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Dualities and Dual Pairs in Heyting Algebras

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Abstract We extract the abstract core of finite homomorphism dualities using the techniques of Heyting algebras and (combinatorial) categories.

Keywords Homomorphisms · Structural theorems in combinatorics · Good characterization · Finite duality

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1 Introduction

Finite dualities appeared in [32] in the categorical context of dual characterizations of various classes of structures. It is a simple idea: characterize a given class both

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by forbidden substructures (associated with subobjects) and by decompositions (associated with factorobjects); this proved to be surprisingly fruitful. In retrospect, it was also a timely concept as it coincided with the introduction (in the logical and artificial intelligence contexts) of the paradigm of Constraint Satisfaction [24, 26].

Only later it was realized (in the context of complexity theory) that these notions are two aspects of the same general problem, the study of homomorphisms of relational structures [12].

Finite dualities represent an extremal case of the above mentioned Constraint Satisfaction Problem. Provided we have a finite duality, the problem in question is polynomially decidable. Furthermore, in a broad context such problems coincide with the decision problem for classes of structures that are first-order decidable [3, 21, 38]. For general relational structures, finite dualities were characterized in [35] and a number of interesting particular cases were investigated as well [14, 18, 29, 36].

Here, following [33] we return to the original motivation and discuss finite dualities in the categorical context. We aim at pointing out those categories in which one can describe finite dualities using the interplay of general categorical and order theoretical concepts and techniques.

2 Background: Dualities in Graphs and Similar Categories

2.1 The categories we will work with are *finitely concrete*, that is, the objects are finite sets endowed with structures, and morphisms are maps respecting the structures in a specified way.

Typically we have in mind categories such as that of (finite) symmetric graphs, or oriented graphs, with edge preserving homomorphisms (or more general relations resp. relational systems), with relation preserving maps. Some of the results can be applied for other choices of morphisms (strong or full homomorphisms).

2.2 We will assume that our categories admit finite sums (coproducts)

$$\iota_j : A_j \rightarrow A = \coprod_{i=1}^n A_i$$

(characterized by the property that for each system $f_j : A_j \rightarrow B$, $j = 1, \dots, n$ there is exactly one morphism $A \rightarrow B$ such that $f \iota_j = f_j$ for all j).

This is the minimal assumption; for more involved facts we will assume also the existence of finite products

$$\pi_j : A = \prod_{i=1}^n A_i \rightarrow A_j$$

(characterized by the property that for each system $f_j : B \rightarrow A_j$ there is exactly one morphism $B \rightarrow A$ such that $\pi_j f = f_j$ for all j), and also the Heyting property (see Section 2.6 below).

2.3 The *Constraint Satisfaction Problem* (briefly, *CSP*) in a category \mathcal{C} is the membership problem of the class

$$\mathbf{CSP}(\mathcal{B}) = \{X \in \mathcal{C} \mid X \rightarrow B \text{ for some } B \in \mathcal{B}\}$$

where $X \rightarrow Y$ stands for “there exists a morphism $f : X \rightarrow Y$ ” and \mathcal{B} is a class of objects. Here we are concerned with the situation in which this class can be represented as

$$\mathbf{Forb}(\mathcal{A}) = \{X \in \mathcal{C} \mid A \not\rightarrow X \text{ for all } A \in \mathcal{A}\}$$

where $X \not\rightarrow Y$ stands for “there exists no morphism $f : X \rightarrow Y$ ” and \mathcal{A} is a finite class of objects (with an infinite \mathcal{A} this is always possible). In fact the classes \mathcal{B} we are interested in are also finite. Thus, we investigate the situations of finite systems A_1, \dots, A_n and B_1, \dots, B_m of objects such that

$$\forall i, A_i \not\rightarrow X \text{ iff } \exists j, X \rightarrow B_j \tag{2.0.1}$$

and in this case we speak of a *finite duality*.

Note Instead of forbidding morphisms from the objects A_i , one is sometimes interested in forbidding subobjects from a finite family of isomorphism classes. If we have (2.0.1), it is easy to replace the A_i ’s by finitely many other objects providing such a “subobject forbidding characterization”. Similarly, the $X \rightarrow B$ type requirements can typically be replaced by requirements of epimorphisms.

Note The name *Constraint Satisfaction Problem* originates in the computational setting, where the description involves variables and constraints (the elements and structures of X) and a domain with relations (the structures in \mathcal{B}).

2.4 The Poset $\tilde{\mathcal{C}}$

Given a category \mathcal{C} , consider the set of objects ordered by

$$A \leq B \quad \equiv_{\text{df}} \quad \exists f : A \rightarrow B$$

and denote the obtained (pre-)ordered set by

$$\tilde{\mathcal{C}}.$$

In fact, we usually think of $\tilde{\mathcal{C}}$ as the poset of the obvious equivalence classes.

Note that if we assume the existence of sums as we did in Section 2.2 above, $\tilde{\mathcal{C}}$ is a join-semilattice. If we have, moreover, also the products, $\tilde{\mathcal{C}}$ is a lattice.

2.5 Heyting Categories

A *Heyting algebra* is a bounded lattice with an extra operation \Rightarrow satisfying

$$a \wedge b \leq c \quad \text{iff} \quad a \leq (b \Rightarrow c).$$

A *Heyting category* is a category \mathcal{C} such that $\tilde{\mathcal{C}}$ is a Heyting algebra.

Note Trivially, any Cartesian closed category (that is, a category with exponentiation $\langle X, Y \rangle$ such that the sets of morphisms

$$A \times B \rightarrow C \quad \text{and} \quad A \rightarrow \langle B, C \rangle$$

are naturally equivalent—see, e.g. [23]) is Heyting. However, Cartesian closedness is not in general necessary, because in a Heyting category the requirement is weaker: we just require that there exist a morphism $A \times B \rightarrow C$ iff there exists a morphism $A \rightarrow \langle B \Rightarrow C \rangle$ and drop the requirement of natural equivalence. An example of a Heyting category which is not Cartesian closed is the category of loopless graphs (with an additional terminal object). For this and more examples, see [33].

2.6 Cores

In our categories the objects can be canonically reduced to make the relation $A \rightarrow B$ antisymmetric (up to isomorphism).

Lemma 2.6.1 *Let X be a finite object in a concrete category. Then each bijective morphism $\phi : X \rightarrow X$ is an isomorphism.*

Proof There is a $k \neq 0$ and n such that $\phi^{n+k} = \phi^n$. Thus, $\phi^k = \text{id}$ and $\phi \cdot \phi^{k-1} = \phi^{k-1} \cdot \phi = \text{id}$. □

Proposition 2.6.2 *Let X be a finite object in a concrete category. Then the smallest subobject $Y \subseteq X$ such that there is a morphism $f : X \rightarrow Y$*

- (1) *is a retract of X , and*
- (2) *is uniquely determined, up to isomorphism.*

Proof

- (1) Let $j : Y \rightarrow X$ be the embedding morphism. Then, by minimality, $\phi = fj : Y \rightarrow Y$ is bijective and by Lemma 2.6.1 it is an isomorphism, and we have the retraction $r = \phi^{-1} f$.
- (2) Now if $j' : Z \rightarrow X$ is an embedding of another subobject with the property, we have mutually inverse isomorphisms $r' j$ and $r j'$. □

The object Y from Proposition 2.6.2 is called the *core* of X ; denote it by cX . An object X is called a *core* if it is the core of some object. Note that

- a core is the core of itself, and
- in a concrete category \mathcal{C} with finite objects, A and B are equivalent in $\tilde{\mathcal{C}}$ (that is $A \leq B$ and $B \leq A$) iff cA and cB are isomorphic.

Thus, if we restrict ourselves to cores and representatives of isomorphism classes,

- the pre-ordered set $\tilde{\mathcal{C}}$ becomes actually a poset.

Furthermore, any duality

$$\forall i, A_i \dashrightarrow X \quad \text{iff} \quad \exists j, X \rightarrow B_j$$

can be replaced by the duality in the cores

$$\forall i, cA_i \dashv X \text{ iff } \exists j, X \rightarrow cB_j$$

[16, 17].

3 Transversals and Weak Right Duals

3.1 In a poset (X, \leq) we will use the standard notation, for a subset $M \subseteq X$,

$$\downarrow M = \{x \in X \mid \exists m \in M, x \leq m\},$$

$$\uparrow M = \{x \in X \mid \exists m \in M, x \geq m\},$$

and for an element $m \in X$, let $\downarrow m = \downarrow\{m\}$ and $\uparrow m = \uparrow\{m\}$.

3.2 Connected Elements

A element a of a lattice L is *connected* if

$$a \leq b \vee c \implies a \leq b \text{ or } a \leq c. \tag{3.2.1}$$

Note Another (and perhaps more frequently used) term is *join-prime* or \vee -*prime*. We use “connected” because of the interpretation in the posets \tilde{C} (recall Section 2.4) we are primarily interested in. Note that a core graph satisfies (3.2.1) (in any choice of morphisms at least as demanding as the standard graph homomorphism) iff it is connected in the usual sense, i.e., there is a path between any two of its vertices.

The set of all connected elements of a semilattice L will be denoted by

$$\text{Cn}L \text{ or simply by } \text{Cn}.$$

3.3 Connected Decompositions

The upper semilattices we will be working with will possess *finite connected decompositions*, that is

$$\text{for each } a \in L, \text{ there is finite } F \subseteq \text{Cn}L \text{ such that } a = \bigvee F. \tag{3.3.1}$$

Note In a semilattice L satisfying (3.3.1), the set $\downarrow a \cap \text{Cn}L$ has only finitely many maximal elements for every $a \in L$, and

$$a = \bigvee \text{Max}(\downarrow a \cap \text{Cn}).$$

The latter is then the only irredundant connected decomposition of a .

A *connected component* of an element $a \in L$ is a $c \in \text{Cn}L$ such that $a = c$ or there exists a decomposition $a = b \vee c \neq b$.

For a subset $A \subseteq L$ write

$$A_{\text{Cn}}$$

for the set of all connected components of the elements from A . Clearly, a connected element has exactly one connected component: itself. Moreover, in a lattice with finite connected decompositions, A_{Cn} is finite for every finite subset $A \subseteq L$.

Note In a (semi)lattice satisfying (3.3.1), we have

$$\{a\}_{\text{Cn}} = \text{Max}(\downarrow a \cap \text{Cn}).$$

3.4 Finite Dualities

A *duality pair* (l, r) in L is a pair of elements such that

$$\downarrow r = L \setminus \uparrow l$$

(that is,

$$l \not\leq x \text{ iff } x \leq r.)$$

The element r (obviously uniquely determined by l) is then called the *right dual* of l , and similarly l is called the *left dual* of r . If the other element of the pair is not specified we speak of a *right* or a *left dual*.

Note The elements l that are left duals are always connected. In fact, whenever $l \leq \bigvee x_i$ for any join $\bigvee x_i$, then necessarily $l \leq x_j$ for some j (indeed, if for all $j, l \not\leq x_j$ then for all $j, x_j \leq r$, and hence $\bigvee x_i \leq r$ and $l \not\leq \bigvee x_i$). This property is usually called *supercompactness*, for obvious reasons.

A *finite duality* in L is a pair (A, B) of finite subsets of L such that

- (1) distinct elements in A resp. B are incomparable, and
- (2) $x \in \uparrow A$ iff $x \notin \downarrow B$ (in other words, $\downarrow B = L \setminus \uparrow A$).

(Compare with Section 2.3.) An element l (resp. r) is a *weak left dual* (resp. *weak right dual*) if there is a finite duality (A, B) such that $l \in A$ (resp. $r \in B$).

Fact 3.4.1 *The set B in a finite duality (A, B) is uniquely determined by A , and vice versa.*

Proof If $(A, B_1), (A, B_2)$ are finite dualities then $\downarrow B_1 = \downarrow B_2$, and hence for each $x \in B_1$ there is an $\alpha(x) \in B_2$ such that $x \leq \alpha(x)$, and similarly for each $x \in B_2$ there is a $\beta(x) \in B_1$ such that $x \leq \beta(x)$. Thus, $x \leq \beta\alpha(x)$ and $x \leq \alpha\beta(x)$ and by incomparability $\alpha\beta = \text{id}$ and $\beta\alpha = \text{id}$, and finally $x \leq \alpha(x) \leq x$ and $\alpha(x) = x$ and similarly $\beta(x) = x$. \square

3.5 Transversals

For antichains $M, N \subseteq L$, we will write

$$M \preceq N \text{ for } N \subseteq \uparrow M. \tag{3.5.1}$$

One sometimes speaks of M as of a *refinement* of N ; thus if $M \preceq N$, then N is *coarser*.

A subset $M \subseteq A_{C_n}$ is said to be a *transversal* of $A \subseteq L$ if

- (T1) distinct elements of M are incomparable,
- (T2) $A \subseteq \uparrow M$, and
- (T3) in the refinement order \preceq , M is maximal with respect to the properties (T1) and (T2).

A subset satisfying only (T1) and (T2) is called a *quasitransversal* of A .

3.6 Let (A, B) be a finite duality. If M is a quasitransversal of A , then by (T2)

$$\uparrow A \subseteq \uparrow M \quad \text{and hence} \quad L \setminus \uparrow M \subseteq \downarrow B.$$

For a quasitransversal M of A , set

$$\overline{M} = A_{C_n} \setminus \uparrow M.$$

Note that if M, N are quasitransversals such that $M \preceq N$, then $\overline{M} \subseteq \overline{N}$.

In the following, if we speak about M being a (*quasi*)*transversal* for (A, B) , we mean that (A, B) is a finite duality and M is a (*quasi*)*transversal* for A .

Lemma 3.6.1 *Let $\overline{M} = \emptyset$ for a transversal M of (A, B) . Then*

1. $A = M = A_{C_n}$, and
2. B has only one element.

Proof

1. Suppose first that $A \neq A_{C_n}$. Let c be a maximal element of the set $\{c \in A_{C_n} \mid c \text{ is a component of some disconnected } a \in A\}$. Then $c \notin A$ because A is an antichain. As $\overline{M} = \emptyset$ and by the choice of c , the set $M \setminus \{c\}$ is a quasitransversal, and moreover $M \preceq M \setminus \{c\}$; thus $M = M \setminus \{c\}$ and so $c \notin M$. Because $c \notin \overline{M} = \emptyset$, there is some $c' \in M$ such that $c' < c$. Again, since A is an antichain, $c' \notin A$. Let $A' = \{a \in A \mid \downarrow a \cap M = \{c'\}\}$; observe that $c' \notin A'_{C_n}$ because $c' \notin A$. Let $M' = \text{Min}((M \setminus \{c'\}) \cup A'_{C_n})$. Then M' is a quasitransversal and $M < M'$, a contradiction. Thus $A = A_{C_n}$.
Hence A_{C_n} is an antichain, so it is a quasitransversal. Since $A_{C_n} \setminus \uparrow M = \emptyset$ we have $A_{C_n} \subseteq \uparrow M$, and hence $M \preceq A_{C_n}$. By (T3), $A_{C_n} = M$.
2. Let $r_1, r_2 \in B$ be distinct. Then, by the incomparability condition, $r_1 \vee r_2 \not\leq r$ for all $r \in B$ and hence $l \leq r_1 \vee r_2$ for some $l \in A$. Now by 1., l is connected and hence $l \leq r_1$ or $l \leq r_2$, a contradiction. □

Lemma 3.6.2 *Let M be a transversal of (M, B) . Then*

1. $\overline{M} = \emptyset$, and
2. there is no other transversal.

Proof

1. By definition, all elements of a transversal are connected, so $M = M_{C_n}$ and $\overline{M} = M_{C_n} \setminus \uparrow M_{C_n} = \emptyset$.
2. If N is another transversal, then $N \subseteq M_{C_n} = M \subseteq \uparrow M$, and so $M \preceq N$. Thus, by (T3), $N = M$. □

Lemma 3.6.3 *Let M be a transversal of (A, B) . Then there is precisely one $r \in B$ such that*

- (1) $M \cap \downarrow r = \emptyset$, and
- (2) $\overline{M} \subseteq \downarrow r$.

Proof

- I. If $\overline{M} = \emptyset$ for a transversal M , we have $B = \{r\}$ by Lemma 3.6.1, and this r satisfies the conditions.
- II. Now suppose that A is not its own transversal. Then $\overline{M} \neq \emptyset$. Set $s = \bigvee \overline{M}$. We have $s \notin \uparrow M$ (if $x \in M$ and $x \leq s$ then by connectedness $x \leq y \in \overline{M}$), hence $s \notin \uparrow A$, and consequently $s \in \downarrow B$ and we have an $r \in B$ such that $s \leq r$ so that $\overline{M} \subseteq \downarrow s \subseteq \downarrow r$.
Now suppose $c \in M \cap \downarrow r$. By (T3), the set $(M \setminus \{c\})$ is not a quasitransversal, and thus $A \not\subseteq \uparrow (M \setminus \{c\})$. Choose some $l \in A \setminus \uparrow (M \setminus \{c\})$ and let $l = \bigvee l_i$ be a connected decomposition of l . If $l_i \geq b \in M$, then $b = c$ and hence $l_i = c$. Hence for every i , either $l_i = c$ or $l_i \in \overline{M}$, and therefore $l \leq r$, contradicting the duality.
- III. Finally let distinct r_1, r_2 have the property. Then $r_1 \vee r_2 \notin \downarrow B$, hence $r_1 \vee r_2 \in \uparrow M$ and there is a (connected) $x \in M$ such that $x \leq r_1 \vee r_2$; thus, $x \leq r_1$ or $x \leq r_2$, contradicting (1). □

3.7 The uniquely determined r from Lemma 3.6.3 will be denoted by

$$r(M).$$

Note that if A is not its own transversal, then $r(M)$ is determined by the formula

$$\bigvee \overline{M} \leq r(M) \in B. \tag{3.5.2}$$

Lemma 3.7.1 *If M_1, M_2 are distinct transversals, then $\overline{M}_1 \cap M_2 \neq \emptyset$.*

Proof If $M_1 \neq M_2$, then $M_2 \not\preceq M_1$ and hence there is a $c \in M_2 \setminus \uparrow M_1$. Then $c \in (A_{C_n} \setminus \uparrow M_1) \cap M_2$. □

Lemma 3.7.2 *For $r \in B$ set $M = \text{Min}\{x \in A_{C_n} \mid x \not\leq r\}$. Then M is a quasitransversal, and if M_r is a transversal with $M \preceq M_r$, then $r(M_r) = r$.*

Proof Let $l \in A$. Then $l = \bigvee \{x \in A_{C_n} \mid x \leq l\} \not\leq r$ and hence there is an $x \in A_{C_n}$ such that $x \leq l$ and $x \not\leq r$. Thus M is a quasitransversal.

We have $\overline{M}_r = \overline{M}$. If $\overline{M} = \emptyset$, then Lemma 3.6.1 applies. Suppose that $\overline{M} \neq \emptyset$ and $\bigvee \overline{M} \not\leq r$. Then there is an $x \in \overline{M}$ such that $x \not\leq r$, that is, $x \in M$, a contradiction. □

Proposition 3.8 *Let (A, B) be a finite duality in a semilattice with finite connected decompositions. The formulas*

$$M \mapsto r(M) \quad \text{where} \quad \bigvee \overline{M} \leq r(M) \in B \quad (\text{if } \overline{M} \neq \emptyset),$$

$$r \mapsto M_r \quad \text{where} \quad M_r \supseteq M = \{x \in A_{\text{cn}} \mid x \not\leq r\}$$

constitute a one-to-one correspondence between the transversals of (A, B) and elements of B .

Proof We already know that $r(M_r) = r$. Let M_1, M_2 be distinct transversals. By Lemma 3.7.1, there is a $c \in \overline{M_1} \cap M_2$. By Lemma 3.6.3(1), $c \not\leq \downarrow r(M_1)$ and by (3.5.2), $c \leq \downarrow r(M_2)$. Hence $r(M_1) \neq r(M_2)$. □

Proposition 3.9 *Let L be a semilattice with finite connected decompositions and let (A, B) be a finite duality. Then each transversal M of A together with the element $r(M)$ defined in Section 3.7 constitutes a finite duality $(M, \{r(M)\})$.*

Proof Set $r = r(M)$. We have $M \subseteq L \setminus \downarrow r(M)$ and hence $\uparrow M \subseteq L \setminus \downarrow r(M)$ by Lemma 3.6.3.

Now let $x \notin \uparrow M = \bigcup \{\uparrow c \mid c \in M\}$. We want to prove that $x \in \downarrow r(M)$. Let $y = x \vee \bigvee \overline{M}$. We have $c \not\leq x$ for all $c \in M$, and by connectedness $c \not\leq y$ for all $c \in M$. Suppose that $l \leq y$ for some $l \in A$. If $c \in M$ and $c \leq l$ we have $c \leq y$ and hence $c \leq x$, a contradiction. Thus, $y \notin \uparrow A$ and hence $y \leq \downarrow B$, that is, $y \leq r'$ for some $r' \in B$. But then $\bigvee \overline{M} \leq r'$ and hence $r' = r(M)$. Therefore $x \in \downarrow r(M)$. □

4 Connected Components of Weak Left Duals are Left Duals

In Section 3.7 we have seen that given a finite duality (A, B) , each element $r \in B$ is in a duality $(M, \{r\})$. In this section we will obtain dualities in the reversed order. Instead of dualities for elements $l \in A$ we will have them for their connected components $c \in A_{\text{cn}}$. For these, however, we will prove something stronger. Namely, we will show that each such element is a left dual.

Unlike the previous section we will have to assume that the lattice L is Heyting (and hence the categorical interpretation holds for Heyting categories only).

4.1 Gaps

A pair of elements (a, b) of a poset L is a *gap* if $a < b$ and $a \leq c \leq b$ implies that $a = c$ or $b = c$, for every $c \in L$.

4.2 We will need two facts from [33].

Proposition 4.2.1 ([33], 2.6) *The gaps in a Heyting algebra L with connected decompositions are exactly the pairs (a, b) such that for some duality (l, r) ,*

$$l \wedge r \leq a \leq r \quad \text{and} \quad b = a \vee l.$$

Proposition 4.2.2 ([33], 3.3) *Let L be a Heyting algebra with connected decompositions, let $A = \{l_i \mid i \in J\}$ be a subset of L and let $r \in L$. Let either J be finite or L admit infima of sets of the size of J . Then the pair $(A, \{r\})$ is a duality if and only if there are dualities (l_i, r_i) , $i \in J$, such that*

$$r = \bigwedge_{i \in J} r_i.$$

Lemma 4.3 *In a Heyting algebra with finite connected decompositions, every element of a transversal of a finite duality (A, B) is a left dual.*

Proof Let M be a transversal. We have the duality $(M, r(M))$, by Proposition 3.9. Thus, by Proposition 4.2.2 there is a duality (m, r_m) for each $m \in M$. □

Proposition 4.4 *In a Heyting algebra with finite connected decompositions, a connected component of a weak left dual is a left dual.*

Proof Let (A, B) be a duality and let $c \in A_{\text{Cn}}$. Suppose it is not a left dual; then in particular, by Lemma 4.3, it is contained in no transversal.

Set

$$a = \bigvee \{c' \in A_{\text{Cn}} \mid c' < c\} \vee \bigvee \{c \wedge c' \mid c' \in A_{\text{Cn}}, c, c' \text{ incomparable}\}.$$

We have $a < c$ since else by the connectedness of c some of the summands would be equal to c , which they are not. Now the couple (a, c) is not a gap: else we would have, by Proposition 4.2.1 a duality (l, r) such that $c = a \vee l$, and hence $c = l$.

Thus there exists x such that

$$a < x < c.$$

Claim *If $c' \in A_{\text{Cn}}$ and $c' \neq c$ then*

$$\begin{aligned} c' \leq x & \text{ iff } c' < c, \quad \text{and} \\ c' \geq x & \text{ iff } c' > c. \end{aligned}$$

Proof of Claim In the first case: if $c' \leq x$, then $c' \leq x < c$; and if $c' < c$, then $c' \leq a < x$.

Now consider the second case. Trivially if $c < c'$, then $x \leq c'$. Now suppose $x \leq c'$. Then if $c' < c$, we have $x = c'$ by the first equivalence, hence $x = c' \leq a$, a contradiction. If c and c' are incomparable, then $x \leq c \wedge c' \leq a$, a contradiction again. Thus, $c < c'$ is the only alternative left. □

Proof continued Let $l \in A$ be such that c is one of its connected components and let $l = b \vee c \neq b$ be a decomposition witnessing the fact. Set $q = b \vee x$. We cannot have $l \leq q$ since else $c \leq b$ and $b \leq l \leq b$ contradicting the choice of the decomposition. Consequently, we also have $l' \not\leq q$ for any other $l' \in A$ since otherwise $l' \leq l$. Thus, $\forall l \in A, l \not\leq q$ and hence

$$\exists r \in B, \quad q \leq r.$$

Let M be the transversal such that $r = r(M)$, so that in particular

$$\forall m \in M, \quad m \not\leq q.$$

We have $c \notin M$ since c is in no transversal, and hence $m \not\leq x$ for all $m \in M$. By Claim, $m \not\leq c$ for all $m \in M$, and hence $c \leq r$.

Now, since $q \leq r$, we have $l = q \vee c \leq r$ contradicting the duality (A, B) . □

Corollary 4.4.1 *If a Heyting algebra with finite connected decompositions has no non-trivial duality pair, then it admits no nontrivial finite duality.*

4.5 Note

Compare the following two facts (the first obtained by combining Propositions 3.8 and 4.2.2, the second is an immediate consequence of Proposition 4.4) holding in Heyting algebras with connected decompositions:

- each weak right dual is a meet of right duals, and
- each weak left dual is a join of left duals.

(The facts from which these statements follow are, of course, stronger.)

5 The Transversal Construction Reversed: From Dual Pairs to Finite Dualities

In previous sections the notion of a transversal helped to analyze finite dualities (A, B) in terms of the individual elements of A and B . The elements $r \in B$ have been shown to be naturally associated with transversals of (A, B) (in Proposition 3.8), and then the elements of A have been shown to be joins of left duals (see Proposition 4.4). In this section we will use the procedure reversely: namely, for a finite set A of sums of left duals we will construct a finite duality.

Observation 5.1 In any lattice, if (l_i, r_i) , $i = 1, \dots, n$, are dual pairs, then $(\{l_1, \dots, l_n\}, \{\bigwedge_{i=1}^n r_i\})$ is a duality.

(Indeed, $\forall i, l_i \not\leq x$ iff $\forall i, x \leq r_i$ iff $x \leq \bigwedge r_i$.)

Lemma 5.2 *In a distributive lattice L , let c be a connected component of an $a \in L$ and let $a = \bigvee_{i=1}^n a_i$. Then c is a connected component of some a_i .*

Proof Let $a = x \vee c \neq x$. By the connectedness, $c \leq a_i$ for some i . Then $a_i = (x \vee c) \wedge a_i = (x \wedge a_i) \vee c \neq x \wedge a_i$ since otherwise $c \leq x$ and $a = x$. □

Proposition 5.3 *Let L be a Heyting algebra with finite connected decompositions. Let A be a finite set such that each $a \in A$ is a finite join of left duals. Then there exists a finite duality (A, B) .*

Proof Let $a = \bigvee_{i=1}^{n_a} c_i(a)$ be connected decompositions of the $a \in A$. Then

$$A_{Cn} \subseteq \{c_i(a) \mid a \in A, i = 1, \dots, n_a\}.$$

Now if $a = \bigvee_{j=1}^k a_j$ with a_j left duals, then each connected component of a is, by Lemma 5.2, a connected component of some of the a_j , and hence each $c_i(a) \in A_{Cn}$ is, by Proposition 4.4, a left dual. Denote by $r_i(a)$ the corresponding right dual.

Let \mathcal{M} be the set of transversals of A , hence $M \subseteq A_{Cn}$. Trivially, it is finite. For $M \in \mathcal{M}$ set

$$r_M = \bigwedge \{r_i(q) \mid c_i(q) \in M\}$$

and consider

$$B = \{r_M \mid M \in \mathcal{M}\}.$$

Let $x \leq r_M$ for some $M \in \mathcal{M}$. Then for all $c_i(q) \in M$, we have $x \leq r_i(q)$ and hence $c_i(q) \not\leq x$. For an arbitrary $a \in A$ there is a $c_i(q) \leq a$ and hence $a \not\leq x$.

On the other hand, let $a \not\leq x$ for all $a \in A$. Thus, for each $a \in A$ we have a connected component $x_{i_a}(a)$ such that $x_{i_a}(a) \not\leq x$. Set

$$M' = \{x_{i_a}(a) \mid a \in A\}$$

and consider M'' the system of all minimal elements of M' (to satisfy (T1)). Now M'' is a quasitransversal and we have a transversal $M \preceq M''$. Then for each $c_i(q) \in M$, we have $c_i(q) \not\leq x$, hence $x \leq r_i(q)$, and finally $x \leq r$. □

Note The elements r_M corresponding to the transversals M are exactly the minimal elements of $B' = \{\bigwedge_{a \in A} r_{i_a}(a) \mid 1 \leq i_a \leq n_a\}$, and so $B = \text{Min } B'$.

From Propositions 4.4 and 5.3 we now immediately obtain.

Corollary 5.3.1 *Let L be a Heyting algebra with finite connected decompositions. Then the following statements on an element $a \in L$ are equivalent:*

- (1) a is a weak left dual,
- (2) a is a finite join of left duals.

5.4 Define

$$w\text{ld}(L)$$

as the set of all the weak left duals in L (this is the obvious abbreviation, but, by coincidence it also alludes to the German word “Wald” for forest; it so happens that in case of binary relations the weak left duals are precisely the disjoint unions of trees, the forests). Then, by Proposition 5.3 (and Fact 3.4.1) we have

Corollary 5.4.1 *For each antichain $A \subseteq \text{wd}(L)$ there is precisely one finite duality (A, B) .*

5.5 Note

By Observation 5.1 and the definition of r_M , we have the dualities $(M, \{r_M\})$, and hence if $c_i(q) \in \overline{M}$, that is, $c_i(q) \notin \uparrow M$, then $c_i(q) \leq r_M$. Thus,

$$\bigvee \overline{M} \leq r_M$$

and $r_M = r(M)$ as in Proposition 3.8.

5.6 Remarks

As we already mentioned in Section 2.1, typical examples of a Heyting algebra we have in mind are provided by categories of graphs or relational structures. A characterization of finite dualities in the category of relational structures is provided in [14, 35]. The relationship between finite dualities and duality pairs (analogues of our Propositions 4.4 and 5.3) has recently been reproved in the special case of digraphs [11] using the Directed Sparse Incomparability Lemma. Here we would like to point out that sparse incomparability is *not* necessary to achieve these results; in fact, much weaker assumptions suffice. However, we do employ sparse incomparability in the next section, where we study the connection between dualities and maximal antichains.

6 Sparse Incomparability and Antichains

6.1 In [5] one can find the following fact (cf. [34, 37]).

Directed Sparse Incomparability Lemma *Let m, k be positive integers and let H be a directed graph which is not an orientation of a forest. Then there exists a directed graph H' such that*

- (1) *the girth of H' is finite and greater than k ,*
- (2) *for each directed graph G with fewer than m vertices, we have $H' \rightarrow G$ iff $H \rightarrow G$, and*
- (3) *$H \not\rightarrow H'$ and $H' \rightarrow H$.*

It should be clear now why the following assumption will be made in the Heyting context.

Sparse incomparability axiom—briefly, SIA.

This is the assumption that for any $x \in L$, any M, U finite subsets of L such that $(\{x\} \cup \uparrow U) \cap \text{wd}(L) = \emptyset$, there is a $y \in L \setminus \text{wd}(L)$ such that

$$y \in \uparrow \{x\}, \quad y \notin \uparrow (\{x\} \cup U) \quad \text{and} \quad \forall m \in M, \quad y \leq m \text{ iff } x \leq m. \quad (\text{SIA})$$

So for instance, if \mathcal{C} is the category of digraphs (or relational structures with a fixed signature) and homomorphisms, then the poset $\tilde{\mathcal{C}}$ defined in Section 2.4 is a Heyting algebra with SIA.

Observation 6.2 If (A, B) is a finite duality in a lattice L , then

$$A \cup (B \searrow \downarrow A)$$

is a finite maximal antichain in L .

(Indeed, it is an antichain because $a \not\leq b$ for any $a \in A, b \in B$. It is maximal because each $x \in L$ is either in $\uparrow A$ or there is a $b \in B$ with $x \leq B$; in the latter case, if $b \leq a$ for an $a \in A$ we have $x \leq a$.)

6.3 We are now going to show that the antichains of Observation 6.2 are in some sense the typical antichains in Heyting algebras with finite connected decompositions and SIA.

Lemma 6.3.1 *In a Heyting algebra L with SIA and finite connected decompositions, let C be a finite maximal antichain. Set $A = C \cap \downarrow \text{wl}\partial(L)$. Then*

$$\uparrow C \setminus C = \uparrow A \setminus C.$$

Proof The inclusion \supseteq is trivial. Thus, let $x \in \uparrow C \setminus C$ and set $U = C \setminus A$.

If $x \notin \uparrow U$ then $x \in \uparrow A$ and hence $x \in \uparrow A \setminus C$.

If $x \in \uparrow U$ then $x \notin \text{wl}\partial(L)$. We have $\uparrow U \cap \text{wl}\partial(L) = \emptyset$ and hence we can apply SIA to obtain a $y \notin \text{wl}\partial(L)$ such that

$$y \notin \uparrow(\{x\} \cup U) \quad \text{and} \quad \forall m \in C \cup \{x\}, y \leq m \text{ iff } x \leq m. \tag{*}$$

Now $x \in \uparrow C \setminus C$ and hence if $m \in C \cup \{x\}$ then $x \leq m$ only if $x = m$ and consequently $y \notin \downarrow C$. Since C is a maximal antichain, $y \in \uparrow C \setminus C$. By $(*)$, $y \notin U = C \setminus A$, hence $y \in \uparrow A$ and since $y \leq x$ we have, by $(*)$ again, $x \in \uparrow A$. □

Proposition 6.3.2 *In a Heyting algebra L with SIA and finite connected decompositions, let C be a finite maximal antichain such that $A = C \cap \downarrow \text{wl}\partial(L) = C \cap \text{wl}\partial(L)$. Consider the unique finite duality (A, B) . Then*

$$C = A \cup (B \searrow \downarrow A).$$

Proof If $b \in B$ then $b \notin \uparrow A$ and hence, by Lemma 6.3.1, $b \notin \uparrow C \setminus C$. Consequently, since C is a maximal antichain, $b \in \downarrow C$.

Now suppose that, moreover, $b \notin \downarrow A$. We want to prove that $b \in C$. If not, $b < c$ for some $c \in C$ and this means, by our assumption, that $c \in C \setminus A$. Then $c \not\leq b'$ for all $b' \in B$ and hence, by duality, $a \leq c$ for some $a \in A$ contradicting the antichain property. Thus, $b \in C$ and we have $A \cup (B \searrow \downarrow A) \subseteq C$, and since by Observation 6.1 $A \cup (B \searrow \downarrow A)$ is a maximal antichain, $A \cup (B \searrow \downarrow A) = C$. □

6.4 Splitting antichains

Our final Proposition 6.3.2 asserts that every finite maximal antichain C in a Heyting algebra L with SIA either contains an element of $\downarrow \mathfrak{wld}(L) \setminus \mathfrak{wld}(L)$, or has a very special structure. In particular, if $B \cap \downarrow A = \emptyset$, it can be *split* into two subsets (A and B) so that every element of $L \setminus C$ is above A or below B . This fact has a direct connection to the splitting property of posets (see [1, 2, 6, 9–11, 13]). Indeed, in [14] it is proved that finite maximal antichains in the poset arising from the category \mathcal{C} of relational structures with one relation contain no elements of $\downarrow \mathfrak{wld}(\tilde{\mathcal{C}}) \setminus \mathfrak{wld}(\tilde{\mathcal{C}})$. Almost all such antichains split, with finitely many characterized exceptions. However, the proof relies heavily on special properties of the category in question (digraphs) and we do not expect that it could be easily translated into the general setting of Heyting algebras.

6.5 Remarks

The Sparse Incomparability Lemma has a long and interesting history. While it seems to have been formulated specifically in this form first in [34] for $G = K_n$ and then in [37] for general G , it was preceded in the seminal work on sparse graphs with high chromatic number by Erdős and others [4, 7, 8, 15, 19, 22, 25, 27, 28, 39]. This useful lemma is related to an important result in descriptive complexity by Kun [20], and to a recent result on limits in graph sequences [30, 31].

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