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The variety of quasi-Stone algebras does not have the amalgamation property

Sara-Kaja Fischer

Abstract. We give an example showing that the variety of quasi-Stone algebras does not have the amalgamation property.

The question whether the variety **QSA** of quasi-Stone algebras has the amalgamation property is posed as an open problem in [2]. In this note, we show that the answer is negative by providing a counterexample. In particular, this also provides a counterexample to the claim made in [1] that the class of all finite quasi-Stone algebras has the amalgamation property.

An algebra $(L; \wedge, \vee, {}', 0, 1)$ of type $(2, 2, 1, 0, 0)$ is a *quasi-Stone algebra* (in the following: a QSA) if $(L; \wedge, \vee, 0, 1)$ is a bounded distributive lattice and the unary operation $'$ satisfies the following conditions for all $a, b \in L$:

(QS1) $0' = 1$ and $1' = 0$,
(QS2) $(a \vee b)' = a' \wedge b'$ (the $\vee$-DeMorgan law),
(QS3) $(a \wedge b')' = a' \vee b''$ (the weak $\wedge$-DeMorgan law),
(QS4) $a \wedge a'' = a$,
(QS5) $a' \vee a'' = 1$ (the Stone identity).

We write QSA's as pairs $(L, {}')$ where L stands for the underlying bounded distributive lattice. QSA-homomorphisms are defined in the obvious way.

In [1], it is shown that the category of all QS-spaces together with QS-maps is dually equivalent with the category **QSA**. Here, a *QS-space* is a pair $(X, \mathcal{E})$ consisting of a Priestley space X and an equivalence relation $\mathcal{E}$ on X satisfying certain conditions.

For a given QSA $(L, {}')$, its QS-space is constructed as follows: Let $X = D(L)$ be the (standard) Priestley space of all prime filters of L and set

$$\mathcal{E} = \{(P, Q) \in D(L) \times D(L) \mid P \cap B(L) = Q \cap B(L)\},$$

where $B(L) = \{a' \mid a \in L\}$ is the skeleton of L. Then $(X, \mathcal{E})$ is the dual QS-space of $(L, {}')$. We write $[x]_\mathcal{E}$ for the $\mathcal{E}$-class of $x \in X$, and $\mathcal{E}(U) = \bigcup_{x \in U} [x]_\mathcal{E}$ for any subset $U \subseteq X$.

QS-maps are defined as follows: Let $(X, \mathcal{E})$ and $(Y, \mathcal{F})$ be QS-spaces. Then a continuous, order preserving map $\varphi\colon X \to Y$ is a *QS-map* if

$$\mathcal{E}(\varphi^{-1}(U)) = \varphi^{-1}(\mathcal{F}(U))$$

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for each clopen increasing set $U \subseteq Y$. For a given QSA-homomorphism $f\colon L \to K$, its dual QS-map is given by $D(f)\colon D(K) \to D(L)$ with $D(f)(P) = f^{-1}(P)$ for $P \in D(K)$. Crucially for our purposes, QSA-embeddings correspond bijectively to onto QS-maps.

Now we can present our example of a tuple (L, M, N, i, j), where L, M, N are QSA's and $i\colon L \to M$, $j\colon L \to N$ are QSA-embeddings, which can not be amalgamated within **QSA**. The failure of amalgamation will be shown by means of the duality described above.

Let L be the three-element Stone algebra ($0' = 1$ and $a' = 1' = 0$), M the four-element Boolean lattice, and N a six-element Stone algebra as defined in Figure 1:

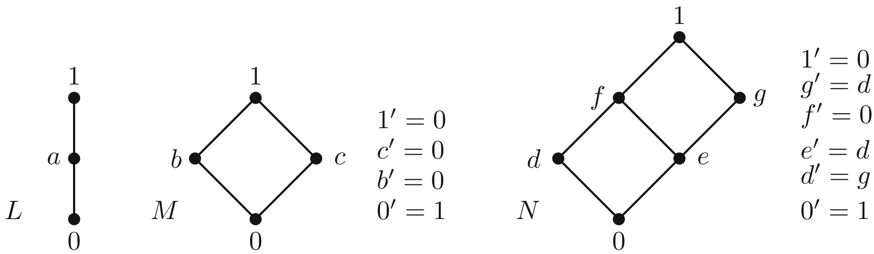


Figure 1

We choose QSA-embeddings $i\colon L \hookrightarrow M$ such that $i(0) = 0$, $i(a) = b$, $i(1) = 1$, and $j\colon L \hookrightarrow N$ such that $j(0) = 0$, $j(a) = f$, $j(1) = 1$. The corresponding QS-spaces are given in Figure 2 (putting $X = D(L)$, $Y = D(M)$, and $Z = D(N)$):

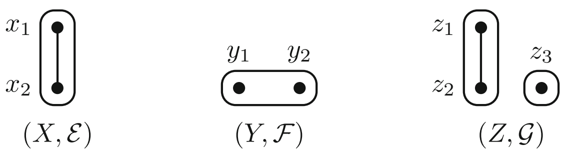
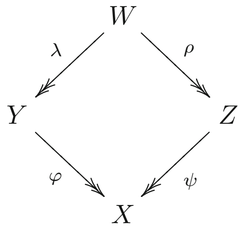


Figure 2

With it we have the onto QS-maps $\varphi = D(i)\colon Y \twoheadrightarrow X$ with $\varphi(y_1) = x_1$, $\varphi(y_2) = x_2$, and $\psi = D(j)\colon Z \twoheadrightarrow X$ with $\psi(z_1) = \psi(z_3) = x_1$, $\psi(z_2) = x_2$.

Assume that there is a QSA K and embeddings $h\colon M \hookrightarrow K$, $k\colon N \hookrightarrow K$ amalgamating (L, M, N, i, j), i.e., such that $h \circ i = k \circ j$. Let $(W, \mathcal{H})$ be the dual space of K with $W = D(K)$, and let $\lambda = D(h)$, $\rho = D(k)$ be the duals of h and k, respectively. Then, by duality, the following diagram commutes:



Since ρ is onto, there is some $w_1 \in W$ such that $\rho(w_1) = z_3$. But then $\rho(w) = z_3$ for all $w \in [w_1]_{\mathcal{H}}$ because $\{z_3\}$ is a clopen increasing set, and therefore $[w_1]_{\mathcal{H}} \subseteq \mathcal{H}(\rho^{-1}(\{z_3\})) = \rho^{-1}(\mathcal{G}(\{z_3\})) = \rho^{-1}(\{z_3\})$. Thus, for all $w \in [w_1]_{\mathcal{H}}$, we have $\psi \circ \rho(w) = x_1$, and by the commutativity of the diagram, it follows that also $\varphi \circ \lambda(w) = x_1$ for all $w \in [w_1]_{\mathcal{H}}$. This implies that $[w_1]_{\mathcal{H}} \subseteq \lambda^{-1}(\{y_1\}) \subseteq \lambda^{-1}(\mathcal{F}(\{y_2\}))$ and that $[w_1]_{\mathcal{H}} \cap \mathcal{H}(\lambda^{-1}(\{y_2\})) = \emptyset$. Hence, $\lambda^{-1}(\mathcal{F}(\{y_2\})) \neq \mathcal{H}(\lambda^{-1}(\{y_2\}))$ which is a contradiction, since $\{y_2\}$ is a clopen increasing set.

A different counterexample has been obtained independently by S. Solovjov (private communication).

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Sara-Kaja Fischer
Mathematical Institute, University of Bern, CH-3012 Bern, Switzerland
e-mail: sara.fischer@math.unibe.ch