Multibody Syst Dyn (2012) 27:437–454
DOI 10.1007/s11044-011-9276-5

# Rigid body dynamics with a scalable body, quaternions and perfect constraints

**Michael Möller · Christoph Glocker**

Received: 11 January 2011 / Accepted: 16 September 2011 / Published online: 5 November 2011


**Abstract** In this paper, we present a formulation of the quaternion constraint for rigid body rotations in the form of a standard perfect bilateral mechanical constraint, for which the associated Lagrangian multiplier has the meaning of a constraint force. First, the equations of motion of a scalable body are derived. A scalable body has three translational, three rotational, and one uniform scaling degree of freedom. As generalized coordinates, an unconstrained quaternion and a displacement vector are used. To the scalable body, a perfect bilateral constraint is added, restricting the quaternion to unit length and making the body rigid. This way a quaternion based differential algebraic equation (DAE) formulation for the dynamics of a rigid body is obtained, where the $7 \times 7$ mass matrix is regular and the unit length restriction of the quaternion is enforced by a mechanical constraint. Finally, the equations of motion in the form of a DAE are linked to the Newton–Euler equations of motion of a rigid body. The rigid body DAE formulation is useful for the construction of (energy) consistent integrators.

**Keywords** Quaternion · Rotation · Scaling · Constraint · Mass matrix · Rigid body · Differential algebraic equation · Equations of motion

## 1 Introduction

The unit quaternion, also known as Euler parameters, is a well-known parameterization of finite rotations and often used to represent a body's orientation in rigid body dynamics. When formulating the equations of motion of a rigid body using quaternions, one usually starts with the Newton–Euler equations in terms of the translational and angular velocity. Subsequently, the derivative of the unit quaternion is related to the angular velocity, yielding

M. Möller (✉) · C. Glocker
Institute of Mechanical Systems, Center of Mechanics, Department of Mechanical and Process Engineering, ETH Zurich, 8092 Zurich, Switzerland
e-mail: michael.moeller@imes.mavt.ethz.ch

C. Glocker
e-mail: christoph.glocker@imes.mavt.ethz.ch

a kinematic equation which enforces the unity of the quaternion on velocity level (cf. [14]). To prevent drift of the length of the quaternion in numerical simulations, the quaternion has therefore to be resized to unit length after each integration step (e.g., as in [10]) or, alternatively, a discretization scheme which preserves the unit length constraint has to be employed. One way to achieve the latter is to extend the description to a differential algebraic equation (DAE) formulation, where the unity of the quaternion is explicitly contained in the set of equations. Unfortunately, the extension of the equations of motion of a rigid body to a DAE yields a cumbersome formulation for which the Lagrange multiplier and the equation enforcing the unit length of the quaternion have no direct physical meaning. Additionally, the resulting mass matrix is either singular or uses an arbitrarily chosen mass (cf. [13–15]).

In the present paper, a different approach to formulate the DAE is taken. First, the infinite dimensional dynamics of the underlying continuum is reduced to a scalable body, by using perfect bilateral constraints. A scalable body has three translational, three rotational, and one uniform scaling degree of freedom. The displacement of the center of mass and an unconstrained quaternion are used as generalized coordinates. By introducing an additional perfect bilateral constraint one can force the scalable body to become a rigid body, i.e., the scaling degree of freedom is suppressed by this constraint. Without reducing the set of coordinates, this yields naturally a DAE description of the dynamics of a rigid body where the $7 \times 7$ mass matrix is positive definite and the unit length restriction is enforced by a mechanical constraint.

In contrast to previous works [2, 12–14, 16, 18–20], the quaternion in this paper is not assumed to be of unit length while deriving the equations of motion. The unit length restriction is added only in a last step in the form of a perfect bilateral constraint to reduce the scalable body to a rigid body. Besides deriving the nonsingular $7 \times 7$ mass matrix for a quaternion based rigid body formulation, we also discuss the associated mechanical model in the form of the unconstrained scalable body.

Since the present paper is similar to the recent work by Betsch and Siebert [2], a few of the differences will be outlined in the following. In [2], the assumption of a unit length quaternion is introduced right at the beginning, but still a nonsingular $7 \times 7$ mass matrix is obtained. This is achieved by using a director-based formulation for the kinetic energy of a rigid body [3], which is identical to the kinetic energy of a body with all twelve affine degrees of freedom. This kinetic energy still contains the contributions from the scaling degree of freedom, contrary to the kinetic energy based on the angular velocity. By reducing the directors based generalized coordinates to a unit quaternion—without using the unit length property in a harmful way—a nonsingular $7 \times 7$ quaternion based rigid body mass matrix is obtained. This mass matrix is identical to the one associated with the full unrestricted quaternion degrees of freedom of a scalable body, but the link to the mechanical model of a scalable body is not shown in [2]. When formulating the inverse mass matrix, [2] uses a simplification valid only for unit quaternions. This leads to a DAE formulation where the Lagrange multiplier is not the mechanical scaling constraint force: Setting the multiplier to zero does not recover the full dynamics of a scalable body. Beside this, in [2], the Lagrange multiplier is always zero for a free rigid body, even when the body is rotating. For a rotating rigid body, one would expect a nonzero scaling constraint force preventing the body from getting larger.

In the work by Vadali [20], the quaternion unit length restriction is not imposed everywhere in the derivation of the equations of motion, but still the kinetic energy of the rigid body is used as a starting point. This kinetic energy based on the angular velocity does not contain the contributions from the scaling degree of freedom anymore, thus it is valid only under the quaternion unit length assumption. The result is a singular $4 \times 4$ mass matrix for the rotational dynamics of a rigid body and a Lagrange multiplier which is always zero.

O'Reilly and Varadi [16] first derive the equations of motion of a scalable body in terms of a position vector, a scaling factor and a rotation matrix. The resulting equations of motion are then applied to the dynamics of Hoberman's sphere which is as a nice application of the scalable body model. In [16], the derivation of the equations of motion of a scalable body is based on the equations of motion of a body with all twelve affine degrees of freedom, while the present work relies directly on the principle of virtual work and the principle of d'Alembert–Lagrange. While the equations of motion of a scalable body in terms of quaternions are given in [16] only in the abstract form of Lagrange's equations of the second kind, the present work also derives two simplified representations where all partial derivatives have been evaluated. In the second part of [16], the equations of motion of a rigid body in terms of quaternions are derived. This derivation considers only the rotational dynamics of the body and omits the contributions from the scaling dynamics which results in a singular $4 \times 4$ mass matrix associated with the quaternion. Omitting the contributions from the scaling dynamics corresponds to a unit length assumption for the quaternion. The singular mass matrix problem is avoided in the present paper by considering the contributions of all degrees of freedom of a scalable body.

In the recent work by Udwadia and Schutte [18], the kinetic energy of a rotating rigid body is used as starting point. This kinetic energy already contains the quaternion unit length assumption, which is also used at different points in the derivation of the equations of motion. To avoid a singular mass matrix, an arbitrary positive mass is inserted into the equations. This additional arbitrary mass has been used as well by Morton [13]. In [19], the method is generalized to a certain class of constrained mechanical systems with positive semidefinite mass matrices, and again demonstrated for the quaternion based formulation of the rotational dynamics of a rigid body. The resulting equations of motion do not have a direct physical model motivating the arbitrarily chosen mass. In the resulting equations of motion, there is no clear separation into the unconstrained dynamics and the perfect quaternion constraint.

The DAE formulation of the dynamics of a rigid body is useful for the construction of (energy) consistent integration schemes. With respect to consistent integrators, important properties of the DAE formulation presented in this work are the constant (and regular) mass matrix when generalized velocities are used, the singularity free parameterization of the rotation, as well as the small set of generalized coordinates and velocities. The quaternion based DAE uses only one redundant coordinate and one redundant velocity, while the director based formulations [3] use six redundant coordinates and six redundant velocities. The smaller set of coordinates can be of advantage when no techniques can be applied to reduce the set of coordinates after discretization. For further details on energy consistent integrators, the reader is referred to [2, 3, 6, 11].

The paper is organized as follows: In Sect. 2, an overview on quaternions and their matrix and vector representation is given. The parameterization of rotations and uniform scaling with quaternions is described in Sect. 3. The kinematics of the scalable body on displacement and velocity level is derived in Sect. 4. The principle of virtual work is used in Sect. 5 to derive the variational equations of motion of a system for which the kinematics is given by a set of generalized coordinates and enforced by perfect bilateral constraints. In Sect. 6, the variational equations of motion from Sect. 5 are then evaluated with the kinematics of the scalable body from Sect. 4. To the resulting equations of motion, a perfect bilateral constraint is added in Sect. 7 yielding the DAE formulation of a rigid body. In Sect. 8, the derivative of the quaternion is replaced by a scaling velocity and a generalized angular velocity, giving the DAE a form which can be directly linked to the Newton–Euler equations of motion of a rigid body.

## 2 Quaternions

In this section, a brief introduction to quaternions and the notation used in this paper is given. For more background on quaternions, the reader is referred to [1, 8, 9, 21].

A quaternion $A \in \mathbb{H}$ is a hypercomplex number with one real and three imaginary parts. The imaginary parts are formed with three real coefficients and three imaginary units $i, j, k$, i.e.,

$$A = a_0 + a_1 i + a_2 j + a_3 k \in \mathbb{H}, \quad a_i \in \mathbb{R}. \tag{1}$$

In this work, the notation $A = (a_0, \boldsymbol{a})$ for a quaternion is used, where $a_0$ is the real part and $\boldsymbol{a} = (a_1, a_2, a_3)^\mathsf{T} \in \mathbb{R}^3$ is a vector consisting of the three coefficients of the imaginary part. The conjugate $A^*$ of a quaternion is defined as

$$A^* = (a_0, -\boldsymbol{a}). \tag{2}$$

The addition of two quaternions is done componentwise and is associative. The real part $\mathrm{Re}(A)$ and the imaginary part $\mathrm{Im}(A)$ of a quaternion is given by

$$\mathrm{Re}(A) = (a_0, 0) = \frac{1}{2}\big(A + A^*\big), \qquad \mathrm{Im}(A) = (0, \boldsymbol{a}) = \frac{1}{2}\big(A - A^*\big). \tag{3}$$

All possible products of the imaginary units can be determined from the definition

$$i^2 = j^2 = k^2 = ijk = -1 \tag{4}$$

as formulated in [8]. This gives the rule

$$AB = (a_0, \boldsymbol{a})\,(b_0, \boldsymbol{b}) = \big(a_0 b_0 - \boldsymbol{a}^\mathsf{T}\boldsymbol{b},\, a_0\boldsymbol{b} + b_0\boldsymbol{a} + \tilde{\boldsymbol{a}}\boldsymbol{b}\big) \tag{5}$$

for the multiplication of two quaternions, where $\tilde{\boldsymbol{a}} \in \mathbb{R}^{3\times 3}$ is the real skew-symmetric matrix associated with the cross product, so that $\tilde{\boldsymbol{a}}\boldsymbol{b} = \boldsymbol{a} \times \boldsymbol{b}$ for any $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^3$. The quaternion multiplication is not commutative. For the addition and multiplication of quaternions, the distributive law holds. The conjugate of a quaternion product is the product of the conjugates in inverse order, i.e.,

$$(AB)^* = B^* A^*. \tag{6}$$

The norm of a quaternion is defined by

$$|A| = \sqrt{a_0^2 + \boldsymbol{a}^\mathsf{T}\boldsymbol{a}}. \tag{7}$$

The product of a quaternion and its conjugate is equal to the square of the norm of the quaternion, i.e.,

$$AA^* = \big(|A|^2, 0\big). \tag{8}$$

This can be used to form the inverse of a quaternion as

$$A^{-1} = \frac{A^*}{|A|^2} \tag{9}$$

for any nonzero quaternion. A quaternion can be mapped to a real $4 \times 4$ matrix with the function

$$\boldsymbol{\varphi} : \mathbb{H} \to \mathbb{R}^{4\times 4}, \quad \boldsymbol{\varphi}\big((a_0, \boldsymbol{a})\big) = \begin{pmatrix} a_0 & -\boldsymbol{a}^\mathsf{T} \\ \boldsymbol{a} & a_0 \boldsymbol{I} + \tilde{\boldsymbol{a}} \end{pmatrix}. \tag{10}$$

The matrix of a conjugated quaternion is then the transposed of the matrix of a quaternion, i.e.

$$\boldsymbol{\varphi}(A^*) = \boldsymbol{\varphi}^\mathsf{T}(A). \tag{11}$$

The matrix of a product of two quaternions is equal to the product of the matrices of the quaternions

$$\boldsymbol{\varphi}(AB) = \boldsymbol{\varphi}(A)\boldsymbol{\varphi}(B). \tag{12}$$

The inverse of the matrix of a quaternion can be obtained by mapping the inverse of the quaternion to a matrix or by normalizing the transposed of the matrix

$$\boldsymbol{\varphi}^{-1}(A) = \boldsymbol{\varphi}\big(A^{-1}\big) = \frac{1}{|A|^2}\boldsymbol{\varphi}^\mathsf{T}(A). \tag{13}$$

Sometimes it is useful to interpret a quaternion as a real 4-dimensional vector, for which the function

$$\boldsymbol{\psi} : \mathbb{H} \to \mathbb{R}^4, \quad \boldsymbol{\psi}\big((a_0, \boldsymbol{a})\big) = \begin{pmatrix} a_0 \\ \boldsymbol{a} \end{pmatrix} \tag{14}$$

is introduced. The vector representation of the product of two quaternions is then equal to the product of the matrix of the first quaternion and the vector of the second quaternion, i.e.,

$$\boldsymbol{\psi}(AB) = \boldsymbol{\varphi}(A)\boldsymbol{\psi}(B). \tag{15}$$

The vector of a conjugated quaternion can be obtained by multiplying the vector of the quaternion with the matrix

$$\boldsymbol{T} := \begin{pmatrix} 1 & 0 \\ 0 & -\boldsymbol{I} \end{pmatrix} \tag{16}$$

which yields

$$\boldsymbol{\psi}(A^*) = \boldsymbol{T}\,\boldsymbol{\psi}(A). \tag{17}$$

## 3 Rotation and scaling

In this section, the parameterization of rotations and uniform scaling by using quaternions is shown. As a starting point, we note that the product of a quaternion $A \in \mathbb{H}$, a purely imaginary quaternion $(0, \boldsymbol{x})$ generated by a vector $\boldsymbol{x} \in \mathbb{R}^3$, and the conjugate $A^*$ of the first quaternion always evaluates to a purely imaginary quaternion. This property can be formulated as

$$(0, \boldsymbol{y}) = A\,(0, \boldsymbol{x})\,A^* \tag{18}$$

and follows directly with the help of (3):

$$\mathrm{Re}\big(A\,(0, \boldsymbol{x})\,A^*\big) = \frac{1}{2}\big(A\,(0, \boldsymbol{x})\,A^* + A\,(0, -\boldsymbol{x})\,A^*\big) = \frac{1}{2}A\big((0, \boldsymbol{x}) + (0, -\boldsymbol{x})\big)A^* = 0. \tag{19}$$

The product in (18) can be rewritten by using (6),

$$(0,\, \boldsymbol{y}) = A\big(A\,(0, \boldsymbol{x})^*\big)^*, \tag{20}$$

and after applying (14), (15), and (17), one obtains the linear relation

$$\begin{pmatrix} 0 \\ \boldsymbol{y} \end{pmatrix} = \boldsymbol{\varphi}(A)\boldsymbol{T}\boldsymbol{\varphi}(A)\boldsymbol{T} \begin{pmatrix} 0 \\ \boldsymbol{x} \end{pmatrix} \tag{21}$$

in vector notation. Multiplication of (21) by $(0 \quad \boldsymbol{I})$ from the left yields

$$\boldsymbol{y} = \underbrace{\big(0\; \boldsymbol{I}\big)\,\boldsymbol{\varphi}(A)}_{\big(\boldsymbol{a}\;\; a_0\boldsymbol{I} + \tilde{\boldsymbol{a}}\big)}\; \underbrace{\boldsymbol{T}\,\boldsymbol{\varphi}(A)\,\boldsymbol{T}\,(\,0\;\boldsymbol{I}\,)^{\mathsf{T}}}_{\begin{pmatrix} \boldsymbol{a}^{\mathsf{T}} \\ a_0\boldsymbol{I} + \tilde{\boldsymbol{a}} \end{pmatrix}} \boldsymbol{x}. \tag{22}$$

By using the abbreviations,

$$\boldsymbol{R} := \frac{1}{|A|^2}\big(\boldsymbol{a}\;\; a_0\boldsymbol{I} + \tilde{\boldsymbol{a}}\big)\begin{pmatrix} \boldsymbol{a}^{\mathsf{T}} \\ a_0\boldsymbol{I} + \tilde{\boldsymbol{a}} \end{pmatrix}, \qquad s := |A|^2, \qquad |A| \neq 0. \tag{23}$$

Equation (22) finally becomes

$$\boldsymbol{y} = s\boldsymbol{R}\boldsymbol{x}. \tag{24}$$

The matrix $\boldsymbol{R}$ is a rotation matrix as it has the properties

$$\boldsymbol{R}\boldsymbol{R}^{\mathsf{T}} = \boldsymbol{R}^{\mathsf{T}}\boldsymbol{R} = \boldsymbol{I}, \quad \mathrm{Det}(\boldsymbol{R}) = 1, \tag{25}$$

which can be verified by using the definition (23) and the identity

$$\tilde{\boldsymbol{x}}\,\tilde{\boldsymbol{y}} \equiv \boldsymbol{y}\boldsymbol{x}^{\mathsf{T}} - \boldsymbol{x}^{\mathsf{T}}\boldsymbol{y}\boldsymbol{I}. \tag{26}$$

The identity (26) is the vector triple product expansion, also known as Lagrange's formula, written in matrix notation. It follows that the product $A\,(0, \boldsymbol{x})\,A^*$ evaluates to a quaternion $(0,\, \boldsymbol{y})$ for which the vector $\boldsymbol{y}$ is the result of rotating and scaling the vector $\boldsymbol{x}$ by $\boldsymbol{R}$ and $s$, respectively. As a consequence, for any quaternion $A \in \mathbb{H}$, $A \neq 0$, there exists a rotation matrix $\boldsymbol{R} \in \mathrm{SO}(3)$ and a scaling factor $s \in \mathbb{R}^+$ such that

$$A\,(0, \boldsymbol{x})\,A^* = (0, s\boldsymbol{R}\boldsymbol{x})\,, \quad \forall \boldsymbol{x} \in \mathbb{R}^3. \tag{27}$$

To deduce the inverse, we take an arbitrary $\boldsymbol{R} \in \mathrm{SO}(3)$ and show the existence of an $A \in \mathbb{H}$ such that (27) holds. Any $\boldsymbol{R} \in \mathrm{SO}(3)$ may be represented as

$$\boldsymbol{R} = \boldsymbol{I} + \tilde{\boldsymbol{n}}\sin\varphi + \tilde{\boldsymbol{n}}^2(1 - \cos\varphi) \tag{28}$$

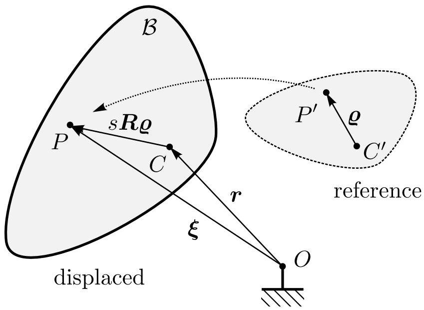
with $\boldsymbol{n} \in \mathbb{R}^3$ being the axis of rotation ($|\boldsymbol{n}| = 1$) and $\varphi$ the rotation angle. Since

$$(0, s\boldsymbol{R}) = \big(0, s\big(\boldsymbol{I} + \tilde{\boldsymbol{n}}\sin\varphi + \tilde{\boldsymbol{n}}^2(1 - \cos\varphi)\big)\big) = A\,(0, \boldsymbol{x})\,A^*, \quad \forall \boldsymbol{x} \in \mathbb{R}^3 \tag{29}$$

is fulfilled for any of the two quaternions

$$A = \pm\left(\sqrt{s}\cos\frac{\varphi}{2}, \boldsymbol{n}\sqrt{s}\sin\frac{\varphi}{2}\right), \tag{30}$$

**Fig. 1** Kinematics




one already has proven this assertion, i.e., that for any rotation matrix $\boldsymbol{R} \in \mathrm{SO}(3)$ and scaling factor $s \in \mathbb{R}^+$ there exists a quaternion $A \in \mathbb{H}$ such that

$$(0, s\boldsymbol{R}\boldsymbol{x}) = A\,(0, \boldsymbol{x})\,A^*, \quad \forall \boldsymbol{x} \in \mathbb{R}^3. \tag{31}$$

It has to be noted that the associated mapping $s\boldsymbol{R} \to A$ is not unique, because the two quaternions $A$ and $-A$ yield the same rotation and scaling.

## 4 Kinematics

In this section, the kinematics of the scalable body with three translational, three rotational, and one uniform scaling degree of freedom is described. Every point $P'$ of the scalable body in a reference configuration can be addressed by a fixed vector $\boldsymbol{\varrho} \in \mathbb{R}^3$ ($\boldsymbol{\varrho} = \text{const.}$) starting from a reference point $C'$ (cf. Fig. 1). The actual position $P$ of a point in a displaced configuration is described by the vector $\boldsymbol{\xi} \in \mathbb{R}^3$ starting at the inertial point $O$. The vector $\boldsymbol{\xi}$ can be obtained by applying a rotation $\boldsymbol{R} \in \mathrm{SO}(3)$ and a scaling $s \in \mathbb{R}^+$ on the vector $\boldsymbol{\varrho}$ and adding a displacement $\boldsymbol{r} \in \mathbb{R}^3$ for the translational degrees of freedom. This yields the kinematic relation

$$\boldsymbol{\xi} = s\boldsymbol{R}\boldsymbol{\varrho} + \boldsymbol{r} \tag{32}$$

for the scalable body. To parameterize the rotation $\boldsymbol{R}$ and the scaling $s$, a quaternion

$$A = (a_0, \boldsymbol{a}) \in \mathbb{H}, \quad |A| \neq 0 \tag{33}$$

is used, which allows to reformulate the kinematic relation (32) in quaternion notation

$$(0, \boldsymbol{\xi}) = A\,(0, \boldsymbol{\varrho})\,A^* + (0, \boldsymbol{r}) \tag{34}$$

by using the results from Sect. 3. The displacement $\boldsymbol{r}$ and the components of the quaternion $A$ are grouped into a generalized coordinates vector

$$\boldsymbol{q} := \begin{pmatrix} \boldsymbol{r} \\ a_0 \\ \boldsymbol{a} \end{pmatrix}. \tag{35}$$

The absolute velocity $\dot{\boldsymbol{\xi}}$ of a point on the body is obtained by differentiating equation (34) with respect to time,

$$
\begin{aligned}
(0, \dot{\boldsymbol{\xi}}) &= \dot{A}\,(0, \boldsymbol{\varrho})\,A^* + A\,(0, \boldsymbol{\varrho})\,\dot{A}^* + (0, \dot{\boldsymbol{r}}) \\
&= \frac{AA^*}{|A|^2}\dot{A}\,(0, \boldsymbol{\varrho})\,A^* + A\,(0, \boldsymbol{\varrho})\,\dot{A}^*\frac{AA^*}{|A|^2} + (0, \dot{\boldsymbol{r}}) \\
&= \frac{A}{|A|}\big((0, \boldsymbol{\varrho})\,\dot{A}^*A - A^*\dot{A}\,(0, -\boldsymbol{\varrho})\big)\frac{A^*}{|A|} + (0, \dot{\boldsymbol{r}}) \\
&= \frac{A}{|A|}\,\mathrm{Im}\big((0, \boldsymbol{\varrho})\,\big(2A^*\dot{A}\big)^*\big)\frac{A^*}{|A|} + (0, \dot{\boldsymbol{r}})\,.
\end{aligned} \tag{36}
$$

Here, it is useful to introduce new variables

$$
\boldsymbol{v} := \dot{\boldsymbol{r}}, \qquad (\nu, \boldsymbol{\omega}) := 2A^*\dot{A} \tag{37}
$$

for the terms containing the derivatives of the generalized coordinates. A geometric interpretation of $\boldsymbol{v}$, $\nu$, and $\boldsymbol{\omega}$ is given at the end of this section. The new variables $\boldsymbol{v}$, $\nu$, and $\boldsymbol{\omega}$ can then be taken to define the vector of generalized velocities as

$$
\boldsymbol{u} := \begin{pmatrix} \boldsymbol{v} \\ \nu \\ \boldsymbol{\omega} \end{pmatrix} = \underbrace{\begin{pmatrix} \boldsymbol{I} & 0 \\ 0 & 2\boldsymbol{\varphi}(A^*) \end{pmatrix}}_{=:\,\boldsymbol{Q}} \dot{\boldsymbol{q}} \tag{38}
$$

and relate them to the derivative of the generalized coordinates as shown. The matrix $\boldsymbol{Q} \in \mathbb{R}^{7\times 7}$ is regular for any $A \neq 0$ and can be obtained by rewriting (37) with the help of (15). In terms of the generalized velocities $\boldsymbol{u}$, (36) becomes

$$
\begin{aligned}
(0, \dot{\boldsymbol{\xi}}) &= \frac{A}{|A|}\,\mathrm{Im}\big((0, \boldsymbol{\varrho})\,(\nu, -\boldsymbol{\omega})\big)\frac{A^*}{|A|} + (0, \boldsymbol{v}) \\
&= \frac{A}{|A|}\,\mathrm{Im}\big((\boldsymbol{\varrho}^{\mathsf{T}}\boldsymbol{\omega}, \boldsymbol{\varrho}\nu - \tilde{\boldsymbol{\varrho}}\boldsymbol{\omega})\big)\frac{A^*}{|A|} + (0, \boldsymbol{v}) \\
&= \frac{A}{|A|}\,(0, \boldsymbol{\varrho}\nu - \tilde{\boldsymbol{\varrho}}\boldsymbol{\omega})\,\frac{A^*}{|A|} + (0, \boldsymbol{v}) \\
&= (0, \boldsymbol{R}\boldsymbol{\varrho}\nu - \boldsymbol{R}\tilde{\boldsymbol{\varrho}}\boldsymbol{\omega}) + (0, \boldsymbol{v}) \\
&= \big(0, (\boldsymbol{I} \;\; \boldsymbol{R}\boldsymbol{\varrho} \;\; -\boldsymbol{R}\tilde{\boldsymbol{\varrho}})\,\boldsymbol{u}\big)\,,
\end{aligned} \tag{39}
$$

where the quaternion product has been removed with the help of (27). Two purely imaginary quaternions are equal when their imaginary components are equal. This yields the equation

$$
\dot{\boldsymbol{\xi}} = (\boldsymbol{I} \;\; \boldsymbol{R}\boldsymbol{\varrho} \;\; -\boldsymbol{R}\tilde{\boldsymbol{\varrho}})\,\boldsymbol{Q}\dot{\boldsymbol{q}} \tag{40}
$$

relating the absolute velocity $\dot{\boldsymbol{\xi}}$ of a point of the scalable body to the derivative of the generalized velocities $\dot{\boldsymbol{q}}$.

The generalized velocity defined in (38) consists of the vector $\boldsymbol{v}$, the scalar $\nu$ and the vector $\boldsymbol{\omega}$. The velocity $\boldsymbol{v}$ is the absolute velocity of the point $C$, which can be seen directly from the definition. To get an interpretation of $\nu$ and $\boldsymbol{\omega}$, their definition (37) can be reformulated in terms of $\dot{s}$ and $\dot{\boldsymbol{R}}$. First, the absolute velocity $\dot{\boldsymbol{\xi}}$ expressed in $\dot{s}$ and $\dot{\boldsymbol{R}}$ is obtained by differentiating (32) with respect to time, yielding

$$\begin{aligned}\dot{\boldsymbol{\xi}} &= \dot{\boldsymbol{r}} + \dot{s}\boldsymbol{R}\boldsymbol{\varrho} + s\dot{\boldsymbol{R}}\boldsymbol{\varrho} \\ &= \dot{\boldsymbol{r}} + \boldsymbol{R}\big(\dot{s}\boldsymbol{\varrho} + s\boldsymbol{R}^{\mathsf{T}}\dot{\boldsymbol{R}}\boldsymbol{\varrho}\big).\end{aligned} \tag{41}$$

On the other hand, the absolute velocity $\dot{\boldsymbol{\xi}}$ in terms of $\nu$ and $\boldsymbol{\omega}$ can be obtained from the last line of (39),

$$\dot{\boldsymbol{\xi}} = \boldsymbol{v} + \boldsymbol{R}(\nu\boldsymbol{\varrho} + \tilde{\boldsymbol{\omega}}\boldsymbol{\varrho}). \tag{42}$$

Of course, both representations of the velocity field have to be equal for any point of the body, i.e.,

$$\boldsymbol{v} + \boldsymbol{R}\nu\boldsymbol{\varrho} + \boldsymbol{R}\tilde{\boldsymbol{\omega}}\boldsymbol{\varrho} = \dot{\boldsymbol{r}} + \boldsymbol{R}\dot{s}\boldsymbol{\varrho} + \boldsymbol{R}s\boldsymbol{R}^{\mathsf{T}}\dot{\boldsymbol{R}}\boldsymbol{\varrho}, \quad \forall\boldsymbol{\varrho} \in \mathbb{R}^3. \tag{43}$$

Solving this variational equation for $\boldsymbol{\varrho} = 0$ and $\boldsymbol{\varrho} \neq 0$ yields

$$\boldsymbol{v} = \dot{\boldsymbol{r}}, \qquad \nu = \dot{s}, \qquad \tilde{\boldsymbol{\omega}} = s\boldsymbol{R}^{\mathsf{T}}\dot{\boldsymbol{R}}. \tag{44}$$

Obviously $\nu$ is the scalar scaling velocity associated with the scaling factor $s$. The absolute angular velocity $\boldsymbol{\Omega}$ associated with a rotation $\boldsymbol{R}$ is given by

$$\tilde{\boldsymbol{\Omega}} := \dot{\boldsymbol{R}}\boldsymbol{R}^{\mathsf{T}}. \tag{45}$$

Expressing $\tilde{\boldsymbol{\omega}}$ in terms of $\tilde{\boldsymbol{\Omega}}$,

$$\tilde{\boldsymbol{\omega}} = s\boldsymbol{R}^{\mathsf{T}}\dot{\boldsymbol{R}} = s\boldsymbol{R}^{\mathsf{T}}\dot{\boldsymbol{R}}\boldsymbol{R}^{\mathsf{T}}\boldsymbol{R} = s\boldsymbol{R}^{\mathsf{T}}\tilde{\boldsymbol{\Omega}}\boldsymbol{R}, \tag{46}$$

together with the rotational invariance of the cross product

$$(\boldsymbol{R}\boldsymbol{x})^{\sim} = \boldsymbol{R}\tilde{\boldsymbol{x}}\boldsymbol{R}^{\mathsf{T}}, \quad \forall\boldsymbol{x} \in \mathbb{R}^3,\ \boldsymbol{R} \in \mathrm{SO}(3) \tag{47}$$

yields the matrix relation

$$\tilde{\boldsymbol{\omega}} = s\big(\boldsymbol{R}^{\mathsf{T}}\boldsymbol{\Omega}\big)^{\sim}. \tag{48}$$

Removing the cross product operator one gets the equation

$$\boldsymbol{\omega} = s\boldsymbol{R}^{\mathsf{T}}\boldsymbol{\Omega}. \tag{49}$$

This means the vector $\boldsymbol{\omega}$ is the angular velocity $\boldsymbol{\Omega}$ associated with the rotation $\boldsymbol{R}$, scaled by the factor $s$ and rotated with $\boldsymbol{R}^{\mathsf{T}}$ from the displaced configuration back to the reference configuration.

## 5 Principle of virtual work

The equations of motion of an infinite-dimensional mechanical system are classically described by the principle of virtual work

$$\delta W = \int_{S} \delta\boldsymbol{\xi}^{\mathsf{T}}(\ddot{\boldsymbol{\xi}}\,\mathrm{d}m - \mathrm{d}\boldsymbol{F}) = 0, \quad \forall\delta\boldsymbol{\xi} \tag{50}$$

where $\mathrm{d}m$ is the mass distribution and $\boldsymbol{\xi}$ describes the displacement at each point of the system as already shown in Fig. 1. For more background on the principle of virtual work, the reader is referred to [4, 5, 7, 17]. The force distribution $\mathrm{d}\boldsymbol{F}$ in classical dynamics is composed of the constraint forces $\mathrm{d}\boldsymbol{F}_z$ and some remaining forces $\mathrm{d}\boldsymbol{F}_q$, i.e.,

$$\mathrm{d}\boldsymbol{F} = \mathrm{d}\boldsymbol{F}_z + \mathrm{d}\boldsymbol{F}_q. \tag{51}$$

All constraints acting on a mechanical system may implicitly be taken into account by a minimal parameterization of the surviving degrees of freedom via the so-called generalized coordinates $\boldsymbol{q}$. If the constraints are bilateral and perfect, they fulfill by definition the principle of d'Alembert–Lagrange which reads

$$\boldsymbol{\xi} = \boldsymbol{\xi}(\boldsymbol{\varrho}, \boldsymbol{q}, t), \quad \int_{\mathcal{S}} \delta\boldsymbol{\xi}^{\mathsf{T}} \mathrm{d}\boldsymbol{F}_z = 0, \quad \delta\boldsymbol{\xi} = \frac{\partial \boldsymbol{\xi}}{\partial \boldsymbol{q}} \delta\boldsymbol{q}, \quad \forall \delta\boldsymbol{q}. \tag{52}$$

The principle of d'Alembert–Lagrange is the force law of the perfect bilateral constraints that restrict the motion to the remaining degrees of freedom taken into account by $\boldsymbol{q}$. It states that the virtual work of the constraint forces has to vanish for any virtual displacements compatible with the constraint. If the variation $\delta\boldsymbol{\xi}$ is restricted to variations induced by $\delta\boldsymbol{q}$ then the constraint forces $\mathrm{d}\boldsymbol{F}_z$ disappear in (50) and one gets the variational equations of motion

$$\delta\boldsymbol{q}^{\mathsf{T}} \int_{\mathcal{S}} \left(\frac{\partial \boldsymbol{\xi}}{\partial \boldsymbol{q}}\right)^{\mathsf{T}} \ddot{\boldsymbol{\xi}}\, \mathrm{d}m - \delta\boldsymbol{q}^{\mathsf{T}} \underbrace{\int_{\mathcal{S}} \left(\frac{\partial \boldsymbol{\xi}}{\partial \boldsymbol{q}}\right)^{\mathsf{T}} \mathrm{d}\boldsymbol{F}_q}_{=:\, \boldsymbol{f}_q} = 0, \quad \forall \delta\boldsymbol{q} \tag{53}$$

for the system in terms of the coordinates $\boldsymbol{q}$. Classically, the acceleration terms in (53) are reformulated as a difference

$$\left(\frac{\partial \boldsymbol{\xi}}{\partial \boldsymbol{q}}\right)^{\mathsf{T}} \ddot{\boldsymbol{\xi}} = \frac{\mathrm{d}}{\mathrm{d}t}\left[\left(\frac{\partial \boldsymbol{\xi}}{\partial \boldsymbol{q}}\right)^{\mathsf{T}} \dot{\boldsymbol{\xi}}\right] - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \boldsymbol{\xi}}{\partial \boldsymbol{q}}\right)^{\mathsf{T}} \dot{\boldsymbol{\xi}} \tag{54}$$

by using Leibniz's law. The partial derivatives of $\boldsymbol{\xi}$ in (54) can be expressed as partial derivative of the absolute velocity,

$$\frac{\partial \boldsymbol{\xi}}{\partial \boldsymbol{q}} = \frac{\partial \dot{\boldsymbol{\xi}}}{\partial \dot{\boldsymbol{q}}}, \qquad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \boldsymbol{\xi}}{\partial \boldsymbol{q}}\right) = \frac{\partial \dot{\boldsymbol{\xi}}}{\partial \boldsymbol{q}}, \tag{55}$$

where the absolute velocity function $\dot{\boldsymbol{\xi}}$ is given by

$$\dot{\boldsymbol{\xi}}(\boldsymbol{\varrho}, \dot{\boldsymbol{q}}, \boldsymbol{q}, t) := \frac{\partial \boldsymbol{\xi}}{\partial \boldsymbol{q}} \dot{\boldsymbol{q}} + \frac{\partial \boldsymbol{\xi}}{\partial t}. \tag{56}$$

Combining (53), (54), and (55) yields the variational equations of motion

$$\delta\boldsymbol{q}^{\mathsf{T}} \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathcal{S}} \left(\frac{\partial \dot{\boldsymbol{\xi}}}{\partial \dot{\boldsymbol{q}}}\right)^{\mathsf{T}} \dot{\boldsymbol{\xi}}\, \mathrm{d}m - \delta\boldsymbol{q}^{\mathsf{T}} \int_{\mathcal{S}} \left(\frac{\partial \dot{\boldsymbol{\xi}}}{\partial \boldsymbol{q}}\right)^{\mathsf{T}} \dot{\boldsymbol{\xi}}\, \mathrm{d}m - \delta\boldsymbol{q}^{\mathsf{T}} \boldsymbol{f}_q = 0, \quad \forall \delta\boldsymbol{q} \tag{57}$$

in terms of the absolute velocity $\dot{\boldsymbol{\xi}}$ and its derivatives. Instead of using the absolute velocity $\dot{\boldsymbol{\xi}}$ one can introduce the kinetic energy

$$T(\dot{\boldsymbol{q}}, \boldsymbol{q}, t) := \frac{1}{2} \int_{\mathcal{S}} \dot{\boldsymbol{\xi}}^{\mathsf{T}} \dot{\boldsymbol{\xi}}\, \mathrm{d}m, \tag{58}$$

and replace the partial derivatives in (57) by partial derivatives of the kinetic energy

$$\delta \boldsymbol{q}^{\mathsf{T}} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{\boldsymbol{q}}}\right)^{\mathsf{T}} - \delta \boldsymbol{q}^{\mathsf{T}}\left(\frac{\partial T}{\partial \boldsymbol{q}}\right)^{\mathsf{T}} - \delta \boldsymbol{q}^{\mathsf{T}} \boldsymbol{f}_{q} = 0, \quad \forall \delta \boldsymbol{q}. \tag{59}$$

This variational equation of motion describes the dynamics of a mechanical system with kinematics $\boldsymbol{\xi}(\boldsymbol{\varrho}, \boldsymbol{q}, t)$ enforced by perfect bilateral constraints of d'Alembert–Lagrange type. The variational equations (53) and (57) describe exactly the same dynamics as (59) but in terms of the function $\boldsymbol{\xi}$ and its derivatives. Equation (59) is very close to Lagrange's equations of the second kind, except that it is formulated as a variational equation and it still contains a general force term $\boldsymbol{f}_q$, in which any additional forces can be considered.

## 6 Equations of motion

The variational equations of motion described in Sect. 5 can be combined with the kinematics from Sect. 4 to obtain the equations of motion of the scalable body. In the following, the variational formulation (59) based on the kinetic energy will be used to derive the equations of motion. Alternatively, one could also use (53) or (57) directly. As a first step, the definition of the kinetic energy (58) is evaluated with the absolute velocity (40), which yields

$$\begin{aligned} T &= \frac{1}{2}\int_{\mathcal{B}} \dot{\boldsymbol{\xi}}^{\mathsf{T}} \dot{\boldsymbol{\xi}} \,\mathrm{d}m \\ &= \frac{1}{2}\dot{\boldsymbol{q}}^{\mathsf{T}} \boldsymbol{Q}^{\mathsf{T}} \underbrace{\int_{\mathcal{B}} \begin{pmatrix} \boldsymbol{I} & \boldsymbol{R}\boldsymbol{\varrho} & -\boldsymbol{R}\tilde{\boldsymbol{\varrho}} \\ \boldsymbol{\varrho}^{\mathsf{T}}\boldsymbol{R}^{\mathsf{T}} & \boldsymbol{\varrho}^{\mathsf{T}}\boldsymbol{\varrho} & 0 \\ \tilde{\boldsymbol{\varrho}}\boldsymbol{R}^{\mathsf{T}} & 0 & -\tilde{\boldsymbol{\varrho}}^2 \end{pmatrix} \mathrm{d}m}_{=:\,\boldsymbol{M}} \boldsymbol{Q}\dot{\boldsymbol{q}} \\ &= \frac{1}{2}\dot{\boldsymbol{q}}^{\mathsf{T}} \boldsymbol{Q}^{\mathsf{T}} \boldsymbol{M} \boldsymbol{Q}\dot{\boldsymbol{q}}. \end{aligned} \tag{60}$$

The arbitrary reference point $C'$ introduced in Sect. 4 is chosen now to be identical with the center of mass of the mass distribution in the reference configuration. This means that the integral

$$\int_{\mathcal{B}} \boldsymbol{\varrho} \,\mathrm{d}m = 0 \tag{61}$$

is always zero. Two abbreviations

$$m := \int_{\mathcal{B}} \mathrm{d}m, \qquad \boldsymbol{\Theta} := \int_{\mathcal{B}} -\tilde{\boldsymbol{\varrho}}^2 \,\mathrm{d}m \tag{62}$$

are introduced to represent the mass distribution in the body. The symbol $m$ is the total mass and $\boldsymbol{\Theta}$ is the classical inertia tensor with respect to the center of mass in the reference configuration. The remaining integral

$$\int_{\mathcal{B}} \boldsymbol{\varrho}^{\mathsf{T}} \boldsymbol{\varrho} \,\mathrm{d}m = \frac{1}{2} \operatorname{Tr} \boldsymbol{\Theta} \tag{63}$$

is half the trace of the classical inertia tensor $\boldsymbol{\Theta}$ as can be verified easily. This finally yields the constant and symmetric mass matrix

$$\boldsymbol{M} = \begin{pmatrix} m\boldsymbol{I} & 0 & 0 \\ 0 & \frac{1}{2}\operatorname{Tr}\boldsymbol{\Theta} & 0 \\ 0 & 0 & \boldsymbol{\Theta} \end{pmatrix}. \tag{64}$$

If the classical inertia tensor $\boldsymbol{\Theta}$ of the body is positive definite, then the mass matrix $\boldsymbol{M}$ is positive definite as well. For the lower right submatrix of $\boldsymbol{M}$ the abbreviation

$$\hat{\boldsymbol{\Theta}} := \begin{pmatrix} \frac{1}{2}\operatorname{Tr}\boldsymbol{\Theta} & 0 \\ 0 & \boldsymbol{\Theta} \end{pmatrix} \tag{65}$$

is introduced. The kinetic energy $T$ as given in (60) is a quadratic form in $\dot{\boldsymbol{q}}$, from which the partial derivative $\partial T/\partial\dot{\boldsymbol{q}}$ can be obtained directly. To calculate the partial derivative $\partial T/\partial\boldsymbol{q}$ it is useful to note that in this case the kinetic energy can be formulated as the sum of a quadratic form in $\boldsymbol{q}$ plus a term not depending on $\boldsymbol{q}$. One gets

$$\begin{aligned} T &= \frac{1}{2}\dot{\boldsymbol{q}}^{\mathsf{T}}\boldsymbol{Q}^{\mathsf{T}}\boldsymbol{M}\,\boldsymbol{Q}\dot{\boldsymbol{q}} \\ &= \frac{m}{2}\dot{\boldsymbol{r}}^{\mathsf{T}}\dot{\boldsymbol{r}} + 2\,\boldsymbol{\psi}^{\mathsf{T}}(\dot{A})\,\boldsymbol{\varphi}(A)\,\hat{\boldsymbol{\Theta}}\,\boldsymbol{\varphi}^{\mathsf{T}}(A)\,\boldsymbol{\psi}(\dot{A}) \\ &= \frac{m}{2}\dot{\boldsymbol{r}}^{\mathsf{T}}\dot{\boldsymbol{r}} + 2\,\boldsymbol{\psi}^{\mathsf{T}}(A)\,\boldsymbol{\varphi}(\dot{A})\,\boldsymbol{T}^{\mathsf{T}}\hat{\boldsymbol{\Theta}}\,\boldsymbol{T}\,\boldsymbol{\varphi}^{\mathsf{T}}(\dot{A})\,\boldsymbol{\psi}(A) \\ &= \frac{m}{2}\dot{\boldsymbol{r}}^{\mathsf{T}}\dot{\boldsymbol{r}} + 2\,\boldsymbol{\psi}^{\mathsf{T}}(A)\,\boldsymbol{\varphi}(\dot{A})\,\hat{\boldsymbol{\Theta}}\,\boldsymbol{\varphi}^{\mathsf{T}}(\dot{A})\,\boldsymbol{\psi}(A) \\ &= \frac{m}{2}\dot{\boldsymbol{r}}^{\mathsf{T}}\dot{\boldsymbol{r}} + \frac{1}{2}\boldsymbol{q}^{\mathsf{T}}\dot{\boldsymbol{Q}}^{\mathsf{T}}\boldsymbol{M}\,\dot{\boldsymbol{Q}}\boldsymbol{q} \end{aligned} \tag{66}$$

where the identity

$$\boldsymbol{\varphi}^{\mathsf{T}}(A)\boldsymbol{\psi}(B) = \boldsymbol{\psi}\big(A^*B\big) = \boldsymbol{\psi}\big((B^*A)^*\big) = \boldsymbol{T}\,\boldsymbol{\psi}\big(B^*A\big) = \boldsymbol{T}\,\boldsymbol{\varphi}^{\mathsf{T}}(B)\boldsymbol{\psi}(A) \tag{67}$$

has been used. Using these two representations of the kinetic energy, one obtains

$$\left(\frac{\partial T}{\partial\dot{\boldsymbol{q}}}\right)^{\mathsf{T}} = \boldsymbol{Q}^{\mathsf{T}}\boldsymbol{M}\,\boldsymbol{Q}\dot{\boldsymbol{q}}, \qquad \left(\frac{\partial T}{\partial\boldsymbol{q}}\right)^{\mathsf{T}} = \dot{\boldsymbol{Q}}^{\mathsf{T}}\boldsymbol{M}\,\dot{\boldsymbol{Q}}\boldsymbol{q} \tag{68}$$

for the partial derivatives. Inserting them into the variational equations of motion (59) yields

$$\delta\boldsymbol{q}^{\mathsf{T}}\left(\frac{\mathrm{d}}{\mathrm{d}t}\big(\boldsymbol{Q}^{\mathsf{T}}\boldsymbol{M}\,\boldsymbol{Q}\dot{\boldsymbol{q}}\big) - \dot{\boldsymbol{Q}}^{\mathsf{T}}\boldsymbol{M}\,\dot{\boldsymbol{Q}}\boldsymbol{q} - \boldsymbol{f}_q\right) = 0 \quad \forall\delta\boldsymbol{q}. \tag{69}$$

By evaluating the time derivative and the variation, one gets the equations of motion

$$\boldsymbol{Q}^{\mathsf{T}}\boldsymbol{M}\,\boldsymbol{Q}\ddot{\boldsymbol{q}} + \boldsymbol{Q}^{\mathsf{T}}\boldsymbol{M}\,\dot{\boldsymbol{Q}}\dot{\boldsymbol{q}} + \dot{\boldsymbol{Q}}^{\mathsf{T}}\boldsymbol{M}\big(\boldsymbol{Q}\dot{\boldsymbol{q}} - \dot{\boldsymbol{Q}}\boldsymbol{q}\big) - \boldsymbol{f}_q = 0 \tag{70}$$

of the scalable body. The only thing that remains to do, is to specify the force distribution $\mathrm{d}\boldsymbol{F}_q$ and to calculate from it the associated generalized force $\boldsymbol{f}_q$,

$$\boldsymbol{f}_q = \int_{\mathcal{B}}\left(\frac{\partial\boldsymbol{\xi}}{\partial\boldsymbol{q}}\right)^{\mathsf{T}}\mathrm{d}\boldsymbol{F}_q. \tag{71}$$

Before doing this, the force distribution is split up once more

$$\mathrm{d}\boldsymbol{F}_q = \mathrm{d}\boldsymbol{F}_e + \mathrm{d}\boldsymbol{F}_g \tag{72}$$

into a portion $\mathrm{d}\boldsymbol{F}_e$ and a force distribution $\mathrm{d}\boldsymbol{F}_g$ which will be used in Sect. 7 to realize an additional perfect bilateral constraint that makes the scalable body rigid. According to (71), the generalized force associated with $\mathrm{d}\boldsymbol{F}_g$ is denoted by

$$\boldsymbol{f}_g := \int_{\mathcal{B}} \left(\frac{\partial \boldsymbol{\xi}}{\partial \boldsymbol{q}}\right)^{\mathsf{T}} \mathrm{d}\boldsymbol{F}_g. \tag{73}$$

The partial derivative occurring in (71) can be obtained via relation (55) and the absolute velocity (40),

$$\frac{\partial \boldsymbol{\xi}}{\partial \boldsymbol{q}} = \frac{\partial \dot{\boldsymbol{\xi}}}{\partial \dot{\boldsymbol{q}}} = (\boldsymbol{I} \quad \boldsymbol{R}\boldsymbol{\varrho} \quad -\boldsymbol{R}\tilde{\boldsymbol{\varrho}})\,\boldsymbol{Q}. \tag{74}$$

The three integrals that result for $\mathrm{d}\boldsymbol{F}_e$ when putting (74) into (71) are

$$\boldsymbol{F} := \int_{\mathcal{B}} \mathrm{d}\boldsymbol{F}_e, \qquad S_C := \int_{\mathcal{B}} \boldsymbol{\varrho}^{\mathsf{T}} \boldsymbol{R}^{\mathsf{T}} \mathrm{d}\boldsymbol{F}_e, \qquad \boldsymbol{M}_C := \int_{\mathcal{B}} \tilde{\boldsymbol{\varrho}} \boldsymbol{R}^{\mathsf{T}} \mathrm{d}\boldsymbol{F}_e \tag{75}$$

and have the following meaning: The vector $\boldsymbol{F}$ is the resultant external force and the vector $\boldsymbol{M}_C$ is the resultant moment with respect to the point $C$. The scalar $S_C$ is the resultant scaling force with respect to point $C$. The resultant scaling force and the resultant moment are formed by rotating the external forces with $\boldsymbol{R}^{\mathsf{T}}$ back from the displaced configuration to the reference configuration. With these abbreviations, one gets the complete generalized force (71) as

$$\boldsymbol{f}_q = \boldsymbol{Q}^{\mathsf{T}} \begin{pmatrix} \boldsymbol{F} \\ S_C \\ \boldsymbol{M}_C \end{pmatrix} + \boldsymbol{f}_g. \tag{76}$$

Setting the resultant forces and the additional generalized force $\boldsymbol{f}_g$ equal to zero results in a generalized force $\boldsymbol{f}_q$ which is equal to zero as well. In this case, the equations of motion (70) would describe the dynamics of a free scalable body.

## 7 Rigid body

In this section, an additional perfect bilateral constraint is applied on the scalable body in order to make it rigid. The scaling is the only additional degree of freedom that makes the scalable body different from a rigid body. For a rigid body, the scaling $s$ as introduced in Sect. 4 is always equal to one. The corresponding constraint equation can be written by (23) as

$$g(\boldsymbol{q}) = |A|^2 - 1 = 0. \tag{77}$$

A force law of d'Alembert/Lagrange type

$$\delta \boldsymbol{q}^{\mathsf{T}} \boldsymbol{f}_g = 0 \quad \forall \delta \boldsymbol{q} \mid \delta g = 0 \tag{78}$$

will now be formulated to complete the description of the perfect bilateral constraint. The product $\delta \boldsymbol{q}^{\mathsf{T}} \boldsymbol{f}_g$ is the virtual work done by the constraint force (cf. Sect. 5). The virtual

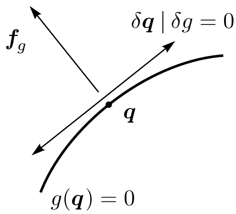



**Fig. 2** Constraint

work has to vanish for any virtual displacements induced by $\delta \boldsymbol{q}$ that are compatible with the constraint (i.e., $\delta g = 0$). A simplified illustration of this situation is shown in Fig. 2. The relation between variations $\delta g$ of the constraint and variations $\delta \boldsymbol{q}$ of the generalized coordinates is classically given by

$$\delta g = \frac{\partial g}{\partial \boldsymbol{q}} \delta \boldsymbol{q}. \tag{79}$$

Combining this with the force law, (78) yields

$$\delta \boldsymbol{q}^{\mathsf{T}} \boldsymbol{f}_g = 0 \quad \forall \delta \boldsymbol{q} \mid \delta \boldsymbol{q}^{\mathsf{T}} \left( \frac{\partial g}{\partial \boldsymbol{q}} \right)^{\mathsf{T}} = 0. \tag{80}$$

Evaluating the variation reveals that the generalized constraint force $\boldsymbol{f}_g$ lies in the linear subspace spanned by the vector $(\partial g / \partial \boldsymbol{q})^{\mathsf{T}}$. This can be formulated with (35) and (14) as

$$\boldsymbol{f}_g = \left( \frac{\partial g}{\partial \boldsymbol{q}} \right)^{\mathsf{T}} \lambda = \begin{pmatrix} 0 \\ 2\boldsymbol{\psi}(A) \end{pmatrix} \lambda, \quad \lambda \in \mathbb{R}, \tag{81}$$

where $\lambda$ is the scalar constraint force associated with the constraint. Inserting the generalized constraint force $\boldsymbol{f}_g$ into (70) finally yields with the help of (76) the DAE description

$$\boldsymbol{Q}^{\mathsf{T}} \boldsymbol{M} \boldsymbol{Q} \ddot{\boldsymbol{q}} + \boldsymbol{Q}^{\mathsf{T}} \boldsymbol{M} \dot{\boldsymbol{Q}} \dot{\boldsymbol{q}} + \dot{\boldsymbol{Q}}^{\mathsf{T}} \boldsymbol{M} \big( \boldsymbol{Q} \dot{\boldsymbol{q}} - \dot{\boldsymbol{Q}} \boldsymbol{q} \big) - \boldsymbol{Q}^{\mathsf{T}} \begin{pmatrix} \boldsymbol{F} \\ S_C + \lambda \\ \boldsymbol{M}_C \end{pmatrix} = 0, \quad |A|^2 = 1 \tag{82}$$

of the dynamics of a rigid body. In this formulation, the unit length restriction of the quaternion is explicitly contained as algebraic constraint. The associated constraint force $\lambda$ is mechanically consistent, as setting it to zero and dropping the constraint equation restores the unrestricted dynamics of the scalable body.

## 8 Generalized velocities

The DAE formulation of the dynamics of a rigid body obtained in the last section can be further simplified by replacing the derivative of the generalized coordinates $\dot{\boldsymbol{q}}$ with the generalized velocities $\boldsymbol{u}$ as introduced in Sect. 4. As a first step, the equations of motion (69) based on the principle of virtual work are replaced by an equivalent formulation based on the principle of virtual power

$$\delta \dot{\boldsymbol{q}}^{\mathsf{T}} \left( \frac{\mathrm{d}}{\mathrm{d}t} \big( \boldsymbol{Q}^{\mathsf{T}} \boldsymbol{M} \boldsymbol{Q} \dot{\boldsymbol{q}} \big) - \dot{\boldsymbol{Q}}^{\mathsf{T}} \boldsymbol{M} \dot{\boldsymbol{Q}} \boldsymbol{q} - \boldsymbol{f}_q \right) = 0 \quad \forall \delta \dot{\boldsymbol{q}}. \tag{83}$$

Next, the kinematic relation (38) is solved for the derivative of the generalized coordinates

$$\dot{\boldsymbol{q}} = \boldsymbol{Q}^{-1}\boldsymbol{u}. \tag{84}$$

The inverse $\boldsymbol{Q}^{-1}$ can be obtained by inverting each block on its diagonal. With the help of relation (13) one gets

$$\boldsymbol{Q}^{-1} = \begin{pmatrix} \boldsymbol{I} & 0 \\ 0 & \frac{1}{2|A|^2}\boldsymbol{\varphi}(A) \end{pmatrix}. \tag{85}$$

Equation (84) applies in the same form for the virtual velocities,

$$\delta\dot{\boldsymbol{q}} = \boldsymbol{Q}^{-1}\delta\boldsymbol{u}. \tag{86}$$

Inserting (84) and (86) into (83) yields

$$\delta\boldsymbol{u}^{\mathsf{T}}\boldsymbol{Q}^{-\mathsf{T}}\frac{\mathrm{d}}{\mathrm{d}t}\big(\boldsymbol{Q}^{\mathsf{T}}\boldsymbol{M}\boldsymbol{u}\big) - \delta\boldsymbol{u}^{\mathsf{T}}\boldsymbol{Q}^{-\mathsf{T}}\dot{\boldsymbol{Q}}^{\mathsf{T}}\boldsymbol{M}\dot{\boldsymbol{Q}}\boldsymbol{q} - \delta\boldsymbol{u}^{\mathsf{T}}\boldsymbol{Q}^{-\mathsf{T}}\boldsymbol{f}_q = 0 \quad \forall\delta\boldsymbol{u}. \tag{87}$$

After having carried out the derivatives with respect to time, one gets

$$\delta\boldsymbol{u}^{\mathsf{T}}\boldsymbol{M}\dot{\boldsymbol{u}} + \delta\boldsymbol{u}^{\mathsf{T}}\boldsymbol{Q}^{-\mathsf{T}}\dot{\boldsymbol{Q}}^{\mathsf{T}}\boldsymbol{M}\big(\boldsymbol{u} - \dot{\boldsymbol{Q}}\boldsymbol{q}\big) - \delta\boldsymbol{u}^{\mathsf{T}}\boldsymbol{Q}^{-\mathsf{T}}\boldsymbol{f}_q = 0 \quad \forall\delta\boldsymbol{u}. \tag{88}$$

To further simplify this equation, the product $\boldsymbol{Q}^{-\mathsf{T}}\dot{\boldsymbol{Q}}^{\mathsf{T}}$ is evaluated in terms of the generalized velocities $\boldsymbol{u}$ and the generalized coordinates $\boldsymbol{q}$. The matrix $\boldsymbol{Q}^{-\mathsf{T}}$ can be obtained by transposing (85) and applying relation (11). Transposing the time derivative of (38) yields the matrix $\dot{\boldsymbol{Q}}^{\mathsf{T}}$. For the product, one obtains

$$\begin{aligned}\boldsymbol{Q}^{-\mathsf{T}}\dot{\boldsymbol{Q}}^{\mathsf{T}} &= \begin{pmatrix} \boldsymbol{I} & 0 \\ 0 & \frac{1}{2|A|^2}\boldsymbol{\varphi}(A^*) \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 2\boldsymbol{\varphi}(\dot{A}) \end{pmatrix} = \frac{1}{2|A|^2}\begin{pmatrix} 0 & 0 \\ 0 & \boldsymbol{\varphi}(2A^*\dot{A}) \end{pmatrix} \\ &= \frac{1}{2|A|^2}\begin{pmatrix} 0 & 0 \\ 0 & \boldsymbol{\varphi}((\nu,\boldsymbol{\omega})) \end{pmatrix} = \frac{1}{2|A|^2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & \nu & -\boldsymbol{\omega}^{\mathsf{T}} \\ 0 & \boldsymbol{\omega} & \nu\boldsymbol{I} + \tilde{\boldsymbol{\omega}} \end{pmatrix}\end{aligned} \tag{89}$$

by using relations (10) and (12) as well as the kinematic relation (37) in quaternion notation. Similarly, the product $\dot{\boldsymbol{Q}}\boldsymbol{q}$ can be evaluated

$$\dot{\boldsymbol{Q}}\boldsymbol{q} = \begin{pmatrix} 0 & 0 \\ 0 & 2\boldsymbol{\varphi}(\dot{A}^*) \end{pmatrix}\begin{pmatrix} \boldsymbol{r} \\ \boldsymbol{\psi}(A) \end{pmatrix} = \begin{pmatrix} 0 \\ \boldsymbol{\psi}(2\dot{A}^*A) \end{pmatrix} = \begin{pmatrix} 0 \\ \boldsymbol{\psi}((\nu,\boldsymbol{\omega})^*) \end{pmatrix} = \begin{pmatrix} 0 \\ \nu \\ -\boldsymbol{\omega} \end{pmatrix}. \tag{90}$$

Combining the results from (89) and (90), one gets the simplification

$$\begin{aligned}&\boldsymbol{Q}^{-\mathsf{T}}\dot{\boldsymbol{Q}}^{\mathsf{T}}\boldsymbol{M}(\boldsymbol{u} - \dot{\boldsymbol{Q}}\boldsymbol{q}) \\ &\quad = \frac{1}{2|A|^2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & \nu & -\boldsymbol{\omega}^{\mathsf{T}} \\ 0 & \boldsymbol{\omega} & \nu\boldsymbol{I} + \tilde{\boldsymbol{\omega}} \end{pmatrix}\begin{pmatrix} m\boldsymbol{I} & 0 & 0 \\ 0 & \frac{1}{2}\operatorname{Tr}\boldsymbol{\Theta} & 0 \\ 0 & 0 & \boldsymbol{\Theta} \end{pmatrix}\begin{pmatrix} \boldsymbol{v} \\ 0 \\ 2\boldsymbol{\omega} \end{pmatrix} \\ &\quad = \frac{1}{2|A|^2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & \nu & -\boldsymbol{\omega}^{\mathsf{T}} \\ 0 & \boldsymbol{\omega} & \nu\boldsymbol{I} + \tilde{\boldsymbol{\omega}} \end{pmatrix}\begin{pmatrix} m\boldsymbol{v} \\ 0 \\ 2\boldsymbol{\Theta}\boldsymbol{\omega} \end{pmatrix} = \frac{1}{|A|^2}\begin{pmatrix} 0 \\ -\boldsymbol{\omega}^{\mathsf{T}} \\ \nu\boldsymbol{I} + \tilde{\boldsymbol{\omega}} \end{pmatrix}\boldsymbol{\Theta}\,\boldsymbol{\omega}\end{aligned} \tag{91}$$

for the product occurring in (88). Inserting the simplification (91) and the generalized force (76) together with the constraint force (81) into (88), one gets the variational equation

$$\delta\boldsymbol{u}^\mathsf{T}\boldsymbol{M}\dot{\boldsymbol{u}} + \delta\boldsymbol{u}^\mathsf{T}\frac{1}{|A|^2}\begin{pmatrix} 0 \\ -\boldsymbol{\omega}^\mathsf{T} \\ \nu\boldsymbol{I} + \tilde{\boldsymbol{\omega}} \end{pmatrix}\boldsymbol{\Theta}\,\boldsymbol{\omega} - \delta\boldsymbol{u}^\mathsf{T}\begin{pmatrix} \boldsymbol{F} \\ S_C + \lambda \\ \boldsymbol{M}_C \end{pmatrix} = 0 \quad \forall\delta\boldsymbol{u}. \tag{92}$$

Eliminating the variation and completing the set of equations with the kinematic relation (84) and the constraint equation (77), one gets the full DAE formulation for the dynamics of a rigid body

$$\begin{cases} m\dot{\boldsymbol{v}} = \boldsymbol{F}, \\ \dfrac{1}{2}\operatorname{Tr}\boldsymbol{\Theta}\,\dot{\nu} - \dfrac{1}{|A|^2}\boldsymbol{\omega}^\mathsf{T}\boldsymbol{\Theta}\boldsymbol{\omega} = S_C + \lambda, \\ \boldsymbol{\Theta}\dot{\boldsymbol{\omega}} + \dfrac{1}{|A|^2}(\nu\boldsymbol{I} + \tilde{\boldsymbol{\omega}})\boldsymbol{\Theta}\boldsymbol{\omega} = \boldsymbol{M}_C, \end{cases} \qquad \begin{cases} \dot{\boldsymbol{r}} = \boldsymbol{v}, \\ \dot{A} = \dfrac{1}{2|A|^2}A\,(\nu, \boldsymbol{\omega}), \end{cases} \qquad |A|^2 = 1 \tag{93}$$

in terms of the generalized velocities $\boldsymbol{v}$, $\nu$, and $\boldsymbol{\omega}$. This formulation is equivalent to (82). If the constraint force $\lambda$ is set to zero and the constraint equation $|A|^2 = 1$ is removed, one recovers the full seven degree of freedom dynamics of the scalable body. The rigid body DAE (93) can be simplified to the classical Newton–Euler equations when the constraint is differentiated twice

$$s = |A|^2 = 1 \quad \Rightarrow \quad \dot{s} = \nu = 0, \quad \dot{\nu} = 0 \tag{94}$$

and all occurrences of $|A|^2$, $\nu$ and $\dot{\nu}$ are eliminated. One then gets the ODE of a rigid body

$$\begin{cases} m\dot{\boldsymbol{v}} = \boldsymbol{F}, \\ \boldsymbol{\Theta}\dot{\boldsymbol{\omega}} + \tilde{\boldsymbol{\omega}}\boldsymbol{\Theta}\boldsymbol{\omega} = \boldsymbol{M}_C, \end{cases} \qquad \begin{cases} \dot{\boldsymbol{r}} = \boldsymbol{v}, \\ \dot{A} = \dfrac{1}{2}A\,(0, \boldsymbol{\omega}) \end{cases} \tag{95}$$

and an equation to calculate the scaling constraint force as

$$\lambda = -\boldsymbol{\omega}^\mathsf{T}\boldsymbol{\Theta}\boldsymbol{\omega} - S_C. \tag{96}$$

The scaling constraint force $\lambda$ has to balance the external resultant scaling force $S_C$ and a term depending on the angular velocity. If there is no external resultant scaling force $S_C$ and the body is not rotating, then a scalable body is identical to a rigid body.

Note that while it is very simple to reduce the rigid body DAE formulation (93) to the rigid body ODE formulation (95), the inverse way going from the ODE to the DAE based on the scalable body is not directly possible. In the Newton–Euler equations used in the ODE description of a rigid body, the scaling dynamics is no longer present. Exactly this scaling dynamics and its coupling to the Newton–Euler equations is missing when one tries to recover the scalable body based DAE of a rigid body from the Newton–Euler equations. Correspondingly, the mass term $1/2\operatorname{Tr}\boldsymbol{\Theta}$ associated with the scaling dynamics in the second equation of (93) can be replaced by any other value if only the rigid body dynamics is to be described correctly.

## 9 Conclusion

In this paper, the quaternion based equations of motion of a scalable body have been derived. For a scalable body, an unconstrained quaternion was used to parameterize the rotational and scaling degrees of freedom. A perfect bilateral constraint has been added to the scalable body to make the body rigid. In this way, a DAE formulation for the dynamics of a rigid body with a regular mass matrix and a quaternion unit length restriction in the form of a standard mechanical constraint has been obtained. By using quaternions, the singularity problems of Euler angles are avoided, while keeping the number of redundant variables to a minimum.

The mechanical model of a scalable body itself might be rarely used in a technical application, due to the fact that a scalable body is rather complicated to build in reality. Nevertheless, the equations of motion of a scalable body can be valuable for the interpretation of the quaternion based rigid body DAE. The regular $7 \times 7$ mass matrix used in the rigid body DAE formulation is exactly the mass matrix of the scalable body. The Lagrangian multiplier associated with the quaternion unit length constraint has the meaning of a constraint force, preventing the scaling body from changing its size. While already (82) completely describes the dynamics of a rigid body in DAE form, it is difficult to see the connection to the Newton–Euler equations (95). This is the reason why the equations of motion of a scalable body have been reformulated to (93) using angular and scaling velocities. The rigid body DAE formulation (93) can be directly recognized as a DAE generalization of the Newton–Euler equations (95), while at the same time being the complete description of the dynamics of a scalable body when removing the explicitly contained constraint.

For DAE based energy consistent integrators, the most important results are the mass matrix (64) and the kinetic energy (60) of a scalable body. With the model of the scalable body, the kinetic energy is valid without any additional restrictions on the quaternion and thus can be used directly in any Lagrangian (for example, [11]) or Hamiltonian based approaches. To get a rigid body, the quaternion unit length constraint can be added in a second step like any other additional perfect bilateral constraint of a multibody system as fits best the employed integration scheme.

## References


1. Altmann, S.L.: Rotations, Quaternions and Double Groups. Clarendon Press, Oxford (1986)
2. Betsch, P., Siebert, R.: Rigid body dynamics in terms of quaternions: Hamiltonian formulation and conserving numerical integration. Int. J. Numer. Methods Eng. **79**, 444–473 (2009)
3. Betsch, P., Steinmann, P.: Constrained integration of rigid body dynamics. Comput. Methods Appl. Mech. Eng. **191**(3–5), 467–488 (2001)
4. Bremer, H.: Dynamik und Regelung mechanischer Systeme. Teubner, Stuttgart (1988)
5. Glocker, Ch.: Set-Valued Force Laws: Dynamics of Non-Smooth Systems. Lecture Notes in Applied Mechanics, vol. 1. Springer, Berlin (2001)
6. Gonzalez, O.: Time integration and discrete Hamiltonian systems. J. Nonlinear Sci. **6**, 449–467 (1996)
7. Hamel, G.: Theoretische Mechanik. Springer, Berlin (1967)
8. Hamilton, W.R.: On quaternions; or on a new system of imaginaries in algebra. Philos. Mag., 25–36 (1844–1850)
9. Kuipers, J.B.: Quaternions and Rotation Sequences. Princeton University Press, Princeton (1999)
10. Le Saux, C., Leine, R.I., Glocker, Ch.: Dynamics of a rolling disk in the presence of dry friction. J. Nonlinear Sci. **15**(1), 27–61 (2005)
11. Lens, E., Cardona, A., Géradin, M.: Energy preserving time integration for constrained multibody systems. Multibody Syst. Dyn. **11**, 41–61 (2004)
12. Maciejewski, A.J.: Hamiltonian formalism for Euler parameters. Celest. Mech. **37**, 47–57 (1985)
13. Morton, H.S. Jr.: Hamiltonian and Lagrangian formulations of rigid-body rotational dynamics based on the Euler parameters. J. Astronaut. Sci. **41**(4), 569–591 (1993)
14. Nikravesh, P.E.: Computer-Aided Analysis of Mechanical Systems. Prentice-Hall, London (1988)
15. Nikravesh, P.E., Chung, I.S.: Application of Euler parameters to the dynamic analysis of three-dimensional constrained mechanical systems. J. Mech. Des. **104**, 785–791 (1982)
16. O'Reilly, O.M., Varadi, P.C.: Hoberman's sphere, Euler parameters and Lagrange's equations. J. Elast. **56**, 171–180 (1999)

17. Papastavridis, J.G.: Analytical Mechanics: A Comprehensive Treatise on the Dynamics of Constrained Systems; for Engineers, Physicists, and Mathematicians. Oxford University Press, Oxford (2002)
18. Udwadia, F.E., Schutte, A.D.: An alternative derivation of the quaternion equations of motion for rigid-body rotational dynamics. J. Appl. Mech. **77**(4) (2010)
19. Udwadia, F.E., Schutte, A.D.: Equations of motion for general constrained systems in Lagrangian mechanics. Acta Mech. **213**, 111–129 (2010)
20. Vadali, S.R.: On the Euler parameter constraint. J. Astronaut. Sci. **36**(3), 259–265 (1988)
21. Ward, J.P.: Quaternions and Cayley Numbers. Kluwer Academic, Dordrecht (1997)