there exists some constant $C_1 \geq 0$ such that

$$\begin{aligned}\left|\xi_1^{\tau_1} \ldots \xi_n^{\tau_n} \cdot \partial_\xi^{\underline{\tau}} \psi^X(\xi)\right| &\leq C_1 \cdot \left|\int_{B_1(0)} (1 - \cos\langle \xi, x\rangle) k^0(x) dx\right| + D \\ &\leq C_1 \cdot \psi^{X^0}(\xi) + D \\ &\leq C_1 \cdot C_2 \cdot \sum_{i=1}^{n} (1 + |\xi_i|^2)^{\frac{\alpha_i}{2}} + D,\end{aligned}$$

where the last line follows from Lemma 4.1 with some suitable constant $C_2 \geq 0$. Merging the constants thus implies

$$\left|\xi_1^{\tau_1} \ldots \xi_n^{\tau_n} \cdot \partial_\xi^{\underline{\tau}} \psi^X(\xi)\right| \lesssim \sum_{i=1}^{n} (1 + |\xi_i|^2)^{\frac{\alpha_i}{2}}, \quad \text{for all } \xi \in \mathbb{R}^n.$$

Hence, setting $\underline{\tau} = 0 \in \mathbb{N}_0^n$ implies (4.8). For any $\underline{\tau} \in \mathbb{N}_0^n$, estimate (4.9) follows from division by $|\xi_1|^{\tau_1} \ldots |\xi_n|^{\tau_n}$. □

5. Sparse Tensor Product Approximation of Anisotropic Operators

In this section we study the numerical solution of the original integrodifferential equation (1.1),

$$\mathcal{A}u = f,$$

with $\mathcal{A} = \mathcal{A}_p$, $p \in \Gamma^{\underline{\alpha}}(\mathbb{R}^n)$ for some $\underline{\alpha} \in \mathbb{R}^n$. For the numerical solution of (1.1), we restrict the state space $\mathbb{R}^n$ to a bounded subdomain $\square := [0,1]^n$, say, and employ the Galerkin finite element method with respect to a hierarchy of conforming trial spaces $\widehat{V}_J \subset \widehat{V}_{J+1} \subset \ldots \subset H^{\underline{\alpha}/2}(\square)$, where

$$H^{\underline{\alpha}/2}(\square) := \left\{ u|_{\square} \,:\, u \in H^{\underline{\alpha}/2}(\mathbb{R}^n),\, u|_{\mathbb{R}^n \setminus \square} = 0 \right\}.$$

For an analysis of the error introduced by the localization of $\mathbb{R}^n$ to $\square$, we refer to [39, Section 4.5]. Now, the variational problem of interest reads: Find $u_J \in \widehat{V}_J$ such that,

$$\mathcal{E}(u_J, v_J) := \langle \mathcal{A}u_J,\, v_J \rangle = \langle f,\, v_J \rangle \quad \text{for all } \; v_J \in \widehat{V}_J. \tag{5.1}$$

The index J represents the meshwidth of order 2^{-J}. In order to ensure that there exists a unique solution to (5.1), in addition to the continuity (3.6) of $\mathcal{E}(\cdot,\cdot)$ we assume that the bilinear form satisfies a Gårding inequality in $H^{\underline{\alpha}/2}$, i.e. there exist constants $c > 0$, $c' \geq 0$ such that

$$\mathcal{E}(u,u) \geq c\|u\|^2_{H^{\underline{\alpha}/2}} - c'\|u\|^2_{L^2}, \qquad \text{for all } u \in H^{\underline{\alpha}/2}. \tag{5.2}$$

The nested trial spaces $\widehat{V}_J \subset \widehat{V}_{J+1}$ we employ in (5.1) shall be sparse tensor product spaces based on a wavelet multiresolution analysis described in the next sections.

5.1. Wavelets on the unit interval

On the unit interval $[0,1]$ we shall employ scaling functions and wavelets based on the construction of [12, 13, 35] and the references therein.

The trial spaces $\mathcal{V}_j$ are spanned by single-scale bases $\Phi_j = \{\phi_{j,k} : k \in \Delta_j\}$, where Δ_j denote suitable index sets. The approximation order of the trial spaces we denote by d, i.e.

$$d = \sup\left\{ s \in \mathbb{R} : \sup_{j\geq 0} \left\{ \frac{\inf_{v_j \in \mathcal{V}_j} \|v - v_j\|_0}{2^{-js}\|v\|_s} \right\} < \infty\,,\ \forall\, v \in H^s([0,1]) \right\}. \tag{5.3}$$

To these single-scale bases there exist biorthogonal complement or *wavelet* bases $\Psi_j = \{\psi_{j,k} : k \in \nabla_j\}$, where $\nabla_j := \Delta_{j+1} \backslash \Delta_j$. Denoting by $\mathcal{W}_j$ the span of Ψ_j, there holds

$$\mathcal{V}_{j+1} = \mathcal{W}_{j+1} \oplus \mathcal{V}_j, \quad \text{for all } j \geq 0, \tag{5.4}$$

and

$$\mathcal{V}_j = \mathcal{W}_0 \oplus \ldots \oplus \mathcal{W}_j, \quad \text{for all } j \geq 0. \tag{5.5}$$

Crucial for the following analysis is that the wavelets on $[0,1]$ satisfy the following norm estimates (cf. e.g. [13, 14], for the one-sided estimates we refer to [50]): For an arbitrary $u \in H^t([0,1])$, $0 \le t \le d$, with wavelet decomposition

$$u = \sum_{j=0}^{\infty} \sum_{k \in \nabla_j} u_{j,k} \psi_{j,k},$$

there holds the norm equivalence,

$$\sum_{(j,k)} 2^{2tj} |u_{j,k}|^2 \sim \|u\|^2_{H^t([0,1])}\,, \quad \text{if } 0 \le t < d - 1/2, \tag{5.6}$$

or the one-sided estimate,

$$\sum_{(j,k)} 2^{2tj} |u_{j,k}|^2 \lesssim \|u\|^2_{H^t([0,1])}\,, \quad \text{if } d - 1/2 \le t < d. \tag{5.7}$$

In case $t = d$ there only holds,

$$\sum_{\substack{(j,k)\\ j \le J}} 2^{2tj} |u_{j,k}|^2 \lesssim J\,\|u\|^2_{H^t([0,1])}\,, \quad \text{if } t = d. \tag{5.8}$$

For concrete examples of wavelet bases we refer to [11, 21].

5.2. Sparse tensor product spaces

For $x = (x_1, \ldots, x_n) \in [0,1]^n$, we denote,

$$\psi_{\mathbf{j},\mathbf{k}}(x) := \psi_{j_1,k_1} \otimes \ldots \otimes \psi_{j_n,k_n}(x_1, \ldots, x_n) = \psi_{j_1,k_1}(x_1) \ldots \psi_{j_n,k_n}(x_n).$$

On $[0,1]^n =: \square$, we define the subspace $V_J \subset H^{\underline{\alpha}/2}(\square)$ as the (full) tensor product of the spaces defined on $[0,1]$

$$V_J := \bigotimes_{i=1}^{n} \mathcal{V}_J, \tag{5.9}$$

which can be written using (5.5) as

$$\begin{aligned} V_J &= \operatorname{span}\{\psi_{\mathbf{j},\mathbf{k}} : k_i \in \nabla_{j_i},\, 0 \le j_i \le J,\, i = 1, \ldots, n\} \\ &= \sum_{j_1,\ldots,j_n=0}^{J} \mathcal{W}_{j_1} \otimes \ldots \otimes \mathcal{W}_{j_n}. \end{aligned}$$

We define the regularity $\gamma > |\underline{\alpha}|_\infty/2$ of the trial spaces by

$$\gamma = \sup\{s \in \mathbb{R} : V_J \subset H^s(\square)\}. \tag{5.10}$$

The sparse tensor product spaces $\widehat{V}_J$ are defined by,

$$\begin{aligned} \widehat{V}_J :=& \operatorname{span}\{\psi_{\mathbf{j},\mathbf{k}} : k_i \in \nabla_{j_i},\, i = 1, \ldots, n;\, 0 \le |\mathbf{j}|_1 \le J\} \\ =& \sum_{0 \le |\mathbf{j}|_1 \le J} \mathcal{W}_{j_1} \otimes \ldots \otimes \mathcal{W}_{j_n}. \end{aligned} \tag{5.11}$$

One readily infers that $N_J := \dim(V_J) = \mathcal{O}(2^{nJ})$ whereas $\widehat{N}_J := \dim(\widehat{V}_J) = \mathcal{O}(2^J J^{n-1})$ as J tends to infinity. However, both spaces have similar approximation properties in terms of the finite element meshwidth $h = 2^{-J}$, provided the function to be approximated is sufficiently smooth. To characterize the necessary extra smoothness we introduce the spaces $\mathcal{H}^{\underline{s}}([0,1]^n)$, $\underline{s} \in \mathbb{N}_0^n$, of all measurable functions $u : [0,1]^n \to \mathbb{R}$, such that the norm,

$$\|u\|_{\mathcal{H}^{\underline{s}}(\square)} := \Bigg(\sum_{\substack{0 \le \alpha_i \le s_i,\\ i=1,\dots,n}} \|\partial_1^{\alpha_1} \dots \partial_n^{\alpha_n} u\|^2_{L^2(\square)} \Bigg)^{1/2},$$

is finite. That is

$$\mathcal{H}^{\underline{s}}([0,1]^n) = \bigotimes_{i=1}^n H^{s_i}([0,1]). \tag{5.12}$$

For arbitrary $\underline{s} \in \mathbb{R}^n_{\ge 0}$, we define $\mathcal{H}^{\underline{s}}$ by interpolation. By (5.9), one may decompose any $u \in L^2(\square)$ into

$$u(x) = \sum_{\substack{j_i \ge 0\\ i=1,\dots,n}} \sum_{k_i \in \nabla_{j_i}} u_{\mathbf{j},\mathbf{k}} \psi_{\mathbf{j},\mathbf{k}}(x) = \sum_{\substack{j_i \ge 0\\ i=1,\dots,n}} \sum_{k_i \in \nabla_{j_i}} u_{\mathbf{j},\mathbf{k}} \psi_{j_1,k_1}(x_1) \dots \psi_{j_n,k_n}(x_n).$$

In this style, the sparse grid projection $\widehat{P}_J : L^2(\square) \to \widehat{V}_J$ is defined by truncation of the wavelet expansion:

$$(\widehat{P}_J u)(x) := \sum_{0 \le |\mathbf{j}|_1 \le J} \sum_{\mathbf{k} \in \nabla_{\mathbf{j}}} u_{\mathbf{j},\mathbf{k}} \psi_{\mathbf{j},\mathbf{k}}(x), \tag{5.13}$$

where $\nabla_{\mathbf{j}} = \nabla_{(j_1,\dots,j_n)} := \nabla_{j_1} \times \dots \times \nabla_{j_n}$.

5.3. Convergence rates

Denoting by u and u_J the solutions of (1.1) and the corresponding variational problem (5.1), we need to analyze the error

$$\|u - u_J\|_{\mathcal{E}} \sim \|u - u_J\|_{H^{\underline{\alpha}/2}(\square)}.$$

For this, at first we derive an anisotropic version of the approximation property of the sparse tensor product projection $\widehat{P}_J$, see [49, Proposition 3.2] for its isotropic properties.

Theorem 5.1. *For $i = 1, \dots, n$ suppose $0 \le \frac{\alpha_i}{2} < \gamma$ and let $\frac{\alpha_i}{2} < t_i \le d$ with γ and d given by (5.10) and (5.3). For $u \in H^{\underline{\alpha}/2}(\square)$ there holds*

$$\|u - \widehat{P}_J u\|_{H^{\underline{\alpha}/2}(\square)} \lesssim \begin{cases} 2^{\overline{(\frac{\alpha}{2}-t)}J} \|u\|_{\mathcal{H}^{\underline{t}}(\square)} & \text{if } \begin{cases} \overline{\alpha} \ne 0 \text{ or} \\ t_i \ne d \text{ for all } i, \end{cases} \\ 2^{\overline{(\frac{\alpha}{2}-t)}J} J^{\frac{n-1}{2}} \|u\|_{\mathcal{H}^{\underline{t}}(\square)} & \text{otherwise,} \end{cases} \tag{5.14}$$

where we denote $\underline{t} = (t_1, \dots, t_n)$ and $\overline{(\frac{\alpha}{2} - t)} = \max\{\frac{\alpha_1}{2} - t_1, \dots, \frac{\alpha_n}{2} - t_n\}$.

Proof. At first recall that, as shown in [36], in contrast to the tensor product structure of

$$\mathcal{H}^{\underline{s}} = \bigotimes_{i=1}^{n} H^{s_i}([0,1]),$$

for each $\underline{s} \in \mathbb{R}^n$ the spaces $H^{\underline{s}}(\square)$ admit an intersection structure

$$H^{\underline{s}}(\square) = \bigcap_{i=1}^{n} H_i^{s_i}(\square),$$

in the sense of equivalent norms. Therefore, due to the norm equivalences (5.6) one infers that if $0 \le s_i < \gamma$, $i = 1, \ldots, n$, there holds for each $v \in H^{\underline{s}}$,

$$\|v\|^2_{H^{\underline{s}}(\square)} \sim \sum_{j_1,\ldots,j_n=0}^{\infty} (1 + 2^{2s_1 j_1} + \ldots + 2^{2s_n j_n}) \|Q_{j_1} \otimes \ldots \otimes Q_{j_n} v\|^2, \tag{5.15}$$

where the mappings $Q_{j_i} : L^2([0,1]) \to \mathcal{W}_{j_i}$, $i = 1, \ldots, n$, denote the projections onto the increments spaces $\mathcal{W}_{j_i}$ defined in Section 5.1. Furthermore, because of the tensor product structure of $\mathcal{H}^{\underline{t}}(\square)$, for each $v \in \mathcal{H}^{\underline{t}}(\square)$ there also holds the one-sided estimate

$$\sum_{j_1,\ldots,j_n=0}^{\infty} 2^{2\sum_{i=1}^{d} t_i j_i} \|Q_{j_1} \otimes \ldots \otimes Q_{j_n} v\|^2 \lesssim \|v\|^2_{\mathcal{H}^{\underline{t}}(\square)}, \tag{5.16}$$

provided that $t_i < d$ for all $i = 1, \ldots, n$. Combining (5.15) and (5.16), setting $\underline{s} = \underline{\alpha}/2$, and writing $w_{\mathbf{j}} = Q_{j_1} \otimes \ldots \otimes Q_{j_n} u$, one obtains in case $t_i < d$ for all $i = 1, \ldots, n$,

$$\begin{aligned}
&\|u - \widehat{P}_J u\|^2_{H^{\underline{\alpha}/2}} \\
&\lesssim \sum_{|\mathbf{j}|_1 > J} (1 + 2^{\alpha_1 j_1} + \ldots + 2^{\alpha_n j_n}) \|w_{\mathbf{j}}\|^2 \\
&\lesssim \sum_{|\mathbf{j}|_1 > J} (2^{-2\sum_{i=1}^{n} t_i j_i} + 2^{(\alpha_1 - 2t_1) j_1} + \ldots + 2^{(\alpha_n - 2t_n) j_n}) 2^{2\sum_{i=1}^{n} t_i j_i} \|w_{\mathbf{j}}\|^2 \\
&\lesssim \max_{|\mathbf{j}|_1 > J} \left\{ (2^{-2\sum_{i=1}^{n} t_i j_i} + 2^{(\alpha_1 - 2t_1) j_1} + \ldots + 2^{(\alpha_n - 2t_n) j_n}) \right\} \|u\|^2_{\mathcal{H}^{\underline{t}}(\square)} \\
&\lesssim 2^{\overline{(\alpha - 2t)} J} \|u\|^2_{\mathcal{H}^{\underline{t}}(\square)},
\end{aligned}$$

with $\overline{(\alpha - 2t)} = \max\{\alpha_1 - 2t_1, \ldots, \alpha_n - 2t_n\}$. In case the set

$$\mathcal{I} := \{i \in \{1, \ldots, n\} : t_i = d\},$$

is non-empty, one may assume without loss of generality that for each $i \in \mathcal{I}$ there holds $\alpha_i = \overline{\alpha}$ and

$$\overline{\alpha - 2t} = \overline{\alpha} - 2d = \alpha_i - 2t_i \quad \text{for all } i \in \mathcal{I}, \tag{5.17}$$

because otherwise one can replace t_i with some suitable $t_i' < t_i = d$ and argue as above to obtain the same convergence rate and smoothness requirements on u, since $H^{t_i}([0,1]) \subset H^{t_i'}([0,1])$.

Because for each coordinate direction $i \in \mathcal{I}$, i.e. $t_i = d$, there only holds the weaker one-sided norm estimate (5.8), instead of (5.16) one obtains

$$2^{2\sum_{i\in\mathcal{I}} t_i j_i} \Big\| \bigotimes_{i\in\mathcal{I}} Q_{j_i} \otimes \bigotimes_{i\notin\mathcal{I}} \mathrm{id}_{[0,1]}\, v \Big\|^2 \lesssim \|v\|^2_{\mathcal{H}^{\underline{\tau}}(\square)}, \tag{5.18}$$

with $\tau_i := t_i$ if $i \in \mathcal{I}$ and $\tau_i := 0$ otherwise. Here $\mathrm{id}_{[0,1]}$ denotes the identity on $L^2([0,1])$. Employing the stronger norm estimates deduced from (5.6) and (5.7) in all directions $i \notin \mathcal{I}$ first, one infers exactly as above,

$$\begin{aligned}
&\|u - \widehat{P}_J u\|^2_{H^{\underline{\alpha}/2}} \\
&\lesssim \sum_{|\mathbf{j}|_1 > J} (1 + 2^{\alpha_1 j_1} + \ldots + 2^{\alpha_n j_n}) \|Q_{j_1} \otimes \ldots \otimes Q_{j_n} u\|^2 \\
&\lesssim \max_{\substack{|\mathbf{j}|_1 > J \\ j_i \,:\, i\notin\mathcal{I}}} \Big\{ 2^{\alpha_i j_i - 2\sum_{k\notin\mathcal{I}} t_k j_k} \\
&\quad \times \sum_{j_k : k\in\mathcal{I}} 2^{\max_k\{\alpha_k j_k\}} \Big\| \bigotimes_{k} Q_{j_k} \otimes \bigotimes_{k\notin\mathcal{I}} \mathrm{id}_{[0,1]}\, u \Big\|^2_{\mathcal{H}^{\underline{t}-\underline{\tau}}(\square)} \Big\} \\
&\lesssim \max_{\substack{|\mathbf{j}|_1 > J \\ j_i \,:\, i\notin\mathcal{I}}} \Big\{ 2^{\alpha_i j_i - 2\sum_{k\notin\mathcal{I}} t_k j_k} \sum_{j_k : k\in\mathcal{I}} 2^{\overline{\alpha}\max_k\{j_k\}} 2^{-2d\sum_k j_k} \|u\|^2_{\mathcal{H}^{\underline{t}}(\square)} \Big\},
\end{aligned} \tag{5.19}$$

where in the last line (5.17) was employed in conjunction with (5.18). To estimate the remaining sum one may now proceed as in the proof of [49, Proposition 3.2]. If $\overline{\alpha} > 0$, herewith one obtains

$$\sum_{j_k : k\in\mathcal{I}} 2^{\overline{\alpha}\max_k\{j_k\}} 2^{-2d\sum_k j_k} \lesssim \max_{j_k : k\in\mathcal{I}} \Big\{ 2^{(\overline{\alpha}-2d)\sum_k j_k} \Big\}, \tag{5.20}$$

where the j_k run through the set of all indices that are admissible in the last sum of (5.19). Finalizing the argument one obtains

$$\begin{aligned}
&\|u - \widehat{P}_J u\|^2_{H^{\underline{\alpha}/2}} \\
&\lesssim \max_{|\mathbf{j}|_1 > J} \Big\{ 2^{\max_{i\notin\mathcal{I}}\{\alpha_i j_i\} - 2\sum_{k\notin\mathcal{I}} t_k j_k} 2^{(\overline{\alpha}-2d)\sum_{k\in\mathcal{I}} j_k} \Big\} \|u\|^2_{\mathcal{H}^{\underline{t}}(\square)} \\
&\lesssim \max_{|\mathbf{j}|_1 > J} \Big\{ 2^{\sum_{k\notin\mathcal{I}} \alpha_k j_k - 2\sum_{k\notin\mathcal{I}} t_k j_k} 2^{(\overline{\alpha}-2d)\sum_{k\in\mathcal{I}} j_k} \Big\} \|u\|^2_{\mathcal{H}^{\underline{t}}(\square)} \\
&\lesssim \max_{|\mathbf{j}|_1 > J} \Big\{ 2^{\overline{(\alpha-2t)}\sum_{k=1}^n j_k} \Big\} \|u\|^2_{\mathcal{H}^{\underline{t}}(\square)} \\
&\lesssim 2^{\overline{(\alpha-2t)}J} \|u\|^2_{\mathcal{H}^{\underline{t}}(\square)}.
\end{aligned}$$

In case $\overline{\alpha} = 0$, instead of (5.20) one obtains

$$\sum_{j_i : i\in\mathcal{I}} 2^{\overline{\alpha}\max_{\{i\}}\{j_i\}} 2^{-2d\sum_i j_i} \lesssim \max_{j_i : i\in\mathcal{I}} \Big\{ 2^{(\overline{\alpha}-2d)\sum_i j_i} \Big(\sum_i j_i\Big)^{n-1} \Big\}. \tag{5.21}$$

Then analogous arguments as in the case $\overline{\alpha} > 0$ yield the required result. □

Herewith one immediately obtains the desired minimal regularity sparse tensor product convergence result:

Proposition 5.2. *For a Lévy copula process with tempered stable margins defined by* (2.5) *and* $\overline{\alpha}$ *as in* (2.6) *the solutions* u *and* u_J *of* (1.1) *and* (5.1) *satisfy*

$$\|u - u_J\|_{\mathcal{E}} \sim \|u - u_J\|_{H^{\underline{\alpha}/2}(\square)} \lesssim 2^{-(d-\frac{\overline{\alpha}}{2})J} \|u\|_{\mathcal{H}^{\underline{\rho}}(\square)}, \tag{5.22}$$

provided $u \in \mathcal{H}^{\underline{\rho}}(\square)$. *The smoothness parameter* $\underline{\rho} \in \mathbb{R}^n_{>0}$ *is given by*

$$\rho_i = d - (\frac{\overline{\alpha}}{2} - \frac{\alpha_i}{2}), \tag{5.23}$$

for each $i = 1, \dots, n$.

Proof. With this choice of $\underline{\rho}$ there holds $\overline{(\alpha - 2\rho)} = \alpha_i - 2\rho_i$ for all $i \in \{1, \dots, n\}$. Hence the smoothness requirement on u in each coordinate direction is minimal and the result follows from Theorem 5.1. □

Remark 5.3. In case $\alpha_i = \overline{\alpha}$ for all $i = 1, \dots, n$, Proposition 5.2 coincides with the sparse tensor product convergence result for isotropic operators (cf. [49]).

Appendix A. Proof of Proposition 4.5

The goal of this Section is the proof of

Proposition 4.5. *Suppose for any* $\underline{\tau} \in \mathbb{N}_0^n$ *the underlying Lévy copula* F *satisfies*

$$\partial_x^{\underline{\tau}}\big(\partial_1 \dots \partial_n F(x)\big) = \partial_1 \dots \partial_n F(x) \cdot \prod_{i=1}^{n} \frac{1}{|x_i|^{\tau_i}} \cdot b_{\underline{\tau}}(x), \qquad \text{for all } x \in \mathbb{R}^n, \tag{A.1}$$

where $b_{\underline{\tau}} : \mathbb{R}^n \to \mathbb{R}$ *is uniformly bounded. Then for any* $\underline{\tau} \in \mathbb{N}_0^n$ *and* $x \in \mathbb{R}^n$, $|x| \le 1$, *there holds*

$$\big|\partial_x^{\underline{\tau}}\big(x_1^{\tau_1} \dots x_n^{\tau_n} k^{\underline{\beta}}(x_1, \dots, x_n)\big)\big| \lesssim k^0(x_1, \dots, x_n). \tag{A.2}$$

By the quasi self-reproductive structure of the derivatives of F in (A.1), it suffices to show that for any $i = 1, \dots, n$ there holds

$$\big|\partial_{x_i}^{\tau_i}\big(x_i^{\tau_i} k^{\underline{\beta}}(x_1, \dots, x_n)\big)\big| \lesssim k^0(x_1, \dots, x_n), \quad |x| \le 1.$$

Without loss of generality we assume $i = 1$. The proof comprises of the following lemmas. Throughout, we assume $x_1 \ne 0$. Since we are only interested in derivatives with respect to x_1, we simplify some notation and assume that $x_2, \dots, x_n \in \mathbb{R}$ are fixed unless indicated otherwise. With the tail integrals $U_1^{\beta_1}, \dots, U_n^{\beta_n}$ as in (2.7), we set

$$G(x_1) := G(x_1, \dots, x_n) := \partial_1 \dots \partial_n F(x_1, \dots, x_n),$$
$$H(x_1) := H(x_1, \dots, x_n) := G(U_1^{\beta_1}(x_1), \dots, U_n^{\beta_n}(x_n)).$$

Furthermore, we denote by $G^{(k)}$, $H^{(k)}$ the k-th derivative of G and H with respect to x_1. In order to estimate the derivatives of

$$k^{\underline{\beta}}(x_1,\dots,x_n)=H(x_1,\dots,x_n)k_1^{\beta_1}(x_1)\dots k_n^{\beta_n}(x_n),$$

we begin by analyzing the marginal tail integral $U_1^{\beta_1}$:

Lemma A.1. *Let $s\in\mathbb{N}$. For any $\nu_j\in\mathbb{N}$, $p_j\in\mathbb{N}_0$, $j=1,\dots s$, the derivative*

$$\partial_{x_1}\biggl(\prod_{j=1}^{s}(\partial^{\nu_j}U_1^{\beta_1})^{p_j}\biggr),$$

of $U_1^{\beta_1}$ is a linear combination of terms of the form $\prod_{j=1}^{s'}(\partial^{\mu_j}U_1^{\beta_1})^{\pi_j}$, with

$$\sum_{j=1}^{s'}\pi_j=\sum_{j=1}^{s}p_j,\quad \sum_{j=1}^{s'}\mu_j\pi_j=1+\sum_{j=1}^{s}\nu_j p_j.$$

Proof. The claim is proved by induction on s. For $s=1$ there holds

$$\partial_{x_1}\bigl((\partial^{\nu}U_1^{\beta_1})\bigr)^{\mu}=\mu\bigl((\partial^{\nu}U_1^{\beta_1})\bigr)^{\mu-1}\cdot(\partial^{\nu+1}U_1^{\beta_1}),$$

which proves the basis. To show that the validity of the hypothesis for some $s\in\mathbb{N}$ implies its validity for $s+1$ one finds

$$\begin{aligned}\partial_{x_1}\biggl(\prod_{j=1}^{s+1}(\partial^{\nu_j}U_1^{\beta_1})^{p_j}\biggr)&=\partial_{x_1}\biggl(\prod_{j=1}^{s}(\partial^{\nu_j}U_1^{\beta_1})^{p_j}\biggr)(\partial^{\nu_{s+1}}U_1^{\beta_1})^{p_{s+1}}\\&\quad+\prod_{j=1}^{s}(\partial^{\nu_j}U_1^{\beta_1})^{p_j}\cdot p_{s+1}(\partial^{\nu_{s+1}}U_1^{\beta_1})^{p_{s+1}-1}(\partial^{\nu_{s+1}+1}U_1^{\beta_1}).\end{aligned}\tag{A.3}$$

Since the hypothesis is valid for s, one obtains that the first summand in (A.3) is indeed a linear combination of terms of the required form. The sum of its powers satisfies

$$\sum_{j=1}^{s'}\pi_j+p_{s+1}=\sum_{j=1}^{s}p_j+p_{s+1}=\sum_{j=1}^{s+1}p_j,$$

as required. For the weighted sums there holds

$$\sum_{j=1}^{s'}\mu_j\pi_j+\nu_{s+1}p_{s+1}=1+\sum_{j=1}^{s}\nu_j p_j+\nu_{s+1}p_{s+1}=1+\sum_{j=1}^{s+1}\nu_j p_j.$$

One readily infers that the second summand of (A.3) can be represented as a suitable product of derivatives of g. The powers of these derivatives satisfy

$$\sum_{j=1}^{s}p_j+(p_{s+1}-1)+1=\sum_{j=1}^{s+1}p_j.$$

For the weighted sums one finally obtains

$$\sum_{j=1}^{s} \nu_j p_j + (p_{s+1} - 1)\nu_{s+1} + (\nu_{s+1} + 1) = \sum_{j=1}^{s+1} \nu_j p_j + 1. \qquad \square$$

Lemma A.1 enables us to show

Lemma A.2. *For any $k \in \mathbb{N}$ there holds*

$$H^{(k)}(x_1) = \partial_{x_1}^k H(x_1, \dots, x_n) = \sum_{l=1}^{k} c_{l,k} G^{(l)}(U_1^{\beta_1}(x_1), \dots, U_n^{\beta_n}(x_n)) J_{l,k}(x_1),$$

where

$$J_{l,k} = \sum_m c_{l,k,m} \prod_{j=1}^{s(l,k)} (\partial^{\nu_{j,m}} U_1^{\beta_1})^{p_{j,m}}, \tag{A.4}$$

with suitable $\nu_{j,m} \in \mathbb{N}$, $p_{j,m} \in \mathbb{N}_0$ and constants $c_{l,k}$, $c_{l,k,m} \in \mathbb{R}$. Furthermore, for each m there holds

$$\sum_{j=1}^{s(l,k)} p_{j,m} = l, \qquad \sum_{j=1}^{s(l,k)} \nu_{j,m} p_{j,m} = k. \tag{A.5}$$

Proof. We proceed by induction on k. For $k = 1$, with $J_{1,1} = \partial U_1^{\beta_1}$ the induction basis is obvious. Assuming the validity of the hypothesis for some $k \in \mathbb{N}$ one obtains its validity for $k + 1$ as follows:

$$\begin{aligned} H^{(k+1)}(x_1) = &\sum_{l=1}^{k} c_{l,k} G^{(l+1)}(U_1^{\beta_1}(x_1), \dots, U_n^{\beta_n}(x_n)) \cdot (\partial U_1^{\beta_1}) J_{l,k}(x_1) \\ &+ \sum_{l=1}^{k} c'_{l,k} G^{(l)}(U_1^{\beta_1}(x_1), \dots, U_n^{\beta_n}(x_n)) \cdot \partial_{x_1} (J_{l,k}(x_1)), \end{aligned} \tag{A.6}$$

where $c_{l,k}$, $c'_{l,k}$ denote some suitable constants. By the hypothesis, $J_{l,k}$ is a linear combination of products as in (A.4). Thus, any "pure" summand (i.e. it does not contain any further sub-summands) in the first summand of (A.6) is of the form

$$c \cdot G^{(l+1)}(U_1^{\beta_1}(x_1), \dots, U_n^{\beta_n}(x_n)) \cdot \underbrace{(\partial U_1^{\beta_1}) \prod_{j=1}^{s} (\partial^{\nu_j} U_1^{\beta_1})^{p_j}}_{=:A},$$

where c denotes some constant. Using the validity of the hypothesis for k, the additional factor A defines $J_{l+1,k+1}$ and satisfies (A.5) for $k + 1$.

For the second summand of (A.6) one needs to show that for each $l = 1, \dots, k$ the factor $\partial_{x_1} (J_{l,k}(x_1))$ provides a suitable additive contribution to $J_{l,k+1}$. By the hypothesis, each "pure" summand of $J_{l,k}$ is of the form

$$F := \prod_{j=1}^{k} (\partial^{\nu_j} U_1^{\beta_1})^{p_j}.$$

By Lemma A.1, its derivative $\partial_{x_1} F$ is a linear combination of terms of the form $\prod_j (\partial^{\mu_j} U_1^{\beta_1})^{\pi_j}$ with

$$\sum_j \pi_j = \sum_{j=1}^{k} \nu_j = l,$$

$$\sum_j \mu_j \pi_j = \sum_{j=1}^{k} \nu_j p_j + 1 = k+1,$$

where in both equations the induction hypothesis was applied to obtain the last equality. Thus, $\partial_{x_1} (J_{l,k}(x_1))$ indeed provides an additional additive term to the representation of $J_{l,k+1}$ that satisfies (A.5). □

The following lemma will finally enable us to give the proof of Proposition 4.5 below.

Lemma A.3. *If* (A.1) *holds then*

$$\left|\partial_{x_1}^k H(x_1, \dots, x_n)\right| \lesssim \frac{1}{|x_1|^k} H(x_1, \dots, x_n),$$

for all $(x_1, \dots, x_n) \in \mathbb{R}^n$, $|x_1| \leq 1$.

Proof. Denoting by $c_{l,k}$ some suitable constants, Lemma A.2 implies

$$\begin{aligned} &\left|\partial_{x_1}^k H(x_1, \dots, x_n)\right| \\ &\leq \sum_{l=1}^{k} c_{l,k} \cdot \left| G^{(l)}(U_1^{\beta_1}(x_1), \dots, U_n^{\beta_n}(x_n)) \right| \cdot |J_{l,k}(x_1)| \\ &\leq \sum_{l=1}^{k} c_{l,k} \cdot \left| G^{(l)}(U_1^{\beta_1}(x_1), \dots, U_n^{\beta_n}(x_n)) \right| \cdot \prod_{j=1}^{s(l,k)} (\partial^{\nu_j} U_1^{\beta_1})^{p_j}, \end{aligned}$$

where the powers p_j and the orders of differentiation ν_j still depend on l and k in such a way that $\sum_j p_j = l$ and $\sum_j \nu_j p_j = k$. Note that

$$(\partial^{\nu_j} U_1^{\beta_1})(x_1) = \frac{e^{-\beta_1 x_1}}{|x_1|^{\nu_j + \alpha_1}} \cdot P_{\nu_j}(x_1),$$

where P_{ν_j} is some suitable polynomial of degree $\nu_j - 1$ in x_1 that does not vanish at $x_1 = 0$. One therefore obtains

$$\begin{aligned} &\left|\partial_{x_1}^k H(x_1, \dots, x_n)\right| \\ &\leq \sum_{l=1}^{k} c'_{l,k} \cdot \left| G^{(l)}(U_1^{\beta_1}(x_1), \dots, U_n^{\beta_n}(x_n)) \right| \cdot \prod_{j=1}^{s(l,k)} \frac{1}{x_1^{p_j(\nu_j + \alpha_1)}} \\ &\leq \sum_{l=1}^{k} c'_{l,k} \cdot \left| G^{(l)}(U_1^{\beta_1}(x_1), \dots, U_n^{\beta_n}(x_n)) \right| \cdot \frac{1}{|x_1|^{k + l\alpha_1}}. \end{aligned}$$

By (A.1) there holds

$$\left|G^{(l)}(U_1^{\beta_1}(x_1),\dots,U_n^{\beta_n}(x_n))\right| \lesssim \frac{G(U_1^{\beta_1}(x_1),\dots,U_n^{\beta_n}(x_n))}{(U_1^{\beta_1}(x_1))^l},$$

for all $(x_1,\dots,x_n)\in\mathbb{R}^n$. Thus, since for each $x_1\in\mathbb{R}$ with $|x_1|\le 1$ there holds $(U_1^{\beta_1}(x_1))^{-l}\lesssim |x_1|^{l\alpha_1}$, one obtains

$$\left|\partial_{x_1}^k H(x_1,\dots,x_n)\right| \lesssim G(U_1^{\beta_1}(x_1),\dots,U_n^{\beta_n}(x_n))\cdot\frac{1}{|x_1|^k},\qquad \text{for } |x_1|\le 1. \qquad \square$$

Using the above lemmas one can now prove Proposition 4.5:

Proof. Using Leibniz' rule,

$$\begin{aligned}\left|\partial_{x_1}^\tau\left(x_1^\tau k^{\underline{\beta}}(x_1,\dots,x_n)\right)\right| &= \left|\sum_{j=0}^{\tau} c_j\partial_{x_1}^j\left(k^{\underline{\beta}}(x_1,\dots,x_n)\right)\partial^{\tau-j}\left(x_1^\tau\right)\right| \\ &= \left|\sum_{j=0}^{\tau} c_j' x_1^j\partial_{x_1}^j\left(k^{\underline{\beta}}(x_1,\dots,x_n)\right)\right|.\end{aligned}$$

Since $\partial_{x_1}^\tau(k_1^{\beta_1}(x_1))\lesssim |x_1|^{-(\tau+1+\alpha_1)}$ for all $x_1\in\mathbb{R}$ with $|x_1|\le 1$, Lemma A.3 implies

$$\begin{aligned}&\left|\partial_{x_1}^\tau\left(x_1^\tau k^{\underline{\beta}}(x_1,\dots,x_n)\right)\right| \\ &\le \sum_{j=0}^{\tau} c_j'|x_1|^j\sum_{i=0}^{j}\frac{c_i}{|x_1|^{i+1+\alpha_1}}\cdot\prod_{s=2}^{n}\frac{1}{|x_s|^{1+\alpha_s}}\cdot\left|\partial_{x_1}^{j-i}H(x_1,\dots,x_n)\right| \\ &\le \sum_{j=0}^{\tau} c_j'|x_1|^j\sum_{i=0}^{j}\frac{c_i}{|x_1|^{1+\alpha_1}}\cdot\prod_{s=2}^{n}\frac{1}{|x_s|^{1+\alpha_s}}\cdot G(U_1^{\beta_1}(x_1),\dots,U_n^{\beta_n}(x_n))\cdot\frac{1}{|x_1|^j} \\ &\le c\cdot k^0(x_1,\dots,x_n). \qquad \square\end{aligned}$$

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Nils Reich
ETH Zürich
Seminar for Applied Mathematics
8092 Zürich
Switzerland
e-mail: reich@math.ethz.ch

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