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On the stability of travelling waves with vorticity obtained by minimization

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Abstract. We modify the approach of Burton and Toland Comm. Pure Appl. Math. LXIV. 975–1007 (2011) to show the existence of periodic surface water waves with vorticity in order that it becomes suited to a stability analysis. This is achieved by enlarging the function space to a class of stream functions that do not correspond necessarily to travelling profiles. In particular, for smooth profiles and smooth stream functions, the normal component of the velocity field at the free boundary is not required a priori to vanish in some Galilean coordinate system. Travelling periodic waves are obtained by a direct minimization of a functional that corresponds to the total energy and that is therefore preserved by the time-dependent evolutionary problem (this minimization appears in Comm. Pure Appl. Math. LXIV. 975–1007 (2011) after a first maximization). In addition, we not only use the circulation along the upper boundary as a constraint, but also the total horizontal impulse (the velocity becoming a Lagrange multiplier). This allows us to preclude parallel flows by choosing appropriately the values of these two constraints and the sign of the vorticity. By stability, we mean conditional energetic stability of the set of minimizers as a whole, the perturbations being spatially periodic of given period. Our proofs depend on the assumption that the surface offers some resistance to stretching and bending.

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1. Introduction

For a fixed Hölder exponent $\gamma \in (0, 1)$, period $P > 0$ and average height $Q > 0$, we shall consider domains $\Omega \subset \mathbb{R}^2$ and curves $\mathscr{S}$ such that there exists a $C^{1,\gamma}$-map $F : \mathbb{R}^2 \to \mathbb{R}^2$ satisfying the following properties:

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- F restricted to $\mathbb{R} \times [0, Q]$ is a diffeomorphism from $\mathbb{R} \times [0, Q]$ onto $\overline{\mathit{\Omega}}$,
- $\text{meas}(\mathit{\Omega} \cap ((0, P) \times \mathbb{R})) = PQ$,
- $F(x_1, 0) = (x_1, 0)$ for all $x_1 \in \mathbb{R}$,
- $\mathscr{S} \subset \mathbb{R} \times (0, \infty)$ and F restricted to $\mathbb{R} \times \{Q\}$ is a homeomorphism from $\mathbb{R} \times \{Q\}$ onto $\mathscr{S}$,
- $F(x_1 + P, x_2) = (F_1(x_1 + P, x_2), F_2(x_1 + P, x_2)) = (F_1(x_1, x_2) + P, F_2(x_1, x_2))$ for all $x = (x_1, x_2) \in \mathbb{R} \times [0, Q]$.

As a consequence the curve $\mathscr{S}$ is of class $C^{1,\gamma}$ in the open upper half plane, P-periodic and is a connected component of the boundary of the region Ω. Let $\mathcal{S}$ and $\mathit{\Omega}$ denote one period of $\mathscr{S}$ and Ω. We denote by $\mathfrak{O}$ the set of all domains Ω defined in this way, and we write $\Omega \in \mathfrak{O}$ or $\mathit{\Omega} \in \mathfrak{O}$. Thus $\mathscr{S}$ must be a simple curve (without self-intersection or self-touching) but it need not be the graph of a function. While this is fairly general, it excludes some cases of physical interest. For example, a row of rolling beads of mercury constitutes a travelling wave with a disconnected free surface whose components are not graphs of functions, and beads of mercury can touch without coalescing.

If $\mathbb{R}^2$ is identified with the complex plane $\mathbb{C}$, the point (x_1, x_2) corresponding to the complex number $x_1 + ix_2$, it can be shown (see e.g. the appendix A of the paper by Constantin and Varvaruca [10]) that there exists a holomorphic map

$$\widetilde{\phi} + i\widetilde{\psi} : \Omega \to \mathbb{R} \times (0, 1) \tag{1.1}$$

such that

- $\widetilde{\phi} + i\widetilde{\psi}$ can be extended into a diffeomorphism from $\overline{\mathit{\Omega}}$ onto $\mathbb{R} \times [0, 1]$,
- $\widetilde{\psi}, \widetilde{\phi}$ are real-valued functions of class $C^{1,\gamma}$ on $\overline{\mathit{\Omega}}$ and their gradients never vanish on $\overline{\Omega}$,
- $\widetilde{\psi}|_{\{x_2=0\}} = 0$ and $\widetilde{\psi}|_{\mathscr{S}} = 1$,
- $\widetilde{\phi}(x + P) + i\widetilde{\psi}(x + P) = \widetilde{\phi}(x) + i\widetilde{\psi}(x) + \widetilde{P}$ for all $x = x_1 + ix_2 \in \mathbb{R} \times [0, 1]$, where

$$\widetilde{P} = \int_0^P \partial_1 \widetilde{\phi}(x_1, 0) dx_1 = \int_0^P \partial_2 \widetilde{\psi}(x_1, 0) dx_1 = \int_{\mathcal{S}} \nabla \widetilde{\psi} \cdot n \, dS \tag{1.2}$$

and n is the outward normal to Ω at a point of $\mathcal{S}$.

We shall write $\xi \in H^{1/2}_{per}(\mathcal{S})$ or $\xi \in H^{1/2}_{per}(\mathscr{S})$ if ξ is the trace on $\mathscr{S}$ of some $\psi_\xi \in H^1_{loc}(\mathit{\Omega})$ that is P-periodic in x_1.

Analogously, we shall write $\zeta \in L^2_{per}(\mathit{\Omega})$ if $\zeta \in L^2_{loc}(\mathit{\Omega})$ is P-periodic in x_1.

Given $\mathit{\Omega}$, $\mathscr{S}$, $\xi \in H^{1/2}_{per}(\mathscr{S})$ and $\zeta \in L^2_{per}(\mathit{\Omega})$, let $\psi \in H^1_{\text{loc}}(\mathit{\Omega})$ be the weak solution of the boundary value problem

$$-\Delta\psi = \zeta \quad \text{ on } \mathit{\Omega}, \tag{1.3a}$$

$$\psi(x_1, 0) = 0, \tag{1.3b}$$

$$\psi = \xi \quad \text{ on } \mathscr{S}, \tag{1.3c}$$

$$\psi \text{ is } P\text{-periodic in } x_1, \text{ written } \psi \in H^1_{per}(\Omega) \text{ or } \psi \in H^1_{per}(\mathit{\Omega}). \tag{1.3d}$$

On one period, the circulation C and the total horizontal impulse I are given by

$$C = C(\Omega, \xi, \zeta) := \int_{\mathcal{S}} \nabla\psi \cdot n \, dS,$$
$$I = I(\Omega, \xi, \zeta) := \int_{\Omega} \partial_2 \psi dx = \int_{\Omega} \nabla x_2 \cdot \nabla\psi dx.$$

By $C(\Omega, \xi, \zeta) = \int_{\mathcal{S}} \nabla\psi \cdot n \, dS$, we mean

$$C(\Omega, \xi, \zeta) = \int_{\Omega} \nabla\psi \cdot \nabla\widehat{\psi} \, dx - \int_{\Omega} \zeta\widehat{\psi} \, dx,$$

where $\widehat{\psi}$ is any function in $H^1_{per}(\Omega)$ such that $\widehat{\psi}|_{\{x_2=0\}} = 0$ and $\widehat{\psi}|_{\mathcal{S}} = 1$. For example we can choose $\widehat{\psi} = \widetilde{\psi}$. When ψ is regular enough, these two ways of defining $C(\Omega, \xi, \zeta)$ agree, but the latter one requires less regularity. We can also write, if there is enough regularity available,

$$I(\Omega, \xi, \zeta) = \int_{\mathcal{S}} x_2 \nabla\psi \cdot n \, dS + \int_{\Omega} x_2 \zeta \, dx.$$

Let us fix μ and ν in $\mathbb{R}$. Then (Ω, ξ, ζ) defines a travelling water wave with stream function ψ, circulation μ, total horizontal impulse ν and vorticity ζ, if, in addition,

$$C(\Omega, \xi, \zeta) = \mu, \quad I(\Omega, \xi, \zeta) = \nu, \tag{1.3e}$$

$$\xi = \lambda_1 x_2 + \lambda_2|_{\mathscr{S}} \quad \text{for some } \lambda_1, \lambda_2 \in \mathbb{R}, \tag{1.3f}$$

$$\zeta = \lambda \circ (\psi - \lambda_1 x_2) \text{ almost everywhere for some function } \lambda \tag{1.3g}$$

and

$$\frac{1}{2}|\nabla\psi - (0, \lambda_1)|^2 + g\, x_2 = \text{constant on } \mathscr{S}, \tag{1.3h}$$

where g is gravity. The travelling wave is moving with speed λ_1 to the right and Eq. (1.3g) reflects the fact that vorticity in steady flows is constant on streamlines. The constants λ_1, λ_2 in (1.3f) and the function λ in (1.3g) are not prescribed.

If the surface reacts to stretching and bending, the Bernoulli condition (1.3h) is replaced by

$$\begin{aligned} \frac{1}{2}|\nabla\psi - (0, \lambda_1)|^2 + g\, x_2 - T\beta\big(\ell(\mathcal{S}) - P\big)^{\beta-1}\sigma & \\ + E\Big(2\sigma'' + \sigma^3\Big) = \text{ constant on } \mathscr{S}, & \end{aligned} \tag{1.3h$'$}$$

where $'$ denotes differentiation with respect to arc length along the surface, $\sigma(x)$ is the curvature of the surface at $x \in \mathscr{S}$, $\ell(\mathcal{S})$ is the length of $\mathcal{S}$, $E \geq 0$ is a coefficient of bending resistance and $\beta \geq 1$. See [15]. The case $E = 0$ and $\beta = 1$ corresponds to simple surface tension with coefficient T.

The total energy $\mathcal{L}(\Omega,\xi,\zeta)$ of a solution of (1.3)(a–d) in one period is the sum of the kinetic energy, the gravitational potential energy and the surface energy:

$$\mathcal{L}(\Omega,\xi,\zeta) := \frac{1}{2}\int_\Omega |\nabla\psi|^2\, dx + g\int_\Omega x_2\; dx + \mathcal{E}(\mathcal{S}), \tag{1.4}$$

where ψ is the solution to the corresponding boundary value problem (1.3)(a–d),

$$\mathcal{E}(\mathcal{S}) = T(\ell(\mathcal{S}) - P)^\beta + E\int_0^{\ell(\mathcal{S})} |\sigma|^2 ds, \tag{1.5}$$

and s is the arc length.[1] Hence we are lead to the minimization problem

$$\min\{\mathcal{L}(\Omega,\xi,\zeta) : \Omega\in\mathfrak{O},\ \xi\in H^{1/2}_{per}(\mathcal{S}), \zeta\in\mathcal{R}(\Omega), C=\mu, I=\nu\},$$

where $\mathfrak{O}$ is the class of domains Ω described above and $\mathcal{R}(\Omega)\subset L^2(\Omega)$ is the set of rearrangements supported in Ω of a given function $\zeta_Q\in L^2(\Omega_Q)$, where $\Omega_Q=(0,P)\times(0,Q)$. Note that $\Omega\neq\Omega_Q$ is allowed and ζ_Q does not depend on Ω. However, in general, $\mathcal{R}(\Omega)$ is not weakly closed in $L^2(\Omega)$ and we shall work instead with its weak closure $\overline{\mathcal{R}(\Omega)}^w$ in $L^2(\Omega)$, which is a convex subset of $L^2(\Omega)$; see the discussion in [6, p. 979, 3rd parag.]. Hence, as in [6], we shall rather consider

$$\min\{\mathcal{L}(\Omega,\xi,\zeta) : \Omega\in\mathfrak{O},\ \xi\in H^{1/2}_{per}(\mathcal{S}), \zeta\in\overline{\mathcal{R}(\Omega)}^w, C=\mu, I=\nu\}. \tag{1.6}$$

Observe that $\mathit{\Omega}_Q := \mathbb{R}\times(0,Q)\in\mathfrak{O}$. We write $\Omega\in\mathfrak{O}$ or $\mathit{\Omega}\in\mathfrak{O}$, and we assume that $\mathcal{L}(\Omega,\xi,\zeta)=+\infty$ is allowed, for example if the surface energy is infinite because the boundary is not regular enough. We assume $T>0$, $\beta\geq 1$ and $E>0$ in order to obtain compactness for the above minimization problem.

In (1.6), the boundary condition (1.3f) is not prescribed, but we will show that it holds for minimizers. Hence, in (1.6), any stream function ψ that is compatible with the vorticity function ζ is allowed (by choosing $\xi=\psi|_{\mathscr{S}}$). This feature will be crucial in the stability analysis of Sect. 5.

A way of avoiding parallel flows. When $\Omega=\Omega_Q$, by taking $\widehat{\psi}=x_2/Q$ we get

$$I(\Omega,\xi,\zeta) = Q\int_\Omega \nabla(x_2/Q)\cdot\nabla\psi dx = QC(\Omega,\xi,\zeta) + \int_\Omega x_2\zeta\, dx.$$

Hence, if ζ_Q is essentially one-signed and not trivial, then $I(\Omega_Q,\xi,\zeta) - QC(\Omega_Q,\xi,\zeta)\neq 0$ has the same sign as ζ_Q. Thus, to avoid parallel flows, it seems natural to choose μ,ν so that $(\nu-Q\mu)\zeta_Q\leq 0$ a.e. (or $\nu-Q\mu\neq 0$ if ζ_Q vanishes a.e.).

[1] If p is a parametrisation of $\mathscr{S}$ such that $|\frac{d}{dx}p|$ is constant and $p(x+P)=p(x)+(P,0)$, then

$$\int_0^{\ell(\mathcal{S})} |\sigma|^2 ds = \left(\frac{P}{\ell(\mathcal{S})}\right)^3 \int_0^P \left|\frac{d^2}{dx^2}p(x)\right|^2 dx.$$

In [6], the power 3 is wrongly omitted in several places, without invalidating the main results.

In [6], parallel flows were precluded by choosing μ large enough. They were proved to be saddle points of the energy, and thus different from any minimizer (there, the energy functional was obtained after a first maximization). For related works on global minimization in hydrodynamical problems and stability, see [3,5,7–9]. In particular, the paper [8] by Constantin, Sattinger and Strauss contains two variational formulations for gravity water waves with vorticity. In their first formulation, instead of considering the constraint $\zeta \in \overline{\mathcal{R}(\Omega)}^w$ for a given $\zeta_Q \in L^2(\Omega_Q)$ (among other constraints), they subtract from the energy functional a term of the form $\int_\Omega F(\zeta)dx$, where $F : \mathbb{R} \to \mathbb{R}$ is a given C^2-function such that F'' never vanishes. As a result, for any critical point, $(F')^{-1}$ turns out to be the so-called vorticity function. They do not apply their approach to existence results, but it leads to an elegant linear stability analysis in [9].

Overview of the paper. Section 2 discusses the solution by minimization of the elliptic equation $-\Delta\psi = \zeta$ for fixed Ω, ζ, μ and ν and establishes the unknown boundary data ξ. In Sect. 3 it is shown that that the Bernoulli boundary condition is satisfied by constrained minimizers when Ω is allowed to vary. Sect. 4 proves compactness of minimizing sequences and establishes the existence of constrained minimizers. The main stability result is Theorem 5.2 which is proved using compactness of minimizing sequences together with some theory of transport equations summarised in the Appendix.

Some open questions.

– Is there a criterion that ensures uniqueness of the constrained minimizer (up to translational invariance)? In such a case, the present notion of stability would be related to "orbital" stability.
– If ζ_Q is smooth, what can be said about the regularity of the minimizers?
– Is there an explicit ζ_Q for which the free boundaries of the minimizers are not graphs?
– For an initial profile near the one of a minimizer, is the solution to the evolutionary problem defined for small enough positive times? A stability result like Theorem 5.2 stated under this assumption is qualified as "conditional" (see [14] and, for well-posedness issues for related settings, see e.g. [11]). We therefore raise the question whether such a solution to the evolutionary problem is defined for all positive times.

2. Minimization on fixed domain

We begin with a useful lemma.

Lemma 2.1. *Suppose that $\Omega \in \mathfrak{D}\backslash\{\Omega_Q\}$ and $\zeta \in L^2(\Omega)$. Then*

$$C(\Omega, 1, 0) = \int_\Omega |\nabla\widetilde{\psi}|^2 dx > P/Q$$

(see (1.1) for the definition of $\widetilde{\psi}$) and, for all $\mu, \nu \in \mathbb{R}$, there exist $\lambda_1 = \lambda_{1,\Omega,\zeta}$ and $\lambda_2 = \lambda_{2,\Omega,\zeta}$ such that

$$C(\Omega, \lambda_1 x_2 + \lambda_2, \zeta) = \mu, \;\; I(\Omega, \lambda_1 x_2 + \lambda_2, \zeta) = \nu.$$

Moreover $\lambda_1, \lambda_2 \in \mathbb{R}$ are unique.

Proof. We require

$$\begin{aligned} \mu &= C(\Omega, \lambda_1 x_2 + \lambda_2, \zeta) = \lambda_2 C(\Omega, 1, 0) + \lambda_1 C(\Omega, x_2, 0) + C(\Omega, 0, \zeta) \\ &= \lambda_2 C(\Omega, 1, 0) + \lambda_1 P + C(\Omega, 0, \zeta), \\ \nu &= I(\Omega, \lambda_1 x_2 + \lambda_2, \zeta) = \lambda_2 I(\Omega, 1, 0) + \lambda_1 I(\Omega, x_2, 0) + I(\Omega, 0, \zeta) \\ &= \lambda_2 P + \lambda_1 PQ, \end{aligned}$$

because

$$C(\Omega, x_2, 0) = \int_{\mathcal{S}} \nabla x_2 \cdot n \, dS = \int_\Omega \mathrm{div}(\nabla x_2) dx + \int_0^P \partial_2 x_2 \, dx_1 = \int_0^P \partial_2 x_2 \, dx_1 = P,$$

$$I(\Omega, x_2, 0) = \int_\Omega \partial_2 x_2 dx = PQ,$$

$$\begin{aligned} I(\Omega, 1, 0) &= \int_\Omega \nabla x_2 \cdot \nabla \widetilde{\psi} \, dx = \int_\Omega div(\widetilde{\psi} \nabla x_2) dx = \int_\Omega div\left((\widetilde{\psi} - 1) \nabla x_2 \right) dx \\ &= \int_{\partial\Omega} (\widetilde{\psi} - 1) \nabla x_2 \cdot n \, dS = \int_0^P \partial_2 x_2 \, dx_1 = P \end{aligned}$$

and

$$I(\Omega, 0, \zeta) = \int_\Omega \nabla x_2 \cdot \nabla \psi \, dx = \int_\Omega div(\psi \nabla x_2) dx = \int_{\partial\Omega} \psi \nabla x_2 \cdot n \, dS = 0,$$

where ψ is the solution to the system (1.3a) to (1.3d) with $\xi = 0$.

Let $\widetilde{\psi}$ be, as in (1.1), the harmonic function on Ω that vanishes on $\{x_2 = 0\}$, is 1 on $\mathscr{S}$ and is P-periodic in x_1. Then, by (1.2), $\widetilde{P} = C(\Omega, 1, 0) = \int_\Omega |\nabla \widetilde{\psi}|^2 dx$. Let us check that

$$C(\Omega, 1, 0) \geq P/Q \text{ with equality exactly when } \Omega = \Omega_Q. \tag{2.1}$$

In order to do this, consider as in (1.1) the harmonic conjugate $\widetilde{\phi}$ of $\widetilde{\psi}$, that is, $\nabla \widetilde{\phi}$ is obtained from $\nabla \widetilde{\psi}$ by a clockwise rotation through $\pi/2$. Then $\widetilde{\phi}(x + P) - \widetilde{\phi}(x)$ is a constant equal to $\widetilde{P} = C(\Omega, 1, 0)$ (see above) and the map $(\widetilde{\phi}, \widetilde{\psi})$ is a diffeomorphism from Ω to $\mathbb{R} \times (0, 1)$.

We denote by (u, v) the Euclidean coordinates in $\mathbb{R} \times (0, 1)$ and by $(u, v) \to x_2(u, v)$ the map that associates with (u, v) the x_2 coordinate of the corresponding point in Ω. Observe that

$$\partial_{u,v} x_2 = (\partial_u x_2, \partial_v x_2) = \partial_{x_1, x_2} x_2 \, (\partial(x_1, x_2)/\partial(u, v))$$

(Jacobian matrix),

$$\partial_{u,v} v = \partial_{x_1, x_2} \tilde{\psi} \, (\partial(x_1, x_2)/\partial(u, v)),$$

$$(\partial(x_1, x_2)/\partial(u, v))(\partial(x_1, x_2)/\partial(u, v))^T = \{\det(\partial(x_1, x_2)/\partial(u, v))\} I \tag{2.2}$$

(multiple of the identity matrix; this is a consequence of the Cauchy-Riemann equations) and thus

$$\begin{aligned}\partial_{u,v}x_2\cdot\partial_{u,v}x_2 &= \partial_{x_1,x_2}x_2\cdot\partial_{x_1,x_2}x_2\det(\partial(x_1,x_2)/\partial(u,v))\\ &= \det(\partial(x_1,x_2)/\partial(u,v))\end{aligned}$$

and

$$\partial_{u,v}x_2\cdot\partial_{u,v}v = \partial_{x_1,x_2}x_2\cdot\partial_{x_1,x_2}\tilde{\psi}\,\det(\partial(x_1,x_2)/\partial(u,v))$$

As a consequence, we get that

$$\int_{u=0}^{\widetilde{P}}\int_{v=0}^{1}|\nabla x_2(u,v)|^2dudv = \int_\Omega dx = PQ$$

and

$$\int_0^{\widetilde{P}}\partial_2x_2(u,1)du \overset{\text{Gauss}}{=} \int_0^{\widetilde{P}}\int_0^1 div(v\nabla x_2(u,v))dudv$$

$$= \int_0^{\widetilde{P}}\int_0^1\nabla x_2(u,v)\cdot\nabla v\,dudv = \int_\Omega\partial_{x_1,x_2}x_2\cdot\partial_{x_1,x_2}\widetilde{\psi}\,dx = I(\Omega,1,0) = P.$$

Hence

$$PQ \geq \min\left\{\int_0^{\widetilde{P}}\int_0^1|\nabla y(u,v)|^2dudv : y\in H^1_{per}((0,\widetilde{P})\times(0,1)), y(\cdot,0)=0,\right.$$
$$\left.\int_0^{\widetilde{P}}\partial_2y(u,1)du = P\right\}.$$

The minimum depends on $\tilde{P}$ and therefore it depends on the shape of the domain Ω, because $\tilde{P} = C(\Omega,1,0)$. The minimum is reached exactly at the function $y(u,v) = (P/\tilde{P})v$, which shows that the value of the minimum is $(P/\tilde{P})^2\tilde{P} = P^2/C(\Omega,1,0)$. Hence $PQ \geq P^2/C(\Omega,1,0)$ and $C(\Omega,1,0) \geq P/Q$ with equality exactly when $\Omega = \Omega_Q$. Since $\Omega \neq \Omega_Q$ we now have $QC(\Omega,1,0) - P > 0$, so the equations for λ_1 and λ_2 can be solved uniquely. $\square$

Proposition 2.2. *Given $\Omega \in \mathfrak{O}\backslash\{\Omega_Q\}$, $\zeta \in L^2(\Omega)$ and $\mu,\nu\in\mathbb{R}$, the minimizer $\xi_{\Omega,\zeta}$ for the kinetic energy over $\{\xi \in H^{1/2}_{per}(\mathcal{S}) : C(\Omega,\xi,\zeta) = \mu, I(\Omega,\xi,\zeta) = \nu\}$ exists and is unique, and there exist λ_1 and λ_2 in $\mathbb{R}$ such that*

$$\xi = \xi_{\Omega,\zeta} = (\lambda_1x_2+\lambda_2)|_{\mathcal{S}}. \tag{2.3}$$

Proof. We consider the minimum of the functional $\psi \to \frac{1}{2}\int_\Omega|\nabla\psi|^2dx$ over $\psi\in H^1_{per}(\Omega)$ such that

$$-\Delta\psi = \zeta \quad\text{on}\quad \Omega,\ \ \psi(\cdot,0) = 0,$$

$$\int_\Omega\nabla\psi\cdot\nabla\widetilde{\psi}\,dx - \int_\Omega\zeta\widetilde{\psi}\,dx = \mu \quad\text{and}\quad \int_\Omega\nabla\psi\cdot\nabla x_2\,dx = \nu,$$

where $\widetilde{\psi}$ is defined in (1.1) (such a ψ exists, by Lemma 2.1). A standard convexity argument gives a minimizer ψ and it suffices to set $\xi = \psi|_{\mathcal{S}}$.

Consider any $h \in H^1_{per}(\Omega)$ such that

$$\Delta h = 0,\ h|_{\{x_2=0\}} = 0, \quad \int_\Omega \nabla h \cdot \nabla x_2 \, dx = 0 \quad \text{and} \quad \int_\Omega \nabla h \cdot \nabla \widetilde{\psi} \, dx = 0.$$

For all $t \neq 0$, we get

$$\begin{aligned}\frac{1}{2}\int_\Omega |\nabla\psi|^2 dx &\leq \frac{1}{2}\int_\Omega |\nabla(\psi + th)|^2 dx \\ &= \frac{1}{2}\int_\Omega |\nabla\psi|^2 dx + t\int_\Omega \nabla\psi \cdot \nabla h \, dx + \frac{1}{2}t^2 \int_\Omega |\nabla h|^2 dx\end{aligned}$$

and thus $\int_\Omega \nabla\psi \cdot \nabla h \, dx = 0$. More generally, if $h \in H^1_{per}(\Omega)$ only satisfies $\Delta h = 0$ and $h|_{\{x_2=0\}} = 0$, we consider instead of h the function

$$\begin{aligned} h &- \frac{P\int_\Omega \nabla h \cdot \nabla\widetilde{\psi}\, dx - \int_\Omega |\nabla\widetilde{\psi}|^2\, dx \int_\Omega \nabla h \cdot \nabla x_2 \, dx}{P^2 - PQ\int_\Omega |\nabla\widetilde{\psi}|^2\, dx} x_2 \\ &- \frac{P\int_\Omega \nabla h \cdot \nabla x_2 \, dx - PQ\int_\Omega \nabla h \cdot \nabla\widetilde{\psi}\, dx}{P^2 - PQ\int_\Omega |\nabla\widetilde{\psi}|^2\, dx}\widetilde{\psi}, \end{aligned}$$

which satisfies the two additional constraints, in view of the relations

$$\int_\Omega \nabla x_2 \cdot \nabla\widetilde{\psi}\, dx = P; \qquad \int_\Omega |\nabla x_2|^2 dx = PQ.$$

Instead of $0 = \int_\Omega \nabla\psi \cdot \nabla h \, dx$, we get

$$\begin{aligned} 0 &= \int_\Omega \nabla\psi \cdot \nabla h \, dx - \frac{P\int_\Omega \nabla h \cdot \nabla\widetilde{\psi}\, dx - \int_\Omega |\nabla\widetilde{\psi}|^2\, dx \int_\Omega \nabla h \cdot \nabla x_2 \, dx}{P^2 - PQ\int_\Omega |\nabla\widetilde{\psi}|^2\, dx} \\ &\quad \times \int_\Omega \nabla x_2 \cdot \nabla\psi \, dx - \frac{P\int_\Omega \nabla h \cdot \nabla x_2 \, dx - PQ\int_\Omega \nabla h \cdot \nabla\widetilde{\psi}\, dx}{P^2 - PQ\int_\Omega |\nabla\widetilde{\psi}|^2\, dx} \int_\Omega \nabla\widetilde{\psi} \cdot \nabla\psi \, dx \\ &= \int_\Omega \nabla\psi \cdot \nabla h \, dx + \frac{\int_\Omega |\nabla\widetilde{\psi}|^2\, dx \int_\Omega \nabla x_2 \cdot \nabla\psi \, dx - P\int_\Omega \nabla\widetilde{\psi} \cdot \nabla\psi \, dx}{P^2 - PQ\int_\Omega |\nabla\widetilde{\psi}|^2\, dx} \\ &\quad \times \int_\Omega \nabla x_2 \cdot \nabla h \, dx + \frac{PQ\int_\Omega \nabla\widetilde{\psi} \cdot \nabla\psi \, dx - P\int_\Omega \nabla x_2 \cdot \nabla\psi \, dx}{P^2 - PQ\int_\Omega |\nabla\widetilde{\psi}|^2\, dx} \int_\Omega \nabla\widetilde{\psi} \cdot \nabla h \, dx \\ &= \int_\Omega \nabla \left\{ \psi + \frac{\int_\Omega |\nabla\widetilde{\psi}|^2\, dx \int_\Omega \nabla x_2 \cdot \nabla\psi \, dx - P\int_\Omega \nabla\widetilde{\psi} \cdot \nabla\psi \, dx}{P^2 - PQ\int_\Omega |\nabla\widetilde{\psi}|^2\, dx} x_2 \right. \\ &\quad \left. + \frac{PQ\int_\Omega \nabla\widetilde{\psi} \cdot \nabla\psi \, dx - P\int_\Omega \nabla x_2 \cdot \nabla\psi \, dx}{P^2 - PQ\int_\Omega |\nabla\widetilde{\psi}|^2\, dx}\widetilde{\psi} \right\} \cdot \nabla h \, dx \end{aligned}$$

for all $h \in H^1_{per}(\Omega)$ such that $\Delta h = 0$ and $h|_{\{x_2=0\}} = 0$. Hence, as we explain below, there exist λ_1 and λ_2 in $\mathbb{R}$ satisfying (2.3), namely

$$\lambda_1 = \lambda_{1,\Omega,\zeta} = -\frac{\int_\Omega |\nabla\widetilde{\psi}|^2\, dx \int_\Omega \nabla x_2 \cdot \nabla\psi \, dx - P\int_\Omega \nabla\widetilde{\psi} \cdot \nabla\psi \, dx}{P^2 - PQ\int_\Omega |\nabla\widetilde{\psi}|^2\, dx}$$

and

$$\lambda_2 = \lambda_{2,\Omega,\zeta} = -\frac{PQ\int_\Omega \nabla\widetilde{\psi}\cdot\nabla\psi\,dx - P\int_\Omega \nabla x_2\cdot\nabla\psi\,dx}{P^2 - PQ\int_\Omega |\nabla\widetilde{\psi}|^2\,dx}.$$

Observe that these values must be equal to those obtained in Lemma 2.1, but here they are expressed with the help of the minimal stream function ψ. Hence the uniqueness statement in Lemma 2.1 gives the desired uniqueness of the minimizer ξ.

Let us briefly explain why $\psi - \lambda_1 x_2 - \lambda_2\widetilde{\psi} = 0$ on $\mathscr{S}$ if

$$\int_\Omega \nabla(\psi - \lambda_1 x_2 - \lambda_2\widetilde{\psi})\cdot\nabla h\,dx = 0$$

for all $h \in H^1_{per}(\Omega)$ such that $\Delta h = 0$ and $h|_{\{x_2=0\}} = 0$. Consider the holomorphic map $\widetilde{\phi}+i\widetilde{\psi}$ in (1.1) and write $\psi = \psi_0\circ(\widetilde{\phi}+i\widetilde{\psi})$ and $h = h_0\circ(\widetilde{\phi}+i\widetilde{\psi})$. We also use the notation (u,v) for the coordinates in $(0,\widetilde{P})\times(0,1)$ and $(u,v)\to x_2(u,v)$ for the map that associates with (u,v) the x_2 coordinate of the corresponding point in Ω. We get

$$\int_0^{\widetilde{P}}\int_0^1 \nabla(\psi_0(u,v) - \lambda_1 x_2(u,v) - \lambda_2 v)\cdot\nabla h_0(u,v)\,dudv = 0$$

for all $h_0 \in H^1_{per}((0,\widetilde{P})\times(0,1))$ such that $\Delta h_0 = 0$ and $h_0|_{\{v=0\}} = 0$, changing variables with the aid of (2.2). The upper boundary $\{v_2 = 1\}$ being regular, we can deduce that $\psi_0(u,1) - \lambda_1 x_2(u,1) - \lambda_2 = 0$ for almost all u. $\square$

If ψ denotes the corresponding stream function then equation (2.3) is a weak formulation of the condition that the modified velocity field $(\partial_2\psi - \lambda_1, -\partial_1\psi)$ be tangent to the upper boundary and correspond to a stationary wave that travels with speed λ_1 to the right. This tangency condition would hold classically if the free surface were of class C^2. However our existence theorem in Sect. 4 below does not yield enough regularity for this to be asserted at present.

Proposition 2.3. *Let $\Omega \in \mathfrak{O}\backslash\{\Omega_Q\}$ be given and let (Ω,ξ,ζ) be a minimizer of $\mathcal{L}$ over all $(\Omega,\widetilde{\xi},\widetilde{\zeta})$ such that $\mathcal{L}(\Omega,\xi,\zeta) < \infty$, $\widetilde{\xi} \in H^{1/2}_{per}(\mathscr{S})$, $\widetilde{\zeta} \in \overline{\mathcal{R}(\Omega)}^w$, $C(\Omega,\widetilde{\xi},\widetilde{\zeta}) = \mu$ and $I(\Omega,\widetilde{\xi},\widetilde{\zeta}) = \nu$.*

Then there exist λ_1 and λ_2 in $\mathbb{R}$ such that $\xi = (\lambda_1 x_2 + \lambda_2)|_{\mathcal{S}}$ and a decreasing function λ such that

$$\zeta = \lambda\circ(\psi - \lambda_1 x_2) \quad \text{a.e. on } \Omega,$$

where ψ is the stream function related to (Ω,ξ,ζ).

If ζ_Q is essentially one-signed then $\zeta \in \mathcal{R}(\Omega)$.

Remark. Proposition 2.3 contains no assertion concerning existence of minimizers. Sufficient conditions for their existence will be given later.

Proof. Only the last statement need be proved. For $h \in L^2(\Omega)$ define $\psi_h \in H^1_{per}(\Omega)$ by

$$-\Delta\psi_h = h,$$

$$\psi_h = 0 \quad \text{on } \{x_2 = 0\},$$

$$\psi_h|_{\mathcal{S}} \text{ is a linear combination of 1 and } x_2,$$

$$\mu = \int_{\mathcal{S}} \nabla\psi_h \cdot n\, dS,\ \nu = \int_\Omega \partial_2\psi_h\, dx.$$

Because $\Omega \neq \Omega_Q$ it follows that ψ_h is well defined and $\psi_h|_{\mathcal{S}} = \lambda_{1,\Omega,h}x_2 + \lambda_{2,\Omega,h}$ in terms of the unique constants given by Lemma 2.1. In particular we take $\lambda_1 = \lambda_{1,\Omega,\zeta}$, $\lambda_2 = \lambda_{2,\Omega,\zeta}$ and observe that $\psi_\zeta|_{\mathcal{S}}$ is equal to the optimal $\xi_{\Omega,\zeta}$ of Proposition 2.2. Then $\xi = \xi_{\Omega,\zeta}$ and, for fixed Ω, ζ minimizes the function

$$h \to \frac{1}{2}\int_\Omega |\nabla\psi_h|^2 dx$$

over all $h \in L^2(\Omega)$ such that h is in $\overline{\mathcal{R}(\Omega)}^w$. As in [6], for such a h and all $t \in [0,1]$, we set $h_t = (1-t)\zeta + th \in \overline{\mathcal{R}(\Omega)}^w$ and get that $\psi_{h_t} = (1-t)\psi_\zeta + t\psi_h$ and that

$$\begin{aligned}
0 &\leq \frac{1}{2}\int_\Omega |\nabla\psi_{h_t}|^2 dx - \frac{1}{2}\int_\Omega |\nabla\psi_\zeta|^2 dx = t\int_\Omega \nabla(\psi_h - \psi_\zeta)\cdot\nabla\psi_\zeta\, dx + o(t) \\
&= t\int_\Omega \nabla(\psi_h - \psi_\zeta)\cdot\nabla(\psi_\zeta - \lambda_1 x_2 - \lambda_2\widetilde{\psi})\, dx \\
&\quad + t\lambda_2\int_\Omega \nabla(\psi_h - \psi_\zeta)\cdot\nabla\widetilde{\psi}\, dx + t\lambda_1\int_\Omega \nabla(\psi_h - \psi_\zeta)\cdot\nabla x_2\, dx + o(t) \\
&= t\int_\Omega (h-\zeta)(\psi_\zeta - \lambda_1 x_2 - \lambda_2\widetilde{\psi})\, dx + t\lambda_2\int_\Omega (h-\zeta)\widetilde{\psi}\, dx + o(t) \\
&= t\int_\Omega (h-\zeta)(\psi_\zeta - \lambda_1 x_2) dx + o(t)
\end{aligned}$$

because

$$(\psi_\zeta - \lambda_1 x_2 - \lambda_2\widetilde{\psi})|_{\partial\Omega} = 0,$$

$$\int_\Omega \nabla(\psi_h - \psi_\zeta)\cdot\nabla\widetilde{\psi}\, dx - \int_\Omega (h-\zeta)\widetilde{\psi}\, dx = C(\Omega, \psi_h|_{\mathcal{S}}, h) - C(\Omega, \psi_\zeta|_{\mathcal{S}}, \zeta) = 0,$$

$$\int_\Omega \nabla(\psi_h - \psi_\zeta)\cdot\nabla x_2\, dx = I(\Omega, \psi_h|_{\mathcal{S}}, h) - I(\Omega, \psi_\zeta|_{\mathcal{S}}, \zeta) = 0.$$

Hence $\int_\Omega (h-\zeta)(\psi_\zeta - \lambda_1 x_2) dx \geq 0$ and the map

$$h \to \int_\Omega h(\psi_\zeta - \lambda_1 x_2) dx$$

reaches its minimum at ζ, where $h \in \overline{\mathcal{R}(\Omega)}^w$. As moreover $-\Delta(\psi_\zeta - \lambda_1 x_2) = \zeta$, the same argument as in [6, Lemma 2.3] ensures that there exists a decreasing function λ such that

$$\zeta = \lambda \circ (\psi_\zeta - \lambda_{1,\Omega,\zeta} x_2) \text{ a.e. on } \Omega.$$

If ζ_Q is one-signed except on a set of zero measure then it follows as in [6, Lemma 2.3] that $\zeta \in \mathcal{R}(\Omega)$. $\square$

3. The Bernoulli boundary condition

In what follows, we consider some fixed minimizer $(\underline{\Omega}, \underline{\xi}, \underline{\zeta})$ and outline how to adapt the method in [6] to show that the Bernoulli condition (1.3h) or (1.3h′) holds in some weak sense. Let $\lambda_{1,\underline{\Omega},\underline{\zeta}}$, $\lambda_{2,\underline{\Omega},\underline{\zeta}}$ and λ be the constants and decreasing function given by Proposition 2.3.

Theorem 3.1. *Suppose that the upper boundary $\underline{\mathscr{S}}$ of $\underline{\Omega}$ is given by an H^2 regular curve and*

$$\underline{\Omega} \neq \Omega_Q.$$

We set $\psi_0 = \underline{\psi} - \lambda_{1,\underline{\Omega},\underline{\zeta}} x_2$ and we let $p : \mathbb{R} \to \mathbb{R}^2$ such that $|p'(s)| = 1$ on $\mathbb{R}$ be an H^2-parametrisation of $\underline{\mathscr{S}}$. Then, for all solenoidal smooth vector fields ω defined in a neighbourhood of $\overline{\underline{\Omega}}$, vanishing on $\{x_2 = 0\}$ and P-periodic in x_1, any minimizer $(\underline{\Omega}, \underline{\xi}, \underline{\zeta})$ satisfies

$$\begin{aligned} 0 = & \int_{\underline{\Omega}} \nabla\psi_0 \cdot D\omega \nabla\psi_0 dx + g \int_{\underline{\Omega}} \nabla \cdot (x_2 \omega) dx \\ & + \beta T(\ell(\mathcal{S}) - P)^{\beta-1} \int_0^{\ell(\underline{\mathcal{S}})} (\omega \circ p)'(s) \cdot p'(s) ds \\ & + E \int_0^{\ell(\underline{\mathcal{S}})} \left(2(w \circ p)'' \cdot p'' - 3|p''|^2 (\omega \circ p)' \cdot p'\right) ds. \end{aligned}$$

If p and ψ_0 are regular enough, this can be written

$$\begin{aligned} 0 = & \int_{\mathcal{S}} \left(\frac{1}{2}|\nabla\psi_0|^2 + \Lambda(\psi_0)\right)(\omega \cdot n) dS + g \int_{\mathcal{S}} x_2 (\omega \cdot n) dS \\ & - \beta T(\ell(\mathcal{S}) - P)^{\beta-1} \int_{\underline{\mathcal{S}}} \sigma(\omega \cdot n) dS + E \int_{\underline{\mathcal{S}}} (\sigma^3 + 2\sigma'')(\omega \cdot n) dS \end{aligned}$$

where Λ is a primitive of λ and σ is the curvature, and thus

$$\frac{1}{2}|\nabla\psi_0|^2 + g x_2 - \beta T(\ell(\mathcal{S}) - P)^{\beta-1}\sigma + E(\sigma^3 + 2\sigma'')$$

is constant on $\underline{\mathscr{S}}$.

Proof. We only explain how to get the term

$$\int_{\underline{\Omega}} \nabla\psi_0 \cdot D\omega \nabla\psi_0 dx$$

by following the method of [6, Subsection 2.3], since the other terms do not involve ψ_0 so the calculations are the same as in [6]. For small $t \geq 0$ let the diffeomorphims τ be defined on Ω by $\tau(t)(x) = X(t)$, where

$$\dot{X}(t) = \omega(X(t)), \; X(0) = x,$$

and

$$\Omega(t) = \tau(t)\underline{\Omega}, \quad \zeta(t) = \underline{\zeta} \circ \kappa(t) \in \overline{\mathcal{R}(\Omega(t))}^w,$$

where $\kappa(t)$ denotes the inverse of $\tau(t)$. We denote by $\overline{\psi}(t)$ the solution of (1.3a) to (1.3f) corresponding to $\Omega(t)$ and $\zeta(t)$, and we set

$$\begin{aligned}
\xi(t) &= \overline{\psi}(t)|_{\mathscr{S}(t)}\\
\overline{\psi}_0(t) &= \overline{\psi}(t) - \lambda_{1,\underline{\Omega},\underline{\zeta}}x_2\\
\xi_0(t) &= \overline{\psi}_0(t)|_{\mathscr{S}(t)}\\
\Psi_0(t) &= \overline{\psi}_0(t)\circ\tau(t)\\
\Gamma(t) &= [D\kappa(t)\circ\tau(t)]^T = [(D\tau(t))^{-1}]^T
\end{aligned}$$

($\Gamma(t)$ at x is the transpose of the spatial derivative of κ evaluated at $\tau(t)(x)$).

Note that $\overline{\psi}(0) = \underline{\psi}$ and $\overline{\psi}_0(0) = \psi_0$. Moreover the dependence of $\overline{\psi}(t)\circ\tau(t) \in H^1_{per}(\overline{\Omega})$ with respect to t is smooth, because $C(\Omega(t), 1, 0)$, $\lambda_{1,\Omega(t),\zeta(t)}$ and $\lambda_{2,\Omega(t),\zeta(t)}$ are smooth in t, as can be checked with the help of the formulae following (2.3) and by arguing in the fixed domain $\underline{\Omega}$ (via the map $\tau(t)$) as in [6, after (1.14)]. Then the map $t \to \mathcal{L}(\Omega(t), \xi(t), \zeta(t))$ reaches its minimum at $t = 0$ and therefore its derivative vanishes at $t = 0$. Let us compute the derivative of the term corresponding to the kinetic energy.

First note that

$$C(\Omega(t), \xi_0(t), \zeta(t)) = \mu - \lambda_{1,\underline{\Omega},\underline{\zeta}}P,\ I(\Omega(t), \xi_0(t), \zeta(t)) = \nu - \lambda_{1,\underline{\Omega},\underline{\zeta}}PQ,\\ \det D\tau(t) = 1,\ \det D\kappa(t) = 1$$

and

$$\int_{\Omega(t)} \nabla\overline{\psi}_0(t)\cdot\nabla(\psi_0\circ\kappa(t))dx = C(\Omega(t), \xi_0(t), \zeta(t))\lambda_{2,\underline{\Omega},\underline{\zeta}} + \int_{\Omega(t)} \zeta(t)(\psi_0\circ\kappa(t))dx$$

because $\psi_0\circ\kappa(t)|_{\{x_2=0\}} = 0$, $\psi_0\circ\kappa(t)|_{\mathscr{S}(t)} = \lambda_{2,\underline{\Omega},\underline{\zeta}}$ and $\Delta\overline{\psi}_0(t) = -\zeta(t)$. Hence

$$\begin{aligned}
\int_{\underline{\Omega}} \Gamma(t)\nabla\Psi_0(t)\cdot\Gamma(t)\nabla\psi_0 dx &= \int_{\Omega(t)} \nabla(\Psi_0(t)\circ\kappa(t))\cdot\nabla(\psi_0\circ\kappa(t))dx\\
&= \int_{\Omega(t)} \nabla\overline{\psi}_0(t)\cdot\nabla(\psi_0\circ\kappa(t))dx = C(\Omega(t), \xi_0(t), \zeta(t))\lambda_{2,\underline{\Omega},\underline{\zeta}}\\
&\quad + \int_{\Omega(t)} \zeta(t)(\psi_0\circ\kappa(t))dx = (\mu - \lambda_{1,\underline{\Omega},\underline{\zeta}}P)\lambda_{2,\underline{\Omega},\underline{\zeta}} + \int_{\underline{\Omega}} \underline{\zeta}\,\psi_0\,dx.
\end{aligned}$$

By differentiating with respect to t at $t = 0$ in the equation

$$\int_{\underline{\Omega}} \Gamma(t)\nabla\Psi_0(t)\cdot\Gamma(t)\nabla\psi_0\,dx = (\mu - \lambda_{1,\underline{\Omega},\underline{\zeta}}P)\lambda_{2,\underline{\Omega},\underline{\zeta}} + \int_{\underline{\Omega}} \underline{\zeta}\,\psi_0\,dx,$$

we get

$$\int_{\underline{\Omega}} \nabla\dot{\Psi}_0(0)\cdot\nabla\psi_0\,dx_1dx_2 + 2\int_{\underline{\Omega}} \nabla\psi_0\cdot\dot{\Gamma}(0)\nabla\psi_0\,dx = 0. \tag{3.1}$$

Let

$$K(t) = \frac{1}{2}\int_{\Omega(t)} |\nabla\overline{\psi}(t)|^2\,dx = \frac{1}{2}\int_{\Omega(t)} |\nabla\overline{\psi}_0(t)|^2\,dx + \nu\lambda_{1,\underline{\Omega},\underline{\zeta}} + \frac{1}{2}\lambda^2_{1,\underline{\Omega},\underline{\zeta}}PQ.$$

Then

$$\begin{aligned}\dot K(0) &= \frac{d}{dt}\left(\frac{1}{2}\int_{\Omega(t)} |\nabla\overline{\psi}_0(t)|^2\,dx\right)\bigg|_{t=0}\\ &= \frac{d}{dt}\left(\frac{1}{2}\int_{\Omega} |\Gamma(t)\nabla\Psi_0(t)|^2\,dx\right)\bigg|_{t=0}\\ &= \int_{\underline{\Omega}} \nabla\psi_0\cdot\dot\Gamma(0)\nabla\psi_0 dx + \int_{\underline{\Omega}} \nabla\psi_0\cdot\nabla\dot\Psi_0(0) dx\\ &= -\int_{\underline{\Omega}} \nabla\psi_0\cdot\dot\Gamma(0)\nabla\psi_0\,dx\end{aligned}$$

by (3.1). Now

$$\Gamma(t)(x_1,x_2) = (D\tau(t)[x_1,x_2]^T)^{-1} = I - tD\omega[x_1,x_2]^T + o(t) \text{ as } t\to 0,$$

$\dot\Gamma(0) = -D\omega^T$ and

$$\dot K(0) = \int_{\underline{\Omega}} \nabla\psi_0\cdot D\omega\nabla\psi_0\,dx.$$

The end of the proof is as in [6]. □

4. Minimization

In what follows, the Hölder exponent γ is equal to $1/4$, so that in particular $H^2_{loc}(\mathbb{R})\subset C^{1,\gamma}(\mathbb{R})$.

Let $\mathcal{P}$ be the set of all injective H^2_{loc}-functions $p:\mathbb{R}\to\mathbb{R}\times(0,\infty)$ such that $p(x+P) = p(x) + (P,0)$ for all x, $p_1(0) = 0$ and $|p'|$ is constant. The length ℓ_p of $p([0,P])$ is equal to $\ell_p = \int_0^P |p'(x)|\,dx$ and thus $|p'(x)| = \ell_p/P$ everywhere. We shall use the notation

$$\mathscr{S}_p = p(\mathbb{R}) \quad \text{and} \quad \mathcal{S}_p = p((0,P)).$$

For $p\in\mathcal{P}$, we shall write $p\in\mathcal{P}_Q$ if there exists $\mathit{\Omega}\in\mathfrak{O}$ such that the corresponding upper boundary $\mathscr{S}$ satisfies $\mathscr{S} = \mathscr{S}_p$. We shall then write

$$\mathit{\Omega}_p = \mathit{\Omega} \quad \text{and} \quad \Omega_p = ((0,P)\times\mathbb{R})\cap\mathit{\Omega}.$$

We supplement the definition of $\mathcal{L}$ (see (1.4) and (1.5)) by setting

$$\mathcal{L}(\Omega_p,\xi,\zeta) = +\infty \quad \text{for } p\notin\mathcal{P}_Q.$$

In particular $\mathcal{L}(\Omega_p,\xi,\zeta) = +\infty$ if $p\in\mathcal{P}$ is such that the area of Ω_p is different from PQ.

Also, if $\mathcal{P}_Q\ni p_i \rightharpoonup p\in\mathcal{P}_Q$ in $H^2_{\mathrm{loc}}(\mathbb{R},\mathbb{R}^2)$, then

$$\ell_p = \lim_{i\to\infty}\ell_{p_i} \quad \text{and} \quad \int_0^P |p''|^2 ds \le \liminf_{i\to\infty}\int_0^P |p_i''|^2 ds.$$

The next lemma leads to an explicit criterion for the free surface to remain away from the bottom.

Lemma 4.1. *For any $p \in \mathcal{P}_Q$,*

$$Q \leq \min p_2(\mathbb{R}) + \frac{P}{2\pi} a\left(\frac{2\pi}{P}\ell_p\right), \tag{4.1}$$

where $2\pi a(\ell)$ (when $\ell > 2\pi$) is the area enclosed between a circular arc of length ℓ and a chord of length 2π, and thus

$$\frac{P^2}{2\pi} a\left(\frac{2\pi}{P}\ell\right)$$

is the area enclosed between a circular arc of length ℓ and a chord of length P.

Moreover

$$\int_{\Omega_p} x_2 dx_1 dx_2 \geq PQ^2/2. \tag{4.2}$$

Proof. See [6]. □

As a consequence, if $T + E > 0$, then $\mathcal{L} \geq gPQ^2/2$ with equality exactly when $\Omega_p = \Omega_Q$ and the fluid is at rest (see (1.4) and (1.5)).

The following lemma, taken from [6], provides an explicit way of ensuring that the free surface is without double points, namely, it is sufficient to check that inequality (4.3) below does not hold.

Lemma 4.2. *Suppose that $p \in H^2_{loc}(\mathbb{R}, \mathbb{R}^2)$ is not injective and satisfies $p(x + P) = p(x) + (P, 0)$ for all x. Then $p(\mathbb{R})$ contains a closed loop with arc length no greater than $\ell_p - P$ (see [13]). Let*

$$p'(x) = |p'(x)|(\cos\vartheta(s), \sin\vartheta(s)) = P^{-1}\ell_p(\cos\vartheta(s), \sin\vartheta(s)),$$

where $s = x\ell_p/P$ denotes arc length. Then, on the loop, the range of ϑ must exceed π and thus, for some $0 \leq s_2 - s_1 \leq \ell_p - P$,

$$\pi \leq |\vartheta(s_2) - \vartheta(s_1)| \leq P\ell_p^{-1} \int_{s_1 P/\ell_p}^{s_2 P/\ell_p} |p''(x)| dx$$

$$\leq P\ell_p^{-1}\sqrt{P\ell_p^{-1}|s_2 - s_1|}\|p''\|_{L^2(0,P)} \leq \sqrt{\ell_p - P}\left(\frac{P}{\ell_p}\right)^{3/2} \|p''\|_{L^2(0,P)},$$

hence

$$\pi \leq \sqrt{\ell_p - P}\left(\frac{P}{\ell_p}\right)^{3/2} \|p''\|_{L^2(0,P)}. \tag{4.3}$$

Let

$$W = \{(\Omega, \xi, \zeta) : \ p \in \mathcal{P}_Q, \ \Omega = \Omega_p \in \mathfrak{O}, \ \xi \in H^{1/2}_{per}(\mathscr{S}_p), \ \zeta \in L^2(\Omega)\},$$

$$V := \{(\Omega, \xi, \zeta) \in W : \ \zeta \in \overline{\mathcal{R}(\Omega)}^w, \ C(\Omega, \xi, \zeta) = \mu, \ I(\Omega, \xi, \zeta) = \nu\}.$$

By (1.4), (1.5) and (4.2), if $T > 0$ there is a bounded subset of $(0, P) \times (0, \infty)$ that contains all domains Ω such that, for some ξ and ζ, $(\Omega, \xi, \zeta) \in W$ and $\mathcal{L}(\Omega, \xi, \zeta) < \inf_V \mathcal{L} + 1$; hence

$$\exists R > 0 \ \forall (\Omega, \xi, \zeta) \in W \ \left(\mathcal{L}(\Omega, \xi, \zeta) < \inf_V \mathcal{L} + 1 \Rightarrow \overline{\Omega} \subset [0, P] \times [0, R)\right). \tag{4.4}$$

Let

$$\mathcal{R} = \{\zeta \in L^2((0,P) \times (0,R)) : \ \zeta \text{ is a rearrangement of } \zeta_Q\}$$

and $\overline{\mathcal{R}}^w$ be its weak closure in $L^2((0,P) \times (0,R))$.

Hypothesis (M2) in the following existence result is related to the various inequalities arising in the two previous lemmata.

Theorem 4.3. *Assume that*

(M1) V *does not contain any* (Ω, ξ, ζ) *with* $\Omega = \Omega_Q$,

(M2) *there exist* $\mathcal{L}_0 > \inf_V \mathcal{L}$, $T > 0$, $\beta \geq 1$ *and* $E > 0$ *such that*

$$\frac{P}{2\pi} a \left(\frac{2\pi}{P} \left\{ \frac{\mathcal{L}_0 - \frac{g}{2} PQ^2}{T} \right\}^{1/\beta} + 2\pi \right) < Q, \tag{4.5}$$

and

$$\left(\mathcal{L}_0 - \frac{g}{2} PQ^2 \right) \left\{ \frac{\mathcal{L}_0 - \frac{g}{2} PQ^2}{T} \right\}^{1/\beta} < E\pi^2 \tag{4.6}$$

(see (1.5) *for the meaning of* T, β *and* E*).*

Then $\inf_V \mathcal{L}$ *is attained.*

Remarks. (1) If we allow $\mathcal{L}_0 = \inf_V \mathcal{L}$ in (M2) or require $\mathcal{L}_0 = \inf_V \mathcal{L}$, we do not change the meaning of (M2); however $\mathcal{L}_0 > \inf_V \mathcal{L}$ will be used in the proof of Theorem 4.4. Note that, by Lemma 2.1, $V \neq \emptyset$.

(2) Assumption (M1) holds if ζ_Q is essentially one-signed and not trivial, and $(\nu - Q\mu)\zeta_Q \leq 0$ a.e. (or $\nu - Q\mu \neq 0$ if ζ_Q vanishes a.e.). See the paragraph "A way of avoiding parallel flows" in the Introduction.

(3) To see that all assumptions can be fulfilled, choose any $T > 0$, $\beta \geq 1$ and $E > 0$, and then choose $\mathcal{L}_0 > \frac{g}{2} PQ^2$ near enough to $\frac{g}{2} PQ^2$ so that (4.5) and (4.6) hold (this is possible because $a(s) \to 0$ as $s \to 2\pi$ from the right). Choose $p \in \mathcal{P}_Q$ near enough to $(0,Q)$ in H^2_{loc} and such that $\Omega_p \neq \Omega_Q$. We know that

$$I(\Omega_p, 1, 0) - QC(\Omega_p, 1, 0) = P - QC(\Omega_p, 1, 0) < 0$$

(see (2.1)). Choose ζ_Q essentially non-negative and small enough in $L^2(\Omega_Q)$, and $\zeta \in \mathcal{R}(\Omega_p)$ such that $I(\Omega_p, 1, \zeta) - QC(\Omega_p, 1, \zeta) < 0$. For $\epsilon > 0$, we have $I(\Omega_p, \epsilon, \epsilon\zeta) - QC(\Omega_p, \epsilon, \epsilon\zeta) < 0$. We then set $\mu_\epsilon = C(\Omega_p, \epsilon, \epsilon\zeta)$ and $\nu_\epsilon = I(\Omega_p, \epsilon, \epsilon\zeta)$. For V_ϵ corresponding $\epsilon\zeta_Q$, μ_ϵ and ν_ϵ, we get that $(\Omega_p, \epsilon, \epsilon\zeta) \in V_\epsilon$ and, if $p - (0,Q) \in H^2_{loc}$ and ϵ are small enough, that $\inf_{V_\epsilon} \mathcal{L} < \mathcal{L}_0$.

(4) For the above choice of V_ϵ, the minimizer turns out to be near Ω_Q (as $p - (0,Q)$ and $\epsilon > 0$ above are chosen small enough). However in order to check (M2) for general $T > 0$, $\beta \geq 1$, $E > 0$ ζ_Q, μ and ν, it is enough to exhibit explicitly an appropriate $(\Omega, \xi, \zeta) \in V$. The observation that the free surface of Ω need not be a graph (but must not touch or intersect itself) was intended to help addressing this question, for example by numerical simulations.

The previous theorem is an immediate consequence of the following one. For convenience write $\overline{\psi}(p,\zeta,\widetilde{\mu},\widetilde{\nu})$ for the solution to (1.3a)–(1.3f) corresponding to the domain $\Omega_p \neq \Omega_Q$ (with $p \in \mathcal{P}_Q$), the vorticity function ζ, circulation $\widetilde{\mu}$ and horizontal impulse $\widetilde{\nu}$, and write $\overline{\xi}(p,\zeta,\widetilde{\mu},\widetilde{\nu}) = \overline{\psi}(p,\zeta,\widetilde{\mu},\widetilde{\nu})|_{\mathscr{S}_p}$. Moreover we write $\overline{\lambda}_1(p,\zeta,\widetilde{\mu},\widetilde{\nu})$ and $\overline{\lambda}_2(p,\zeta,\widetilde{\mu},\widetilde{\nu})$ for the corresponding λ_1 and λ_2 given by Lemma 2.1 applied to $\Omega_p \neq \Omega_Q$, ζ, $\widetilde{\mu}$ and $\widetilde{\nu}$.

Theorem 4.4. *As in Theorem 4.3, assume (M1) and (M2). For each $k \in \mathbb{N}$, let $p_k \in \mathcal{P}_Q$ with $\Omega_{p_k} \neq \Omega_Q$, $\zeta_k \in L^2(\Omega_{p_k}) \subset L^2((0,P)\times(0,\infty))$ and $\mu_k, \nu_k \in \mathbb{R}$. Suppose that*

$$dist_{L^2((0,P)\times(0,\infty))}\left(\zeta_k, \overline{\mathcal{R}}^w\right) \to 0,$$
$$\lim_{k\to\infty} \mu_k = \mu, \quad \lim_{k\to\infty} \nu_k = \nu$$

and

$$\begin{aligned}\limsup_{k\to\infty} &\mathcal{L}(\Omega_{p_k}, \overline{\xi}(p_k,\zeta_k,\mu_k,\nu_k),\zeta_k) \\ &= \limsup_{k\to\infty}\left\{\frac{1}{2}\int_{\Omega_{p_k}} |\nabla\overline{\psi}(p_k,\zeta_k,\mu_k,\nu_k)|^2\,dx + g\int_{\Omega_{p_k}} x_2\,dx + T(\ell_{p_k}-P)^\beta\right. \\ &\quad \left. +E\left(\frac{P}{\ell_{p_k}}\right)^3 \int_0^P |p_k''(x)|^2 dx\right\} \le \inf_V \mathcal{L}. \end{aligned} \tag{4.7}$$

In particular these hypotheses hold true if $\{(\Omega_{p_k}, \overline{\xi}(p_k,\zeta_k,\mu_k,\nu_k),\zeta_k)\}_{k\in\mathbb{N}}$ is a minimizing sequence in V of $\mathcal{L}$ (and thus $\mu_k = \mu$ and $\nu_k = \nu$ for all k).

Then there is a sequence $\{k_j\} \subset \mathbb{N}$ such that $\{p_{k_j}\}$ converges weakly in H^2_{per} to some $p \in \mathcal{P}_Q$ and $\{\zeta_{k_j}\}$ seen in $L^2((0,P)\times(0,\infty))$ converges weakly to some $\zeta \in L^2(\Omega_p)$. Moreover $L^2((0,P)\times(0,\infty))$ can be seen as a subspace of the dual space $\left(H^1((0,P)\times(0,\infty))\right)'$ of $H^1((0,P)\times(0,\infty))$ and

$$\zeta_{k_j} \to \zeta \text{ strongly in } \left(H^1((0,P)\times(0,\infty))\right)'. \tag{4.8}$$

Since $\Omega_p \in \mathfrak{D}$, there exists a $C^{1,\gamma}$-map $F : \mathbb{R}^2 \to \mathbb{R}^2$ satisfying the following properties:

- *F restricted to $\mathbb{R}\times[0,Q]$ is a diffeomorphism from $\mathbb{R}\times[0,Q]$ onto $\overline{\Omega_p}$,*
- *$F(x_1,0) = (x_1,0)$ for all $x_1 \in \mathbb{R}$,*
- *F restricted to $\mathbb{R}\times\{Q\}$ is a homeomorphism from $\mathbb{R}\times\{Q\}$ onto $\mathscr{S}_p$,*
- *$F(x_1+P,x_2) = (F_1(x_1,x_2)+P, F_2(x_1,x_2))$ for all $x = (x_1,x_2) \in \mathbb{R}\times[0,Q]$.*

In the same way as for F, we introduce $F_j : \mathbb{R}^2 \to \mathbb{R}^2$ such that, restricted to $\mathbb{R}\times[0,Q]$, it is a diffeomorphism from $\mathbb{R}\times[0,Q]$ onto $\overline{\Omega_{p_{k_j}}}$.

Then this can be done in such a way that

$$||F_j - F||_{C^1(\mathcal{U})} \to 0 \quad \text{for some open set } \mathcal{U} \text{ containing } \overline{\Omega_Q}, \tag{4.9}$$

$$\overline{\lambda}_1(p_{k_j},\zeta_{k_j},\mu_{k_j},\nu_{k_j}) \to \overline{\lambda}_1(p,\zeta,\mu,\nu),\ \overline{\lambda}_2(p_{k_j},\zeta_{k_j},\mu_{k_j},\nu_{k_j}) \to \overline{\lambda}_2(p,\zeta,\mu,\nu) \tag{4.10}$$

and

$$||\overline{\psi}(p_{k_j}, \zeta_{k_j}, \mu_{k_j}, \nu_{k_j}) - \overline{\psi}(p, \eta, \mu, \nu)||_{H^1_{per}((0,P)\times(0,R))} \to 0, \tag{4.11}$$

where R *is large enough so that the closures of* Ω_p *and all* Ω_{p_k} *are subsets of* $[0, P] \times [0, R)$ *(see* (4.4)*) and where* $\overline{\psi}(p, \zeta, \mu, \nu)$ *and all* $\overline{\psi}(p_k, \zeta_k, \mu_k, \nu_k)$ *have been extended in* $(0, P) \times (0, R)$ *by* $\overline{\lambda}_1(p, \zeta, \mu, \nu)x_2 + \overline{\lambda}_2(p, \zeta, \mu, \nu)$ *and* $\overline{\lambda}_1(p_k, \zeta_k, \mu_k, \nu_k)x_2 + \overline{\lambda}_2(p_k, \zeta_k, \mu_k, \nu_k)$*.*

Finally $(\Omega_p, \overline{\xi}(p, \zeta, \mu, \nu), \zeta) \in V$*,* $\mathcal{L}(\Omega_p, \overline{\xi}(p, \zeta, \mu, \nu), \zeta) = \inf_V \mathcal{L}$*, the lim-sup in* (4.7) *is a limit:*

$$\lim_{k\to\infty} \mathcal{L}(\Omega_{p_k}, \overline{\xi}(p_k, \zeta_k, \mu_k, \nu_k), \zeta_k) = \inf{}_V \mathcal{L} \tag{4.12}$$

and

$$p_{k_j} \to p \quad \textit{strongly in} \quad H^2_{per}\ . \tag{4.13}$$

Proof. Let $p_k \in \mathcal{P}_Q$, $\zeta_k \in L^2(\Omega_{p_k})$ and $\mu_k, \nu_k \in \mathbb{R}$ be such that

$$\mathrm{dist}_{L^2((0,P)\times(0,\infty))}\left(\zeta_k, \overline{\mathcal{R}}^w\right) \to 0,$$
$$\lim_{k\to\infty} \mu_k = \mu, \ \lim_{k\to\infty} \nu_k = \nu$$

and

$$\limsup_{k\to\infty} \left\{ \frac{1}{2} \int_{\Omega_{p_k}} |\nabla\overline{\psi}(p_k, \zeta_k, \mu_k, \nu_k)|^2\, dx + g \int_{\Omega_{p_k}} x_2\, dx + T(\ell_{p_k} - P)^\beta \right.$$
$$\left. + E\left(\frac{P}{\ell_{p_k}}\right)^3 \int_0^P |p_k''(x)|^2 dx \right\} \le \inf{}_V \mathcal{L},$$

For simplicity, we set

$$\overline{\psi}_k = \overline{\psi}(p_k, \zeta_k, \mu_k, \nu_k),\ \overline{\xi}_k = \overline{\xi}(p_k, \zeta_k, \mu_k, \nu_k)$$
$$\overline{\lambda}_{1,k} = \overline{\lambda}_1(p_k, \zeta_k, \mu_k, \nu_k) \ \text{ and } \ \overline{\lambda}_{2,k} = \overline{\lambda}_2(p_k, \zeta_k, \mu_k, \nu_k)$$

(remember that we write $\overline{\lambda}_1(p_k, \zeta_k, \mu_k, \nu_k)$ and $\overline{\lambda}_2(p_k, \zeta_k, \mu_k, \nu_k)$ for the corresponding λ_1 and λ_2 given by Lemma 2.1 applied to $\Omega_{p_k} \neq \Omega_Q$).

We get, for all $k \in \mathbb{N}$ large enough,

$$\frac{1}{2} \int_{\Omega_{p_k}} |\nabla\overline{\psi}_k|^2\, dx + \frac{g}{2} PQ^2$$
$$+ T(\ell_{p_k} - P)^\beta + E\left(\frac{P}{\ell_{p_k}}\right)^3 \int_0^P |p_k''(x)|^2 dx \overset{(4.2)}{\le} \mathcal{L}(\Omega_{p_k}, \overline{\xi}_k, \zeta_k) \le \mathcal{L}_0, \tag{4.14}$$

$$\ell_{p_k} - P \overset{(4.14)}{\le} \left\{ \frac{\mathcal{L}_0 - \frac{g}{2}PQ^2}{T} \right\}^{1/\beta}, \tag{4.15}$$

$$\frac{P}{2\pi} a\left(\frac{2\pi}{P}\ell_{p_k}\right) \overset{(4.5)}{<} Q,\ \min p_{k,2}(\mathbb{R}) \overset{(4.1)}{>} 0, \tag{4.16}$$

$$(\ell_{p_k} - P)\left(\frac{P}{\ell_{p_k}}\right)^3 ||p_k''||^2_{L^2(0,P)} \overset{(4.15),(4.6)}{<} \pi^2 \frac{E\left(\frac{P}{\ell_{p_k}}\right)^3 ||p_k''||^2_{L^2(0,P)}}{\mathcal{L}_0 - \frac{g}{2}PQ^2} \overset{(4.14)}{\leq} \pi^2 \tag{4.17}$$

uniformly in k large enough. Observe that $\{p_k\}$ is bounded in H^2_{per} because $\mathcal{L}(\Omega_{p_k}, \overline{\xi}_k, \zeta_k)$ is bounded and $T, E > 0$. So there is a strictly increasing sequence $\{k_j\} \subset \mathbb{N}$ such that $\{p_{k_j}\}$ converges weakly in H^2_{per} to some p and $\{\zeta_{k_j}\}$ seen in $L^2((0,P) \times (0,\infty))$ converges weakly to some $\zeta \in L^2(\Omega_p)$.

Remember the constant $R > 0$ introduced in (4.4). As in fact $\{\zeta_{k_j}\} \subset L^2((0,P)\times(0,R))$ and as the inclusion map $L^2((0,P)\times(0,R)) \subset \big(H^1((0,P)\times(0,R))\big)'$ is compact, we get (4.8).

By lemma 4.2 and (4.17), p is injective and, by (4.16), $p(\mathbb{R}) \subset \mathbb{R}\times(0,\infty)$. Hence $p \in \mathcal{P}_Q$ (see [6]).

Let F be as in the statement. Then F restricted to some open neighbourhood $\mathcal{U}$ of $\mathbb{R} \times [0, Q]$ is still a diffeomorphism onto the open set $F(\mathcal{U})$ containing $\overline{\Omega_p}$. As a consequence, for large enough j, $\Omega_{p_{k_j}} \subset F(\mathcal{U})$ and $F^{-1}(\mathscr{S}_{p_{k_j}})$ is the graph of a map $x_1 \to H_j(x_1)$ that is C^1-close to the constant map $x_1 \to x_2 = Q$. Define

$$G_j(x_1, x_2) = (x_1, x_2\, H_j(x_1)/Q) \quad \text{and} \quad F_j = (F_{j1}, F_{j2}) = F \circ G_j \quad \text{for all } j.$$

Then (4.9) holds. Extend $\overline{\psi}_k$ on $(0,P) \times (0,R)$, as in the statement. Observe that

$$\sup{}_{j\in\mathbb{N}} \int_{\Omega_{p_{k_j}}} |\nabla\overline{\psi}_{k_j}|^2 dx < \infty$$

because $\mathcal{L}(\Omega_{p_{k_j}}, \overline{\xi}_{k_j}, \zeta_{k_j})$ is finite. Hence we get successively

$$\sup{}_{j\in\mathbb{N}} \int_{\Omega_Q} |\nabla\left(\overline{\psi}_{k_j} \circ F_j\right)|^2 dx < \infty,$$
$$\sup_{j\in\mathbb{N}} ||\overline{\psi}_{k_j} \circ F_j||_{H^1(\Omega_Q)} < \infty$$

by Poincaré's inequality,

$$\sup{}_{j\in\mathbb{N}} \int_0^P \left|\overline{\psi}_{k_j} \circ F_j\big|_{x_2=Q}\right|^2 dx_1 < \infty,$$

or, equivalently,

$$\sup{}_{j\in\mathbb{N}} \int_0^P |\overline{\lambda}_{1,k_j} F_{j2}(x_1, Q) + \overline{\lambda}_{2,k_j}|^2 dx_1 < \infty.$$

Suppose first that $\{\overline{\lambda}_{1,k_j}\}$ is unbounded. Taking a subsequence if necessary, $F_{j2}(\cdot, Q) + (\overline{\lambda}_{2,k_j}/\overline{\lambda}_{1,k_j})$ would converge to 0 in $L^2(0,P)$, and therefore $F_2(x_1, Q) = Q$ for all $x_1 \in (0,P)$ (this follows from (4.9)). Hence $\Omega_p = \Omega_Q$.

Let $\widetilde{Q} \in (Q/2, Q)$. From the Poincaré inequality, it follows that the sequence $\{\overline{\psi}_{k_j}\}$ seen in $H^1_{per}((0,P) \times (0,\widetilde{Q}))$ is bounded too and therefore, up to a subsequence, it converges weakly to some $\psi_{\widetilde{Q}} \in H^1_{per}((0,P)\times(0,\widetilde{Q}))$. Moreover this can be achieved in such a way that there exists $\psi \in H^1_{per}(\Omega_Q)$ independent of $\widetilde{Q}$ such that $\psi_{\widetilde{Q}}$ and ψ are equal on $(0,P)\times(0,\widetilde{Q})$. Also, up to a subsequence, the sequence $\{\zeta_{k_j}\}$ seen in $L^2((0,P)\times(0,R))$ converges weakly to some ζ that belongs in fact to $L^2(\Omega_p) = L^2(\Omega_Q)$, that is, ζ vanishes almost everywhere outside Ω_Q. Moreover $\zeta \in \overline{\mathcal{R}}^w$ because

$$\mathrm{dist}_{L^2((0,P)\times(0,\infty))}\left(\zeta_{k_j}, \overline{\mathcal{R}}^w\right) \to 0.$$

In fact ζ even belongs to the convex set $\overline{\mathcal{R}(\Omega_Q)}^w$, as it can be seen from the characterisation of $\overline{\mathcal{R}(\Omega)}^w$ for any open bounded set Ω of measure $m > 0$ in terms of decreasing rearrangements on $[0,m]$. See e.g. Lemma 2.2 in [4].[2]

Let $\xi = \psi|_{(0,P)\times\{Q\}}$. Then, in a weak sense, $-\Delta\psi = \zeta$ on Ω_Q, $\psi(\cdot,0) = 0$ and $\psi(\cdot,Q) = \xi$.

By choosing $\widehat{\psi} \in H^1_{per}((0,P)\times(0,R))$ such that $\widehat{\psi}$ restricted to $\{x_2 = 0\}$ vanishes and such that $\widehat{\psi} = 1$ on $(0,P)\times(Q/3,R)$, we get that

$$\begin{aligned}
\mu &= \lim_{j\to\infty}\mu_{k_j} = \lim_{j\to\infty} C(\Omega_{p_{k_j}}, \overline{\xi}_{k_j}, \zeta_{k_j}) = \lim_{j\to\infty}\int_{\Omega_{p_{k_j}}} \{\nabla\overline{\psi}_{k_j}\cdot\nabla\widehat{\psi} - \zeta_{k_j}\widehat{\psi}\}dx \\
&= \lim_{j\to\infty}\int_{(0,P)\times(0,Q/2)} \nabla\overline{\psi}_{k_j}\cdot\nabla\widehat{\psi}dx - \lim_{j\to\infty}\int_{(0,P)\times(0,R)} \zeta_{k_j}\widehat{\psi}dx \\
&= \int_{\Omega_Q}\{\nabla\psi\cdot\nabla\widehat{\psi} - \zeta\widehat{\psi}\}dx = C(\Omega_Q,\xi,\zeta)
\end{aligned} \tag{4.18}$$

and

$$\begin{aligned}
\nu &= \lim_{j\to\infty}\nu_{k_j} = \lim_{j\to\infty}\int_{\Omega_{p_{k_j}}} \nabla\overline{\psi}_{k_j}\cdot\nabla x_2\,dx \\
&= \lim_{\widetilde{Q}\to Q-}\lim_{j\to\infty}\int_{(0,P)\times(0,\widetilde{Q})} \nabla\overline{\psi}_{k_j}\cdot\nabla x_2\,dx = \int_{\Omega_Q}\nabla\psi\cdot\nabla x_2\,dx = I(\Omega_Q,\xi,\zeta).
\end{aligned} \tag{4.19}$$

Hence $(\Omega_Q,\xi,\zeta)\in V$, which contradicts (M1). As a consequence $\{\overline{\lambda}_{1,k_j}\}$ is bounded. We now apply some of the above arguments again.

From the Poincaré inequality, it follows that the sequence $\{\overline{\psi}_{k_j}\}$ seen now in $H^1_{per}((0,P)\times(0,R))$ is bounded and therefore, up to a subsequence,

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[2] Indeed let $g_1 : (0,PQ)\to\mathbb{R}$ be the right-continuous and decreasing rearrangement of $\zeta\in L^2(\Omega_Q)$. If ζ is seen in $L^2((0,P)\times(0,R))$ instead, we can also consider its right-continuous and decreasing rearrangement $g_2 : (0,PR)\to\mathbb{R}$.

Note that g_2 vanishes on an interval Z_ζ of length at least $PR-PQ$. Moreover the graph of g_1 is obtained from the one of g_2 by deleting from Z_ζ an interval of length $PR-PQ$ and shifting to the left the part of the graph of g_2 that is to the right of Z_ζ.

We note by G_1 and G_2 the rearrangements corresponding to ζ_Q.

With the partial ordering $\prec$ of Burton-McLeod (see their lemma 2.2), we get successively $\zeta\in\overline{\mathcal{R}}^w$, $g_2\prec G_2$, $g_1\prec G_1$ and therefore $\zeta\in\overline{\mathcal{R}(\Omega_Q)}^w$.

it converges weakly to some $\psi \in H^1_{per}((0,P)\times(0,R))$. In particular it follows that $\{\overline{\lambda}_{2,k_j}\}$ is bounded. Again, up to a subsequence, the sequence $\{\zeta_{k_j}\}$ seen in $L^2((0,P)\times(0,R))$ converges weakly to some ζ that belongs to $\overline{\mathcal{R}(\Omega_p)}^w$.

By choosing again $\widehat{\psi} \in H^1_{per}((0,P)\times(0,R))$ such that $\widehat{\psi}$ restricted to $\{x_2=0\}$ vanishes and such that $\widehat{\psi}=1$ on some open set containing $\mathscr{S}_P$ and all $\mathscr{S}_{p_{k_j}}$, we get that

$$\mu = \lim_{j\to\infty} C(\Omega_{p_{k_j}}, \overline{\xi}_{k_j}, \zeta_{k_j}) = \lim_{j\to\infty}\int_{\Omega_{p_{k_j}}} \{\nabla\overline{\psi}_{k_j}\cdot\nabla\widehat{\psi} - \zeta_{k_j}\widehat{\psi}\}dx$$

$$= \int_{\Omega_p}\{\nabla\psi\cdot\nabla\widehat{\psi} - \zeta\widehat{\psi}\}dx = C(\Omega_p, \psi|_{\mathscr{S}}, \zeta)$$

and

$$\nu = \lim_{j\to\infty}\int_{\Omega_{p_{k_j}}} \nabla\overline{\psi}_{k_j}\cdot\nabla x_2\,dx = \lim_{j\to\infty}\int_{\Omega_p}\nabla\overline{\psi}_{k_j}\cdot\nabla x_2\,dx$$

$$= \int_{\Omega_p}\nabla\psi\cdot\nabla x_2\,dx = I(\Omega_p,\psi|_{\mathscr{S}},\zeta).$$

By convexity, for all $\widetilde{Q} > Q$ and $q \in \mathcal{P}_{\widetilde{Q}}$ such that $\Omega_p \subset \Omega_q \subset (0,P)\times(0,R)$ and $\mathscr{S}_q \cap \mathscr{S}_p = \emptyset$, we have

$$\int_{\Omega_q}|\nabla\psi|^2dx \le \liminf_{j\to\infty}\int_{\Omega_q}|\nabla\overline{\psi}_{k_j}|^2dx$$

$$\le \liminf_{j\to\infty}\left(\int_{\Omega_{p_{k_j}}}|\nabla\overline{\psi}_{k_j}|^2dx + \mathrm{Const\,meas}(\Omega_q\backslash\Omega_{p_{k_j}})\right)$$

(because the sequence $\{\overline{\lambda}_{1,k_j}\}$ is bounded) and therefore

$$\int_{\Omega_p}|\nabla\psi|^2dx \le \lim_{j\to\infty}\int_{\Omega_{p_{k_j}}}|\nabla\overline{\psi}_{k_j}|^2dx.$$

It follows that $\mathcal{L}(\Omega_p,\psi|_{\mathscr{S}_p},\zeta) \le \inf_V \mathcal{L}$ and $(\Omega_p,\psi|_{\mathscr{S}_p},\zeta) \in V$ so $\mathcal{L}(\Omega_p,\psi|_{\mathscr{S}_p},\zeta) = \inf_V \mathcal{L}$ and $\psi = \overline{\psi}(p,\zeta,\mu,\nu)$. Hence (4.10) holds and

$$\int_{\Omega_p}|\nabla\psi|^2dx = \lim_{j\to\infty}\int_{\Omega_{p_{k_j}}}|\nabla\overline{\psi}_{k_j}|^2dx. \tag{4.20}$$

By (4.10), for all $\widetilde{Q} > Q$ and $q \in \mathcal{P}_{\widetilde{Q}}$ such that $\Omega_p \subset \Omega_q \subset (0,P)\times(0,R)$ and $\mathscr{S}_p \cap \mathscr{S}_q = \emptyset$, we have

$$\int_{((0,P)\times(0,R))\backslash\Omega_q}|\nabla\psi|^2dx = \lim_{j\to\infty}\int_{((0,P)\times(0,R))\backslash\Omega_q}|\nabla\overline{\psi}_{k_j}|^2dx.$$

Hence

$$\int_{(0,P)\times(0,R)}|\nabla\psi|^2dx = \lim_{j\to\infty}\int_{(0,P)\times(0,R)}|\nabla\overline{\psi}_{k_j}|^2dx$$

and (4.11) holds too.

Together with (4.20), the fact that

$$\mathcal{L}(\Omega_{p_{k_j}}, \overline{\xi}_{k_j}, \zeta_{k_j}) \to \mathcal{L}(\Omega_p, \psi|_{\mathscr{S}_p}, \zeta)$$

implies that

$$\int_0^P |p''_{k_j}|^2 dx \to \int_0^P |p''|^2 dx.$$

Hence $p_{k_j} \to p$ strongly in H^2_{per}. □

5. On stability

In this section, we assume that hypotheses (M1) and (M2) in Theorem 4.3 hold true. Moreover the Hölder exponent γ is still equal to $1/4$.

For smooth flows, the evolutionary problem reads as follows (see e.g. [9]). Let $\psi(t,\cdot,\cdot) \in C^\infty_{per}(\Omega(t))$ be the stream function at time t on the domain $\Omega(t) \in \mathfrak{O}$, that is, the velocity field is given by $u = (u_1, u_2) = (\partial_{x_2}\psi, -\partial_{x_1}\psi)$ on $\Omega(t)$. The Euler equation for an inviscid flow becomes

$$\begin{cases} \partial_t u_1 + u_1\, \partial_{x_1} u_1 + u_2\, \partial_{x_2} u_1 = -\partial_{x_1} \mathrm{Pr} \\ \\ \partial_t u_2 + u_1\, \partial_{x_1} u_2 + u_2\, \partial_{x_2} u_2 = -\partial_{x_2} \mathrm{Pr} - g \end{cases} \quad \text{on } \Omega(t),$$

where $\mathrm{Pr}(t, x_1, x_2)$ is the pressure. The kinematic boundary conditions are

$$\psi(t, x_1, 0) = 0$$

on the bottom and

$$\partial_t p - (\partial_{x_2}\psi, -\partial_{x_1}\psi) \in \mathrm{span}\{\partial_s p\}$$

on the upper boundary $\mathscr{S}(t)$ of $\Omega(t)$ that we assume of the form

$$\mathscr{S}(t) = \{p(t,s) \in \mathbb{R} \times (0,\infty) : s \in \mathbb{R}\}$$

with p smooth such that $p(t,\cdot) \in \mathcal{P}_Q$ for all $t \in \mathbb{R}$. The kinematic boundary condition on the top can also be written

$$\nabla\psi \cdot \partial_s p = \det p'$$

where ∇ is the gradient with respect to (x_1, x_2) and p' is the matrix of the first order partial derivatives with respect to t and s. The dynamic boundary condition on the top reads (compare with (1.3h$'$))

$$\mathrm{Pr} = -T\beta\big(\ell(\mathcal{S}(t)) - P\big)^{\beta-1}\sigma + E\Big(2\sigma'' + \sigma^3\Big) + \text{ function of } t \text{ only}$$

on $\mathscr{S}(t)$, where $'$ denotes differentiation with respect to arc length along the surface $\mathscr{S}(t)$, $\sigma(t,x)$ is the curvature of the surface at $x \in \mathscr{S}(t)$ and $\ell(\mathcal{S}(t))$ is the length of $\mathcal{S}(t)$.

It is a standard result of classical hydrodynamics that the vorticity function $\zeta = \partial_{x_1}u_2 - \partial_{x_2}u_1 = -\Delta\psi$ is convected by the flow, where Δ is the Laplacian with respect to (x_1, x_2). Similarly the circulation along the bottom is preserved, thanks to the equation $\partial_t u_1 + (1/2)\partial_{x_1}(u_1^2) = -\partial_{x_1}\mathrm{Pr}$ available at the bottom (because $u_2 = 0$ there). Hence the circulation C along one period of the free boundary is preserved too. These considerations have been the motivation for the variational problems studied in this paper.

Let us begin our study of stability by defining a distance dist_0 between $(\Omega_1, \xi_1, \zeta_1)$ and $(\Omega_2, \xi_2, \zeta_2)$ in the set

$$W^* = \{(\Omega, \xi, \zeta) : \; p \in \mathcal{P}_Q, \; \Omega = \Omega_p \in \mathfrak{O}\backslash\{\Omega_Q\}, \\ \xi \in H^{1/2}_{per}(\mathscr{S}_p), \; \zeta \in L^2(\Omega) \subset L^2((0,P) \times (0,\infty))\}$$

(in the definition of W^*, $\Omega = \Omega_Q$ is forbidden). Let $R > 0$ be given by (4.4). For $i \in \{1,2\}$, we write $\psi_{\Omega_i,\xi_i,\zeta_i}$ for the solution to (1.3a)–(1.3d) corresponding to the domain Ω_i, ξ_i and the vorticity function ζ_i. Moreover we write $\overline{\psi}(\Omega_i, \zeta_i, \mu_i, \nu_i)$ for the solution to (1.3a)–(1.3f) corresponding to Ω_i, ζ_i $\mu_i = C(\Omega_i, \xi_i, \zeta_i)$ and $\nu_i = I(\Omega_i, \xi_i, \zeta_i)$ (see Lemma 2.1 for the existence of such a solution), and then extended on $\{(0,P) \times (0,\infty)\}\backslash\Omega_i$ in a way that is independent of x_1 and affine in x_2 (see Eq. (1.3f) on the free boundary, ξ in (1.3a)–(1.3f) being now not given a priori).

If $\Omega_1 = \Omega_2$ and $\overline{\psi}(\Omega_1, \zeta_1, \mu_1, \nu_1) = \overline{\psi}(\Omega_2, \zeta_2, \mu_2, \nu_2)$, we set

$$\mathrm{dist}_0((\Omega_1, \xi_1, \zeta_1), (\Omega_2, \xi_2, \zeta_2)) = ||\zeta_1 - \zeta_2||_{(H^1((0,P)\times\mathbb{R}))'} \\ + ||\nabla\psi_{\Omega_1,\xi_1,\zeta_1} - \nabla\psi_{\Omega_2,\xi_2,\zeta_2}||_{L^2(\Omega_1)},$$

(when actually $\zeta_1 = \zeta_2$) and in all other cases write

$$\mathrm{dist}_0((\Omega_1, \xi_1, \zeta_1), (\Omega_2, \xi_2, \zeta_2)) = \inf{}_{s\in[0,P]} ||p_1(s + \cdot) - p_2||_{H^2_{per}} \\ + ||\zeta_1 - \zeta_2||_{(H^1((0,P)\times\mathbb{R}))'} + ||\nabla\psi_{\Omega_1,\xi_1,\zeta_1} - \nabla\overline{\psi}(\Omega_1, \zeta_1, \mu_1, \nu_1)||_{L^2(\Omega_1)} \\ + ||\nabla\psi_{\Omega_2,\xi_2,\zeta_2} - \nabla\overline{\psi}(\Omega_2, \zeta_2, \mu_2, \nu_2)||_{L^2(\Omega_2)} \\ + ||\nabla\overline{\psi}(\Omega_1, \zeta_1, \mu_1, \nu_1) - \nabla\overline{\psi}(\Omega_2, \zeta_2, \mu_2, \nu_2)||_{L^2((0,P)\times(0,R))}$$

for some parameterisations p_1 and p_2 of the free boundaries (that is, $p_1, p_2 \in \mathcal{P}_Q$, $\Omega_1 = \Omega_{p_1}$ and $\Omega_2 = \Omega_{p_2}$). Observe that $p_1, p_2 : \mathbb{R} \to \mathbb{R}^2$ are uniquely defined only up to translations in s.

Theorem 4.4 implies that the set $D(\mu, \nu, \zeta_Q)$ of minimizers of $\mathcal{L}|_V$ endowed with the distance dist_0 is compact (see (4.8)–(4.13)).

Lemma 5.1. *Let $((\Omega_n, \xi_n, \zeta_n) : n \in \mathbb{N}) \subset W$ be such that*

$$dist_{L^2((0,P)\times(0,\infty))}\left(\zeta_n, \overline{\mathcal{R}}^w\right) \to 0, \\ C(\Omega_n, \xi_n, \zeta_n) \to \mu, \; I(\Omega_n, \xi_n, \zeta_n) \to \nu, \quad \limsup_{n\to\infty} \mathcal{L}(\Omega_n, \xi_n, \zeta_n) \le \inf{}_V \mathcal{L}.$$

Then $\Omega_n \neq \Omega_Q$ for all n sufficiently large, the distance $dist_0$ of $(\Omega_n, \xi_n, \zeta_n)$ to the set $D(\mu, \nu, \zeta_Q)$ of minimizers converges to 0 *and* $\lim_{n\to\infty} \mathcal{L}(\Omega_n, \xi_n, \zeta_n) = \inf_V \mathcal{L}$.

Proof. Let us first suppose that $\Omega_n \neq \Omega_Q$ for all $n \in \mathbb{N}$. For each n, let $\mu_n = C(\Omega_n, \xi_n, \zeta_n)$, $\nu_n = I(\Omega_n, \xi_n, \zeta_n)$ and $p_n \in \mathcal{P}_Q$ be such that $\Omega_n = \Omega_{p_n}$.

We write $\overline{\psi}_n$ for the solution to (1.3a)–(1.3f) corresponding to the domain $\Omega_n \neq \Omega_Q$, the vorticity function ζ_n, circulation μ_n and horizontal impulse ν_n, and write $\overline{\xi}_n$ for the trace of $\overline{\psi}_n$ to the upper boundary of Ω_n. In particular

$$C(\Omega_n, \overline{\xi}_n, \zeta_n) = \mu_n \;\text{ and }\; I(\Omega_n, \overline{\xi}_n, \zeta_n) = \nu_n. \tag{5.1}$$

Moreover we write $\overline{\lambda}_{1n}$ and $\overline{\lambda}_{2n}$ for the corresponding λ_1 and λ_2 given by Lemma 2.1 applied to Ω_n, ζ_n, μ_n and ν_n.

As

$$\mathcal{L}(\Omega_n, \overline{\xi}_n, \zeta_n) \leq \mathcal{L}(\Omega_n, \xi_n, \zeta_n),$$

(see Proposition 2.2), we can apply Theorem 4.4 to the sequence $\{(\Omega_n, \overline{\xi}_n, \zeta_n)\}_{n\geq 1}$: the distance dist_0 of $(\Omega_n, \overline{\xi}_n, \zeta_n)$ to the set $D(\mu, \nu, \zeta_Q)$ of minimizers converges to 0 (see (4.8)–(4.13)). We also have proved that there is at least one minimizer.

This implies that the distance dist_0 of $(\Omega_n, \xi_n, \zeta_n)$ to the set $D(\mu, \nu, \zeta_Q)$ of minimizers converges to 0. To see it, we write $\psi_{\Omega,\xi,\zeta}$ for the solution to (1.3a)–(1.3d) corresponding to the domain $\Omega \neq \Omega_Q$, ξ and the vorticity function ζ (however ξ is not assumed to satisfy (1.3f)). We let $\widetilde{\psi}_n$ be, as in (1.1), the harmonic function on Ω_n that vanishes on $\{x_2 = 0\}$, is 1 on $\mathscr{S}_n$ and is P-periodic in x_1.

Looking for a contradiction, assume that some subsequence, still denoted by $\{(\Omega_n, \xi_n, \zeta_n)\}$, is such that its distance dist_0 to $D(\mu, \nu, \zeta_Q)$ remains away from 0. Taking a further subsequence if needed, we may also assume that $(\Omega_n, \overline{\xi}_n, \zeta_n)$ tends to some $(\Omega, \xi, \zeta) \in D(\mu, \nu, \zeta)$. We get

$$\begin{aligned}
\int_{\Omega_n} |\nabla(\psi_{\Omega_n,\xi_n,\zeta_n} - \overline{\psi}_n)|^2 dx &= \int_{\Omega_n} |\nabla\psi_{\Omega_n,\xi_n,\zeta_n}|^2 dx - \int_{\Omega_n} |\nabla\overline{\psi}_n|^2 dx \\
&\quad -2\int_{\Omega_n} \nabla(\overline{\psi}_n - \overline{\lambda}_{1,n}x_2 - \overline{\lambda}_{2,n}\widetilde{\psi}_n)\cdot\nabla(\psi_{\Omega_n,\xi_n,\zeta_n} - \overline{\psi}_n)\, dx \\
&\quad -2\int_{\Omega_n} \nabla(\overline{\lambda}_{1,n}x_2 + \overline{\lambda}_{2,n}\widetilde{\psi}_n)\cdot\nabla(\psi_{\Omega_n,\xi_n,\zeta_n} - \overline{\psi}_n)\, dx \\
&\overset{(5.1)}{=} \int_{\Omega_n} |\nabla\psi_{\Omega_n,\xi_n,\zeta_n}|^2 dx - \int_{\Omega_n} |\nabla\overline{\psi}_n|^2 dx - 2\cdot 0 \\
&\quad -2\overline{\lambda}_{1,n}\{I(\Omega_n, \xi_n, \zeta_n) - \mu_n\} - 2\overline{\lambda}_{2,n}\{C(\Omega_n, \xi_n, \zeta_n) - \nu_n\} \\
&= \int_{\Omega_n} |\nabla\psi_{\Omega_n,\xi_n,\zeta_n}|^2 dx - \int_{\Omega_n} |\nabla\overline{\psi}_n|^2 dx \\
&= 2\{\mathcal{L}(\Omega_n, \xi_n, \zeta_n) - \mathcal{L}(\Omega_n, \overline{\xi}_n, \zeta_n)\} \to 0
\end{aligned}$$

because $\limsup_{n\to\infty} \mathcal{L}(\Omega_n, \xi_n, \zeta_n) \leq \inf_V \mathcal{L}$, $\lim_{n\to\infty} \mathcal{L}(\Omega_n, \overline{\xi}_n, \zeta_n) = \inf_V \mathcal{L}$ by (4.12), and $\overline{\psi}_n - \overline{\lambda}_{1,n}x_2 - \overline{\lambda}_{2,n}\widetilde{\psi}_n$ has zero boundary data so it may be treated as a test function. As a further consequence, $\lim_{n\to\infty} \mathcal{L}(\Omega_n, \xi_n, \zeta_n) = \inf_V \mathcal{L}$. Hence the distance dist_0 of $\{(\Omega_n, \xi_n, \zeta_n)\}$ to $D(\mu, \nu, \zeta_Q)$ tends to 0, which is a contradiction.

We have assumed $\Omega_n \neq \Omega_Q$ for all $n \in \mathbb{N}$. If $\Omega_n \neq \Omega_Q$ for all $n \in \mathbb{N}$ sufficiently large, the argument is the same. On the other hand if $\Omega_n = \Omega_Q$ for infinitely many n, we can assume by extracting a subsequence that $\Omega_n = \Omega_Q$ for all $n \in \mathbb{N}$. This case leads to a contradiction as follows, and therefore cannot occur. Taking a further subsequence if needed, we can assume that $\zeta_n \rightharpoonup \zeta$ weakly in $L^2(\Omega_Q)$ and $\psi_{\Omega_Q,\xi_n,\zeta_n} \to \psi$ in $H^1(\Omega_Q)$ for some ζ and ψ. We get

$\zeta \in \overline{\mathcal{R}(\Omega_Q)}^w$ (see the footnote 2), $C(\Omega_Q, \xi, \zeta) = \mu$ and $I(\Omega_Q, \xi, \zeta) = \nu$, where ξ is the trace of ψ. Hence $(\Omega_Q, \xi, \zeta) \in V$, which is in contradiction with (M1). $\square$

We now let t denote time and prove the following stability result, after first giving a definition.

Definition: regular flow Given $\bar{t} \in (0, \infty]$, we call $\{\Omega(t), \xi(t), \zeta(t)\}_{t \in [0,\bar{t})}$ a *regular flow* if, for all t, $\Omega(t) \in \mathfrak{O}$, $\xi(t) \in H^{1/2}_{per}(\mathcal{S}(t))$ with $\mathcal{S}(t) = \partial\Omega(t) \setminus ((0, P) \times \{0\})$, $\zeta(t) \in L^2(\Omega(t)) \subset L^2((0, P) \times (0, \infty))$ and there exists a stream function $\psi \in L^\infty((0, \bar{t}), H^2_{per}((0, P) \times (0, \infty)))$[3] such that $\psi(t) = \psi(t, \cdot)|_{\Omega(t)}$ is a solution to (1.3)(a–d) for almost all $t \in [0, \bar{t})$. Let ψ give rise to the velocity field $u = (\partial_{x_2}\psi, -\partial_{x_1}\psi)$ on $(0, \bar{t}) \times (0, P) \times (0, \infty)$. Concerning the dependence of the domain $\Omega(t)$ on t, we suppose that $\bigcup_{t \in [0,\bar{t})} \Omega(t)$ is bounded, we let $\widetilde{\chi}(t)$ be the characteristic function of $\Omega(t)$, and we assume that the mapping $t \to \widetilde{\chi}(t) \in L^2((0, P) \times (0, \infty))$ is continuous on $[0, \bar{t})$ and that $\widetilde{\chi} \in L^\infty((0, \bar{t}) \times (0, P) \times (0, \infty))$ satisfies the linear transport equation

$$\partial_t \widetilde{\chi} + div(\widetilde{\chi} u) = 0 \quad \text{on} \quad (0, \bar{t}) \times \mathbb{R} \times (0, \infty)$$

(in the sense of distributions, where $\widetilde{\chi}$ and u are extended periodically in x_1). In addition the mapping $t \to \zeta(t) \in L^2((0, P) \times (0, \infty))$ is supposed continuous on $[0, \bar{t})$ and u satisfies the time-dependent hydrodynamic problem (Euler equation or vorticity equation), which takes the form of convection of $\zeta = -\widetilde{\chi}\Delta\psi$ by u according to

$$\partial_t \zeta + div(\zeta u) = 0$$

(in the same sense as above). Finally $\mathcal{L}, I$ and C are all assumed to be conserved, that is, at all $t \in (0, \bar{t})$ they have the same values as at $t = 0$.

For smooth functions these conditions are weaker than those of the full evolutionary problem, for we do not need to be more precise in the statement of the following theorem.

Our main stability result now follows. Whilst this is formulated in terms of dist_0, the subsequent Remarks will discuss alternatives to dist_0 which some readers may consider to be more natural.

Theorem 5.2. *For all $\epsilon > 0$, there exists $\delta > 0$ such that if*

$$\begin{aligned}
&(\Omega_0, \xi_0, \zeta_0) \in W, \ \mathcal{L}(\Omega_0, \xi_0, \zeta_0) < \delta + \min_V \mathcal{L}, \\
&\quad dist_{L^2((0,P)\times(0,\infty))}\left(\zeta_0, \overline{\mathcal{R}(\Omega_0)}^w\right) < \delta, \ |C(\Omega_0, \xi_0, \zeta_0) - \mu| < \delta, \\
&\quad |I(\Omega_0, \xi_0, \zeta_0) - \nu| < \delta,
\end{aligned}$$

and if

$$t \to (\Omega(t), \xi(t), \zeta(t)) \in W$$

[3] By definition of this space, $\psi \in L^1_{loc}((0, \bar{t}) \times (0, P) \times (0, \infty))$ and, for almost all t, $\psi(t, \cdot) \in H^2_{per}((0, P) \times (0, \infty))$. Moreover all the derivatives up to order 2 with respect to x_1 and x_2 are in $L^1_{loc}((0, \bar{t}) \times (0, P) \times (0, \infty))$ and the function $t \to ||\psi(t, \cdot)||_{H^2((0,P)\times(0,\infty))}$ is in L^∞.

is a regular flow on the time interval $[0,\bar{t})$ *such that* $(\Omega(0),\xi(0),\zeta(0)) = (\Omega_0,\xi_0,\zeta_0)$ *(for some* $\bar{t}\in(0,\infty]$ *that is not prescribed), then*

$$\Omega(t)\neq\Omega_Q \text{ and } dist_0\Big(\big(\Omega(t),\xi(t),\zeta(t)\big),D(\mu,\nu,\zeta_Q)\Big)<\epsilon$$

for all $t\in[0,\bar{t})$*,*

Proof. If not, there exist $\epsilon>0$ and, for each n, a regular flow $\{\Omega_n(t), \xi_n(t),\zeta_n(t)\}_{t\in[0,\bar{t}_n)}$ such that

$$\mathcal{L}(\Omega_n(0),\xi_n(0),\zeta_n(0))<\frac{1}{n}+\min_V\mathcal{L},\ \mathrm{dist}_{L^2((0,P)\times(0,\infty))}\left(\zeta_n(0),\overline{\mathcal{R}(\Omega_0)}^w\right)<\frac{1}{n},$$
$$|C(\Omega_n(0),\xi_n(0),\zeta_n(0))-\mu|<\frac{1}{n},\ |I(\Omega_n(0),\xi_n(0),\zeta_n(0))-\nu|<\frac{1}{n}$$

and $t_n\in[0,\bar{t}_n)$ such that either $\Omega_n(t_n)=\Omega_Q$ or

$$\mathrm{dist}_0((\Omega_n(t_n),\xi_n(t_n),\zeta_n(t_n)),D(\mu,\nu,\zeta_Q))\geq\epsilon.$$

Therefore

$$\begin{aligned}\mathcal{L}(\Omega_n(t_n),\xi_n(t_n),\zeta_n(t_n))&=\mathcal{L}(\Omega_n(0),\xi_n(0),\zeta_n(0)),\\ C(\Omega_n(t_n),\xi_n(t_n),\zeta_n(t_n))&=C(\Omega_n(0),\xi_n(0),\zeta_n(0)),\\ I(\Omega_n(t_n),\xi_n(t_n),\zeta_n(t_n))&=I(\Omega_n(0),\xi_n(0),\zeta_n(0)).\end{aligned}$$

We get

$$\mathrm{dist}_{L^2((0,P)\times(0,\infty))}\left(\zeta_n(t_n),\overline{\mathcal{R}}^w\right)<\frac{1}{n};$$

to see this, we introduce as in [5] a "follower" $\chi_n(t)\in\overline{\mathcal{R}}^w$ for $\zeta_n(t)$ as follows. For each $n\in\mathbb{N}$ choose $\chi_n(0)\in\overline{\mathcal{R}(\Omega_n(0))}^w\subset\overline{\mathcal{R}}^w$ with $||\chi_n(0)-\zeta_n(0)||_{L^2(\Omega_n(0))}<1/n$ and let $t\to\chi_n(t)\in L^2((0,P)\times(0,\infty))$ be the unique solution of the linear transport equation $\partial_t\chi_n+\mathrm{div}_x(\chi_n u_n)=0$ that is continuous in $t\in[0,\bar{t}_n)$ (with periodicity condition in x_1), where the velocity $u_n(t)$, as envisaged in the definition of regular flow, is assumed to lie in $L^\infty((0,\bar{t}_n),H^1_{per}((0,P)\times(0,\infty)))$.

The results of DiPerna and Lions [12] and of Bouchut [2] guarantee that, for all $t\in(0,\bar{t}_n)$, $\chi_n(t)$ and $\zeta_n(t)$ are convected by the incompressible flow and thus are rearrangements of $\chi_n(0)$ and $\zeta_n(0)$ respectively vanishing outside $\Omega_n(t)$. See the Appendix for a brief account of the theory in [2,12] that is needed on transport equations, and in particular for the existence and uniqueness of χ_n.

As in [5] we have $\chi_n(t)\in\overline{\mathcal{R}}^w$ and $\chi_n-\zeta_n$ is a solution of the transport equation, so

$$||\chi_n(t_n)-\zeta_n(t_n)||_{L^2(\Omega_n(t))}=||\chi_n(0)-\zeta_n(0)||_{L^2(\Omega_n(0))}<1/n.$$

If $\Omega_n(t_n)=\Omega_Q$ for infinitely many n, we would get a contradiction with the previous lemma. If $\Omega_n(t_n)=\Omega_Q$ for finitely many n, the fact that, for large n, $(\Omega_n(t_n),\xi_n(t_n),\zeta_n(t_n))$ stays away from $D(\mu,\nu,\zeta_Q)$ (with respect to dist_0) would again lead to a contradiction with the previous lemma. $\square$

Remarks. 1. In the statement, the hypotheses

$$\mathcal{L}(\Omega_0,\xi_0,\zeta_0) < \delta + \min_V \mathcal{L},\ |C(\Omega_0,\xi_0,\zeta_0) - \mu| < \delta,\ |I(\Omega_0,\xi_0,\zeta_0) - \nu| < \delta$$

can be replaced by

$$\Omega_0 \neq \Omega_Q \quad \text{and} \quad \mathrm{dist}_0\Big((\Omega_0,\xi_0,\zeta_0), D(\mu,\nu,\zeta_Q)\Big) < \delta$$

because

$$\mathcal{L}(\Omega_0,\xi_0,\zeta_0) \to \min_V \mathcal{L},\ C(\Omega_0,\xi_0,\zeta_0) \to \mu,\ I(\Omega_0,\xi_0,\zeta_0) \to \nu$$

as $\mathrm{dist}_0((\Omega_0,\xi_0,\zeta_0), D(\mu,\nu,\zeta_Q)) \to 0$.

2. Solutions to the evolutionary problem that are considered are supposed regular enough, but nothing is claimed about their existence. This is why the stability result is said to be "conditional". The choice of the distance in the statement is crucial for its meaning. Conditional stability is here with respect to the distance dist_0, that is, the distance dist_0 to the set of minimizers is controlled for subsequent times if it is well enough controlled initially. However nothing is said about other distances and it could be that some other significant distance blows up whereas dist_0 remains under control; as a consequence the solution would nevertheless cease to exist in the considered functional space. On the other hand, a control on dist_0 could be the starting point of a well-posedness analysis (well-posedness of the Cauchy problem for related settings is discussed in many papers, see e.g. [11]).
3. In the statement of the theorem, dist_0 can be replaced by the simpler distance

$$\begin{aligned}\mathrm{dist}_1((\Omega_1,\xi_1,\zeta_1),(\Omega_2,\xi_2,\zeta_2)) &= \inf{}_{s\in[0,P]} ||p_1(s+\cdot) - p_2||_{H^2_{per}} \\ &\quad + ||\zeta_1 - \zeta_2||_{(H^1((0,P)\times\mathbb{R}))'} + ||\nabla\psi_{\Omega_1,\xi_1,\zeta_1} - \nabla\psi_{\Omega_2,\xi_2,\zeta_2}||_{L^2((0,P)\times(0,R))},\end{aligned}$$

where $\nabla\psi_{\Omega_i,\xi_i,\zeta_i}$ has been trivially extended on $((0,P)\times(0,R))\backslash\Omega_i$ (thus dist_1 is defined in terms of vorticity and velocity). Indeed, for all $\epsilon_1 > 0$, there exists $\epsilon_0 > 0$ such that, for all $(\Omega,\xi,\zeta) \in W^*$,

$$\mathrm{dist}_0\Big((\Omega,\xi,\zeta)\,,\, D(\mu,\nu,\zeta_Q)\Big) < \epsilon_0 \Rightarrow \mathrm{dist}_1\Big((\Omega,\xi,\zeta)\,,\, D(\mu,\nu,\zeta_Q)\Big) < \epsilon_1.$$

Otherwise there would exist $\epsilon_1 > 0$ and two sequences $\{(\Omega_{1,n},\xi_{1,n},\zeta_{1,n})\} \subset W^*$ and $\{(\Omega_{2,n},\xi_{2,n},\zeta_{2,n})\} \subset D(\mu,\nu,\zeta_Q)$ such that

$$\lim_{n\to\infty} \mathrm{dist}_0\Big((\Omega_{1,n},\xi_{1,n},\zeta_{1,n})\,,\,(\Omega_{2,n},\xi_{2,n},\zeta_{2,n})\Big) = 0$$

and

$$\begin{aligned}&\inf_{n\in\mathbb{N}} \mathrm{dist}_1\Big((\Omega_{1,n},\xi_{1,n},\zeta_{1,n})\,,\,(\Omega_{2,n},\xi_{2,n},\zeta_{2,n})\Big) \\ &\qquad \geq \inf_{n\in\mathbb{N}} \mathrm{dist}_1\Big((\Omega_{1,n},\xi_{1,n},\zeta_{1,n})\,,\, D(\mu,\nu,\zeta_Q)\Big) \geq \epsilon_1.\end{aligned}$$

Taking subsequences if necessary, we can assume that

$$\lim_{n\to\infty} \mathrm{dist}_0\Big((\Omega_{2,n},\xi_{2,n},\zeta_{2,n})\,,\,(\Omega,\xi,\zeta)\Big) = 0$$

for some $(\Omega, \xi, \zeta) \in D(\mu, \nu, \zeta_Q)$ and thus

$$\lim_{n\to\infty} \mathrm{dist}_0 \left((\Omega_{1,n}, \xi_{1,n}, \zeta_{1,n})\, , \, (\Omega, \xi, \zeta)\right) = 0.$$

If $\Omega_{1,n} = \Omega$ and

$$\begin{aligned}&\nabla\overline{\psi}(\Omega_{1,n}, \zeta_{1,n}, C(\Omega_{1,n}, \xi_{1,n}, \zeta_{1,n}), I(\Omega_{1,n}, \xi_{1,n}, \zeta_{1,n}))\\ &\quad = \nabla\overline{\psi}(\Omega, \zeta, C(\Omega, \xi, \zeta), I(\Omega, \xi, \zeta))\end{aligned}$$

then

$$\mathrm{dist}_1 \left((\Omega_{1,n}, \xi_{1,n}, \zeta_{1,n}), \, (\Omega, \xi, \zeta)\right) = \mathrm{dist}_0 \left((\Omega_{1,n}, \xi_{1,n}, \zeta_{1,n}), \, (\Omega, \xi, \zeta)\right),$$

whereas otherwise,

$$\begin{aligned}&\|\nabla\psi_{\Omega_{1,n},\xi_{1,n},\zeta_{1,n}} - \nabla\psi_{\Omega,\xi,\zeta}\|_{L^2((0,P)\times(0,R))}\\
&\quad \le \|\nabla\psi_{\Omega_{1,n},\xi_{1,n},\zeta_{1,n}} - \nabla\psi_{\Omega,\xi,\zeta}\|_{L^2(\Omega_{1,n})} + \|\nabla\psi_{\Omega,\xi,\zeta}\|_{L^2(\Omega\setminus\Omega_{1,n})}\\
&\quad \le \|\nabla\psi_{\Omega_{1,n},\xi_{1,n},\zeta_{1,n}} - \nabla\overline{\psi}(\Omega_{1,n}, \zeta_{1,n}, C(\Omega_{1,n}, \xi_{1,n}, \zeta_{1,n}), I(\Omega_{1,n}, \xi_{1,n}, \zeta_{1,n}))\|_{L^2(\Omega_{1,n})}\\
&\qquad + \|\nabla\overline{\psi}(\Omega_{1,n}, \zeta_{1,n}, C(\Omega_{1,n}, \xi_{1,n}, \zeta_{1,n}), I(\Omega_{1,n}, \xi_{1,n}, \zeta_{1,n}))\\
&\qquad - \nabla\overline{\psi}(\Omega, \zeta, C(\Omega, \xi, \zeta), I(\Omega, \xi, \zeta))\|_{L^2(\Omega_{1,n})}\\
&\qquad + \|\nabla\overline{\psi}(\Omega, \zeta, C(\Omega, \xi, \zeta), I(\Omega, \xi, \zeta)) - \nabla\psi_{\Omega,\xi,\zeta}\|_{L^2(\Omega_{1,n})} + \|\nabla\psi_{\Omega,\xi,\zeta}\|_{L^2(\Omega\setminus\Omega_{1,n})}\\
&\quad \le \|\nabla\psi_{\Omega_{1,n},\xi_{1,n},\zeta_{1,n}} - \nabla\overline{\psi}(\Omega_{1,n}, \zeta_{1,n}, C(\Omega_{1,n}, \xi_{1,n}, \zeta_{1,n}), I(\Omega_{1,n}, \xi_{1,n}, \zeta_{1,n}))\|_{L^2(\Omega_{1,n})}\\
&\qquad + \|\nabla\overline{\psi}(\Omega_{1,n}, \zeta_{1,n}, C(\Omega_{1,n}, \xi_{1,n}, \zeta_{1,n}), I(\Omega_{1,n}, \xi_{1,n}, \zeta_{1,n}))\\
&\qquad - \nabla\overline{\psi}(\Omega, \zeta, C(\Omega, \xi, \zeta), I(\Omega, \xi, \zeta))\|_{L^2((0,P)\times(0,R))}\\
&\qquad + \|\nabla\overline{\psi}(\Omega, \zeta, C(\Omega, \xi, \zeta), I(\Omega, \xi, \zeta)) - \nabla\psi_{\Omega,\xi,\zeta}\|_{L^2(\Omega)}\\
&\qquad + \|\nabla\overline{\psi}(\Omega, \zeta, C(\Omega, \xi, \zeta), I(\Omega, \xi, \zeta))\|_{L^2(\Omega_{1,n}\setminus\Omega)} + \|\nabla\psi_{\Omega,\xi,\zeta}\|_{L^2(\Omega\setminus\Omega_{1,n})},\end{aligned}$$

and hence

$$\begin{aligned}&\mathrm{dist}_1 \left((\Omega_{1,n}, \xi_{1,n}, \zeta_{1,n})\, , \, (\Omega, \xi, \zeta)\right) \le \mathrm{dist}_0 \left((\Omega_{1,n}, \xi_{1,n}, \zeta_{1,n})\, , \, (\Omega, \xi, \zeta)\right)\\ &\qquad + \|\nabla\overline{\psi}(\Omega, \zeta, C(\Omega, \xi, \zeta), I(\Omega, \xi, \zeta))\|_{L^2(\Omega_{1,n}\setminus\Omega)} + \|\nabla\psi_{\Omega,\xi,\zeta}\|_{L^2(\Omega\setminus\Omega_{1,n})}.\end{aligned}$$

Thus in either case we get the contradiction

$$0 = \lim_{n\to\infty} \mathrm{dist}_1 \left((\Omega_{1,n}, \xi_{1,n}, \zeta_{1,n})\, , \, (\Omega, \xi, \zeta)\right) \ge \epsilon_1.$$

4. In the statement of the theorem, the hypotheses

$$\mathcal{L}(\Omega_0, \xi_0, \zeta_0) < \delta + \min_V \mathcal{L}, \; |C(\Omega_0, \xi_0, \zeta_0) - \mu| < \delta, \; |I(\Omega_0, \xi_0, \zeta_0) - \nu| < \delta$$

can also be replaced by

$$\mathrm{dist}_1 \left((\Omega_0, \xi_0, \zeta_0), D(\mu, \nu, \zeta_Q)\right) < \delta$$

(compare with the first remark). Indeed, for all $\delta > 0$, there exists $\delta_1 > 0$ such that all $(\Omega, \xi, \zeta) \in W$ satisfying $\mathrm{dist}_1 \left((\Omega, \xi, \zeta)\, , \, D(\mu, \nu, \zeta_Q)\right) < \delta_1$ also satisfy

$$\mathcal{L}(\Omega, \xi, \zeta) < \delta + \min_V \mathcal{L}, \; |C(\Omega, \xi, \zeta) - \mu| < \delta, \; |I(\Omega, \xi, \zeta) - \nu| < \delta.$$

Otherwise there would exist $\delta > 0$ and two sequences $\{(\Omega_{1,n}, \xi_{1,n}, \zeta_{1,n})\} \subset W$ and $\{(\Omega_{2,n}, \xi_{2,n}, \zeta_{2,n})\} \subset D(\mu, \nu, \zeta_Q)$ such that

$$\lim_{n\to\infty} \mathrm{dist}_1\Big((\Omega_{1,n}, \xi_{1,n}, \zeta_{1,n})\,,\,(\Omega_{2,n}, \xi_{2,n}, \zeta_{2,n})\Big) = 0$$

and such that one of the following inequalities holds:

$$\inf{}_{n\in\mathbb{N}}\mathcal{L}(\Omega_{1,n}, \xi_{1,n}, \zeta_{1,n}) \geq \delta + \min_V \mathcal{L},$$
$$\inf{}_n|C(\Omega_{1,n}, \xi_{1,n}, \zeta_{1,n}) - \mu| \geq \delta,\ \inf{}_n|I(\Omega_{1,n}, \xi_{1,n}, \zeta_{1,n}) - \nu| \geq \delta.$$

Taking subsequences if necessary, we can assume that

$$\lim_{n\to\infty} \mathrm{dist}_0\Big((\Omega_{2,n}, \xi_{2,n}, \zeta_{2,n})\,,\,(\Omega, \xi, \zeta)\Big) = 0$$

for some $(\Omega, \xi, \zeta) \in D(\mu, \nu, \zeta_Q)$. Arguing as above, we get

$$\lim_{n\to\infty} \mathrm{dist}_1\Big((\Omega_{2,n}, \xi_{2,n}, \zeta_{2,n})\,,\,(\Omega, \xi, \zeta)\Big) = 0$$

and thus

$$\lim_{n\to\infty} \mathrm{dist}_1\Big((\Omega_{1,n}, \xi_{1,n}, \zeta_{1,n})\,,\,(\Omega, \xi, \zeta)\Big) = 0.$$

We then get the contradiction

$$\mathcal{L}(\Omega_{1,n}, \xi_{1,n}, \zeta_{1,n}) \to \min_V \mathcal{L},\ C(\Omega_{1,n}, \xi_{1,n}, \zeta_{1,n})$$
$$\to \mu,\ I(\Omega_{1,n}, \xi_{1,n}, \zeta_{1,n}) \to \nu$$

(see (4.18) and (4.19) for similar computations).

6. Appendix: transport equation theory needed to construct the follower

During the proof of Theorem 5.2 we introduce a "follower", in $\overline{\mathcal{R}}^w$, of a regular flow, by convecting a suitable element of $\overline{\mathcal{R}}^w$ using the velocity field of the flow. Here we present the theory of transport equations needed to justify this construction.

Let us consider a regular flow (see the above definition). As $\bigcup_{t\in[0,\bar{t})} \Omega(t)$ is bounded, we can suppose that, for some $R > 0$, $\bigcup_{t\in[0,\bar{t})} \Omega(t) \subset (0, P) \times (0, R)$ and the divergence-free velocity $u \in L^\infty((0, \bar{t}), H^1_{per}((0, P) \times (0, \infty)))$ vanishes for $x_2 > R$. We extend u to all of $\mathbb{R} \times \mathbb{R}^2$ by setting $u(t, x_1, x_2) = 0$ for $t \notin [0, \bar{t})$, $u(t, x_1, x_2) = (u_1(t, x_1, -x_2), -u_2(t, x_1, -x_2))$ for $x_2 < 0$ and by P-periodicity in x_1. We use the notation $u = (u_1, u_2)$ and $u(t) = u(t, \cdot)$. As, for almost all t, the trace of $u_2(t)$ on the set $x_2 = 0$ is trivial (see (1.3b)), u is now well defined in $L^\infty(\mathbb{R}, H^1_{per}(\mathbb{R}^2))$ and still divergence free.

Existence. Consider initial data $\chi(0) \in L^2(\Omega(0)) \subset L^2((0, P) \times (0, \infty))$ and extend it periodically in x_1 so that we can see it in $L^2_{per}(\mathbb{R} \times (0, \infty)) \subset L^2_{per}(\mathbb{R}^2)$ (and $\chi(0)$ vanishes when $x_2 < 0$). Mollify $\chi(0)$ in x to get $\chi_\varepsilon(0)$ and mollify u in x and t to get $u_{\varepsilon,\tau}(t)$ bounded in $H^1_{per}(\mathbb{R}^2)$. This can be done in such a way that the second component of $u_{\varepsilon,\tau}(t)$ vanishes on $x_2 = 0$. Since, for fixed ϵ and τ, $u_{\varepsilon,\tau} \in L^\infty(\mathbb{R} \times \mathbb{R}^2)$, the solution of

$$\partial_t \chi + div(\chi u_{\varepsilon,\tau}) = 0 \quad \text{in } [0, \infty) \times \mathbb{R}^2$$

with initial data $\chi_\varepsilon(0)$ exists for all positive time by using the flow of $u_{\varepsilon,\tau}$; denote it $\chi_{\varepsilon,\tau}(t) \in L^2_{per}(\mathbb{R}^2)$. Notice that $u_{\varepsilon,\tau}(t)$ is still divergence-free and therefore the flow is rearrangement-preserving, hence

$$\|\chi_{\varepsilon,\tau}(t)\|_2 = \|\chi_{\varepsilon,\tau}(t)\|_{L^2((0,P)\times\mathbb{R})} = \|\chi_\varepsilon(0)\|_2 \leq \|\chi(0)\|_2.$$

Then, for any $1 < s < 2$, we have

$$\begin{aligned}\|\chi_{\varepsilon,\tau}(t)u_{\varepsilon,\tau}(t)\|_s &= \|\chi_{\varepsilon,\tau}(t)u_{\epsilon,\tau}(t)\|_{L^s((0,P)\times\mathbb{R})} \leq \|\chi_{\varepsilon,\tau}(t)\|_2\|u_{\varepsilon,\tau}(t)\|_{2s/(2-s)}\\ &\leq \|\chi(0)\|_2\|u(t)\|_{H^1},\end{aligned}$$

so we have $\chi_{\varepsilon,\tau}(t)u_{\varepsilon,\tau}(t)$ bounded in L^s and thus $div(\chi_{\varepsilon,\tau}(t)u_{\varepsilon,\tau})$ bounded in $W^{-1,s}$. Hence, as in Lemma 10 in [5], for $0 \leq t_1 < t_2$,

$$\|\chi_{\varepsilon,\tau}(t_2) - \chi_{\varepsilon,\tau}(t_1)\|_{-1,s} \leq M\|\chi(0)\|_2|t_2 - t_1|$$

where M is a bound on $\|u(t)\|_{H^1}$ for almost all $t \in [t_1, t_2]$.

Let $1/r + 1/s = 1$ (so $2 < r < \infty$). Then $W^{1,r}((0,P) \times (-2R, 2R)) \hookrightarrow L^2((0,P) \times (-2R, 2R))$ compactly and, taking the adjoints, $L^2 \hookrightarrow W^{-1,s}$ compactly. Since the $\chi_{\varepsilon,\tau}(t)$ all lie in a ball in $L^2((0,P) \times (-2R, 2R))$ (for ϵ, τ small enough) and hence lie in a strongly compact set in $W^{-1,s}$, we can apply the Arzelà-Ascoli theorem to let $\varepsilon, \tau \to 0$ (along any particular sequences) and obtain a sequence converging in $L^\infty((0,\bar{t}), W^{-1,s}_{per}(\mathbb{R} \times (-2R, 2R)))$ and weakly in L^2 on any bounded open subset of $(0,\bar{t}) \times \mathbb{R}^2$ to a limit

$$\chi \in C([0,\bar{t}), W^{-1,s}_{per}(\mathbb{R}^2)) \cap L^2_{loc}((0,\bar{t}) \times \mathbb{R}^2) \cap L^\infty((0,\bar{t}), L^2_{per}(\mathbb{R}^2)),$$

where $L^2_{per}(\mathbb{R}^2)$ is endowed with the norm of $L^2((0,P) \times \mathbb{R})$. Moreover χ solves the linear transport equation on $(0,\bar{t}) \times \mathbb{R}^2$ with initial condition $\chi(0)$, $\chi(t)$ is also weakly continuous in L^2 with respect to $t \in [0,\bar{t})$, $\chi(t)$ vanishes for $x_2 < 0$ and for $x_2 > R$ and $\chi(t) \geq 0$ if $\chi(0) \geq 0$ (because of the way $\chi(\cdot)$ has been obtained as a limit; remember that $\chi(0)$ vanishes for $x_2 \notin [0,R]$, and since $u(t)$ vanishes for $x_2 > R$ and the second component of $u(t)$ is odd in x_2 it follows that the trajectories of the approximating flows do not cross the lines $x_2 = 0$ and $x_2 = R + \varepsilon$).

Rearrangement and uniqueness. Let $t \to \chi(t) \in L^2_{per}(\mathbb{R} \times (0,\infty))$ be such that

$$\begin{aligned}\chi \in\ & C([0,\bar{t}), W^{-1,s}_{per}(\mathbb{R} \times (0,\infty))) \cap L^2_{loc}((0,\bar{t}) \times \mathbb{R} \times (0,\infty))\\ &\cap L^\infty_{loc}((0,\bar{t}), L^2_{per}(\mathbb{R} \times (0,\infty))),\end{aligned}$$

the support of χ is uniformly bounded in the x_2 direction and χ satisfies the linear transport equation on $(0,\bar{t}) \times \mathbb{R} \times (0,\infty)$, that is,

$$\int_{(0,\bar{t})\times\mathbb{R}\times(0,\infty)} (\partial_t \varphi + \nabla\varphi \cdot u)\chi \, dtdx = 0 \tag{6.1}$$

for all $\varphi \in \mathscr{D}((0,\bar{t}) \times \mathbb{R} \times (0,\infty))$. Here χ is not necessarily restricted to be the solution obtained just above and, provided that $\chi \in L^2_{loc}((0,\bar{t}) \times \mathbb{R} \times$

$(0,\infty)) \cap L^\infty_{loc}((0,\bar t), L^2_{per}(\mathbb{R}\times(0,\infty)))$, the hypothesis $\chi \in C([0,\bar t), W^{-1,s}_{per}(\mathbb{R}\times(0,\infty)))$ above is equivalent in this context to the requirement that $t \to \chi(t) \in L^2_{per}(\mathbb{R}\times(0,\infty))$ is continuous in $t \geq 0$ with respect to the weak topology on $L^2_{per}(\mathbb{R}\times(0,\infty))$.

Let us check that (6.1) still holds for all $\varphi \in \mathscr{D}((0,\bar t)\times\mathbb{R}^2)$, so that χ is also a solution to the linear transport equation on $(0,\bar t)\times\mathbb{R}^2$ (where χ vanishes if $x_2<0$). Given such a φ, we introduce $f \in C^\infty(\mathbb{R})$ such that $f(x_2)=0$ for $x_2 \leq 0$, $f(x_2)=1$ for $x_2 \geq 1$ and f is increasing. We set $f_\delta(x_2)=f(x_2/\delta)$ and observe that

$$\int_{(0,\bar t)\times\mathbb{R}\times(0,\infty)} (f_\delta(x_2)\partial_t\varphi + f_\delta(x_2)\nabla\varphi\cdot u + f'_\delta(x_2)\varphi u_2)\chi\, dtdx = 0,$$

where $u=(u_1,u_2)$. As $\chi \in L^\infty((0,\bar t), L^2_{per}(\mathbb{R}\times(0,\infty)))$, we get

$$\int_{(0,\bar t)\times\mathbb{R}\times(0,\infty)} (f_\delta(x_2)\partial_t\varphi + f_\delta(x_2)\nabla\varphi\cdot u)\chi\, dtdx \to \int_{(0,\bar t)\times\mathbb{R}^2} (\partial_t\varphi + \nabla\varphi\cdot u)\chi\, dtdx$$

as $\delta \to 0$, by Lebesgue's theorem. Moreover, if φ is supported in $(0,T)\times(-A,A)^2$ with $0<T<\bar t$, then

$$\begin{aligned}&\left|\int_{(0,\bar t)\times\mathbb{R}\times(0,\infty)} f'_\delta(x_2)\varphi u_2\chi\, dtdx\right|\\ &\leq \delta^{-1}\text{const}\, \|\varphi\chi\|_{L^2((0,T)\times(-A,A)\times(0,\delta))}\|u_2\|_{L^\infty((0,T),L^2((-A,A)\times(0,\delta)))}\\ &\overset{\text{Poincaré}}{\leq} \text{const}\, \|\varphi\chi\|_{L^2((0,T)\times(-A,A)\times(0,\delta))}\|\nabla u_2\|_{L^\infty((0,T),L^2((-A,A)\times(0,\delta)))}\\ &\to 0\end{aligned}$$

as $\delta \to 0$, because $\chi \in L^2_{loc}((0,\bar t)\times\mathbb{R}\times(0,\infty))$ (Poincaré's inequality is available thanks to the fact that the trace of $u_2(t)$ on $x_2=0$ vanishes for almost all t; see e.g. [1], sect. 6.26 in the 1st edition or 6.30 in the 2nd). Thus (6.1) holds for the more general φ as desired.

Now that we know that

$$\chi \in C([0,\bar t), W^{-1,s}_{per}(\mathbb{R}^2)) \cap L^2_{loc}((0,\bar t)\times\mathbb{R}^2) \cap L^\infty_{loc}((0,\bar t), L^2_{per}(\mathbb{R}^2))$$

is a solution to the linear transport equation on $(0,\bar t)\times\mathbb{R}^2$, we mollify in x to get $\chi_\varepsilon \in C([0,\bar t), L^\infty_{per}(\mathbb{R}^2))$. We also assume that χ vanishes if $x_2 \notin [0,R]$.

Choose any $T \in (0,\bar t)$. Then, for bounded $g \in C^1(\mathbb{R})$, by Bouchut [2], proof of Thm 3.2(ii) (especially Lemma 3.1(ii) applied to eq. (3.23)), we have

$$\partial_t g(\chi_\varepsilon) + div(g(\chi_\varepsilon)u) = r_\varepsilon \to 0 \text{ in } L^1((0,T), L^1_{loc}(\mathbb{R}^2)) \text{ as } \varepsilon \to 0.$$

Integrating against a smooth test function of the form $h(t)f(x)$ we have

$$\begin{aligned}&\left|\int_{\mathbb{R}^3} h' f g(\chi_\varepsilon)dtdx + \int_{\mathbb{R}^3} h\nabla f\cdot u g(\chi_\varepsilon)dtdx\right| = \left|\int_{\mathbb{R}^3} h f r_\varepsilon dtdx\right| \\ &\quad\leq \|r_\varepsilon\|_{L^1((0,T)\times(-P,2P)\times(-2R,2R))}\end{aligned} \tag{6.2}$$

provided $\sup_{t\in\mathbb{R}}|h(t)| \leq 1$, $\sup_{x\in\mathbb{R}^2}|f(x)| \leq 1$, h is compactly supported in $(0,T)$ and f is compactly supported in $(-P,2P)\times(-2R,2R)$.

Choose $f = f_\delta \in \mathscr{D}(\mathbb{R}^2)$ of the form $f_\delta(x_1, x_2) = f_1(x_1)f_2(x_2)$ where f_1 vanishes outside $[0, P+\delta]$ and is identically equal to 1 on $[\delta, P]$, while f_2 is compactly supported in $(-R-\delta, R+\delta)$ and is identically equal to 1 on $[-R, R]$. We assume $0 < \delta < \min\{P/2, R\}$. By approximations, the class of allowed f_1 can be enlarged to continuous functions that are piecewise C^1, and therefore we can choose f_1 such that $f_1(x_1) = x_1/\delta$ on $[0, \delta]$ and $f_1(x_1) = 1-(x_1-P)/\delta$ on $[P, P+\delta]$. Then

$$\int_{\mathbb{R}^2} g(\chi_\varepsilon)\nabla f_\delta \cdot u \, dx = \int_{\mathbb{R}^2} g(\chi_\varepsilon) f_1'(x_1)f_2(x_2)u_1 \, dx,$$

because $u = (u_1, u_2)$ vanishes if $x_2 \notin [-R, R]$ and thus $f_2'(x_2)u_2$ vanishes almost everywhere on $\mathbb{R}^2$, where t is fixed in a set of full measure in $(0, T)$. The contributions to the integral of the regions $[0, \delta] \times [-2R, 2R]$ and $[P, P+\delta] \times [-2R, 2R]$ are equal and opposite (because χ_ε and u are P-periodic in x_1, and $f'(x_1) = \pm 1/\delta$ there), while $g(\chi_\varepsilon)f_1'(x_1)f_2(x_2)u_1$ vanishes everywhere else. Hence

$$\int_{\mathbb{R}^2} g(\chi_\varepsilon(t, x))\nabla f_\delta(x) \cdot u(t, x)dx = 0. \tag{6.3}$$

For $0 < t_1 < t_2 < T$, now take $h = h_\delta$ in (6.2) to be any test function on $(0, T)$ with $0 \le h_\delta \le 1$, vanishing outside (t_1, t_2), equal to 1 on $[t_1+\delta, t_2-\delta]$, with $0 \le h_\delta' \le 2/\delta$ on $(t_1, t_1+\delta)$ and $0 \le -h_\delta' \le 2/\delta$ on $(t_2-\delta, t_2)$ $(0 < \delta < (t_2-t_1)/2)$. Applying (6.3) and letting $\delta \to 0$, we obtain

$$\begin{aligned}&\left| \int_{(0,P)\times(-R,R)} g(\chi_\varepsilon(t_2))dx - \int_{(0,P)\times(-R,R)} g(\chi_\varepsilon(t_1))dx \right| \\ &\quad \le \|r_\varepsilon\|_{L^1((0,T)\times(-P,2P)\times(-2R,2R))}\end{aligned}$$

because $g(\chi_\varepsilon) \in C([0, \bar{t}), L^\infty_{per}(\mathbb{R}^2))$. Letting $\varepsilon \to 0$ yields

$$\int_{(0,P)\times(-R,R)} g(\chi(t_2))dx = \int_{(0,P)\times(-R,R)} g(\chi(t_1))dx$$

and we deduce that $\chi(t_2)$ is a rearrangement of $\chi(t_1)$ in $L^2((0, P) \times (-R, R))$. As a consequence $\chi(t_2)$ is a rearrangement of $\chi(t_1)$ in $L^2((0, P) \times (0, R))$ and hence $\|\chi(t, \cdot)\|_{L^2((0,P)\times(0,\infty))}$ is constant in time. As $T \in (0, \bar{t})$ is arbitrary, this proves any solution

$$\begin{aligned}\chi \ &\in C([0, \bar{t}), W^{-1,s}_{per}(\mathbb{R} \times (0, \infty))) \cap L^2_{loc}((0, \bar{t}) \times \mathbb{R} \times (0, \infty)) \\ &\cap L^\infty_{loc}((0, \bar{t}), L^2_{per}(\mathbb{R} \times (0, \infty)))\end{aligned}$$

of the linear transport equation on $(0, \bar{t}) \times \mathbb{R} \times (0, \infty)$ such that χ vanishes for all $x_2 \notin (0, R)$ is strongly continuous with respect to $L^2_{per}(\mathbb{R}\times(0, \infty))$ (because it is weakly continuous and the L^2-norm is preserved). In addition $\chi(t)$ is a rearrangement of $\chi(0)$ for all $t \in (0, \bar{t})$ and therefore if $\chi(0) = 0$ then $\chi(t) = 0$ for all $t \in (0, \bar{t})$. If $\chi(0)$ is not necessarily trivial, this implies by linearity that $t \to \chi(t)$ is unique given $\chi(0)$ (more precisely, unique in this class).

Let $\Omega(t)$ for $t \in (0, \bar{t})$ and $\widetilde{\chi}$ be as in the definition of a regular flow in the previous section, and assume moreover that $\chi(0)$ vanishes outside $\Omega(0)$. Then

$\chi^2/(1+\chi^2) \in [0,1)$ is a solution to the linear transport equation on $(0,\bar{t}) \times \mathbb{R}^2$ (see Thm 3.2(ii) in [2]) and so is $\widetilde{\chi} - \chi^2/(1+\chi^2)$ (by linearity). As $\widetilde{\chi}(0) - \chi(0)^2/(1+\chi(0)^2) \geq 0$ almost everywhere, we get $\widetilde{\chi}(t) - \chi(t)^2/(1+\chi(t)^2) \geq 0$ for all $t \in [0,\bar{t})$ and thus $\chi(t)$ is supported by $\Omega(t)$ for all $t \in [0,\bar{t})$.

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