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Extreme values of a portfolio of Gaussian processes and a trend

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Abstract We consider the extreme values of a portfolio of independent continuous Gaussian processes $\sum_{i=1}^{k} w_i X_i(t)$ ($w_i \in \mathbb{R}$, $k \in \mathbb{N}$) which are asymptotically locally stationary, with expectations $E[X_i(t)] = 0$ and variances $Var[X_i(t)] = d_i t^{2H_i}$ ($d_i \in \mathbb{R}^+$, $0 < H_i < 1$), and a trend $-ct^\beta$ for some constants $\beta, c > 0$ with $\beta > H_i$. We derive the probability $P\{\sup_{t>0} \sum_{i=1}^{k} w_i X_i(t) - ct^\beta > u\}$ for $u \to \infty$, which may be interpreted as ruin probability.

Keywords Gaussian processes, Extreme values, Portfolio of assets, Tail behavior, Ruin probability, Large deviations

AMS 2000 Subject Classification Primary— 60G15, 62G32, 91B28

1. Introduction

The tail behavior of stochastic processes is important e.g., for calculating ruin probabilities in insurance or finance. In this context we consider in this paper particular Gaussian processes, to determine the probability that a Gaussian process $Y(t)$ exceeds a certain boundary $u \in \mathbb{R}$ in an interval $T \in \mathbb{R}$

$$P(u) = P\left\{\sup_{t \in T} Y(t) > u\right\}.$$

In general, it is almost impossible to find the distribution of this supremum. Precise formulas are only known for a couple of stationary processes in a finite or infinite interval (cf. Adler (1990)). The best we can do in general, is to derive the asymptotic behavior of $P(u)$ when $u \to \infty$. This asymptotic behavior is sufficiently

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interesting for its own. However, we are going to derive the probability that the Gaussian process $Y(t) = X(t) - ct^{\beta}$, with trend, exceeds some boundary u

$$P\left\{\sup_{t>0}(X(t) - ct^{\beta}) > u\right\},$$

as $u \to \infty$ with $c, \beta > 0$. In insurance we may note that $X(t)$ represents the sum of the claims up to time t, ct^{β} represents the sum of the premium payments up to time t and u the initial reserve of the firm. The ruin occurs if at some time t the sum of the claims is larger than the sum of premium payments and the reserve.

This problem is investigated for a class of Gaussian processes $X(t)$ (including fractional Brownian motion and self-similar Gaussian processes) in Hüsler and Piterbarg (1999) and for integrated Gaussian processes in Debicki (2002) as well as Hüsler and Piterbarg (2004). In these cases the probability $P(u)$ is approximated by exceedances of $Y(t)$ in a small neighborhood of a unique point where the boundary $(u + ct^{\beta})/\sigma(t)$ has smallest value. This boundary is the result of usual standardization of the process $X(t)$ by $\sigma(t)$ where $\sigma^2(t)$ denotes its variance. Often, only the (unique) point of maximal variance plays the important role. Here the trend has to be considered also, which results in the mentioned minimal boundary value.

In this paper we deal with another particular class of Gaussian processes. We think that a portfolio consists of many different processes $X_i(\cdot)$ which can be modelled e.g., as fractional Brownian motions with $E[X_i(t)] = 0$ and $Var[X_i(t)] = d_i t^{2H_i}$ and possibly different parameters $H_i \in (0, 1)$ and $d_i > 0$. Therefore, we consider $X(t) = \sum_{i=1}^{k} w_i X_i(t)$ as the portfolio of all risks at time t, with $w_i (\in \mathbb{R})$ some weights. As mentioned, the biggest liability of the firm after its start of economic activities at time $t = 0$ is then denoted by $\sup_{t>0}(X(t) - ct^{\beta})$. Thus

$$P\left\{\sup_{t>0}\left(\sum_{i=1}^{k} w_i X_i(t) - ct^{\beta}\right) > u\right\}$$

will be investigated. We assume that the $X_i(\cdot)$ are independent processes. This probability is well-defined if $H_i < \beta$ for all i. As mentioned above, the minima of the boundary function $(u + ct^{\beta})/\sigma(t)$ have to be analyzed together with the path behaviour of $X(t)$ in the neighborhood of possible minima.

Note that $X(t)$ is thus a centered Gaussian process with variance

$$Var[X(t)] = \sum_{i=1}^{k} w_i^2 d_i t^{2H_i} = \sum_{i=1}^{k} W_i t^{2H_i}$$

where $W_i = w_i^2 d_i$. Hence, we might set w.l.o.g. $d_i = 1$ or $w_i = 1$, since in the following only the W_i's are used. It is not necessary to assume that the Gaussian processes $X_i(\cdot)$ are fractional Brownian motions. But certain regularity conditions will be assumed. E.g., we assume that the Gaussian processes are asymptotically locally stationary (see (8) in Condition (A1)) for large u. The processes with largest H_i are important. Hence, let w.l.o.g. $H = H_1 \geq H_2 \geq \ldots \geq H_k$ and define $m(\geq 1)$ as largest index such that $H_m = H$.

In the next section we introduce the sufficient conditions on the Gaussian process $X(t)$ and the main result which is proved in the third section. For its proof we need to investigate the local behaviour of the boundary function in the vicinity of the points with minimum value. This will be combined with the behavior of the weighted sum $X(t)$ of Gaussian processes in these vicinities.

2. Weighted sum of Gaussian processes and main result

The portfolio $X(t)$ is modelled as weighted sum of centered independent Gaussian processes $X_i(t)$ with $Var[X_i(t)] = t^{2H_i}$ with the mentioned numeration $H = H_1 \geq H_2 \geq \ldots \geq H_k$. Defining the standardized process

$$\widetilde{X}(t) = X(t)\Big/\sqrt{Var[X(t)]} = \sum_{i=1}^{k} w_i X_i(t)\Bigg/\sqrt{\sum_{i=1}^{k} W_i t^{2H_i}},$$

we analyze

$$P\left\{\sup_{t>0}(X(t) - ct^{\beta}) > u\right\} = P\left\{\exists t > 0 : \widetilde{X}(t) > \widetilde{f}_u(t)\right\},$$

where

$$\widetilde{f}_u(t) = \frac{u + ct^{\beta}}{\sqrt{\sum_{i=1}^{k} W_i t^{2H_i}}} = \frac{u + ct^{\beta}}{\sqrt{Wt^{2H}}\sqrt{1 + \sum_{j>m} W^{-1} W_j t^{2(H_j - H)}}}$$

is the boundary function and

$$W = \sum_{i=1}^{m} W_i. \tag{1}$$

We make an appropriate time transformation such that the points where the boundary values are minimal, remain finite as $u \to \infty$. Let for each u

$$s = W^{\frac{1}{2H}} u^{-\frac{1}{\beta}} t. \tag{2}$$

The transformed centered Gaussian processes, depending on u, are denoted by

$$X_i^{(u)}(s) = X_i(u^{\frac{1}{\beta}} W^{-\frac{1}{2H}} s), \quad i \leq k, \tag{3}$$

$$X^{(u)}(s) = \sum_{i=1}^{k} w_i X_i^{(u)}(s) = X(u^{\frac{1}{\beta}} W^{-\frac{1}{2H}} s). \tag{4}$$

The time transformation results in the corresponding boundary function $f_u(s)$:

$$f_u(s) = \widetilde{f}_u(W^{-\frac{1}{2H}} u^{\frac{1}{\beta}} s) = u^{1-\frac{H}{\beta}} v(s)(1 + \delta_u(s)) \tag{5}$$

consisting of three factors where

$$v(s) = \frac{1 + \widetilde{c}s^{\beta}}{s^H} \tag{6}$$

and

$$\delta_u(s) = \left(1 + \sum_{j>m} W_j W^{-\frac{H_j}{H}} u^{\frac{2}{\beta}(H_j - H)} s^{2(H_j - H)}\right)^{-\frac{1}{2}} - 1 \tag{7}$$

with $\widetilde{c} = cW^{-\frac{\beta}{2H}}$. Note that $\delta_u(s) \to 1$ as $u \to \infty$. The boundary $f_u(s)$ may have several points with minimal value, depending on u. The smallest of these points is denoted by $s_u^* = \inf\{\operatorname{argmin} f_u(s)\}$. We will show that these points with minimal value converge to the point of minimal value of $v(s)$. Hence we have to investigate the approximation of the probability:

$$P\left\{\sup_{t>0}(X(t)-ct^{\beta})>u\right\}=P\left\{\exists s>0:\widetilde{X}^{(u)}(s)>f_u(s)\right\},$$

where $\widetilde{X}^{(u)}(s)=X^{(u)}(s)/\sqrt{Var[X^{(u)}(s)]}$ denotes the standardized process $X^{(u)}(s)$.

We need the following assumptions for the main result:

(A1) We assume that each $\widetilde{X}_i^{(u)}(s)=X_i^{(u)}(s)/\sqrt{Var[X_i^{(u)}(s)]}$ is asymptotically locally stationary for $i\leq k$, i.e., there exists a function $K_i^2(\cdot)$, regular varying (at 0) with parameter $\alpha_i\in(0,2)$, such that

$$\lim_{u\to\infty}\frac{E[\widetilde{X}_i^{(u)}(s)-\widetilde{X}_i^{(u)}(s')]^2}{K_i^2(|s-s'|)}=D_i \tag{8}$$

uniformly for $s,s'\in S_u=[s_u^*-\epsilon(u),s_u^*+\epsilon(u)]$ with $D_i>0$, $\epsilon(u)=u^{\frac{H}{\beta}-1}\log u$ and s_u^* denotes the smallest point with minimal boundary value: $s_u^*=\inf\{\arg\min f_u(s)\}$.

This condition implies that each Gaussian process $X_i(\cdot)$ has continuous paths in the crucial interval. We could assume this regularity condition to hold for all s, but this is not necessary.

For the variance of the increments of $\widetilde{X}^{(u)}(s)$ we consider the weighted sum of the regularly varying functions $K_i(\cdot)$. By (A1) there exists $K^2(\cdot)$, regularly varying at 0 with index $\alpha=\min\{\alpha_i\}$, such that for some positive $\widetilde{W}$

$$\lim_{s\to s'}\frac{\sum_{i\leq m}W_iD_iK_i^2(|s-s'|)}{K^2(|s-s'|)}=\widetilde{W}.$$

We denote the inverse of $K(\cdot)$ by $K^{-1}(\cdot)$ where $K^{-1}(y)=\inf\{s:K(s)\geq y\}$. Since $K(\cdot)$ is regularly varying at 0 with index $\alpha/2$, $K^{-1}(\cdot)$ is regularly varying at 0 with index $2/\alpha$.

Note that for fractional Brownian motions with Hurst parameter H_i, we have $\alpha_i=2H_i$ and $K_i^2(h)=|h|^{2H_i}$.

(A2) For $j>m$, let $K_j^2(\cdot)$ be such that $\lim\sup_{h\downarrow 0}K_j(h)/K(h)<\infty$, as $u\to\infty$. This condition implies that $\alpha\leq\alpha_j$ also for $j>m$. Note that (A2) holds for $j\leq m$ also by (A1). We use in the following Pickands constant defined by

$$H_\alpha=\lim_{T\to\infty}\frac{1}{T}E\left[\exp\left(\max_{0\leq t\leq T}\chi(t)\right)\right],$$

where $\chi(t)$, $t\geq 0$, is a fractional Brownian motion with drift $E[\chi(t)]=-t^\alpha$ and covariance function $Cov[\chi(s),\chi(t)]=t^\alpha+s^\alpha-|t-s|^\alpha$, hence with Hurst parameter $H=\alpha/2$. Note that here the fractional Brownian $\chi(\cdot)$ motion has variance $2t^\alpha$.

Now we can state our main result on the extreme values of a portfolio of the Gaussian processes $X_i^{(u)}(s)(i=1,\ldots,k)$ as defined in (3). Condition (A1) restricts the behavior of these processes in the small interval S_u. For the other time points s we suppose a quite common Hölder condition for the increments. We assume that for each $i \leq k$

$$\limsup_{u\to\infty} E\Big[X_i^{(u)}(s) - X_i^{(u)}(s')\Big]^2 \leq G_i|s-s'|^{\gamma_i}. \tag{9}$$

holds for any s, s' with $|s-s'| \leq C$, for some $G_i, \gamma_i > 0$ and some large C. For the asymptotic expression we use the constants A and B:

$$A = \left(\frac{H}{\widetilde{c}(\beta-H)}\right)^{-\frac{H}{\beta}} \frac{\beta}{\beta-H}$$

$$B = \left(\frac{H}{\widetilde{c}(\beta-H)}\right)^{-\frac{H+2}{\beta}} H\beta,$$

which are $v(s_0)$ and $v''(s_0)$, respectively, with $s_0 = \operatorname{argmin} v(s) = (H/(\widetilde{c}(\beta-H))^{1/\beta}$, the unique point of minima of $v(\cdot)$ (see Lemma 3.3 below). We can now state the main theorem.

Theorem 2.1: *Let $X_i(t), t > 0, (i = 1, \ldots, k)$ be independent centered continuous Gaussian processes with variance $d_i t^{2H_i}$ and $-ct^\beta$ a trend where $\beta, c, d_i > 0$ and $0 < H_k \leq \ldots \leq H_1 < \min\{1, \beta\}$. Let $w_i \in \mathbb{R}$ denote the weights. Assume the conditions (A1) and (A2) with $0 < \alpha_i < 2$ and* (9). *Then, the tail behavior is given by*

$$P\left\{\sup_{t>0}\left(\sum_{i=1}^k w_i X_i(t) - ct^\beta\right) > u\right\} \sim \frac{\left(\sqrt{\frac{\widetilde{W}}{W}}A\right)^{\frac{2}{\alpha}} H_\alpha 2^{-\frac{1}{\alpha}} \exp\{-\frac{1}{2} f_u^2(s_u^*)\}}{A\sqrt{AB}u^{2-\frac{2H}{\beta}} K^{-1}\left(u^{\frac{H}{\beta}-1}\right)},$$

as $u \to \infty$, where $\alpha = \min_{i\leq m}\alpha_i$, m the number of $H_i = H$, $f_u(s)$ defined in (5) *and* $s_u^* = \inf\{\operatorname{argmin} f_u(s)\}$.

Remark 2.1: In some particular cases we have explicit expressions for s_u^* *and* $f_u(s_u^*)$. For example, if $k = 1$, we have $H = H_1$, $\alpha = \alpha_1$, $K(\cdot) = K_1(\cdot)$, $W = W_1 = d_1 w_1^2$, $\widetilde{W} = W_1 D_1$, $A = (H/(\widetilde{c}(\beta-H)))^{-\frac{H}{\beta}}\beta/(\beta-H)$ and $B = (H/(\widetilde{c}(\beta-H)))^{\frac{-H-2}{\beta}} H\beta$. $K^{-1}(s)$ is regularly varying at 0 with index $2/\alpha$ $(0 < \alpha < 2)$ and $f_u(s)$ reduces to $f_u(s) = u^{1-\frac{H}{\beta}}v(s)$. Hence $f_u^2(s_u^*) = u^{2-\frac{2H}{\beta}}v^2(s_0) = u^{2-\frac{2H}{\beta}}A^2$. We note that $\widetilde{c} = cW^{-\frac{\beta}{2H}} = c(d_1 w_1^2)^{-\frac{\beta}{2H}} = c$, by choosing $d_1 = w_1 = 1$. Hence

$$P\left\{\sup_{t>0}\left(X(t) - ct^\beta\right) > u\right\} \sim \frac{(\sqrt{D_1}A)^{\frac{2}{\alpha}} H_\alpha 2^{-\frac{1}{\alpha}} \exp\left\{-\frac{1}{2}A^2 u^{2-\frac{2H}{\beta}}\right\}}{A\sqrt{AB}u^{2-\frac{2H}{\beta}} K^{-1}\left(u^{\frac{H}{\beta}-1}\right)},$$

which is the result of Hüsler and Piterbarg (1999). But this result holds also for a more general Gaussian process, not only for a fractional Brownian motion, if the stated assumptions (A1) and (9) hold.

Remark 2.2 Further Examples: Let us consider some other simple examples with $X(t)$ satisfying the conditions of the theorem. We do not need to assume that $X_i(t)$ are fractional Brownian motions.

We begin with two processes $X_i(t), i = 1, 2$. *Let* $H = H_1 > H_2$. It implies that $m = 1$ and $W = W_1$. Also $K(\cdot) = K_1(\cdot)$ with $\alpha = \alpha_1$ and $\widetilde{W} = W_1 D_1$ and the same result holds as in Remark 2.1.

This simple situation holds also with more than two processes, if $H_1 = H_2 = \ldots = H_m$ $(k = m \geq 2)$ and $\alpha_1 < \alpha_j, j > 1$. Then $\widetilde{W} = W_1 D_1$, $\alpha = \alpha_1$ and $K = K_1$ in the asymptotic formula.

If in addition to H_j also some of the α_j's are equal to $\alpha_1 = \alpha$, then the result depend on the possible domination of one of the K_j's. For example, let $H_j = H_1$, for all $j \leq m = k$, and $\alpha_1 = \alpha_j$ for $j \leq m' \leq m$ with $\alpha_1 < \alpha_j$ *for* $j > m'$. In addition, assume that $K_j \sim c_j K_1$ *for* $j \leq m'$ and some $c_j \geq 0$, *as* $h \to 0$. Then $\widetilde{W} = \sum_{j \leq m'} W_j D_j c_j$ with $c_1 = 1$, and the result of Theorem 2.1 holds with $K = K_1$, $\alpha = \alpha_1$ and $W = \sum_j W_j$.

If one of the K_j*'s* $(j \leq m')$ dominates the others, by renumbering let this be K_1, then this would mean that $c_j = 0$ for all $1 < j \leq m'$. The result holds then also with such c_j*'s*.

3. Proof

Idea of the proof:

Applying the time transformation (2), the original problem gets

$$\begin{aligned} P\Big\{\sup_{t>0}(X(t) - ct^\beta) > u\Big\} &= P\{\exists t > 0 : X(t) > u + ct^\beta\} \\ &= P\Big\{\exists t > 0 : \widetilde{X}(t) > \widetilde{f}_u(t)\Big\} \\ &= P\Big\{\exists s > 0 : \widetilde{X}^{(u)}(s) > f_u(s)\Big\}. \end{aligned}$$

In Proposition 3.2 we show that all minima of $f_u(s)$ occur in the interval $[s_u^* - \epsilon(u), s_0]$, where $s_u^* = \inf\{\operatorname{argmin} f_u(s)\}$. Therefore we split $P\{\exists s > 0 : \widetilde{X}^{(u)}(s) > f_u(s)\}$ into the probabilities

$$P\Big\{\exists s \in S_u : \widetilde{X}^{(u)}(s) > f_u(s)\Big\} \quad \text{and} \quad P\Big\{\exists s \notin S_u : \widetilde{X}^{(u)}(s) > f_u(s)\Big\}.$$

where $S_u = [s_u^* - \epsilon(u), s_u^* + \epsilon(u)]$. Then we will show that for $u \to \infty$

$$P\Big\{\exists s \notin S_u : \widetilde{X}^{(u)}(s) > f_u(s)\Big\} = o\Big(P\Big\{\exists s \in S_u : \widetilde{X}^{(u)}(s) > f_u(s)\Big\}\Big).$$

We choose $\epsilon(u) = u^{\frac{H}{\beta}-1}\log u$, since we need later that $\epsilon(u)u^{1-\frac{H}{\beta}} \to \infty$ $(u \to \infty)$. Hence it remains to analyze the asymptotic behavior of the leading probability term, where $s \in S_u$. For this proof we need to know the behavior of the portfolio process $X^{(u)}(s)$ or $\widetilde{X}^{(u)}(s)$ and the boundary function $f_u(s)$.

3.1. Properties of the portfolio process

By the definition of the portfolio process its behavior can be characterized as follows.

Lemma 3.1: *For $i \le k$, the means and variances are given by*

$$\begin{aligned} E\big[X_i^{(u)}(s)\big] &= E\big[X^{(u)}(s)\big] = 0, \\ Var[X_i^{(u)}(s)] &= d_i u^{\frac{2H}{\beta}} W^{-1} s^{2H} && \text{for } i \le m \\ Var[X_j^{(u)}(s)] &= d_j s^{2H_j} W^{-\frac{H_j}{H}} u^{\frac{2}{\beta}H_j} = o(u^{\frac{2H}{\beta}}) && \text{for } j > m \\ Var[X^{(u)}(s)] &= u^{2H/\beta} s^{2H} (1+\delta_u(s))^{-2} = u^{2H/\beta} s^{2H}\Big(1 + O\Big(u^{\frac{2}{\beta}(H_{m+1}-H)}\Big)\Big) \end{aligned}$$

as $u \to \infty$.

Proof: The processes are obviously centered. For any $i \le m$ and $j > m$, the variances are simply

$$\begin{aligned} Var[X_i^{(u)}(s)] &= d_i(u^{\frac{1}{\beta}} W^{-\frac{1}{2H}} s)^{2H_i} = d_i u^{\frac{2H}{\beta}} W^{-1} s^{2H}, \quad \text{for } i \le m \\ Var[X_j^{(u)}(s)] &= d_j(u^{\frac{1}{\beta}} W^{-\frac{1}{2H}} s)^{2H_j} = d_j u^{2H_j/\beta} W^{-H_j/H} s^{2H_j} = o(u^{\frac{2H}{\beta}}), \end{aligned}$$

as $u \to \infty$, and

$$\begin{aligned} Var[X^{(u)}(s)] &= \sum_{i \le m} w_i^2 Var[X_i^{(u)}(s)] + \sum_{j>m} w_j^2 Var[X_j^{(u)}(s)] \\ &= u^{\frac{2H}{\beta}} s^{2H}\left(1 + \sum_{j>m} W_j W^{-\frac{H_j}{H}} u^{\frac{2}{\beta}(H_j-H)} s^{2(H_j-H)}\right) \\ &= u^{\frac{2H}{\beta}} s^{2H}(1+\delta_u(s))^{-2} \\ &= u^{\frac{2H}{\beta}} s^{2H}\Big(1 + O\Big(u^{\frac{2}{\beta}(H_{m+1}-H)}\Big)\Big) \end{aligned}$$

The correlation of $\widetilde{X}^{(u)}(s)$ is given by the regularly varying function $K^2(\cdot)$, in the interval S_u. ■

Lemma 3.2: *Assume the conditions (A1) and (A2). Then the correlation function of $\widetilde{X}^{(u)}(s) = X^{(u)}(s)/\sqrt{Var[X^{(u)}(s)]}$ is for small lags, for $s, s' \in S_u$*

$$1 - Corr[\widetilde{X}^{(u)}(s), \widetilde{X}^{(u)}(s')] = \frac{1}{2} E[\widetilde{X}^{(u)}(s) - \widetilde{X}^{(u)}(s')]^2 \sim \frac{\widetilde{W}}{2W} K^2(|s-s'|).$$

Proof: Using the fact that $\widetilde{X}^{(u)}(s)$ and $\widetilde{X}_i^{(u)}(s)$ are standardized, we get

$$E\Big[\widetilde{X}^{(u)}(s) - \widetilde{X}^{(u)}(s')\Big]^2 = E\left[\frac{\sum_{i=1}^{k} w_i X_i^{(u)}(s)}{\sqrt{Var[X^{(u)}(s)]}} - \frac{\sum_{i=1}^{k} w_i X_i^{(u)}(s')}{\sqrt{Var[X^{(u)}(s')]}}\right]^2$$

$$= E\left[\sum_{i=1}^{k} w_i^2 \left(\sqrt{\frac{Var[X_i^{(u)}(s)]}{Var[X^{(u)}(s)]}} \widetilde{X}_i^{(u)}(s) - \sqrt{\frac{Var[X_i^{(u)}(s')]}{Var[X^{(u)}(s')]}} \widetilde{X}_i^{(u)}(s')\right)^2\right].$$

To derive the claim, we split the sum $\sum_{i=1}^{k}$ into $\sum_{i\le m}$ and $\sum_{j>m}$. Using Lemma 3.1, we see that for $i \le m$ and $u \to \infty$

$$\sqrt{\frac{Var[X_i^{(u)}(s)]}{Var[X^{(u)}(s)]}} = \sqrt{\frac{d_i}{W}}(1+\delta_u(s))$$

and get

$$\begin{aligned}
&E\left[\sqrt{\frac{Var[X_i^{(u)}(s)]}{Var[X^{(u)}(s)]}}\widetilde{X}_i^{(u)}(s) - \sqrt{\frac{Var[X_i^{(u)}(s')]}{Var[X^{(u)}(s')]}}\widetilde{X}_i^{(u)}(s')\right]^2\\
&= \frac{d_i}{W}E\Big[(1+\delta_u(s))\Big(\widetilde{X}_i^{(u)}(s) - \widetilde{X}_i^{(u)}(s')\Big) + (\delta_u(s)-\delta_u(s'))\widetilde{X}_i^{(u)}(s')\Big]^2\\
&= \frac{d_i}{W}(1+\delta_u(s))^2 E\Big[\widetilde{X}_i^{(u)}(s) - \widetilde{X}_i^{(u)}(s')\Big]^2 + \frac{d_i}{W}(\delta_u(s)-\delta_u(s'))^2\\
&\quad + 2\frac{d_i}{W}(1+\delta_u(s))(\delta_u(s)-\delta_u(s'))E\Big[\Big(\widetilde{X}_i^{(u)}(s) - \widetilde{X}_i^{(u)}(s')\Big)\widetilde{X}_i^{(u)}(s')\Big]\\
&\quad \sim \frac{d_i}{W}E\Big[\widetilde{X}_i^{(u)}(s) - \widetilde{X}_i^{(u)}(s')\Big]^2,
\end{aligned}$$

where for the last step we used that $\widetilde{X}_i^{(u)}(s)$ is asymptotically locally stationary with $\alpha_i \in (0,2)$ and that $\delta_u(s) - \delta_u(s') = (s-s')\delta_u'(\eta) = (s-s')O\Big(u^{\frac{2}{\beta}(H_{m+1}-H)}\Big)$ for some $\eta \in (s', s)$.

For $j > m$ we use Lemma 3.1 again to derive

$$\begin{aligned}
\sqrt{\frac{Var[X_j^{(u)}(s)]}{Var[X^{(u)}(s)]}} &= \sqrt{d_j}s^{H_j}W^{-\frac{H_j}{2H}}u^{H_j/\beta}\Big/(u^{H/\beta}s^H(1+\delta_u(s))^{-1})\\
&= \sqrt{d_j}W^{-\frac{H_j}{2H}}u^{\frac{1}{\beta}(H_j-H)}s^{H_j-H}(1+\delta_u(s)) =: g_j(u,s)
\end{aligned}$$

and

$$\begin{aligned}
&E\left[\sqrt{\frac{Var[X_j^{(u)}(s)]}{Var[X^{(u)}(s)]}}\widetilde{X}_j^{(u)}(s) - \sqrt{\frac{Var[X_j^{(u)}(s')]}{Var[X^{(u)}(s')]}}\widetilde{X}_j^{(u)}(s')\right]^2\\
&= E\Big[g_j(u,s)\Big(\widetilde{X}_j^{(u)}(s) - \widetilde{X}_j^{(u)}(s')\Big) + \big(g_j(u,s) - g_j(u,s')\big)\widetilde{X}_j^{(u)}(s')\Big]^2\\
&= g_j^2(u,s)E\Big[\widetilde{X}_j^{(u)}(s) - \widetilde{X}_j^{(u)}(s')\Big]^2 + \big(g_j(u,s) - g_j(u,s')\big)^2 + 2g_j(u,s)\\
&\quad \times \big(g_j(u,s) - g_j(u,s')\big)E\Big[\Big(\widetilde{X}_j^{(u)}(s) - \widetilde{X}_j^{(u)}(s')\Big)\widetilde{X}_j^{(u)}(s')\Big]\\
&= O\left(g_j^2(u,s)E\Big[\widetilde{X}_j^{(u)}(s) - \widetilde{X}_j^{(u)}(s')\Big]^2\right)\\
&= O\Big(u^{\frac{2}{\beta}(H_j-H)}K_j^2(|s-s'|)\Big),
\end{aligned}$$

as $u \to \infty$, where we used that $g_j(u,s) = O(u^{\frac{1}{\beta}(H_j - H)})$ and $g_j(u,s) - g_j(u,s') = (s-s')g_j'(u,\eta) = (s-s')O(u^{\frac{1}{\beta}(H_j-H)})$ for some $\eta \in (s', s)$, as well as

$$\begin{aligned} &2g_j(u,s)(g_j(u,s) - g_j(u,s'))E[(\widetilde{X}_j^{(u)}(s) - \widetilde{X}_j^{(u)}(s'))\widetilde{X}_j^{(u)}(s')] \\ &\quad = O\left(u^{\frac{2}{\beta}(H_j-H)}|s-s'|K_j^2(|s-s'|)\right) \end{aligned}$$

Putting the various terms together results in

$$\begin{aligned} E\left[\widetilde{X}^{(u)}(s) - \widetilde{X}^{(u)}(s')\right]^2 &= (1+o(1))\sum_{i\le m} w_i^2\left(\frac{d_i}{W}E\left[\widetilde{X}_i^{(u)}(s) - \widetilde{X}_i^{(u)}(s')\right]^2\right) \\ &+ \sum_{j>m} O\left(u^{\frac{2}{\beta}(H_j-H)}K_j^2(|s-s'|)\right) \sim \frac{\widetilde{W}}{W}K^2(|s-s'|), \end{aligned}$$

using that for all $j > m$

$$O\left(u^{\frac{2}{\beta}(H_j-H)}K_j^2(|s-s'|)\right) = o\left(K^2(|s-s'|)\right)$$

since $H_j < H$ and $K_j(h)/K(h) = O(1)$ for h small.

Hence for u large, the correlation function of $\widetilde{X}^{(u)}(s)$ behaves for $s \to s'$ as

$$1 - Corr[\widetilde{X}^{(u)}(s), \widetilde{X}^{(u)}(s')] = \frac{1}{2}E\left[\widetilde{X}^{(u)}(s) - \widetilde{X}^{(u)}(s')\right]^2 \sim \frac{\widetilde{W}}{2W}K^2(|s-s'|).$$

■

3.2. Behavior of the boundary function

We need to analyze the behavior of the boundary function. Examining $\delta_u(s)$, we see that $\lim_{s\to 0}\delta_u(s) = -1$, $\lim_{s\to\infty}\delta_u(s) = 0$ and $s^{2(H_j-H)}$ strictly decreases to zero, as $s \to \infty$, since $H_j < H$ for $j > m$. Hence $\delta_u(s)$ strictly increases from -1 to 0. With $s_0 = argmin\ v(s)$, it is straightforward to prove the following lemma (given also in Hüsler and Piterbarg (2004)) which implies the next proposition by (5).

Lemma 3.3: *We get* $s_0 = (H/(\tilde{c}(\beta - H)))^{\frac{1}{\beta}}$ *as well as* $v'(s) < 0$ *for* $s < s_0$ *and* $v'(s) > 0$ *for* $s > s_0$. *Hence* s_0 *is unique as point of minimal value of* $v(\cdot)$. *Further*

$$\begin{aligned} v(s_0) &= \left(\frac{H}{\tilde{c}(\beta-H)}\right)^{-\frac{H}{\beta}}\frac{\beta}{\beta-H} =: A \quad and \\ v''(s_0) &= \left(\frac{H}{\tilde{c}(\beta-H)}\right)^{-\frac{H+2}{\beta}}H\beta =: B. \end{aligned}$$

Proposition 3.1: *For* $u \to \infty$ *we get*

$$\begin{aligned} f_u(s_0) &= u^{1-\frac{H}{\beta}}A(1+\delta_u(s_0)) = u^{1-\frac{H}{\beta}}A\left(1 + O\left(u^{\frac{2}{\beta}(H_{m+1}-H)}\right)\right), \\ f_u''(s_0) &= u^{1-\frac{H}{\beta}}B\left(1 + O\left(u^{\frac{2}{\beta}(H_{m+1}-H)}\right)\right), \end{aligned}$$

taking the derivative w.r.t. s.

Proof: Taking derivatives (w.r.t. s), we get $(1+\delta_u(s))' = O(u^{\frac{2}{\beta}(H_{m+1}-H)})$ and $(1+\delta_u(s))'' = O(u^{\frac{2}{\beta}(H_{m+1}-H)})$ for s on a compact interval $\subset (0,\infty)$. Thus

$$f_u''(s) = u^{1-\frac{H}{\beta}}[v''(s)(1+\delta_u(s)) + 2v'(s)(1+\delta_u(s))' + v(s)(1+\delta_u(s))'']$$
$$= u^{1-\frac{H}{\beta}} v''(s)\left[1 + O\left(u^{\frac{2}{\beta}(H_{m+1}-H)}\right)\right],$$

as $u \to \infty$, for $s \in (s_0 - \hat{\delta}, s_0 + \hat{\delta})$ with $\hat{\delta} > 0$ such that $v''(s) > 0$. Together with Lemma 3.3 the statements follows. ∎

Remark 3.1: The boundary function $f_u(s)$ is continuous, has the limits $\lim_{s\to 0} f_u(s) = \lim_{s\to\infty} f_u(s) = \infty$ and at least one minimum, tending to ∞, as $u \to \infty$. Unfortunately, $f_u(s)$ is a non-algebraic function and thus explicit solutions for the minimum points of $f_u(s)$ do not exist in the general case $k > 1$. Further it is unclear whether the global minimum is unique.

Because of Remark 3.1 we need an upper and a lower simple approximation function of $f_u(s)$. Let us define

$$\kappa_u = \sum_{j>m} W_j W^{-\frac{H_j}{H}} u^{\frac{2}{\beta}(H_j - H)}.$$

For $\epsilon(u) > 0$, we introduce the functions

$$f_u^+(s) = u^{1-\frac{H}{\beta}} v(s)(1+\delta_u(s_0+\epsilon(u))) \quad (s>0)$$

and for some small $s_1 < \min\{1, s_0 - \epsilon(u)\}$

$$f_u^-(s) = \begin{cases} f_{u,1}^-(s) = u^{1-\frac{H}{\beta}} v(s)(1+\kappa_u s^{2(H_k - H)})^{-\frac{1}{2}} & (0 < s \le s_1) \\ f_{u,2}^-(s) = u^{1-\frac{H}{\beta}} v(s)(1+\delta_u(s_1)) & (s_1 < s \le s_0 - \epsilon(u)) \\ f_{u,3}^-(s) = u^{1-\frac{H}{\beta}} v(s)(1+\delta_u(s_0 - \epsilon(u))) & (s > s_0 - \epsilon(u)). \end{cases}$$

The constant s_1 is chosen such that $f_{u,1}^-(s)$ is strictly decreasing in $(0, s_1)$. This holds because

$$\frac{\partial}{\partial s} f_{u,1}^-(s) = \frac{u^{1-\frac{H}{\beta}}}{s(s^{2H} + \kappa_u s^{2H_k})^{\frac{3}{2}}} \times \left[\beta\tilde{c}s^{\beta+2H} + \beta\tilde{c}\kappa_u s^{\beta+2H_k} - Hs^{2H} - H_k\kappa_u s^{2Hk} - \tilde{c}Hs^{\beta+2H} - \tilde{c}H_k\kappa_u s^{\beta+2H_k}\right]$$

is negative if $\tilde{c}(\beta - H)s^{\beta+2(H-H_k)} + \tilde{c}\kappa_u(\beta - H_k)s^{\beta} - Hs^{2(H-H_k)} - H_k\kappa_u < 0$, which is true if $s < s_1$ for some $s_1 > 0$ small enough.

Since $1+\delta_u(s)$ is strictly increasing, we have

$$f_u^-(s) < f_u(s) < f_u^+(s) \quad \text{if} \quad s \in (0, s_0 + \epsilon(u)) \tag{10}$$

for any $\epsilon(u) \ge 0$, and

$$f_u(s) \ge f_u^+(s) > f_u^-(s) \quad \text{if} \quad s \in [s_0 + \in(u), \infty), \tag{11}$$

with equality for $s = s_0 + \epsilon(u)$.

Lemma 3.4: *The global minimum/minima of $f_u(s)$ is/are in the interval $(s_1, s_0]$.*
Proof: Since $f_u(s)$ is strictly increasing for $s > s_0$, we have $\min_{s>s_0} f_u(s) = f_u(s_0)$ hence the minima are smaller than s_0. Now $f^-_{u,1}(s)$ is strictly decreasing in $(0, s_1)$ and $f^-_{u,1}(s) \to \infty$ for $s \to 0$. Since $s_1 < s_0$ we have with (10)

$$\min_{s \le s_0}\{f_u(s)\} \le \min_{s \le s_1}\left\{f^-_{u,1}(s)\right\} < \min_{s \le s_1}\{f_u(s)\}$$

for any u, which finishes the proof. ■

Even if there are more than one point with minimal boundary value, they all converge to s_0 which is shown next since this holds for the smallest of these points, denoted by s^*_u.

Proposition 3.2: *We have $s^*_u \to s_0$ for $u \to \infty$. Hence the global minimum/minima points are in (s^*_u, s_0).*

Proof: Lemma 3.4 guarantees that $s^*_u \le s_0$. It remains to show for any $\delta > 0$ that if there exists a sequence $u(i) \to \infty$ such that $s^*_{u(i)} < s_0 - \delta$ leads to a contradiction. Because s_0 minimizes $v(s)$ and since $s^*_{u(i)} < s_0 - \delta$ we have $v(s^*_{u(i)})/v(s_0) > 1 + \Delta(\delta)$ for some $\Delta(\delta) > 0$. Further we know that $(1 + \delta_{u(i)}(s^*_{u(i)}))/(1 + \delta_{u(i)}(s_0)) = 1 + o(1)$ since $\delta_{u(i)}(\cdot) = o(1)$, as $u(i) \to \infty$. Hence

$$\begin{aligned} f_{u(i)}(s^*_{u(i)}) &= f_{u(i)}(s_0)\frac{v(s^*_{u(i)})(1 + \delta_{u(i)}(s^*_{u(i)}))}{v(s_0)(1 + \delta_{u(i)}(s_0))} \\ &> f_{u(i)}(s_0)(1 + \Delta(\delta))(1 + o(1)) \\ &> f_{u(i)}(s_0). \end{aligned}$$

So we get $f_{u(i)}(s^*_{u(i)}) > f_{u(i)}(s_0)$, which cannot be true since $s^*_{u(i)}$ is the smallest of the possible minimal points. ■

Combining the proof of Proposition 3.1 with the above Proposition 3.2 gives

Proposition 3.3: *For $u \to \infty$ we get*

$$\begin{aligned} f_u(s^*_u) &\sim u^{1-\frac{H}{\beta}}A \\ f''_u(s^*_u) &\sim u^{1-\frac{H}{\beta}}B. \end{aligned}$$

3.3. Tail behavior of $X^{(u)}(s)$ for $s \in S_u$

We derive the probability that $X^{(u)}(s)$ exceeds $f_u(s)$ for $s \in S_u$ by applying a result of Bräker (1993a), given also in Bräker (1993b), similar to Hüsler and Piterbarg (1999). This probability will be the major contribution to the investigated probability, asymptotically. Bräker's result (formulated below) is given for locally stationary Gaussian processes, being not dependent on u. So a further approximation step is necessary since $X^{(u)}(s)$ depends on u.

Proposition 3.4: *Assume (A1) and (A2). Then with the correlation function $K(\cdot)$ of $X^{(u)}(s)$ we have for $u \to \infty$*

$$P\left\{\exists s \in S_u \colon \widetilde{X}^{(u)}(s) > f_u(s)\right\} \sim \frac{\left(\sqrt{\frac{\widetilde{W}}{W}}A\right)^{\frac{2}{\alpha}} H_\alpha 2^{-\frac{1}{\alpha}} \exp\left(-\frac{1}{2}f_u^2(s_u^*)\right)}{A\sqrt{AB}u^{2-\frac{2H}{\beta}}K^{-1}\left(u^{\frac{H}{\beta}-1}\right)}.$$

Proof: To apply Bräker's Theorem (Bräker, 1993a), we need to approximate $\widetilde{X}^{(u)}(s)$ by Gaussian processes $U_+(s)$ and $U_-(s)$ which are independent of u. The original probability will then be estimated applying Slepian's inequality (Adler, 1990).

As in Hüsler and Piterbarg (1999), standardized Gaussian processes $U_+(s)$ and $U_-(s)$ exist by the assumptions in Lemma 3.2 such that

$$\lim_{s \to s'} \left[\frac{E[U_\pm(s) - U_\pm(s')]^2}{K^2(|s - s'|)}\right] = \frac{\widetilde{W}}{W}(1 \pm \nu)$$

for any $\nu > 0$, with correlation function given by

$$1 - Corr[U_\pm(s), U_\pm(s')] \sim \frac{\widetilde{W}}{2W}(1 \pm \nu)K^2(|s - s'|),$$

for $s \to s'$. By construction we have for $s, s' \in S_u$ and u large

$$Corr[U_\pm(s), U_\pm(s')] \lesseqgtr Corr[\widetilde{X}^{(u)}(s), \widetilde{X}^{(u)}(s')]$$

for any $\nu > 0$, since $\epsilon(u) \to 0$ for $u \to \infty$. Applying Slepian's inequality we get

$$P\left\{\exists s \in S_u : \widetilde{X}^{(u)}(s) > f_u(s)\right\} \leq P\{\exists s \in S_u : U_+(s) > f_u(s)\}$$
$$P\left\{\exists s \in S_u : \widetilde{X}^{(u)}(s) > f_u(s)\right\} \geq P\{\exists s \in S_u : U_-(s) > f_u(s)\}.$$

We now calculate the two probabilities

$$P_\pm(u) = P\{\exists s \in S_u : U_\pm(s) > f_u(s)\}$$

for $s, s' \in S_u$ and show that $P_+(u) = (1 + O(\nu))P_-(u)$ for $u \to \infty$. Hence the probability $P\{\exists s \in S_u : \widetilde{X}^{(u)}(s) > f_u(s)\}$ is asymptotically equal to $P_+(u)$ or $P_-(u)$, letting $\nu \to 0$. ∎

3.3.1. The calculation of $P_+(u)$ and $P_-(u)$

We have to verify that $f_u(s)$ satisfies the assumptions (f1), ..., (f5) of Bräker's Theorem (Bräker, 1993a) (or Bräker, 1993b), for the derivations of $P_+(u)$ and $P_-(u)$. We consider only $P_+(u)$, since the other term is derived in the same way. Bräker's result states that if the following conditions (f1), (f2), (f3), (f4) and (f5) hold, then the $P_+(u)$ can be asymptotically approximated by the expression given bellow in 12.

(f1): Being an elementary function, $f_u(s)$ is continuous, which is the condition (f1).
(f2): Since $\lim_{u\to\infty} f_u(s) = \infty$ for any $s > 0$, we have $\lim_{u\to\infty} \inf_{s \in S_u} f_u(s) = \infty$ which is condition (f2).

(f3): Set $G(x) = K^{-1}(1/x),\ (x > 0)$, and

$$\Delta_u(s) = G\left(\sqrt{\frac{\widetilde{W}(1+\nu)}{W}} f_u(s)\right) \quad \text{for} \quad s \in S_u.$$

With $\psi(x) = \exp(-x^2/2)/\sqrt{2\psi}x \sim \Psi(x) = 1 - \Phi(x)$ as $x \to \infty$, condition (f3) assumes that for any $\epsilon > 0$

$$\int_{S_u} \frac{\psi(\epsilon f_u(s))}{\Delta_u(s)} ds \to 0$$

for $u \to \infty$. Because S_u is bounded, (f2) holds and $K^{-1}(\cdot)$ is regularly varying at 0 with index $2/\alpha$, as mentioned, hence (f3) follows.

(f4): With $\zeta_u(s,\tau) = [f_u(s + \tau\Delta_u(s)) - f_u(s)]f_u(s)$, condition (f4) states that $\zeta_u(s,\tau)$ converges uniformly to a function $\zeta(s,\tau)$ for $s \in S_u$ and $\tau \le \theta$, some $\theta > 0$. This holds since we show that for $u \to \infty, |\tau| \le \theta, 0 \le \theta < \infty$ and $s \in S_u$

$$|\zeta_u(s,\tau)| \longrightarrow 0.$$

For some $\xi \in (s, s + \tau\Delta_u(s))$ we have

$$f_u'(\xi) = \left[\frac{f_u(s + \tau\Delta_u(s)) - f_u(s)}{\tau\Delta_u(s)}\right]$$

and thus $\zeta_u(s,\tau) = \tau\Delta_u(s) f_u'(\xi) f_u(s)$. Now for $\Delta_u(s)$ we get

$$\begin{aligned}\Delta_u(s) &= \left(\frac{W}{\widetilde{W}(1+\nu)}\right)^{\frac{2}{\alpha}} \left(\frac{1}{f_u(s)}\right)^{\frac{2}{\alpha}} \widetilde{L}\left(\sqrt{\frac{W}{\widetilde{W}(1+\nu)}} \frac{1}{f_u(s)}\right) \\ &= O\left(u^{\frac{2}{\alpha}(\frac{H}{\beta}-1)} (1 + \delta_u(s))^{-\frac{2}{\alpha}} \widetilde{L}(1/f_u(s))\right),\end{aligned}$$

as $u \to \infty$ with $\widetilde{L}(\cdot)$ a slowly varying function.

To estimate $f_u'(\xi)$ we use that $f_u'(s_u^*) = 0$ and Proposition 3.3:

$$\begin{aligned}f_u'(\xi) &= f_u'(s_u^*) + f_u''(s_u^*)(\xi - s_u^*) + o(\xi - s_u^*) \\ &= O\left(u^{1-\frac{H}{\beta}}(\xi - s_u^*)\right) = O(\log u),\end{aligned}$$

since $\xi - s_u^* \le \epsilon(u)$ for $\xi \in S_u$, by the choice of $\epsilon(u)$.

Putting together the estimations of $\Delta_u(s)$ and $f_u'(s)$ we have

$$\begin{aligned}|\zeta_u(s,\tau)| &= O(|\tau\Delta_u(s) f_u'(\xi) f_u(s)|) \\ &= O\left(u^{(1-\frac{H}{\beta})(1-\frac{2}{\alpha})} (1 + \delta_u(s))^{1-\frac{2}{\alpha}} \widetilde{L}(1/f_u(s)) \log u\right) = o(1).\end{aligned}$$

since $\alpha < 2$ and $\beta > H$. Hence the sequence $\zeta_u(s,\tau)$ converges to $\zeta(s,\tau) = 0$ for $u \to \infty$, for any τ bounded.

(f5): (f4) implies that the $|\zeta_u(s,\tau)| \to 0$, hence is finite for all $\tau \in \mathbb{R}$, which is condition (f5): $\sup_s |\zeta(s,\tau)| < \infty$ for any τ.

Thus we can apply Bräker's Theorem since the considered stochastic process $U_+(s)$ is a locally stationary Gaussian process with index $\alpha \in (0,2)$, is independent of the parameter u and the sequence of boundary functions $f_u(s)$ satisfies the conditions (f1), ...,(f5). Bräkers' result states that

$$\lim_{u\to\infty} \frac{1}{\Lambda_u} P\{\exists s \in S_u : U_+(s) > f_u(s)\} = \lim_{u\to\infty} \frac{P_+(u)}{\Lambda_u} = 1, \tag{12}$$

where $\Lambda_u = \int_{S_u} \lambda_u(s)ds$ since $g(s,\tau) = 0$ and where for $s \in S_u$

$$\lambda_u(s) = \frac{H_\alpha 2^{-\frac{1}{\alpha}} \psi(f_u(s))}{G\left(\sqrt{\frac{\widetilde{W}(1+\nu)}{W}} f_u(s)\right)} = \frac{H_\alpha 2^{-\frac{1}{\alpha}} \psi(f_u(s))}{K^{-1}\left(1/\left(\sqrt{\frac{\widetilde{W}(1+\nu)}{W}} f_u(s)\right)\right)}.$$

We derive the behavior of the integral Λ_u as $u \to \infty$. Since $K^{-1}(\cdot)$ is regularly varying with index $2/\alpha$, it follows uniformly for $s \in S_u$

$$1/K^{-1}\left(1/\left(\sqrt{\frac{\widetilde{W}(1+\nu)}{W}} f_u(s)\right)\right) \sim \left(\sqrt{\frac{\widetilde{W}(1+\nu)}{W}} A\right)^{\frac{2}{\alpha}} \Big/ K^{-1}(u^{\frac{H}{\beta}-1}).$$

Since $f_u(s)/f_u(s_0) \to 1$, as $u \to \infty$, uniformly for $s \in S_u$, we derive with Proposition 3.1

$$\int_{S_u} \lambda_u(s)ds = H_\alpha 2^{-\frac{1}{\alpha}} \int_{s_u^*-\epsilon(u)}^{s_u^*+\epsilon(u)} \psi(f_u(s))/K^{-1}\left(1/\left(\sqrt{\widetilde{W}(1+\nu)/W} f_u(s)\right)\right) ds$$

$$\sim \frac{H_\alpha 2^{-\frac{1}{\alpha}} \left(\sqrt{\frac{\widetilde{W}(1+\nu)}{W}} A\right)^{\frac{2}{\alpha}}}{\sqrt{2\pi} A u^{1-\frac{H}{\beta}} K^{-1}(u^{\frac{H}{\beta}-1})} \int_{s_u^*-\epsilon(u)}^{s_u^*+\epsilon(u)} e^{-\frac{1}{2} f_u^2(s)} ds.$$

We expand the exponent of the integrand for $s \to s_u^*$

$$f_u^2(s) = \left(f_u(s_u^*) + f_u'(s_u^*)(s - s_u^*) + \frac{1}{2} f_u''(s_u^*)(s - s_u^*)^2 + o\left((s - s_u^*)^2\right)\right)^2$$

$$= f_u^2(s_u^*) + f_u(s_u^*) f_u''(s_u^*)(s - s_u^*)^2 (1 + o(1)).$$

Then we change the variable $x = \sqrt{f_u(s_u^*) f_u''(s_u^*)}(s - s_u^*)$. The bounds $s_u^* \pm \epsilon(u)$ are replaced by

$$\pm\epsilon(u)\sqrt{f_u(s_u^*) f_u''(s_u^*)} \sim \pm(\log u)\sqrt{AB} \to \pm\infty \quad (u \to \infty).$$

Hence, the integral is by this transformation

$$\int_{s_u^*-\epsilon(u)}^{s_u^*+\epsilon(u)} e^{-\frac{1}{2} f_u^2(s)} ds \sim \frac{e^{-\frac{1}{2} f_u^2(s_u^*)}}{\sqrt{f_u(s_u^*) f_u''(s_u^*)}} \int_{-\infty}^{\infty} e^{-\frac{1}{2} x^2 (1+o(1))} dx \sim \frac{\sqrt{2\pi} e^{-\frac{1}{2} f_u^2(s_u^*)}}{\sqrt{AB} u^{1-\frac{H}{\beta}}},$$

and we get for $P_+(u)$ the approximations

$$P_+(u) \sim \int_{S_u} \lambda_u(s)ds \sim \frac{H_\alpha 2^{-\frac{1}{\alpha}}\left(\sqrt{\frac{\widetilde{W}(1+\nu)}{W}}A\right)^{\frac{2}{\alpha}} e^{-\frac{1}{2}f_u^2(s_u^*)}}{A\sqrt{AB}u^{2-\frac{2H}{\beta}}K^{-1}(u^{\frac{H}{\beta}-1})}.$$

In an analogous way we get the same approximation for $P_-(u)$, replacing $+\nu$ by $-\nu$. Taking the limit $\nu \to 0$, finishes the proof of Proposition 3.4. ■

3.4. Tail behavior of $X^{(u)}(s)$ for $s \notin S_u$

The probability of an exceedance outside of S_u is bounded by the sum of the following four terms:

$$\begin{aligned} &P\Big\{\exists s \notin S_u : \widetilde{X}^{(u)}(s) > f_u(s)\Big\} \\ &\quad \le P\Big\{\exists s \in (0, s_1] : \widetilde{X}^{(u)}(s) > f_u(s)\Big\} + P\Big\{\exists s \in (s_1, s_u^* - \epsilon(u)) : \widetilde{X}^{(u)}(s) > f_u(s)\Big\} \\ &\quad + P\Big\{\exists s \in (s_u^* + \epsilon(u), s_2) : \widetilde{X}^{(u)}(s) > f_u(s)\Big\} + P\Big\{\exists s \ge s_2 : \widetilde{X}^{(u)}(s) > f_u(s)\Big\}. \end{aligned}$$

for some large s_2.

Now we make use of the Hölder condition (9) which holds for each $X_i^{(u)}(\cdot)$. This implies by applying similar derivations as in the proof of Lemma 3.2 that also

$$\limsup_{u\to\infty} E\Big[X^{(u)}(s) - X^{(u)}(s')\Big]^2 \le G|s - s'|^\gamma,$$

for any s, s' with $|s - s'| \le C$ where $\gamma = \min\{\gamma_i\} > 0$ and $G > 0$ a suitable constant. It means that the Hölder condition (9) holds also for $X^{(u)}(\cdot)$.

Proposition 3.5: *For s_1 and the lower bound $f_{u,1}^-(s)$ of the boundary function introduced in Section* 3.2, *we get*

$$P\Big\{\exists s \in (0, s_1] : \widetilde{X}^{(u)}(s) > f_u(s)\Big\} \le C_0 s_1 u^{(1-\frac{H}{\beta})\frac{2}{\gamma}} \Psi\Big(f_{u,1}^-(s_1)\Big),$$

as $u \to \infty$ with C_0 only depending on γ and G.

Proof: For $0 < s \le s_1$ we have $\min\{f_u(s)\} > \min\{f_{u,1}^-(s)\} = f_{u,1}^-(s_1)$ since $f_{u,1}^-(s)$ is strictly decreasing on $(0, s_1]$. By Theorem 8.1 of Piterbarg (1996) we get

$$\begin{aligned} P\Big\{\exists s \in (0, s_1] : \widetilde{X}^{(u)}(s) > f_u(s)\Big\} &\le P\Big\{\sup_{s\in(0,s_1]} \widetilde{X}^{(u)}(s) > f_u(s_1)\Big\} \\ &\le C_0 s_1 u^{(1-\frac{H}{\beta})\frac{2}{\gamma}} \Psi\Big(f_{u,1}^-(s_1)\Big) \end{aligned}$$

for some C_0 only depending on γ and G. ■

For the second and third interval $(s_1, s_u^* - \epsilon(u))$ and $(s_u^* + \epsilon(u), s_2)$, respectively, we apply the same arguments since they are also bounded intervals. But we need the appropriate smallest boundary values. Therefore let

$$\underline{s}_u^* = \inf\{\text{argmin}\, f_u(s) | s \in (s_1, s_u^* - \epsilon(u))\} \tag{13}$$

$$\overline{s}_u^* = \inf\{\text{argmin}\, f_u(s) | s \in (s_u^* + \epsilon(u), s_2)\}. \tag{14}$$

Proposition 3.6: *For $u \to \infty$ we have*

$$P\Big\{\exists s \in (s_1, s_u^* - \epsilon(u)) : \widetilde{X}^{(u)}(s) > f_u(s)\Big\} \le C_1 u^{(1-\frac{H}{\beta})\frac{2}{\gamma}} \Psi(f_u(\underline{s}_u^*))$$

$$P\Big\{\exists s \in (s_u^* + \epsilon(u), s_2) : \widetilde{X}^{(u)}(s) > f_u(s)\Big\} \le C_2 u^{(1-\frac{H}{\beta})\frac{2}{\gamma}} \Psi(f_u(\overline{s}_u^*))$$

where C_1, C_2 depend only on γ and G.

The last interval is unbounded but is split up into bounded subintervals where the same idea is applied again.

Proposition 3.7: *For any $s_2 > s_0 + \epsilon(u)$, $u \to \infty$ we have*

$$P\Big\{\exists s \ge s_2 : \widetilde{X}^{(u)}(s) > f_u(s)\Big\} = O\left(u^{(1-\frac{H}{\beta})(\frac{2}{\gamma}-1)} e^{-\frac{1}{2}\tilde{c}^2 s_2^{2(\beta-H)}(1+\delta_u(s_2))^2 u^{2-\frac{2H}{\beta}}}\right).$$

Proof: We split the interval $[s_2, \infty)$ into subintervals $I_j = [s_2 + j - 1, s_2 + j)$, $j \ge 1$, and apply again Piterbarg's theorem for every subinterval I_j. We assume for simplicity that C in (9) is larger than 1, otherwise we would select smaller subintervals or adapt the constants in (9). Since $|s - s'| \le 1$ for all I_j, $|I_j| = 1$, we have

$$P\Big\{\exists s \in I_j : \widetilde{X}^{(u)}(s) > f_u(s)\Big\} \le C_3 u^{(1-\frac{H}{\beta})\frac{2}{\gamma}} \Psi(f_u(s_2 + j - 1)),$$

using that $f_u(s)$ is strictly increasing for $s > s_0$, where C_3 depends also only on γ and G, not on j. Hence

$$\begin{aligned} P\Big\{\exists s \ge s_2 : \widetilde{X}^{(u)}(s) > f_u(s)\Big\} &\le \sum_{j=1}^{\infty} P\Big\{\exists\, s \in I_j : \widetilde{X}^{(u)}(s) > f_u(s)\Big\} \\ &\le \sum_{j=1}^{\infty} C_3 u^{(1-\frac{H}{\beta})\frac{2}{\gamma}} \Psi(f_u(s_2 + j - 1)) \\ &\le \frac{C_3 u^{(1-\frac{H}{\beta})(\frac{2}{\gamma}-1)}}{\sqrt{2\pi} v(s_2)(1 + \delta_u(s_2))} \sum_{j=1}^{\infty} e^{-\frac{1}{2} f_u^2(s_2+j-1)}. \end{aligned}$$

To derive an upper bound of the sum, we use that

$$v^2(s) = (1 + 2\tilde{c}s^\beta + \tilde{c}^2 s^{2\beta})/s^{2H} \ge \tilde{c}^2 s^{2(\beta-H)},$$

$$1 + \delta_u(s_2) \le 1 + \delta_u(s_2 + j - 1).$$

Let $C(u) = \tilde{c}^2(1+\delta_u(s_2))^2 u^{2-\frac{2H}{\beta}}$ to get

$$\begin{aligned}\sum_{j=1}^{\infty} e^{-\frac{1}{2}f_u^2(s_2+j-1)} &\leq \sum_{j=1}^{\infty} e^{-\frac{1}{2}u^{2-\frac{2H}{\beta}} v^2(s_2+j-1)(1+\delta_u(s_2))^2} \\ &\leq \sum_{j=1}^{\infty} e^{-\frac{1}{2}C(u)(s_2+j-1)^{2(\beta-H)}} \\ &\leq \int_{s_2}^{\infty} e^{-\frac{1}{2}C(u)x^{2(\beta-H)}} dx + e^{-\frac{1}{2}C(u)s_2^{2(\beta-H)}} \\ &\leq e^{-\frac{1}{2}C(u)s_2^{2(\beta-H)}} s_2^{1-2(\beta-H)}/(C(u)(\beta-H)) + e^{-\frac{1}{2}C(u)s_2^{2(\beta-H)}}\end{aligned}$$

where the last term is the dominating term which implies the statement. Here we use that for $\alpha, S > 0$ and $l \to \infty$ the integral is of order

$$\int_S^{\infty} e^{-lx^\alpha} dx \sim e^{-lS^\alpha} S^{1-\alpha}/(\alpha l).$$

■

3.5. Dominating probability

To prove that

$$P\Big\{\sup_{t>0} (X(t) - ct^\beta) > u\Big\} \sim P\Big\{\exists s \in S_u : \widetilde{X}^{(u)}(s) > f_u(s)\Big\}$$

it remains to show that

$$P\Big\{\exists s \in J : \widetilde{X}^{(u)}(s) > f_u(s)\Big\} = o\Big(P\Big\{\exists s \in S_u : \widetilde{X}^{(u)}(s) > f_u(s)\Big\}\Big) \tag{15}$$

for the intervals $J = (0, s_1], (s_1, s_u^* - \epsilon(u)), (s_u^* + \epsilon(u), s_2)$ and $[s_2, \infty)$.

Lemma 3.5: *For $\underline{s}_u^*$ and $\overline{s}_u^*$ defined in* (13) *and* (14) *we have for any $\chi > 0$ as $u \to \infty$*

$$\exp\Big(-\frac{1}{2}\big(f_u^2(\underline{s}_u^*) - f_u^2(s_u^*)\big)\Big) = o(u^{-\chi}).$$

$$\exp\Big(-\frac{1}{2}\big(f_u^2(\overline{s}_u^*) - f_u^2(s_u^*)\big)\Big) = o(u^{-\chi}).$$

Proof:

1) The definitions of s_u^* and $\underline{s}_u^*$ imply $f_u(s_u^*) < f_u(\underline{s}_u^*)$ and thus $v(s_u^*)(1+\delta_u(s_u^*)) < v(\underline{s}_u^*)(1+\delta_u(\underline{s}_u^*))$.

   i) If $\underline{s}_u^* - s_u^* \not\to 0$ for $u \to \infty$, we have for some constant $C > 0$

   $$f_u^2(\underline{s}_u^*) - f_u^2(s_u^*) = u^{2-\frac{2H}{\beta}}\Big(v^2(\underline{s}_u^*)(1+\delta_u(\underline{s}_u^*))^2 - v^2(s_u^*)(1+\delta_u(s_u^*))^2\Big) \geq Cu^{2-\frac{2H}{\beta}}$$

   and thus $f_u^2(\underline{s}_u^*) - f_u^2(s_u^*) > 2\chi \log u$ for any $\chi > 0$ and u large.

   ii) If $\underline{s}_u^* - s_u^* \to 0$ for $u \to \infty$, then still $s_u^* - \underline{s}_u^* > \epsilon(u)$. We use the Taylor expansion of the function $g_u(s) = v(s)(1+\delta_u(s))$ around s_u^*. The first derivative of $g_u(s)$ is 0 by definition of s_u^*, and the second derivative is positive (> some positive constant) for all large u, since $v''(s_u^*) > 0$. Hence $g_u(\underline{s}_u^*) = g_u(s_u^*) + g_u''(s_u^*)(s_u^* - \underline{s}_u^*)^2/2 + o((s_u^* - \underline{s}_u^*)^2)$ and

$$\begin{aligned} f_u^2(\underline{s}_u^*) - f_u^2(s_u^*) &= u^{2-\frac{2H}{\beta}}(g_u(\underline{s}_u^*) + g_u(s_u^*))(g_u(\underline{s}_u^*) - g_u(s_u^*)) \\ &\geq u^{2-\frac{2H}{\beta}} 2g_u(s_u^*)(g_u(\underline{s}_u^*) - g_u(s_u^*)) \\ &\geq u^{2-\frac{2H}{\beta}} g_u(s_u^*) g_u''(s_u^*)(s_u^* - \underline{s}_u^*)^2(1+o(1)) \\ &> u^{2-\frac{2H}{\beta}} g_u(s_u^*) g_u''(s_u^*)\epsilon^2(u)(1+o(1)) \\ &= O((\log u)^2) \gg 2\chi \log u \end{aligned}$$

for any $\chi > 0$ and u large. Thus the first statement holds in both cases.

2) The second statement follows in the same way since $s_u^* - \overline{s}_u^* \to 0$ for $u \to \infty$ by Propositions 3.2 and again

$$\begin{aligned} f_u^2(\overline{s}_u^*) - f_u^2(s_u^*) &\geq u^{2-\frac{2H}{\beta}} g_u(s_u^*) g_u''(s_u^*)(s_u^* - \overline{s}_u^*)^2(1+o(1)) \\ &> u^{2-\frac{2H}{\beta}} g_u(s_u^*) g_u''(s_u^*)\epsilon^2(u)(1+o(1)) \\ &= O((log\, u)^2) \gg 2\chi\, log\, u \end{aligned}$$

for any $\chi > 0$ and u large. ■

We now prove Eq. 15 for $J = (0, s_1]$ using Propositions 3.4 and 3.5

$$\begin{aligned} &P\left\{\exists s \in (0, s_1] : \widetilde{X}^{(u)}(s) > f_u(s)\right\} \Big/ P\left\{\exists s \in S_u : \widetilde{X}^{(u)}(s) > f_u(s)\right\} \\ &= O\left(u^{(1-\frac{H}{\beta})\frac{2}{\gamma}} u^{2-\frac{2H}{\beta}} K^{-1}(u^{\frac{H}{\beta}-1})(f_{u,1}^-(s_1))^{-1} e^{-\frac{1}{2}\left(\left(f_{u,1}^-(s_1)\right)^2 - f_u^2(s_u^*)\right)}\right) \\ &= O\left(\widetilde{L}(u^{\frac{H}{\beta}-1}) e^{\left(1-\frac{H}{\beta}\right)\left(\frac{2}{\gamma}+1-\frac{2}{\alpha}\right)\log u} e^{-\frac{1}{2}\left(\left(f_{u,1}^-(s_1)\right)^2 - f_u^2(s_u^*)\right)}\right) = o(1), \end{aligned}$$

as $u \to \infty$, where $\widetilde{L}(\cdot)$ denotes the slowly varying part of $K^{-1}(\cdot)$. The last steps holds since the exponent

$$\left(f_{u,1}^-(s_1)\right)^2 - f_u^2(s_u^*) = u^{2-\frac{2H}{\beta}}\left[v^2(s_1)(1+o(1)) - v^2(s_u^*)(1+o(1))\right] > u^{2-\frac{2H}{\beta}}\hat{\epsilon},$$

using that $v^2(s_u^*) \to A^2$ and $v^2(s_1) > A^2 + \hat{\epsilon}$ for some $\hat{\epsilon} > 0$ and $s_1 > 0$ small.

Equation 15 holds in the same way for $J = (s_1, s_{u-\epsilon(u)}^*)$ by Propositions 3.4 and 3.6,

$$\begin{aligned} &P\left\{\exists s \in (s_1,\, s_u^* - \epsilon(u), s_2) : \widetilde{X}^{(u)}(s) > f_u(s)\right\} \Big/ P\left\{\exists s \in S_u : \widetilde{X}^{(u)}(s) > f_u(s)\right\} \\ &= O\left(u^{(1-\frac{H}{\beta})(\frac{2}{\gamma}+2)} K^{-1}(u^{\frac{H}{\beta}-1}) f_u^{-1}(\underline{s}_u^*) e^{-\frac{1}{2}\left(f_u^2(\underline{s}_u^*) - f_u^2(s_u^*)\right)}\right) \\ &= O\left(\widetilde{L}(u^{\frac{H}{\beta}-1}) e^{\left(1-\frac{H}{\beta}\right)\left(\frac{2}{\gamma}+1-\frac{2}{\alpha}\right)\log u - \chi \log u}\right) = o(1), \end{aligned}$$

by using Lemma 3.5 with χ such that $\chi > (1 - \frac{H}{\beta})(\frac{2}{\gamma} + 1 - \frac{2}{\alpha})$.

Equation 15 holds also for $J = (s_u^* + \epsilon(u), s_2)$, since Propositions 3.4 and 3.6 imply

$$\begin{aligned}
&P\left\{\exists s \in (s_u^* + \epsilon(u), s_2) : \widetilde{X}^{(u)}(s) > f_u(s)\right\} \Big/ P\left\{\exists s \in S_u : \widetilde{X}^{(u)}(s) > f_u(s)\right\}\\
&\quad = O\left(u^{(1-\frac{H}{\beta})(\frac{2}{\gamma}+2)} K^{-1}(u^{\frac{H}{\beta}-1}) f_u^{-1}(\overline{s}_u^*) e^{-\frac{1}{2}\left(f_u^2(\overline{s}_u^*) - f_u^2(s_u^*)\right)}\right)\\
&\quad = O\left(\widetilde{L}(u^{\frac{H}{\beta}-1}) e^{\left(1-\frac{H}{\beta}\right)\left(\frac{2}{\gamma}+1-\frac{2}{\alpha}\right)\log u - \frac{1}{2}\left(f_u^2(\overline{s}_u^*) - f_u^2(s_u^*)\right)}\right)\\
&\quad = o(1),
\end{aligned}$$

again by Lemma 3.5.

Finally, Eq. 15 holds for $J = [s_2, \infty)$ by using Propositions 3.4 and 3.7

$$\begin{aligned}
&P\left\{\exists s \geq s_2 : \widetilde{X}^{(u)}(s) > f_u(s)\right\} \Big/ P\left\{\exists s \in S_u : \widetilde{X}^{(u)}(s) > f_u(s)\right\}\\
&\quad = O\left(u^{(1-\frac{H}{\beta})(\frac{2}{\gamma}+1)} K^{-1}(u^{\frac{H}{\beta}-1}) e^{-\frac{1}{2}(\tilde{c}^2 s_2^{2(\beta-H)}(1+\delta_u(s_2))^2 u^{2-\frac{2H}{\beta}} - f_u^2(s_u^*))}\right)\\
&\quad = O\left(\widetilde{L}(u^{\frac{H}{\beta}-1}) e^{\left(1-\frac{H}{\beta}\right)\left(\frac{2}{\gamma}+1-\frac{2}{\alpha}\right)\log u - \frac{1}{2}u^{2-\frac{2H}{\beta}}\left(\tilde{c}^2 s_2^{2(\beta-H)}(1+\delta_u(s_2))^2 - v^2(s_u^*)(1+\delta_u(s_u^*))^2\right)}\right)\\
&\quad = o(1),
\end{aligned}$$

since s_2 is large and $s_u^* \to s_0$.

It means that the statement Eq. 15 is proved for all four subintervals which finishes the proof of the main result.

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