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“Note on “Flows induced by a plate moving normal to stagnation-point flow” by P. D. Weidman and M. A. Sprague (Acta Mech. 219, 219–229, 2011)”

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Abstract In a recent paper of Weidman and Sprague (Acta Mech., 2011), the *unsteady* flows generated by an impermeable infinite flat plate advancing with constant velocity V toward, or receding from an orthogonal (plane or axisymmetric) stagnation-point flow, have been investigated by an exact similarity reduction of the Navier–Stokes equations. It has been shown that in the co-moving reference frame of the plate, the induced flow appears as a steady flow, with an additional term Rf'' in the governing equation of the similar stream function $f(\eta)$. The Reynolds number R involved in this additional term is proportional to the plate velocity V . The present paper shows, however, that with the aid of a simple transformation, the additional term Rf'' can be removed from the governing equation, its effect being transferred in the boundary condition for $f(\eta)$. As a consequence, the *unsteady* flow problems of Weidman and Sprague reduce to the classical *steady* stagnation-point flow problems for *permeable surfaces* with a uniform lateral suction or injection of the fluid, so that the transpiration parameter $f(0)$ coincides with R for the plane and with $R/2$ for the axisymmetric flow, respectively. The main benefit of this approach is that all the results of the latter well-investigated problems can simply be transcribed for the problems formulated by Weidman and Sprague (Acta Mech, 2011).

1 Introduction and basic equations

In a recent paper of Weidman and Sprague [1], a comprehensive analytical and numerical study of the title problem has been reported. It has been shown that by the similarity transformation of the stream function

$$\psi(x, z) = \sqrt{av}xf(\eta), \quad \eta = \sqrt{\frac{a}{v}}(z - Vt), \quad (1)$$

the Navier–Stokes equations of the plane stagnation-point flow (*Hiemenz flow*),

$$\begin{aligned} \psi_{tz} + \psi_z\psi_{xz} - \psi_x\psi_{zz} &= -\frac{1}{\rho}p_x + \nu(\psi_{xxz} + \psi_{zzz}), \\ \psi_{tx} + \psi_z\psi_{xx} - \psi_x\psi_{xz} &= \frac{1}{\rho}p_x + \nu(\psi_{xxx} + \psi_{xzz}), \end{aligned} \quad (2)$$

reduce for the similar stream function f to the ordinary differential equation

$$f''' + ff'' - f'^2 + 1 + Rf'' = 0. \quad (3)$$

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In the above equations, the stream function ψ has been defined in terms of the x and z components u and w of the plane velocity field in the usual way, $u = \psi_z$, $w = -\psi_x$ and the subscripts denote partial differentiations with respect to the indicated variables. In Eq. (3), the primes denote differentiations with respect to η , $R = Vl/\nu$ is the Reynolds number based on the plate velocity V and on the reference length $l = \sqrt{\nu/a}$, where a is the strain rate of the flow. For $R > 0$, the plate moves into the stagnation flow, and when $R < 0$ the plate recedes from it. The pertinent boundary conditions are [1]

$$f(0) = 0, \quad f'(0) = 0, \quad f'(\infty) = 1. \quad (4)$$

Owing to the choice (1) of the similarity independent variable η , the induced flow appears in the co-moving Galilean reference frame of the plate as a steady flow.

In the case of a plate moving with constant velocity V normal to the axisymmetric stagnation-point flow (*Homann flow*), similar results have been reported [1], so that the governing differential equations of both stagnation-point problems can be encompassed in the same form

$$f''' + mf f'' - f'^2 + 1 + Rf'' = 0, \quad (5)$$

with $m = 1$ and $m = 2$ for the Hiemenz and Homann flows, respectively. The boundary conditions are given in both cases by Eq. (4).

2 Transformation

At first glance, there seems that due to the presence of the additional term Rf'' , the boundary value problems specified by Eqs. (5) and (4) differ substantially from their classical (properly steady) counterparts corresponding to $R = 0$. This, however, is not so. Indeed, substituting in Eq. (5)

$$f(\eta) = F(\eta) - \frac{R}{m}, \quad (6)$$

one immediately arrives at the equations

$$F''' + mFF'' - F'^2 + 1 = 0, \quad F(0) = R/m, \quad F'(0) = 0, \quad F'(\infty) = 1, \quad (7)$$

which describe the classical Hiemenz and Homann stagnation-point flows toward permeable surfaces when a uniform suction/injection of the fluid with transpiration parameter R/m is applied. It is worth mentioning here that the reduction to the boundary value problem (7) could have been obtained also directly, by replacing the stream function (1) posited in [1] from the very beginning by

$$\psi(x, z) = \sqrt{av}x \left[F(\eta) - \frac{R}{m} \right], \quad \eta = \sqrt{\frac{a}{v}}(z - Vt). \quad (8)$$

Therefore, the solution $f(\eta)$ of the unsteady flow problem of Weidman and Sprague [1] can be obtained from the solution $F(\eta)$ of the classical steady flow problem (7) by simply subtracting from the latter one the constant R/m . Accordingly, all the derivatives of $f(\eta)$ and $F(\eta)$ with respect to η are equal and thus the components (u, w) of the velocity field as well as the dimensionless shear stress $f''(\eta)$ (in particular the skin friction $f''(0)$) are obtained from the solution $F(\eta)$ of problem (7) as

$$u(x, z) = axF'(\eta), \quad w(x, z) = -\sqrt{av}[F(\eta) - R], \quad f''(\eta) = F''(\eta), \quad (9)$$

for the Hiemenz flow and

$$u(r, z) = arF'(\eta), \quad w(r, z) = -2\sqrt{av} \left[F(\eta) - \frac{R}{2} \right], \quad f''(\eta) = F''(\eta), \quad (10)$$

for the Homann flow.

The physical meaning of the above results is that (in the co-moving reference frame) the *unsteady* flow induced by the *impermeable* plate moving into the stagnation flow ($R > 0$) is equivalent to the *steady* stagnation flow occurring over a fixed *permeable* plate when a uniform suction of the fluid with (dimensionless) transversal velocity $R/m > 0$ is applied. Similarly, the *unsteady* case of the uniformly receding ($R < 0$) *impermeable* plate is equivalent to the case of a fixed *permeable* plate in the presence of a uniform injection of the fluid with transversal velocity $R/m < 0$. Bearing in mind that (7) is a well-studied (steady) boundary value problem, all the known result can simply be transcribed to the present unsteady case of [1] with the aid of Eqs. (6), (9), and (10). This circumstance will be illustrated in the sequel by a few examples. For the sake of brevity, only the case of plane flow ($m = 1$) will be considered.

3 Examples

An outstanding feature of the stagnation-point flow problems is that in all these cases Prandtl’s boundary layer approximation becomes exact. The reason is that those terms of the Navier–Stokes equation which usually are omitted in boundary layer theory, vanish identically due to the assumptions about the velocity components of stagnation-point flows. Owing to this fact, the stagnation-point flows have attracted very much attention during the long history of the boundary layer theory. Concerning the boundary value problem (7) with nonvanishing mass transfer, $F(0) \neq 0$, the first representative investigation has been published nearly seventy years ago by Schlichting and Bussmann [2]. In the meantime, the problem (7) has become a textbook issue (see e.g. [3,4]). In this sense, the results of [1] for the large Reynolds number asymptotics, which in the present approach correspond to the massive suction ($R \gg 1$) and massive injection ($R < 0, |R| \gg 1$) of the fluid, can immediately be recovered from the respective equations of Section VI.32 of [4]. Thus, in case of massive suction, Eq. (270) of Section VI.32 of [4] yields

$$F''(0) = f''(0) = R + \frac{2}{R} - \frac{13}{2R^3} + \dots \quad (R \gg 1), \tag{11}$$

which coincides to $O(1/R)$ with Eq. (4.11) of [1]. In the case of massive injection, Eq. (5.55) of [3] gives $F'(\eta) = f'(\eta) = \sin(\eta/|R|)$ which in turn yields $F''(0) = f''(0) = 1/|R|$ in full agreement with Eq. (4.23) of [1]. Moreover, as is well known (see e.g. [3,4]) in the case of massive suction, the boundary layer equations admit a *universal solution*, the *asymptotic suction profile* (see e.g. Eqs. (5.51) and (5.52) of [3]) which in the present case is

$$f'(\eta) = F'(\eta) = 1 - e^{-R\eta} \quad (R \gg 1). \tag{12}$$

This equation can also easily be recovered from Eqs. (4.4) and (4.6) of [1].

In spite of the long history of the problem (7), recently in this respect, several new and exciting results have been reported by King and Cox [5]. Concerning the structure of the solution space, King and Cox [5] have pointed out that in addition to the solution branch with positive skin friction, $F''(0) > 0$, which extends over the whole range $-\infty < R < \infty$ of the transpiration parameter and which has already been found by Schlichting and Bussmann [2], further two solution branches exist. These new solution branches correspond to negative values of the skin friction, $F''(0) < 0$, and do exist only above of a critical value $R_c = 2.0512$ of the suction parameter R (the Reynolds number of [1]). The three solution branches described by King and Cox [5] and illustrated (for different scalings of the involved variables) in their Figs. 1 and 2 have been re-plotted in our Fig. 1, as being recalculated numerically by the present author. Owing to the translation invariance of the boundary layer equations (discussed recently in some detail in [6]), the method of “false origin” worked out initially by Bussmann (and applied in [2] for $F''(0) > 0$) could be adapted to the numerical integration also in the cases $F''(0) < 0$. The two new solution branches p and q discovered by King and Cox [5] match

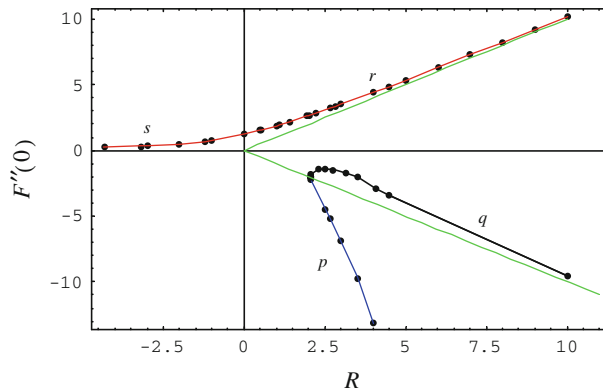


Fig. 1 Structure of the solution space of the boundary value problem (7) according to King and Cox [5]. The branch $s - r$ that includes the Hiemenz solution at $R = 0$ has already been reported by Schlichting and Bussmann [2]. This solution branch is also shown in Fig. 1 of Weidman and Sprague [1]. The new branches p and q discovered by King and Cox [5] match each other at the critical value $R_c = 2.0512$ of the transpiration parameter which represents the lower bound of their domain of existence. The bisectors $\pm R$ are the asymptotes of $F''(0)$ for $R \rightarrow \infty$ (massive suction)

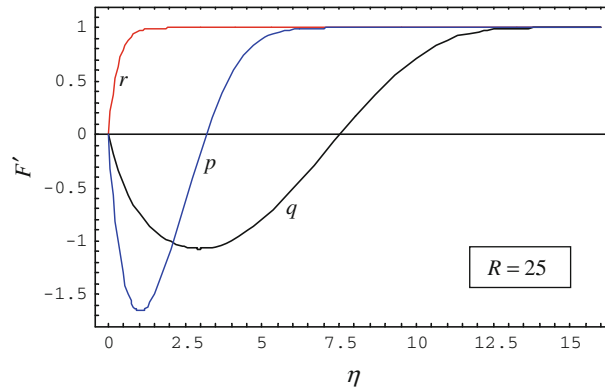


Fig. 2 Three dimensionless velocity profiles $F'(\eta)$ corresponding to the same value $R = 2.5$ of the transpiration parameter R . The corresponding values of the skin friction are $F''(0) = 3.09113$ (r -branch solution), $F''(0) = -4.44823$ (p -branch solution), and $F''(0) = -1.40228$ (q -branch solution). In the latter two cases, backflow domains occur

each other at the critical value $R_c = 2.0512$ of the transpiration parameter which is a saddle-node bifurcation point and represents the lower bound of the existence domain of the corresponding solutions. In this way, in the parameter range $R > R_c$, the boundary value problem (7) admits *triple solutions*. While the similar downstream velocity $f'(\eta) = F'(\eta)$ is everywhere positive for the *Hiemenz-branch* $s - r$ (which has also been found and plotted in Fig. 1 of [1]), in the case of the new solution branches p and q , it always includes also backflow domains, where $F'(\eta) < 0$ holds. This feature is illustrated in our Fig. 2 where the dimensionless velocity profiles $f'(\eta) = F'(\eta)$ of the triple solutions corresponding to $R = 2.5$ have been plotted.

Owing to the relationship (6), also several other results of King and Cox [5] can easily be transcribed for the unsteady problem examined by Weidman and Sprague [1]. In this sense, Eq. (14) of [5] for the skin friction of the solution branch r in the massive suction case (in our notation) becomes

$$F''(0) = f''(0) = R \sum_{n=0}^N \frac{\alpha_n}{R^{2n}} \quad (R \gg 1, \text{ branch } r), \tag{13}$$

where the coefficients α_n , $n = 0, 1, 2, \dots, 11$ are given in Table 1 of [5]. Thus, bearing in mind that $\alpha_0 = 1, \alpha_1 = 2, \alpha_2 = -13/2$, from Eq. (13) one recovers to $O(1/R^2)$ the above relationship (11). In case of the solution branch q , the same quantity is given by Eq. (47) of [5] which (in our notation) reads

$$F''(0) = f''(0) = R \sum_{n=0}^N (-1)^{n+1} \frac{\alpha_n}{R^{2n}} = -R + \frac{2}{R} + \frac{13}{2R^2} - \dots \quad (R \gg 1, \text{ branch } q). \tag{14}$$

This relationship shows clearly that the bisector $F''(0) = -R$ is actually the asymptote of the solution branch q , in agreement with Fig. 1.

The linear stability analysis of King and Cox [5] shows that as expected, all the solutions with negative skin friction associated with the branches p and q are unstable, while the solutions with positive skin friction corresponding to the Hiemenz-branch $s - p$ are linearly stable, in full agreement with the findings of Weidman and Sprague [1] (although in the case of strong injection, i.e. for large negative values of R , the calculation of the eigenvalues becomes increasingly difficult).

4 Conclusions

The *unsteady* flows generated by an impermeable infinite flat plate advancing with constant velocity V toward, or receding from an orthogonal (plane or axisymmetric) stagnation-point flow, are undistinguishable (in the co-moving Galilean reference frame of the plate) from their well-known *steady* counterparts over fixed surfaces when these are *permeable* and a uniform lateral suction or injection of the fluid proportional to V is applied. Owing to this equivalence, all the results of the latter well-investigated problems can simply be transcribed for the unsteady case of the moving impermeable plate.

References

1. Weidman, P.D., Sprague, M.A.: Flows induced by a plate moving normal to stagnation-point flow. *Acta Mech.* doi:[10.1007/s00707-011-0458-2](https://doi.org/10.1007/s00707-011-0458-2), Online 18 February 2011
2. Schlichting, H., Bussmann, K.: Exakte Lösungen für laminare Grenzschicht mit Absaugung und Ausblasen. *Schriften d. dt. Akad. D. Luftfahrtforschung* **7**(2), 25–69 (1943)
3. Schlichting, H., Gersten, K.: *Boundary-Layer Theory*. Springer, Berlin (2000)
4. Rosenhead, L. (ed.): *Laminar Boundary Layers*. Clarendon Press, Oxford (1963)
5. King, J.R., Cox, S.M.: Self-similar “stagnation point” boundary layer flows with suction or injection. *Stud. Appl. Math.* **115**, 73–107 (2005)
6. Magyari, E.: Translation groups of the boundary layer flows induced by continuous moving surfaces. *J. Fluid Mech.* **655**, 327–343 (2010)