J Geom Anal (2008) 18: 1022–1032
DOI 10.1007/s12220-008-9041-z

Eigenvalues Estimate for the Neumann Problem of a Bounded Domain

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Received: 13 November 2007 / Published online: 10 July 2008

Abstract In this note, we investigate upper bounds of the Neumann eigenvalue problem for the Laplacian of a domain Ω in a given complete (not compact a priori) Riemannian manifold (M, g). For this, we use test functions for the Rayleigh quotient subordinated to a family of open sets constructed in a general metric way, interesting for itself. As applications, we prove that if the Ricci curvature of (M, g) is bounded below $\mathrm{Ric}^g \geq -(n-1)a^2$, $a \geq 0$, then there exist constants $A_n > 0$, $B_n > 0$ only depending on the dimension, such that

$$\lambda_k(\Omega) \leq A_n a^2 + B_n \left(\frac{k}{V}\right)^{2/n},$$

where $\lambda_k(\Omega)$ $(k \in \mathbb{N}^*)$ denotes the k-th eigenvalue of the Neumann problem on any bounded domain $\Omega \subset M$ of volume $V = \mathrm{Vol}(\Omega, g)$. Furthermore, this upper bound is clearly in agreement with the Weyl law. As a corollary, we get also an estimate which is analogous to Buser's upper bounds of the spectrum of a compact Riemannian manifold with lower bound on the Ricci curvature.

Keywords Neumann spectrum · Upper bound · Weyl law · Metric geometry

Mathematics Subject Classification (2000) 35P15 · 53C99 · 51F99

Communicated by Peter Li.

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1 Introduction

The goal of this paper is to give upper bounds for the spectrum of the Laplacian acting on compact domains of given volume of a complete Riemannian manifold with Ricci curvature bounded below, and, as far as possible, to make these estimates optimal with respect to the Weyl law.

For compact Riemannian manifolds without boundary, the following result was proved by P. Buser in [3] (Satz 7), [4] (Theorem 6.2 (c)) (see also Li-Yau in [12] (Theorem 16)). If $\{\lambda_k\}_{k=1}^{\infty}$ denote the spectrum of the Laplacian acting on functions, then:

Theorem 1.1 *Let (M^n, g) be a compact n-dimensional Riemannian manifold with Ricci curvature bounded below* $\mathrm{Ric}^g \geq -(n-1)a^2$, $a \geq 0$, *and of volume V.*

There exists a constant $C_n \geq 1$ only depending on the dimension, such that for all $k \in \mathbb{N}^*$, we have

$$\lambda_k(M, g) \leq \frac{(n-1)^2}{4}a^2 + C_n \left(\frac{k}{V}\right)^{2/n}. \tag{1.1}$$

Remarks 1.2 (i) In [12], the constant C_n depends also on the diameter.

(ii) In dimension higher than 2, a normalization on the volume is not enough to control the spectrum: namely, on any compact manifold of dimension higher than 2, one can find a metric of given volume, with arbitrarily large first non-zero eigenvalue λ_2 of the Laplacian, in virtue of the result of B. Colbois and J. Dodziuk [6].

(iii) When $\mathrm{Ric}^g \geq 0$, we deduce that there exists $C_n > 1$ with $\lambda_k(M, g) \leq C_n(\frac{k}{V})^{2/n}$ for all k. However, when Ricci is not supposed positive, then the presence of a term like $\frac{(n-1)^2}{4}a^2$ is necessary: by a result of R. Brooks [2], it is possible to find a family of compact hyperbolic manifolds with volume going to infinity and a positive uniform lower bound on the first nonzero eigenvalue.

The idea of the proof of Theorem 1.1 is to consider k disjoint balls of radius r which almost cover the manifold (M, g), with r around $(\frac{V}{k})^{1/n}$, and to apply then Cheng's theorem [5]. However, such a theorem does not exist on manifolds with boundary, and with Neumann boundary condition. A reason for this is that there is no Bishop-Gromov theorem: indeed, even for a Euclidean domain, it is not possible to control the volume of a ball of radius $2r$ with respect to the volume of a ball of radius r and same center. See also Example 1.4 in [4].

This does not mean that a result in the spirit of Theorem 1.1 does not exist for domains. Namely, P. Kröger [11] proved thanks to harmonic analysis, that on bounded Euclidean domains, the k-th eigenvalue of the Neumann problem was bounded from above by some expression $C_n(k/|\Omega|)^{n/2}$, where C_n only depends upon the dimension. An analogous result can be derived from the much more general and difficult work of N. Korevaar [10] (see also [9]), for bounded domains of non-negative Ricci curvature manifolds, and also for bounded domains of negative Ricci curvature compact manifolds (in this case the bound depends on the diameter).

This naturally leads to the

Question What can be said for bounded domains of a complete Riemannian manifold with Ricci curvature bounded below?

In this note, we consider the Neumann eigenvalue problem for the Laplacian of a bounded domain Ω with smooth boundary, in a given complete (not compact a priori) Riemannian manifold (M, g). More precisely, we search for a couple $(\lambda, u) \in \mathbb{R} \times C^\infty(\overline{\Omega})$ which is a solution of the following boundary elliptic problem

$$\begin{cases} \Delta u = \lambda u & \text{on } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases}$$

where Δ is the non-negative Laplacian of the metric g and ν the outward unit normal of $\partial\Omega$. Since Ω is bounded with smooth boundary, the spectrum of Δ on Ω is an unbounded sequence of real numbers $(\lambda_k(\Omega))_{k\in\mathbb{N}^*}$ which can be increasingly ordered

$$0 = \lambda_1(\Omega) < \lambda_2(\Omega) \le \cdots \le \lambda_k(\Omega) \le \lambda_{k+1}(\Omega) \le \cdots .$$

There exist standard variational characterisations of the spectrum of Δ which can be found for instance in the book of P. Bérard [1] (or in [8]).

The main result of this article is the following.

Theorem 1.3 *Let (M^n, g) be a complete n-dimensional Riemannian manifold with Ricci curvature bounded below* $\mathrm{Ric}^g \ge -(n-1)a^2$, $a \ge 0$.

There exist constants $A_n > 0$, $B_n > 0$ only depending on the dimension, such that for all $k \in \mathbb{N}^*$, $V > 0$ and for each bounded domain $\Omega \subset M$, with smooth boundary and volume V, we have

$$\lambda_k(\Omega) \le A_n a^2 + B_n \left(\frac{k}{V}\right)^{2/n}. \tag{1.2}$$

If the manifold M is compact, an interesting special case is to choose $\Omega = M$, and we recover Theorem 1.1, up to the value of the constant A_n which is not equal to $\frac{(n-1)^2}{4}$ in our paper.

The proof Theorem 1.3 goes in the same spirit as the proof of Theorem 1.1: in order to bound $\lambda_k(\Omega)$, we consider k disjoint sets $A_1, \dots, A_k$ in Ω of measure of the order of $\frac{\mathrm{Vol}(\Omega)}{k}$, and introduce test functions $f_1, \dots, f_k$ subordinated to these sets. We estimate the Rayleigh quotient of these functions by a direct calculation, which gives the theorem. The main improvement of this paper is the construction of an adapted family of sets $A_1, \dots, A_k$, more convenient for our purpose as balls. As this construction is interesting by itself and will be used in other contexts, we present it in a rather abstract (indeed metric) way.

The paper is organised as follows: the metric construction of our sets is done in Sect. 2, and in Sect. 3 we will use them so as to prove Theorem 1.3 by producing some test functions for the variational characterisation of the spectrum.

2 A Metric Approach

In this section, we formalize the geometric situation of Theorem 1.3 (a bounded domain in a complete manifold) in a more general setting (a bounded domain in a complete metric space). More precisely, let (X, d) be a complete, locally compact metric space, $Y \subset X$ a bounded Borelian subset endowed with the induced distance, and μ a Borelian measure with support in $\overline{Y}$ such that $\mu(Y) = \omega$, $0 < \omega < \infty$. We will need in addition the following technical assumptions:

(H1) For each $r > 0$, there exists a constant $C(r) > 0$ such that each ball of radius $4r$ in X may be covered by $C(r)$ balls of radius r. Moreover, $r \mapsto C(r)$ is an increasing function of the radius.

(H2) We suppose that the volume of the r–balls tends to 0 uniformly on X, namely $\lim_{r\to 0} \sup\{\mu(B(x, r)) : x \in X\} = 0$. However, taking (H1) into account, this volume condition is equivalent to $\lim_{r\to 0} \sup\{2C(r)\mu(B(x, r)) : x \in X\} = 0$ which is the (more convenient) condition that will be used in the remainder of the article.

It is important to remark that these hypothesis are quite natural since they make part of the metric properties of the Riemannian manifolds that are involved in Theorem 1.3. These specific metric properties are collected in the following fundamental example.

Example 2.1 A typical example of a couple (X, Y) satisfying the hypothesis (H1), (H2) is to choose X as a complete n-dimensional Riemannian manifold (M, g) with Ricci curvature bounded below $\mathrm{Ric}^g \geq -(n-1)a^2$, $a \geq 0$ (which is the class of manifolds involved in Theorem 1.3), and as Y a bounded domain with smooth boundary in M. The distance d is the distance associated to the Riemannian metric g, the measure μ is the restriction to Y of the Riemannian measure of g. The existence of the constant $C(a, r)$ is given by the classical Bishop-Gromov inequality thanks to the lower bound on the Ricci curvature of g (see [13] p. 156). Precisely, for $0 < r < R$, and for each point $p \in M$, we have

$$\frac{\mathrm{Vol}(B(p, R), g)}{\mathrm{Vol}(B(p, r), g)} \leq \frac{v_a(R)}{v_a(r)}, \tag{2.1}$$

where $v_a(R)$ denotes the volume of a ball of radius R in $\mathbb{M}^n_a$, the simply connected n-dimensional manifold of constant sectional curvature $-a^2$.

This gives a bound on the number of balls of radius r that are necessary to cover a ball of radius $4r$ (this property known as the packing lemma is a consequence of Inequality (2.1)). In fact, fix B_{4r} a $4r$-ball and consider $\{B(x_i, r/2)\}_{i\in I}$ a maximal family of disjoint balls whose center x_i lives in B_{4r}; then the corresponding family of r-balls $\{B(x_i, r)\}_{i\in I}$ cover B_{4r}. In consequence, we can cover a ball of radius $4r$ with $\leq 1 + [\frac{v_a(4r+r/2)}{v_a(r/2)}]$ r-balls. We just define

$$C(a, r) = \max_{t\leq r}\{1 + [\frac{v_a(4t + t/2)}{v_a(t/2)}]\}.$$

The increasing character of $r \mapsto C(a, r)$ is by definition.

Furthermore, as $r \longrightarrow 0$, the ratio $\frac{\mathrm{Vol}(B(p,r),g)}{v_a(r)} \longrightarrow 1$, we obtain

$$\mathrm{Vol}((B(p, R), g) \leq v_a(R),$$

and consequently $\mu(B(p,r)) := \mathrm{Vol}(B(p,r) \cap Y, g)$ goes uniformly to 0 as $r \to 0$.

We prove in the sequel that, under our technical assumptions, one can build some subsets A and D satisfying certain volume conditions.

Lemma 2.2 *Let* (X,d) *be a complete, locally compact metric space,* $Y \subset X$ *a bounded Borelian with the induced distance, and* μ *a Borelian measure with support in* $\overline{Y}$ *such that* $\mu(Y) = \omega$, $0 < \omega < \infty$ *and* $\mu(\overline{Y} \setminus Y) = 0$. *In addition, we make the hypothesis* (H1), (H2). *Let* $0 < \alpha \le \frac{\omega}{2}$. *Thanks to* (H2) *there exists* $r > 0$ *with* $\sup\{2C(r)\mu(B(x,r)) : x \in X\} \le \alpha$.

Then there exist $A, D \subset Y$ *such that* $A \subset D$ *and*

$$\begin{cases} \mu(A) \ge \alpha, \\ \mu(D) \le 2C(r)\alpha, \\ d(A, Y \cap D^c) \ge 3r. \end{cases}$$

Proof We fix the positive numbers r and α. Let us consider any positive integer $m \in \mathbb{N}^*$ and define a non-negative application $\Psi_m : X^m = \underbrace{X \times X \times \cdots \times X}_{m \text{ times}} \longrightarrow \mathbb{R}$ by the relation

$$\Psi_m : \boldsymbol{x} = \left(x^j\right)_{j=1}^{m} \longmapsto \mu\left(\bigcup_{j=1}^{m} B\left(x^j, r\right)\right),$$

which is simply the restriction of the measure μ to $\mathrm{U}_m(r)$ a particular class of open sets which is defined by

$$\mathrm{U}_m(r) := \left\{ \bigcup_{j=1}^{m} B\left(x^j, r\right) / \left(x^j\right)_{j=1}^{m} \in X^m \right\}.$$

Since (X,d) is a complete and locally compact metric space, it is also the case of the finite product X^m when it is endowed with the product distance. Then for each $m \in \mathbb{N}^*$ there exists some $\boldsymbol{x}_{\max,m} \in X^m$ (not necessary unique) such that

$$\Psi_m(\boldsymbol{x}_{\max,m}) = \max_{X^m} \Psi_m = \max_{\mathrm{U}_m(r)} \mu = \mu\left(\bigcup_{j=1}^{m} B\left(x^j_{\max,m}, r\right)\right).$$

We first prove that there exists a finite integer $k \in \mathbb{N}^*$ such that $\Psi_k(\boldsymbol{x}_{\max,k}) \ge \alpha$ and $\Psi_{k-1}(\boldsymbol{x}_{\max,k-1}) \le \alpha$. Indeed, consider the function $\xi : \mathbb{N}^* \longrightarrow \mathbb{R}$ defined by the relation $\xi(m) = \Psi_m(\boldsymbol{x}_{\max,m})$. On one hand, the condition $\sup\{2C(r)\mu(B(x,r)) : x \in X\} \le \alpha$ obviously implies $\xi(1) \le \frac{\alpha}{2C(r)} \le \alpha$. On the other hand, since $\mathrm{Supp}\,\mu \subset \overline{Y}$, there exists a radius $R > 0$ large enough such that $\mu(B(z,R)) \ge 3\omega/4$, for a certain $z \in X$. But it can be clearly deduced from Assumption (H1) that $B(z,R)$ can be finitely covered by $m_0 \in \mathbb{N}^*$ balls of radius r (notice that m_0 depends on R). Consequently it turns out

$$\frac{3\alpha}{2} \leq \frac{3\omega}{4} \leq \mu\left(B(z,R)\right) \leq \max_{\mathrm{U}_{m_0}(r)} \Psi_{m_0} = \xi(m_0).$$

Thereby the function $\xi : \mathbb{N}^* \longrightarrow \mathbb{R}$ satisfies $\xi(1) \leq \alpha$ and $\xi(m_0) \geq \frac{3\alpha}{2}$, which entails the existence of some $k \in \mathbb{N}^*$ such that $\Psi_k(\boldsymbol{x}_{\max,k}) \geq \alpha$ and $\Psi_{k-1}(\boldsymbol{x}_{\max,k-1}) \leq \alpha$.

We now set $U_k := \bigcup_{1\leq j\leq k} B(x^j_{\max,k}, r)$ and $V_k := \bigcup_{1\leq j\leq k} B(x^j_{\max,k}, 4r)$. The next step is to show that

$$\mu(V_k) \leq C(r)\mu(U_k).$$

Still according to Assumption (H1), V_k is covered by $kC(r)$ balls of radius r, namely $V_k \subset \bigcup_{1\leq j\leq kC(r)} B_j$, where the B_j are balls of radius r. But it is quite clear that this union of r-balls can be written as $\bigcup_{1\leq j\leq kC(r)} B_j = \bigcup_{1\leq j\leq C(r)} W_j$ where each $W_j \in \mathrm{U}_k(r)$. It follows

$$\begin{aligned}
\mu(V_k) \leq \mu\left(\bigcup_{1\leq j\leq kC(r)} B_j\right) &= \mu\left(\bigcup_{1\leq j\leq C(r)} W_j\right) \\
&\leq \sum_{j=1}^{C(r)} \mu(W_j) \\
&\leq C(r) \max_{\mathrm{U}_k(r)} \mu = C(r)\xi(k) = C(r)\mu(U_k).
\end{aligned}$$

We finally define the sets $A := Y \cap U_k$ and $D := Y \cap V_k$. We only have to check that they satisfy the properties stated in Lemma 2.2. We observe that $\mu(A) = \mu(U_k)$ since the measure μ is supported in $\overline{Y}$ and $\mu(\overline{Y} \setminus Y) = 0$. Besides, U_k can be written as the union of an element of $\mathrm{U}_{k-1}(r)$ and an element of $\mathrm{U}_1(r)$ so that

$$\mu(A) \leq \xi(k-1) + \xi(1) \leq \alpha\left(1 + \frac{1}{2}\right).$$

Still since $\operatorname{Supp}\mu = \overline{Y}$, we obtain $\mu(D) = \mu(V_k) \leq C(r)\mu(U_k) = C(r)\mu(A) \leq 2C(r)\alpha$. By the definition of U_k and V_k, we straightforwardly have $d(A, Y \cap D^c) \geq 3r$. □

In Sect. 3, we will use the following corollary of Lemma 2.2 to make the proof of Theorem 1.3. We give therein an explicite construction of the domains that were mentioned at the end of the introduction.

Corollary 2.3 *Let (X,d) be a complete, locally compact metric space, $Y \subset X$ a bounded Borelian with the induced distance, and μ a Borelian measure with support in $\overline{Y}$ such that $\mu(Y) = \omega$, $0 < \omega < \infty$ and $\mu(\overline{Y} \setminus Y) = 0$. In addition, we make the hypothesis* (H1), (H2) *as in Lemma 2.2, and take N a positive integer.*

Let $r > 0$ such that $4C^2(r)\mu(B(x,r)) \le \frac{\omega}{N}$ holds for all $x \in X$, and let $\alpha = \frac{\omega}{2C(r)N}$. Then, there exist N measurable subsets $A_1, \dots, A_N \subset Y$ such that $\mu(A_i) \ge \alpha$ and, for each $i \ne j$, $d(A_i, A_j) \ge 3r$.

Proof We construct the family $\left(A_j\right)_{j=1}^N$ by finite induction applying Lemma 2.2.

- $j = 1$. We set $(X_1, d_1, \mu_1) = (X, d, \mu)$ and $Y_1 = Y$, which satisfy the assumptions of Lemma 2.2. Therefore there exist A_1, D_1 such that $A_1 \subset D_1 \subset Y_1 = Y$ and

$$\begin{cases} \mu(A_1) \ge \alpha, \\ \mu(D_1) \le 2C(r)\alpha = \frac{\omega}{N}, \\ d(A_1, Y_1 \cap D_1^c) \ge 3r. \end{cases}$$

- $j = 2$. We set $(X_2, d_2, \mu_2) = (X, d, \mu_{|Y_2})$ and $Y_2 = D_1^c \cap Y_1$, which satisfy the assumptions of Lemma 2.2 with $\omega_2 = \mu_2(Y_2) \ge \omega(1 - \frac{1}{N}) = \omega(\frac{N+1-2}{N}) \ge \alpha$. Therefore there exist A_2, D_2 such that $A_2 \subset D_2 \subset Y_2 = D_1^c \cap Y_1$ and

$$\begin{cases} \mu(A_2) \ge \alpha, \\ \mu(D_2) \le 2C(r)\alpha = \frac{\omega}{N}, \\ d(A_2, Y_2 \cap D_2^c) \ge 3r. \end{cases}$$

  As $A_1 \subset D_1$ and $A_2 \subset Y_1 \cap D_1^c$ we get $d(A_1, A_2) \ge d(A_1, Y_1 \cap D_1^c) \ge 3r$ thanks to the case $j = 1$.

- $j \ge 3$. We suppose that we have already constructed the families $(A_s)_{s=1}^{j-1}$ and $(D_s)_{s=1}^{j-1}$ that satisfy the conditions

$$\begin{cases} A_s \subset D_s \subset Y \cap (D_1 \cup \dots \cup D_{s-1})^c = Y_s, \quad s \le j-1, \\ d(A_s, A_t) \ge 3r, \quad s \ne t, \\ \mu\left(D_1 \cup \dots \cup D_{j-1}\right) \le \omega\left(\frac{j-1}{N}\right). \end{cases}$$

  We set $(X_j, d_j, \mu_j) = (X, d, \mu_{|Y_j})$ and $Y_j = Y \cap (D_1 \cup \dots \cup D_{j-1})^c$, which satisfy the assumptions of Lemma 2.2 with $\omega_j = \mu_j(Y_j) \ge \omega(1 - \frac{j-1}{N}) = \omega(\frac{N+1-j}{N}) \ge \alpha$ if $j \le N$. Therefore there exist A_j, D_j such that $A_j \subset D_j \subset Y_j$ and

$$\begin{cases} \mu(A_j) \ge \alpha, \\ \mu(D_j) \le 2C(r)\alpha = \frac{\omega}{N}, \\ d(A_j, Y_j \cap D_j^c) \ge 3r. \end{cases}$$

  As $A_j \subset Y \cap \left(D_1 \cup \dots \cup D_{j-1}\right)^c \subset Y \cap (D_1 \cup \dots \cup D_{s-1})^c = Y_s$, $s < j$, and $A_s \subset D_s$, we get $d(A_j, A_s) \ge d(A_s, Y_s \cap D_s^c) \ge 3r$ thanks to the case $j = s$. As already said, we can proceed this construction so longer we have enough volume to do it, that is N times. $\square$

3 Proof of Theorem 1.3

Let (M^n, g) be a complete n-dimensional Riemannian manifold with Ricci curvature bounded below $\mathrm{Ric}^g \geq -(n-1)a^2$, and $\Omega \subset M$ a bounded domain of volume V, with smooth boundary.

We observe first that, by renormalisation, it is enough to prove the theorem for the case $a = 1$: namely, if Theorem 1.3 is true for $a = 1$, and if g is a Riemannian metric with $\mathrm{Ric}^g \geq -(n-1)t^2 g$, then $g_0 = t^2 g$ satisfies $\mathrm{Ric}^{g_0} \geq -(n-1)g_0$. Since we have $\lambda_k(g_0) \leq A_n + B_n(\frac{k}{V(g_0)})^{2/n}$, then, because $\lambda_k(g) = t^2\lambda_k(g_0)$ and $V(g) = t^n V(g_0)$, we get $\lambda_k(g) \leq A_n t^2 + B_n(\frac{k}{V})^{2/n}$.

So, let us prove Theorem 1.3 for $a = 1$. As in Example 2.1, let us consider the Borelian measure μ which is the restriction to the domain Ω of the Riemannian volume of (M, g).

In order to prove Theorem 1.3, we will use the classical variational characterization of the spectrum: to estimate λ_k from above, it suffices to construct an $H^1(\Omega)$-orthogonal family of k test functions $\left(f_j\right)_{j=1}^k$, such as each f_j has controlled Rayleigh quotient. In the sequel, we construct test functions with disjoint support related to the sets $A_1, \ldots, A_k$ arising from Corollary 2.3, so that it immediately implies orthogonality in $H^1(\Omega)$.

Lemma 3.1 *Let $A \subset M$ be a subset as in Corollary 2.3. Let $A^r := \{x \in M : d(x, A) \leq r\}$, $r > 0$. There exists a function f supported in A^r whose restriction to Ω is of Rayleigh quotient*

$$R(f) \leq \frac{1}{r^2}\frac{\mu(A^r \setminus A)}{\mu(A)}.$$

Proof Let us define a *plateau* function

$$f(p) = \begin{cases} 1 & \text{if } p \in A, \\ 1 - \frac{d(p,A)}{r} & \text{if } p \in (A^r \setminus A), \\ 0 & \text{if } p \in (A^r)^c. \end{cases}$$

In Corollary 2.3, the domain A is a finite union of metric balls and intersection with complement of balls. The boundary is not smooth, but the function $d(\partial A, \cdot)$ "distance to the boundary of A" is well known to be 1-Lipschitz on M. According to Rademacher's theorem (see Sect. 3.1.2, page 81–84 in [7]), $d(\partial A, \cdot)$ is differentiable L^n almost everywhere (since dVol_g is absolutely continuous with respect to Lebesgue's measure L^n), and its g-gradient satisfies $|\nabla d(\partial A, \cdot)|_g \leq 1$, L^n almost everywhere. It comes out that the gradient of f satisfies L^n almost everywhere

$$|\nabla f(p)|_g \leq \begin{cases} \frac{1}{r} & \text{if } p \in (A^r \setminus A), \\ 0 & \text{if } p \in (A^r \setminus A)^c. \end{cases}$$

We immediately deduce

$$R(f) = \frac{\int_\Omega |\nabla f|_g^2 \,\mathrm{dVol}_g}{\int_\Omega f^2 \,\mathrm{dVol}_g} \le \frac{1}{r^2}\frac{\mu(A^r \setminus A)}{\mu(A)}. \qquad \square$$

Proof of Theorem 1.3. As already said, we apply Corollary 2.3: let $k \in \mathbb{N}^*$ and set $N = 2k$. As the volume of the r-balls uniformly tends to 0 (see assumption (H2)), there exist $r > 0$ with r small enough so that

$$2C(r)\mu(B(x,r)) \le \alpha := \frac{V}{4C(r)k}, \tag{3.1}$$

holds for every $x \in M$. Corollary 2.3 gives the existence of $2k$ measurable subsets $A_1, \dots, A_{2k}$ of measure $\mu(A_i) \ge \frac{V}{4C(r)k}$ with $d(A_i, A_j) \ge 3r$ if $i \ne j$. In particular, the corresponding sets A_i^r and A_j^r are also disjoint.

We can now apply the construction of Lemma 3.1 and we get an $H^1(\Omega)$-orthogonal family of $2k$ test functions $(f_j)_{j=1}^{2k}$, of disjoint supports and whose Rayleigh quotient satisfies

$$R(f_i) \le \frac{1}{r^2}\frac{\mu(A_i^r \setminus A_i)}{\mu(A_i)}.$$

At this point, Corollary 2.3 does not give any control on $\mu(A_i^r)$. Let

$$Q = \sharp\left\{ i \in \{1, \dots, 2k\} : \mu(A_i^r) \ge \frac{V}{k} \right\}.$$

As $\mathrm{Vol}(\Omega, g) = V$, we already see that $Q \le k$, so that for at least k of these $2k$ subsets $A_1, \dots, A_{2k}$, we have $\mu(A_i^r) \le \frac{V}{k}$. For every such A_i, we choose the corresponding function f_i constructed in Lemma 3.1, as test functions. We then have $\mu(A_i^r \setminus A_i) \le \frac{V}{k}$ and $\mu(A_i) \ge \alpha = \frac{V}{4C(r)k}$, such that for such a function f_i

$$R(f_i) \le \frac{1}{r^2}\frac{V/k}{V/4C(r)k} = \frac{4C(r)}{r^2}. \tag{3.2}$$

Our aim now is to prove an upper bound of the kind

$$\lambda_k(g) \le A_n + B_n \left(\frac{k}{V}\right)^{2/n}.$$

Let $\omega_n' > 0$ be the positive constant such that $\mu(B(x,r)) \le \omega_n' r^n$ for radius $r \le 1$ in the hyperbolic space of curvature -1. We then define the integer $k_0 = [\frac{V}{8C(1)^2\omega_n'}] + 1$ (remark that it strongly depends on the volume) and for every $k \ge k_0$, we set

$$r_k = \left(\frac{V}{k}\frac{1}{8C(1)^2\omega_n'}\right)^{1/n}.$$

Clearly, $r_k \leq 1$ and (3.1) holds, since by definition $8C(r_k)^2\mu(B(x,r_k)) \leq 8C(1)^2\omega'_n r_k^n = \frac{V}{k}$. Our Inequality (3.2) now reads as

$$\forall k \geq k_0 \quad \lambda_k \leq \frac{4C(1)}{r_k^2} = 4C(1)\left(8C(1)^2\omega'_n\right)^{2/n}\left(\frac{k}{V}\right)^{2/n}.$$

Now if $k < k_0$, then we obviously have $\lambda_k \leq \lambda_{k_0}$, so that we straightly obtain

$$\forall k \in \mathbb{N}^* \quad \lambda_k \leq \lambda_{k_0} + B_n\left(\frac{k}{V}\right)^{2/n}, \tag{3.3}$$

where we have set $B_n := 4C(1)(8C(1)^2\omega'_n)^{2/n}$. The last thing to do is to estimate the particular eigenvalue λ_{k_0}.

1) If $k_0 = 1$, then $\lambda_{k_0} = \lambda_1 = 0$ and we get Inequality (1.2), with $A_n = 0$.
2) On the contrary, if $k_0 \geq 2$, then we deduce $\frac{V}{8C(1)^2\omega'_n} < k_0 \leq 2\frac{V}{8C(1)^2\omega'_n}$. We can apply Inequality (3.3) with $k = k_0$, which implies

$$\lambda_{k_0} \leq \frac{4C(1)}{r_{k_0}^2} = 4C(1)2^{2/n},$$

and then Inequality (3.3) is nothing but Inequality (1.2) with $A_n = 4C(1)2^{2/n}$, $B_n = 4C(1)(8C(1)^2\omega'_n)^{2/n}$ and $a = 1$. □

Remark 3.2 For the case $a = 0$, a slightly better constant B_n can be obtained by making a direct proof instead of plugging $a = 0$ in Inequality (1.2).

Acknowledgements We thank G. Carron for helpful comments. DM is grateful to the Mathematical Institute of Neuchâtel for financial support during the redaction of a large part of this text (FNRS Grant 20-109130).

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