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ON ALGEBRAIC AUTOMORPHISMS AND THEIR RATIONAL INVARIANTS

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Dedicated to all my family and friends

Abstract. Let X be an affine irreducible variety over an algebraically closed field k of characteristic zero. Given an automorphism Φ, we denote by $k(X)^{\Phi}$ its field of invariants, i.e., the set of rational functions f on X such that $f \circ \Phi = f$. Let $n(\Phi)$ be the transcendence degree of $k(X)^{\Phi}$ over k. In this paper we study the class of automorphisms Φ of X for which $n(\Phi) = dim X - 1$. More precisely, we show that under some conditions on X, every such automorphism is of the form $\Phi = \varphi_g$, where φ is an algebraic action of a linear algebraic group G of dimension 1 on X, and where g belongs to G. As an application, we determine the conjugacy classes of automorphisms of the plane for which $n(\Phi) = 1$.

1. Introduction

Let k be an algebraically closed field of characteristic zero. Let X be an affine irreducible variety of dimension n over k. We denote by $\mathcal{O}(X)$ its ring of regular functions, and by $k(X)$ its field of rational functions. Given an algebraic automorphism Φ of X, denote by Φ^* the field automorphism induced by Φ on $k(X)$, i.e., $\Phi^*(f) = f \circ \Phi$ for any $f \in k(X)$. An element f of $k(X)$ is invariant for Φ (or simply invariant) if $\Phi^*(f) = f$. Invariant rational functions form a field denoted $k(X)^{\Phi}$, and we set

$$n(\Phi) = \operatorname{tr}\deg_k k(X)^{\Phi}$$

In this paper we are going to study the class of automorphisms of X for which $n(\Phi) = n - 1$. There are natural candidates for such automorphisms, such as exponentials of locally nilpotent derivations (see [M] or [Da]). More generally, one can construct such automorphisms by means of algebraic group actions as follows. Let G be a linear algebraic group over k. An algebraic action of G on X is a regular map

$$\varphi : G \times X \longrightarrow X$$

of affine varieties, such that $\varphi(g.g', x) = \varphi(g, \varphi(g', x))$ for any (g, g', x) in $G \times G \times X$. Given an element g of G, denote by φ_g the map $x \mapsto \varphi(g, x)$. Then φ_g clearly defines an

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automorphism of X. Let $k(X)^G$ be the field of invariants of G, i.e., the set of rational functions f on X such that $f \circ \varphi_g = f$ for any $g \in G$. If G is an algebraic group of dimension 1, acting faithfully on X, and if g is an element of G of infinite order, then one can prove by Rosenlicht's theorem (see [Ro]) that

$$n(\varphi_g) = \operatorname{tr}\deg_k k(X)^G = n - 1.$$

We are going to see that, under some mild conditions on X, there are no other automorphisms with $n(\Phi) = n - 1$ than those constructed above. *In what follows, denote by $\mathcal{O}(X)^\nu$ the normalization of $\mathcal{O}(X)$, and by $G(X)$ the group of invertible elements of $\mathcal{O}(X)^\nu$.*

Theorem 1. *Let X be an affine irreducible variety of dimension n over k, such that* $\operatorname{char}(k) = 0$ *and $G(X)^* = k^*$. Let Φ be an algebraic automorphism of X such that $n(\Phi) = n - 1$. Then there exist an abelian linear algebraic group G of dimension 1, and an algebraic action φ of G on X such that $\Phi = \varphi_g$ for some $g \in G$ of infinite order.*

Note that the structure of G is fairly simple. Since every connected linear algebraic group of dimension 1 is either isomorphic to $\mathbf{G}_a(k) = (k, +)$ or $\mathbf{G}_m(k) = (k^*, \times)$ (see [Hum, p. 131]), there exists a finite abelian group H such that G is either equal to $H \times \mathbf{G}_a(k)$ or $H \times \mathbf{G}_m(k)$. Moreover, the assumption on the group $G(X)$ is essential. Indeed, consider the automorphism Φ of $k^* \times k$ given by $\Phi(x, y) = (x, xy)$. Obviously, its field of invariants is equal to $k(x)$. However, it is easy to check that Φ cannot have the form given in the conclusion of Theorem 1.

This theorem is analogous to a result given by Van den Essen and Peretz (see [V-P]). More precisely, they establish a criterion to decide if an automorphism Φ is the exponential of a locally nilpotent derivation, based on the invariants and on the form of Φ. A similar result has been developed by Daigle (see [Da]).

We apply these results to the group of automorphisms of the plane. First, we obtain a classification of the automorphisms Φ of k^2 for which $n(\Phi) = 1$. Second, we derive a criterion on automorphisms of k^2 to have no nonconstant rational invariants.

Corollary 1. *Let Φ be an algebraic automorphism of k^2. If $n(\Phi) = 1$, then Φ is conjugate to one of the following forms*:

- $\Phi_1(x, y) = (a^n x, a^m b y)$, *where $(n, m) \neq (0, 0)$, $a, b \in k$, b is a root of unity but a is not,*
- $\Phi_2(x, y) = (ax, by + P(x))$, *where P belongs to $k[t] - \{0\}$, $a, b \in k$ are roots of unity.*

Corollary 2. *Let Φ be an algebraic automorphism of k^2. Assume that Φ has a unique fixed point p and that $d\Phi_p$ is unipotent. Then $n(\Phi) = 0$.*

We then apply Corollary 2 to an automorphism of $\mathbb{C}^3$ recently discovered by Poloni and Moser-Jauslin (see [M-P]).

We may wonder whether Theorem 1 still holds if the ground field k is not algebraically closed or has positive characteristic. The answer is not known for the moment. In fact, two obstructions appear in the proof of Theorem 1 when k is arbitrary. First, the group $\mathbf{G}_m(k)$ needs to be divisible (see Lemma 8), which is not always the case if k is not algebraically closed. Second, the proof uses the fact that every $\mathbf{G}_a(k)$-action on X

can be reconstructed from a locally nilpotent derivation on $\mathcal{O}(X)$ (see Subsection 4.1), which is no longer true if k has positive characteristic. This phenomenon is due to the existence of different forms for the affine line (see [Ru]). Note that, in case Theorem 1 holds and k is not algebraically closed, the algebraic group G need not be isomorphic to $H \times \mathbf{G}_a(k)$ or $H \times \mathbf{G}_m(k)$, where H is finite. Indeed, consider the unit circle X in the plane $\mathbb{R}^2$, given by the equation $x^2 + y^2 = 1$. Let Φ be a rotation in $\mathbb{R}^2$ with center at the origin and angle $\theta \notin 2\pi\mathbb{Q}$. Then Φ defines an algebraic automorphism of X with $n(\Phi) = 0$, and the subgroup spanned by Φ is dense in $SO_2(\mathbb{R})$. But $SO_2(\mathbb{R})$ is not isomorphic to either $\mathbf{G}_a(\mathbb{R})$ or $\mathbf{G}_m(\mathbb{R})$, even though it is a connected linear algebraic group of dimension 1.

We may also wonder what happens to the automorphisms Φ of X for which $n(\Phi) = \dim X - 2$. More precisely, does there exist an action φ of a linear algebraic group G on X, of dimension 2, such that $\Phi = \varphi_g$ for a given $g \in G$? The answer is no. Indeed, consider the automorphism Φ of k^2 given by $\Phi = f \circ g$, where $f(x, y) = (x + y^2, y)$ and $g(x, y) = (x, y + x^2)$. Let $d(n)$ denote the maximum of the homogeneous degrees of the coordinate functions of the iterate Φ^n. If there existed an action φ of a linear algebraic group G such that $\Phi = \varphi_g$, then the function d would be bounded, which is impossible since $d(n) = 4^n$. A similar argument on the length of the iterates also yields the result. But if we restrict to some specific varieties X, for instance $X = k^3$, one may ask the following question: If $n(\Phi) = 1$, is Φ birationally conjugate to an automorphism that leaves the first coordinate of k^3 invariant? The answer is still unknown.

2. Reduction to an affine curve $\mathcal{C}$

Let X be an affine irreducible variety of dimension n over k. Let Φ be an algebraic automorphism of X such that $n(\Phi) = n - 1$. In this section we are going to construct an irreducible affine curve on which Φ acts naturally. This will allow us to use some well-known results on automorphisms of curves. We set

$$K = \{f \in k(X) \mid \exists m > 0,\ f \circ \Phi^m = f \circ \Phi \circ \cdots \circ \Phi = f\}.$$

It is straightforward that K is a subfield of $k(X)$ containing both k and $k(X)^\Phi$. We begin with some properties of this field.

Lemma 1. *K has transcendence degree $(n-1)$ over k, and is algebraically closed in $k(X)$. In particular, the automorphism Φ of X has infinite order.*

Proof. First we show that K has transcendence degree $(n-1)$ over k. Since K contains the field $k(X)^\Phi$, whose transcendence degree is $(n-1)$, we only need to show that the extension $K/k(X)^\Phi$ is algebraic or, in other words, that every element of K is algebraic over $k(X)^\Phi$. Let f be any element of K. By definition, there exists an integer $m > 0$ such that $f \circ \Phi^m = f$. Let $P(t)$ be the polynomial of $k(X)[t]$ defined as

$$P(t) = \prod_{i=0}^{m-1} (t - f \circ \Phi^i).$$

By construction, the coefficients of this polynomial are all invariant for Φ, and $P(t)$ belongs to $k(X)^\Phi[t]$. Moreover, $P(f) = 0$, f is algebraic over $k(X)^\Phi$, and the first assertion follows.

Second, we show that K is algebraically closed in $k(X)$. Let f be an element of $k(X)$ that is algebraic over K. We need to prove that f belongs to K. By the first assertion of the lemma, f is algebraic over $k(X)^{\Phi}$. Let $P(t) = a_0 + a_1 t + \cdots + a_p t^p$ be a nonzero minimal polynomial of f over $k(X)^{\Phi}$. Since $P(f) = 0$ and all a_i are invariant, we have $P(f \circ \Phi) = P(f) \circ \Phi = 0$. In particular, all elements of the form $f \circ \Phi^i$, with $i \in \mathbb{N}$, are roots of P. Since P has finitely many roots, there exist two distinct integers $m' < m''$ such that $f \circ \Phi^{m'} = f \circ \Phi^{m''}$. In particular, $f \circ \Phi^{m''-m'} = f$ and f belongs to K.

Now if Φ were an automorphism of finite order, then K would be equal to $k(X)$. But this is impossible since K and $k(X)$ have different transcendence degrees. □

Lemma 2. *There exists an integer $m > 0$ such that $K = k(X)^{\Phi^m}$.*

Proof. By definition, $k(X)$ is a field of finite type over k. Since K is contained in $k(X)$, K has also finite type over k. Let $f_1, \dots, f_r$ be some elements of $k(X)$ such that $K = k(f_1, \dots, f_r)$. Let $m_1, \dots, m_r$ be some positive integers such that $f_i \circ \Phi^{m_i} = f_i$, and set $m = m_1 \dots m_r$. By construction, all f_i are invariant for Φ^m. In particular, K is invariant for Φ^m and $K \subseteq k(X)^{\Phi^m}$. Since $k(X)^{\Phi^m} \subseteq K$, the result follows. □

Let L be the algebraic closure of $k(X)$, and let A be the K-subalgebra of L spanned by $\mathcal{O}(X)$. By construction, A is an integral K-algebra of finite type of dimension 1. Let m be an integer satisfying the conditions of Lemma 2. The automorphism $\Psi^* = (\Phi^m)^*$ of $\mathcal{O}(X)$ stabilizes A, hence it defines a K-automorphism of A, of infinite order (see Lemma 1). Let B be the integral closure of A. Then B is also an integral K-algebra of finite type, of dimension 1, and the K-automorphism Ψ^* extends uniquely to B. If $\overline{K}$ stands for the algebraic closure of K, we set

$$C = B \otimes_K \overline{K}.$$

By construction, $\mathcal{C} = \operatorname{Spec}(C)$ is an affine curve over the algebraically closed field $\overline{K}$. Moreover, the automorphism Ψ^* acts on C via the operation

$$\Psi^* : C \longrightarrow C, \quad x \otimes y \longmapsto \Psi^*(x) \otimes y.$$

This makes sense since Ψ^* fixes the field K. Therefore, Ψ^* induces an algebraic automorphism of the curve $\mathcal{C}$. Since K is algebraically closed in $k(X)$ by Lemma 1, C is integral (see [Z-S, Chap. VII, §11, Theorem 38]). But, by construction, B and $\overline{K}$ are normal rings. Since C is a domain and $\operatorname{char}(K) = 0$, C is also integrally closed by a result of Bourbaki (see [Bou, p. 29]). So C is a normal domain and $\mathcal{C}$ is a smooth irreducible curve.

Lemma 3. *Let C be the $\overline{K}$-algebra constructed above. Then either $C = \overline{K}[t]$ or $C = \overline{K}[t, 1/t]$.*

Proof. By Lemma 1, the automorphism Φ of X has infinite order. Since the fraction field of B is equal to $k(X)$, Ψ^* has infinite order on B. But $B \otimes 1 \subset C$, so Ψ^* has infinite order on C. In particular, Ψ acts like an automorphism of infinite order on $\mathcal{C}$. Since $\mathcal{C}$ is affine, it has genus zero (see [Ro2]). Since $\overline{K}$ is algebraically closed, the curve $\mathcal{C}$ is rational (see [Che, p. 23]). Since $\mathcal{C}$ is smooth, it is isomorphic to $\mathbb{P}^1(\overline{K}) - E$, where E is a finite set. Moreover, Ψ acts like an automorphism of $\mathbb{P}^1(\overline{K})$ that stabilizes $\mathbb{P}^1(\overline{K}) - E$.

Up to replacing Ψ by one of its iterates, we may assume that Ψ fixes every point of E. But an automorphism of $\mathbb{P}^1(\overline{K})$ that fixes at least three points is the identity, which is impossible. Therefore, E consists of at most two points, and $\mathcal{C}$ is either isomorphic to $\overline{K}$ or to $\overline{K}^*$. In particular, either $C = \overline{K}[t]$ or $C = \overline{K}[t, 1/t]$. □

3. Normal forms for the automorphism Ψ

Let C and Ψ^* be the $\overline{K}$-algebra and the $\overline{K}$-automorphism constructed in the previous section. In this section we are going to give normal forms for the couple (C, Ψ^*), in case the group $G(X)$ is trivial, i.e., $G(X) = k^*$. We begin with a few lemmas.

Lemma 4. *Let X be an irreducible affine variety over k. Let Ψ be an automorphism of X. Let α, f be some elements of $k(X)^*$ such that $(\Psi^*)^n(f) = \alpha^n f$ for any $n \in \mathbb{Z}$. Then α belongs to $G(X)$.*

Proof. Given an element h of $k(X)^*$ and a prime divisor D on the normalization X^ν, we consider h as a rational function on X^ν, and denote by $\mathrm{ord}_D(h)$ the multiplicity of h along D. This makes sense since the variety X^ν is normal. Fix any prime divisor D on X. Since $(\Psi^*)^n(f) = \alpha^n f$ for any $n \in \mathbb{Z}$, we obtain

$$\mathrm{ord}_D((\Psi^*)^n(f)) = n\,\mathrm{ord}_D(\alpha) + \mathrm{ord}_D(f).$$

Since Ψ is an algebraic automorphism of X, it extends uniquely to an algebraic automorphism of X^ν, which is still denoted Ψ. Moreover, this extension maps every prime divisor to another prime divisor, does not change the multiplicity, and maps distinct prime divisors into distinct ones. If $\mathrm{div}(f) = \sum_i n_i D_i$, where all D_i are prime, then we have

$$\mathrm{div}((\Psi^*)^n(f)) = \sum_i n_i (\Psi^*)^n(D_i),$$

where all $(\Psi^*)^n(D_i)$ are prime and distinct. So the multiplicity of $(\Psi^*)^n(f)$ along D is equal to zero if D is not one of the $(\Psi^*)^n(D_i)$s, and equal to n_i if $D = (\Psi^*)^n(D_i)$. In all cases, if $R = \max\{|n_i|\}$, then we find that $|\mathrm{ord}_D((\Psi^*)^n(f))| \leqslant R$ and $|\mathrm{ord}_D(f)| \leqslant R$, and this implies, for any integer n,

$$|n\,\mathrm{ord}_D(\alpha)| \leqslant 2R.$$

In particular, we find $\mathrm{ord}_D(\alpha) = 0$. Since this holds for any prime divisor D, the support of $\mathrm{div}(\alpha)$ in X^ν is empty and $\mathrm{div}(\alpha) = 0$. Since X^ν is normal, α is an invertible element of $\mathcal{O}(X)^\nu$, hence it belongs to $G(X)$. □

Lemma 5. *Let K be a field of characteristic zero and $\overline{K}$ its algebraic closure. Let C be either equal to $\overline{K}[t]$ or to $\overline{K}[t, 1/t]$. Let Ψ^* be a $\overline{K}$-automorphism of C such that $\Psi^*(t) = at$, where a belongs to $\overline{K}$. Let σ_1 be a K-automorphism of C, commuting with Ψ^*, such that $\sigma_1(\overline{K}) = \overline{K}$. Then $\sigma_1(a)$ is either equal to a or to $1/a$.*

Proof. We distinguish two cases depending on the ring C. First, assume that $C = \overline{K}[t]$. Since σ_1 is a K-automorphism of C that maps $\overline{K}$ to itself, we have $\overline{K}[t] = \overline{K}[\sigma_1(t)]$. In particular, $\sigma_1(t) = \lambda t + \mu$, where λ, μ belong to $\overline{K}$ and $\lambda \neq 0$. Since Ψ^* and σ_1 commute, we obtain

$$\Psi^* \circ \sigma_1(t) = \lambda a t + \mu = \sigma_1 \circ \Psi^*(t) = \sigma_1(a)(\lambda t + \mu).$$

In particular, we have $\sigma_1(a) = a$ and the lemma follows in this case. Second, assume that $C = \overline{K}[t, 1/t]$. Since σ_1 is a K-automorphism of C, we find

$$\sigma_1(t)\sigma_1(1/t) = \sigma_1(t.1/t) = \sigma_1(1) = 1.$$

Therefore, $\sigma_1(t)$ is an invertible element of C, and has the form $\sigma_1(t) = a_1 t^{n_1}$, where $a_1 \in \overline{K}^*$ and n_1 is an integer. Since σ_1 is a K-automorphism of C that maps $\overline{K}$ to $\overline{K}$, we have $\overline{K}[t, 1/t] = \overline{K}[\sigma_1(t), 1/\sigma_1(t)]$. In particular, $|n_1| = 1$ and either $\sigma_1(t) = a_1 t$ or $\sigma_1(t) = a_1/t$. If $\sigma_1(t) = a_1 t$, the relation $\Psi^* \circ \sigma_1(t) = \sigma_1 \circ \Psi^*(t)$ yields $\sigma(a) = a$. If $\sigma_1(t) = a_1/t$, then the same relation yields $\sigma(a) = 1/a$. □

Lemma 6. *Let X be an irreducible affine variety of dimension n over k, such that $G(X) = k^*$. Let Φ be an automorphism of X such that $n(\Phi) = (n-1)$. Let Ψ^* be the automorphism of C constructed in the previous section. If either $C = \overline{K}[t]$ or $C = \overline{K}[t, 1/t]$, and if $\Psi^*(t) = at$, then a belongs to k^*.*

Proof. We are going to prove by contradiction that a belongs to k^*. So assume that $a \notin k^*$. Let σ be any element of $\mathrm{Gal}(\overline{K}/K)$, and denote by σ_1 the K-automorphism of C defined as follows:

$$\forall (x, y) \in B \times \overline{K}, \quad \sigma_1(x \otimes y) = x \otimes \sigma_1(y).$$

Since $\Psi^* \circ \sigma_1(x \otimes y) = \Psi^*(x) \otimes \sigma_1(y) = \sigma_1 \circ \Psi^*(x \otimes y)$ for any element $x \otimes y$ of $B \otimes_K \overline{K}$, Ψ^* and σ_1 commute. Moreover, if we identify $\overline{K}$ with $1 \otimes \overline{K}$, then $\sigma_1(\overline{K}) = \overline{K}$ by construction. By Lemma 5, we obtain

$$\forall \sigma \in \mathrm{Gal}(\overline{K}/K), \quad \sigma(a) = a \text{ or } \sigma(a) = a^{-1}.$$

In particular, the element $(a^i + a^{-i})$ is invariant under the action of $\mathrm{Gal}(\overline{K}/K)$ for any i, and so it belongs to K because $\mathrm{char}(K) = 0$. Now let f be an element of $B - K$. Since f belongs to C, we can express f as follows:

$$f = \sum_{i=r}^{s} f_i t^i.$$

Choose an $f \in B - K$ such that the difference $(s - r)$ is minimal. We claim that $(s - r) = 0$, i.e., $f = f_s t^s$. Indeed, assume that $s > r$. Since f is an element of B, the following expressions:

$$\Psi^*(f) + (\Psi^*)^{-1}(f) - (a^s + a^{-s})f = \sum_{i=r}^{s-1} f_i(a^i + a^{-i} - a^s - a^{-s})t^i,$$

$$\Psi^*(f) + (\Psi^*)^{-1}(f) - (a^r + a^{-r})f = \sum_{i=r+1}^{s} f_i(a^i + a^{-i} - a^r - a^{-r})t^i,$$

also belong to B. By minimality of $(s - r)$, these expressions belong to K. In other words, $f_i(a^i + a^{-i} - a^s - a^{-s}) = 0$ (resp., $f_i(a^i + a^{-i} - a^r - a^{-r}) = 0$) for any $i \neq 0, s$ (resp., for any $i \neq 0, r$). Since k is algebraically closed and $a \notin k^*$ by assumption, $(a^i + a^{-i} - a^s - a^{-s})$ (resp., $(a^i + a^{-i} - a^r - a^{-r})$) is nonzero for any $i \neq s$ (resp., for any $i \neq r$). Therefore, $f_i = 0$ for any $i \neq 0$, and f belongs to K, a contradiction. Therefore, $s = r$ and $f = f_s t^s$. Since f belongs to B, it also belongs to $k(X)$. Since Ψ is an automorphism of X, the element $a^s = \Psi^*(f)/f$ belongs to $k(X)$. Moreover, $(\Psi^*)^n(f) = a^{ns} f$ for any $n \in \mathbb{Z}$. By Lemma 4, a^s belongs to $G(X) = k^*$. Since k is algebraically closed, a belongs to k^*, hence a contradiction, and the result follows. □

Proposition 1. *Let X be an irreducible affine variety of dimension n over k, such that $G(X) = k^*$. Let Φ be an automorphism of X such that $n(\Phi) = (n-1)$. Let C and Ψ^* be the $\overline{K}$-algebra and the $\overline{K}$-automorphism constructed in the previous section. Then up to conjugation, one of the following three cases occurs:*

- *$C = \overline{K}[t]$ and $\Psi^*(t) = t + 1$;*
- *$C = \overline{K}[t]$ and $\Psi^*(t) = at$, where $a \in k^*$ is not a root of unity;*
- *$C = \overline{K}[t, 1/t]$ and $\Psi^*(t) = at$, where $a \in k^*$ is not a root of unity.*

Proof. By Lemma 3, we know that either $C = \overline{K}[t]$ or $C = \overline{K}[t, 1/t]$. We are going to study both cases.

First case: $C = \overline{K}[t]$.

The automorphism Ψ^* maps t to $at + b$, where $a \in \overline{K}^*$ and $b \in \overline{K}$. If $a = 1$, then $b \neq 0$ and up to replacing t with t/b, we may assume that $\Psi^*(t) = t + 1$. If $a \neq 1$, then up to replacing t with $t - c$ for a suitable c, we may assume that $\Psi^*(t) = at$. But then Lemma 6 implies that a belongs to k^*. Since Ψ^* has infinite order, a cannot be a root of unity.

Second case: $C = \overline{K}[t, 1/t]$.

Since $\Psi^*(t)\Psi^*(1/t) = \Psi^*(1) = 1$, $\Psi^*(t)$ is an invertible element of C. So $\Psi^*(t) = at^n$, where $a \in \overline{K}^*$ and $n \neq 0$. Since Ψ^* is an automorphism, n is either equal to 1 or to -1. But if n were equal to -1, then a simple computation shows that $(\Psi^*)^2$ would be the identity, which is impossible. So $\Psi^*(t) = at$, where $a \in \overline{K}^*$. By Lemma 6, a belongs to k^*. As before, a cannot be a root of unity. $\square$

4. Proof of the main theorem

In this section we are going to establish Theorem 1. We will split its proof into two steps depending on the form of the automorphism Ψ^* given in Proposition 1. But before, we begin with a few lemmas.

Lemma 7. *Let Φ be an automorphism of an affine irreducible variety X. Let G be a linear algebraic group and let ψ be an algebraic G-action on X. Let h be an element of G such that the group $\langle h \rangle$ spanned by h is Zariski dense in G. If Φ and ψ_h commute, then Φ and ψ_g commute for any g in G.*

Proof. It suffices to check that Φ^* and ψ_g^* commute for any $g \in G$. For any k-algebra automorphisms α, β of $\mathcal{O}(X)$, denote by $[\alpha, \beta]$ their commutator, i.e., $[\alpha, \beta] = \alpha \circ \beta \circ \alpha^{-1} \circ \beta^{-1}$. For any $f \in \mathcal{O}(X)$, set

$$\lambda(g, f)(x) = [\Phi^*, \psi_g^*](f)(x) - f(x).$$

Since G is a linear algebraic group acting algebraically on the affine variety X, $\lambda(g, f)(x)$ is a regular function on $G \times X$. Since Φ^* and ψ_h^* commute, the automorphisms Φ^* and $\psi_{h^n}^*$ commute for any integer n. So the regular function $\lambda(g, f)(x)$ vanishes on $\langle h \rangle \times X$. Since $\langle h \rangle$ is dense in G by assumption, $\langle h \rangle \times X$ is dense in $G \times X$ and $\lambda(g, f)(x)$ vanishes identically on $G \times X$. In particular, $[\Phi^*, \psi_g^*](f) = f$ for any $g \in G$. Since this holds for any element f of $\mathcal{O}(X)$, the bracket $[\Phi^*, \psi_g^*]$ coincides with the identity on $\mathcal{O}(X)$ for any $g \in G$, and the result follows. $\square$

Lemma 8. *Let Φ be an automorphism of an affine irreducible variety X. Let G be a linear algebraic group and let ψ be an algebraic G-action on X. Let h be an element of G such that the group $\langle h\rangle$ spanned by h is Zariski dense in G. Assume there exists a nonzero integer r such that $\Phi^r = \psi_h$, and that G is divisible. Then there exists an algebraic action φ of $G' = \mathbb{Z}/r\mathbb{Z} \times G$ such that $\Phi = \varphi_{g'}$ for some g' in G'.*

Proof. Fix an element b in G such that $b^r = h$, and set $\Delta = \Phi \circ \psi_{b^{-1}}$. This is possible since G is divisible. By construction, Δ is an automorphism of X. Since $\Phi^r = \psi_h$, Φ and ψ_h commute. By Lemma 7, Φ and ψ_g commute for any $g \in G$. In particular, we have

$$\Delta^r = (\Phi^r) \circ \psi_{b^{-r}} = (\Phi^r) \circ \psi_{h^{-1}} = \mathrm{Id}$$

So Δ is finite, $\Phi = \Delta \circ \psi_b$, and Δ commutes with ψ_g for any $g \in G$. The group G' then acts on X via the map φ defined by

$$\varphi_{(i,g)}(x) = \Delta^i \circ \psi_g(x).$$

Moreover, we have $\Phi = \varphi_{g'}$ for $g' = (1, b)$. □

The proof of Theorem 1 will then go as follows. In the following subsections we are going to exhibit an algebraic action ψ of $\mathbf{G}_a(k)$ (resp., $\mathbf{G}_m(k)$) on X, such that $\Psi = \Phi^m = \psi_h$ for some h. In both cases, the group G we will consider will be linear algebraic of dimension 1, and divisible. Moreover, the element h will span a Zariski dense set because $h \neq 0$ (resp., h is not a root of unity). With these conditions, Theorem 1 will become a direct application of Lemma 8.

4.1. The case $\Psi^*(t) = t + 1$

Assume that $C = \overline{K}[t]$ and $\Psi^*(t) = t + 1$. We are going to construct a nontrivial algebraic $\mathbf{G}_a(k)$-action ψ on X such that $\Psi = \psi_1$. Since $\mathcal{O}(X) \subset C$, every element f of $\mathcal{O}(X)$ can be written as $f = P(t)$, where P belongs to $\overline{K}[t]$. We set $r = \deg_t P(t)$. Since Ψ^* stabilizes $\mathcal{O}(X)$, the expression

$$(\Psi^i)^*(f) = P(t+i) = \sum_{j=0}^{r} P^{(j)}(t)\frac{i^j}{j!}$$

belongs to $\mathcal{O}(X)$ for any integer i. Since the matrix $M = (i^j/j!)_{0\leqslant i,j\leqslant r}$ is invertible in $\mathcal{M}_{r+1}(\mathbb{Q})$, the polynomial $P^{(j)}(t)$ belongs to $\mathcal{O}(X)$ for any $j \leqslant r$. So the $\overline{K}$-derivation $D = \partial/\partial t$ on C stabilizes the k-algebra $\mathcal{O}(X)$. Since $D^{r+1}(f) = 0$, the operator D, considered as a k-derivation on $\mathcal{O}(X)$, is locally nilpotent (see [Van]). Therefore the exponential map

$$\exp uD : \mathcal{O}(X) \longrightarrow \mathcal{O}(X)[u], \quad f \longmapsto \sum_{j\geqslant 0} D^j(f)\frac{u^j}{j!},$$

is a well-defined k-algebra morphism. But $\exp uD$ also defines a K-algebra morphism from C to $C[u]$. Since $\exp uD(t) = t + u$, $\exp D$ coincides with Ψ^* on C. Since C contains the ring $\mathcal{O}(X)$, we have $\exp D = \Psi^*$ on $\mathcal{O}(X)$. So the exponential map induces an algebraic $\mathbf{G}_a(k)$-action ψ on X such that $\Psi = \psi_1$ (see [Van]).

4.2. The case $\Psi^*(t) = at$

Assume that $\Psi^*(t) = at$ and that a is not a root of unity. We are going to construct a nontrivial algebraic $\mathbf{G}_m(k)$-action ψ on X such that $\Psi = \psi_a$. First, note that either $C = \overline{K}[t]$ or $C = \overline{K}[t, 1/t]$. Let f be any element of $\mathcal{O}(X)$. Since $\mathcal{O}(X) \subset C$, we can write f as

$$f = P(t) = \sum_{i=r}^{s} f_i t^i,$$

where the $f_i t^i$ belong a priori to C. Since Ψ^* stabilizes $\mathcal{O}(X)$, the expression

$$(\Psi^j)^*(f) = P(a^j t) = \sum_{i=r}^{s} a^{ji} f_i t^i$$

belongs to $\mathcal{O}(X)$ for any integer j. Since a belongs to k^* and is not a root of unity, the Vandermonde matrix $M = (a^{ij})_{0 \leqslant i,j \leqslant s-r}$ is invertible in $\mathcal{M}_{s-r+1}(k)$. So the elements $f_i t^i$ all belong to $\mathcal{O}(X)$ for any integer i. Consider the map

$$\psi^* : \mathcal{O}(X) \longrightarrow \mathcal{O}(X)[v, 1/v], \quad f \longmapsto \sum_{i=r}^{s} f_i t^i v^i.$$

Then ψ^* is a well-defined k-algebra morphism, which induces a regular map ψ from $k^* \times X$ to X. Moreover we have $\psi_v \circ \psi_{v'} = \psi_{vv'}$ on X for any $v, v' \in k^*$. So ψ defines an algebraic $\mathbf{G}_m(k)$-action on X such that $\Psi = \psi_a$.

5. Proof of Corollary 1

Let Φ be an automorphism of the affine plane k^2, such that $n(\Phi) = 1$. By Theorem 1, there exists an algebraic action φ of an abelian linear algebraic group G of dimension 1 such that $\Phi = \varphi_g$. We will distinguish the cases $G = \mathbb{Z}/r\mathbb{Z} \times \mathbf{G}_m(k)$ and $G = \mathbb{Z}/r\mathbb{Z} \times \mathbf{G}_a(k)$.

First case: $G = \mathbb{Z}/r\mathbb{Z} \times \mathbf{G}_m(k)$.

Then G is linearly reductive and φ is conjugate to a representation in $\mathrm{GL}_2(k)$ (see [Ka] or [Kr]). Since G consists solely of semisimple elements, φ is even diagonalizable. In particular, there exists a system (x, y) of polynomial coordinates, some integers n, m, and some r-roots of unity a, b such that

$$\varphi_{(i,u)}(x, y) = (a^i u^n x, b^i u^m y).$$

Note that, since the action is faithful, the couple (n, m) is distinct from $(0, 0)$. Since k is algebraically closed, we can even reduce $\Phi = \varphi_g$ to the first form given in Corollary 1.

Second case: $G = \mathbb{Z}/r\mathbb{Z} \times \mathbf{G}_a(k)$.

Let ψ and Δ be, respectively, the $\mathbf{G}_a(k)$-action and the finite automorphism constructed in Lemma 8. By Rentschler's theorem (see [Re]), there exists a system (x, y) of polynomial coordinates and an element P of $k[t]$ such that

$$\psi_u(x, y) = (x, y + uP(x)).$$

For any $f \in k[x,y]$, set $\deg_\psi(f) = \deg_u \exp uD(f)$. It is well known that this defines a degree function on $k[x,y]$ (see [Da]). Since ψ and Δ commute, Δ^* preserves the space E_n of polynomials of degree $\leqslant n$ with respect to $\deg_\psi$. In particular, Δ^* preserves $E_0 = k[x]$. So Δ^* induces a finite automorphism of $k[x]$, hence $\Delta^*(x) = ax + b$, where a is a root of unity. Since Δ is finite, either $a \neq 1$ or $a = 1$ and $b = 0$. In any case, up to replacing x by $x - \mu$ for a suitable constant μ, we may assume that $\Delta^*(x) = ax$. Moreover Δ^* preserves the space $E_1 = k[x]\{1, y\}$. With the same arguments as before, we obtain that $\Delta^*(y) = cy + d(x)$, where c is a root of unity and $d(x)$ belongs to $k[x]$. Composing Δ with $\psi_{1/m}$ then yields the second form given in Corollary 1.

6. Proof of Corollary 2

Let Φ be an algebraic automorphism of k^2. We assume that Φ has a unique fixed point p and that $d\Phi_p$ is unipotent. We are going to prove that $n(\Phi) = 0$.

First, we check that $n(\Phi)$ cannot be equal to 2. Assume that $n(\Phi) = 2$. Then $k(x,y)^\Phi$ has transcendence degree 2, and the extension $k(x,y)/k(x,y)^\Phi$ is algebraic, hence finite. Moreover, Φ^* acts like an element of the Galois group of this extension. In particular, Φ^* is finite. By a result of Kambayashi (see [Ka]), Φ can be written as $h \circ A \circ h^{-1}$, where A is an element of $\mathrm{GL}_2(k)$ of finite order and h belongs to $\mathrm{Aut}(k^2)$. Since Φ has a unique fixed point p, we have $h(0,0) = p$. In particular, $d\Phi_p$ is conjugate to A in $\mathrm{GL}_2(k)$. Since $d\Phi_p$ is unipotent and A is finite, A is the identity. Therefore, Φ is also the identity, which contradicts the fact that it has a unique fixed point.

Second we check that $n(\Phi)$ cannot be equal to 1. Assume that $n(\Phi) = 1$. By the previous corollary, up to conjugacy, we may assume that Φ has one of the following forms:

- $\Phi_1(x,y) = (a^n x, a^m by)$, where $(n,m) \neq (0,0)$, b is a root of unity but a is not,
- $\Phi_2(x,y) = (ax, by + P(x))$, where P belongs to $k[t] - \{0\}$ and a, b are roots of unity.

Assume that Φ is an automorphism of type Φ_1. Then $d\Phi_p$ is a diagonal matrix of $\mathrm{GL}_2(k)$, distinct from the identity. But this is impossible since $d\Phi_p$ is unipotent. So assume that Φ is an automorphism of type Φ_2. Then $d\Phi_p$ is a linear map of the form $(u,v) \mapsto (au, bv + du)$, with $d \in k$. Since $d\Phi_p$ is unipotent, we have $a = b = 1$. So (α, β) is a fixed point if and only if $P(\alpha) = 0$. In particular, the set of fixed points is either empty or a finite union of parallel lines. But this is impossible since there is only one fixed point by assumption. Therefore $n(\Phi) = 0$.

7. An application of Corollary 2

In this section we are going to see how Corollary 2 can be applied to the determination of invariants for automorphisms of $\mathbb{C}^3$. Set $Q(x,y,z) = x^2y - z^2 - xz^3$ and consider the following automorphism (see [M-P]):

$$\Phi : \mathbb{C}^3 \longrightarrow \mathbb{C}^3, \quad (x,y,z) \longmapsto \left(x, y(1 - xz) + \frac{Q^2}{4} + z^4, z - \frac{Q}{2}x \right).$$

We are going to show that

$$\mathbb{C}(x,y,z)^\Phi = \mathbb{C}(x) \quad \text{and} \quad \mathbb{C}[x,y,z]^\Phi = \mathbb{C}[x].$$

Let k be the algebraic closure of $\mathbb{C}(x)$. Since $\Phi^*(x) = x$, the morphism Φ^* induces an automorphism of $k[y,z]$, which we denote by Ψ^*. The automorphism Ψ has clearly $(0,0)$ as a fixed point, and its differential at this point is unipotent, distinct from the identity (as can be seen by an easy computation). Moreover, the set of fixed points of Ψ is reduced to the origin. Indeed, if (α,β) is a point of k^2 fixed by Ψ, then $xQ = 0$ and $4\beta^4 - 4x\alpha\beta + Q^2 = 0$. Since x belongs to k^*, we have

$$Q = x^2\alpha - \beta^2 - x\beta^3 = 0 \quad \text{and} \quad \beta^4 - x\alpha\beta = 0.$$

If $\beta = 0$, then $\alpha = 0$ and we find the origin. If $\beta \neq 0$, then dividing by β and multiplying by $-x$ yields the relation

$$x^2\alpha - x\beta^3 = 0.$$

This implies $\beta^2 = 0$ and $\beta = 0$, hence a contradiction. By Corollary 2, the field of invariants of Ψ has transcendence degree zero. So the field of invariants of Φ has transcendence degree $\leqslant 1$ over $\mathbb{C}$. Since this field contains $\mathbb{C}(x)$ and that $\mathbb{C}(x)$ is algebraically closed in $\mathbb{C}(x,y,z)$, we obtain that $\mathbb{C}(x,y,z)^\Phi = \mathbb{C}(x)$. As a consequence, the ring of invariants of Φ is equal to $\mathbb{C}[x]$.

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