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Global Existence for the Generalized Two-Component Hunter–Saxton System

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Dedicated to Professor H. Okamoto on the occasion of his 55th birthday.

Abstract. We study the global existence of solutions to a two-component generalized Hunter–Saxton system in the periodic setting. We first prove a persistence result for the solutions. Then for some particular choices of the parameters (α, κ), we show the precise blow-up scenarios and the existence of global solutions to the generalized Hunter–Saxton system under proper assumptions on the initial data. This significantly improves recent results.

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1. Introduction

In this paper, we shall investigate the global existence of solutions to the periodic boundary value problem for a two-component family of evolutionary systems modeling fluid convection and stretching in one space dimension,

$$\begin{cases} \partial_t m(t,x) + \underbrace{u\partial_x m}_{\text{convection}} + (1-\alpha)\underbrace{\partial_x u\, m}_{\text{stretching}} + \underbrace{\kappa\, \rho\partial_x\rho}_{\text{coupling}} = 0, \\ m(t,x) = -\partial_{xx}^2 u(t,x), \\ \partial_t\rho + \underbrace{u\partial_x\rho}_{\text{convection}} = \alpha u_x\rho, \\ m(0,x) = m^0(x),\ \rho(0,x) = \rho^0(x), \quad x \in \mathbb{S} \simeq \mathbb{R}/\mathbb{Z}, \end{cases} \tag{1.1}$$

where $(1-\alpha) \in \mathbb{R}$ is the ratio of stretching to convection, and κ denotes a real dimensionless constant that measures the impact of the coupling.

This system was first studied in full generality by Wunsch [48], where it was coined the *generalized Hunter–Saxton system*, since for $(\alpha, \kappa) = (-1, \pm 1)$ it becomes the Hunter–Saxton system [47]. The latter is a particular case of the Gurevich–Zybin system pertaining to nonlinear one-dimensional dynamics of dark matter as well as nonlinear ion-acoustic waves (cf. Pavlov [40] and the references therein).

It was noted by Constantin and Ivanov [10] that the Hunter–Saxton system allows for peakon solutions; moreover, Lenells and Lechtenfeld [31] showed that it can be interpreted as the Euler equation on the superconformal algebra of contact vector fields on the $1|2$-dimensional supercircle, which is in accordance with the by now well-known geometric interpretation of the Hunter–Saxton equation as the geodesic flow of the right-invariant $\dot{H}^1(\mathbb{S})$ metric on the space of orientation-preserving circle diffeomorphisms modulo rigid rotations [26,28–31] (see also [11–13,27,32] for related geodesic equations).

The two-component Hunter–Saxton system is a generalization of the Hunter–Saxton equation modeling the propagation of weakly nonlinear orientation waves in a massive nematic liquid crystal (see Hunter and Saxton [24] for a derivation, and also [2–4,44,51]), since the former obviously reduces to the latter if the initial datum ρ^0 is chosen to vanish identically. It turns out that if this choice is made for arbitrary $\alpha \in \mathbb{R}$, one arrives at the generalized Proudman–Johnson equation [15,36,37,39,41,49] with parameter $a = \alpha - 1$. Through this link, the family of systems (1.1) also bridges the rich theories for the axisymmetric Euler flow in $\mathbb{R}^d$ [36,42] if $\alpha = 2/(d-1)$. We also remark that if one sets $\rho = \sqrt{-1}\ u_x$ and $\kappa = -\alpha$, the system (1.1) decouples to give, once again, the generalized Proudman–Johnson equation [7,36,49] with parameter $a = 2\alpha - 1$. Other important special cases of the generalized Hunter–Saxton system (1.1) include the inviscid Kármán–Batchelor flow [5,6,23] for $\alpha = -\kappa = 1$, which admits global strong solutions, and the celebrated Constantin–Lax–Majda equation [16] with $\alpha = -\kappa = \infty$, a one-dimensional model for three-dimensional vorticity dynamics, which has an abundance of solutions blowing up in finite time.

The Hunter–Saxton system (1.1) with parameters $(\alpha, \kappa) = (-1, \pm 1)$ is the *short-wave limit*, obtained via the space-time scaling $(x, t) \mapsto (\varepsilon x, \varepsilon t)$ and letting ε tend to zero in the resulting equation, of the two-component integrable Camassa–Holm system [10,18]. This system, reading as (1.1) with m replaced by $(1 - \partial_{xx}^2)u$, has recently been the object of intensive study (see [10,18–21,31,34,35,52]). Constantin and Ivanov [10] derived the Camassa–Holm system from the Green-Naghdi equations, which themselves originate in the governing equations for water waves (see [25] for a formal analysis, and [1,14] for a rigorous treatment).

One major motivation for studying systems such as the Camassa–Holm system or the system (1.1) lies in their potential exhibition of nonlinear phenomena such as wave-breaking and peaked traveling waves, which are not inherent to small-amplitude models but known to exist in the case of the governing equations for water waves (prior to performing asymptotic expansions in special regimes like the shallow water regime), cf. [8,25,45,46]. In this context, it is of interest to point out that peaked solitons are absent among the solitary wave solutions to the Camassa–Holm system (cf. [35]), while they exist for the Hunter–Saxton system, see [10].

Another reason—and, indeed, the very incentive in [48] and here—for analyzing the family of systems (1.1) has its origin in a paradigm of Okamoto and Ohkitani [37] that the convection term can play a positive role in the global existence problem for hydrodynamically relevant evolution equations (see also [23,38]). The quadratic terms in the first component of (1.1) represent the competition in fluid convection between nonlinear steepening and amplification due to $(1 - \alpha)$-dimensional stretching and κ-dimensional coupling (cf. [22]). The stretching parameter α illustrates the inherent importance of the convection term in delaying or depleting finite-time blow-up, while the coupling constant κ measures the strength of the coupling, and has a strong influence on singularity formation or global existence of the solutions.

Recently, Wunsch [48] proved that the first solution component breaks down in finite time if $(\alpha, \kappa) \in \{-1\} \times \mathbb{R}_-$ and if the initial slope is large enough; moreover, he demonstrated, for $(\alpha, \kappa) \in [-1/2, 0) \times \mathbb{R}_-$, that a sufficiently negative slope at an inflection point of u^0 will become vertical spontaneously. By analogy with the Constantin–Lax–Majda vorticity model equation [16], the case of ∞-dimensional stretching and coupling (i.e., $\alpha = \kappa = \infty$) was shown to lead to catastrophic steepening of the first solution component u as well. Let us finally mention that there are also global weak solutions to the system (1.1) with $\alpha = -\kappa = -1$ (the Hunter–Saxton system, see [50]).

Outline of results. The main purpose of this paper is to broaden our understanding of solutions to (1.1) by proving rigorously that solutions for some particular cases (e.g., $(\alpha, \kappa) \in \{-1, 0\} \times \mathbb{R}_+$) can be global. In the preliminary Sect. 2, we first recall the local-in-time well-posedness result of system (1.1) for $(\alpha, \kappa) \in \mathbb{R} \times \mathbb{R}$ and provide a partial result on the rate of break-down at the origin for $(\alpha, \kappa) \in \{-1\} \times \mathbb{R}_-$. In Sect. 3, we first prove a persistence result of solutions in $H^s \times H^{s-1}, s \geq 2$, for $(\alpha, \kappa) \in \mathbb{R} \times \mathbb{R}$. In Sect. 4, we derive some precise blow-up scenarios for the solutions in the case $(\alpha, \kappa) \in \{-1\} \times \mathbb{R}_+$. In Sect. 5, we first show the existence global solutions in $H^s \times H^{s-1}$ for $(\alpha, \kappa) \in \{-1\} \times \mathbb{R}_+$ under proper assumptions on the initial data, which replaces the artificial assumption made in Sect. 3 that the gradient of the second solution component ρ be bounded. The global existence of sufficiently regular solutions when $(\alpha, \kappa) \in \{0\} \times \mathbb{R}_+$ is also obtained by using the fact that the equation for ρ is a pure transport one.

Notations. Throughout the paper, $\mathbb{S} = \mathbb{R}/\mathbb{Z}$ shall denote the unit circle. By $H^r(\mathbb{S}), r \geq 0$, we will represent the Sobolev spaces of equivalence classes of functions defined on the unit circle $\mathbb{S}$ which have square-integrable distributional derivatives up to order r. The $H^r(\mathbb{S})$-norm will be designated by $\|.\|_{H^r}$ and the norm of a vector $b \in H^r(\mathbb{S}) \times H^{r-1}(\mathbb{S})$ will be written as $\|b\|_{H^r \times H^{r-1}}$. Also, the Lebesgue spaces of order $p \in [1, \infty]$ will be denoted by $L^p(\mathbb{S})$, and the norms of their elements by $\|f\|_{L^p}$. Finally, if $p = 2$, we agree on the convention $\|.\|_{L^2} := \|.\|$; moreover, $\langle ., . \rangle := \langle ., . \rangle_{L^2}$ will denote the L^2 inner product. The relation symbol $\lesssim$ stands for $\leq C$, where C denotes a generic constant.

2. Preliminaries

We rewrite the first equation in (1.1) and consider the following problem with periodic boundary conditions in the remaining part of the paper:

$$\begin{cases} u_{txx} + (1-\alpha)\, u_x\, u_{xx} + u\, u_{xxx} - \kappa\rho\, \rho_x = 0, \quad t > 0,\ x \in \mathbb{R}, \\ \rho_t + u\, \rho_x = \alpha u_x \rho, \quad t > 0,\ x \in \mathbb{R}, \\ u(t, x+1) = u(t, x), \quad \rho(t, x+1) = \rho(t, x) \quad t \geq 0,\ x \in \mathbb{R}, \\ u(0, x) = u^0(x), \quad \rho(0, x) = \rho^0(x), \quad x \in \mathbb{R}. \end{cases} \tag{2.1}$$

Integration in space of the u-equation in (2.1) yields

$$u_{tx} + uu_{xx} - \frac{\alpha}{2} u_x^2 - \frac{\kappa}{2}\rho^2 = a(t), \tag{2.2}$$

where the time-dependent integration constant $a(t)$ is determined by the periodicity of u to be

$$a(t) = -\frac{\kappa}{2} \int_{\mathbb{S}} \rho^2 dx - \frac{\alpha+2}{2} \int_{\mathbb{S}} u_x^2 dx. \tag{2.3}$$

Integrating in space once more, one gets

$$u_t + uu_x = \partial_x^{-1} \left(\frac{\kappa}{2}\rho^2 + \frac{\alpha+2}{2} u_x^2 + a(t) \right) + h(t), \tag{2.4}$$

where $\partial_x^{-1} f(x) := \int_0^x f(y)dy$ and $h(t) : [0, +\infty) \to \mathbb{R}$ is an arbitrary continuous function.

Remark 2.1. While the operator ∂_x^{-1} in general does not preserve the periodicity of the function it acts on, it turns out that solutions to (2.4) are, in fact, periodic. This can be seen as follows. Observe that $g(t, x) = \frac{\kappa}{2}\rho^2 + \frac{\alpha+2}{2} u_x^2 + a(t)$ is periodic in space, so that, due to the definition of a in (2.3), we see that $\int_x^{x+1} g(t, y)dy = \int_0^1 g(t, y)dy = 0$ holds for all t and x. Thus, it is easy to verify that $\partial_x^{-1} g(t, x)$ is also periodic in space, namely, $\int_0^{x+1} g(t, y)dy - \int_0^x g(t, y)dy = 0$.

We first recall a local well-posedness result of system (2.1) (cf. [48, Theorem 2.1], see also [47, Theorem 4.1] for the special case $\alpha = -1, \kappa = 1$).

Theorem 2.2. *Denote $z = (u, \rho)^{tr}$. Given any $z^0 = (u^0, \rho^0)^{tr} \in H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S}), s \geq 2$, for $(\alpha, \kappa) \in \mathbb{R} \times \mathbb{R}$, there exists a maximal life span $T = T(\|z^0\|_{H^s \times H^{s-1}}) > 0$ and a unique solution z to system (2.1) such that*

$$z \in C([0, T); H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S})) \cap C^1([0, T); H^{s-1}(\mathbb{S}) \times H^{s-2}(\mathbb{S})).$$

Remark 2.3. Following the arguments in [51], it is possible to show that the maximal existence time T of the solution in Theorem 2.2 can be chosen independently of the Sobolev order s.

Lemma 2.4. (cf. [48]) *For $(\alpha, \kappa) \in \mathbb{R} \times \mathbb{R}$, let (u, ρ) be a smooth solution to system (2.1). Then*

$$\frac{d}{dt}\, a(t) = -\frac{3}{2}\kappa(\alpha+1) \int_{\mathbb{S}} u_x \rho^2 dx - \frac{(\alpha+1)(\alpha+2)}{2} \int_{\mathbb{S}} u_x^3 dx.$$

In particular, if $(\alpha, \kappa) = \{-1\} \times \mathbb{R}$*, then* $\frac{d}{dt} a(t) = 0$*, which implies that the system enjoys a conservation law, namely,*

$$a(t) \equiv a(0) = -\frac{1}{2}\|u_x^0\|^2 - \frac{\kappa}{2}\|\rho^0\|^2 \tag{2.5}$$

is constant for all $t \geq 0$*.*

In contrast with the cases $(\alpha, \kappa) \in \{-1, 0\} \times \mathbb{R}_+$ where we shall get global existence (see the subsequent sections), a slope of singularity of system (2.1) has been obtained for the case $(\alpha, \kappa) \in \{-1\} \times \mathbb{R}_-$ (cf. [48, Proposition 3.2]), however, no estimate on the blow-up rate was given there. In what follows, we provide a partial blow-up result at the origin $x = 0$ with blow-up rate for solutions to (2.1).

Proposition 2.5. *Let* $z(t, x)$ *be a solution to* (2.1) *with parameters* $(\alpha, \kappa) \in \{-1\} \times \mathbb{R}_-$ *and initial datum* $z^0 \in H^s \times H^{s-1}, s \geq 2$*. In addition, we assume that* u^0 *is odd with* $u_x^0(0) < 0$ *and* ρ^0 *is even with* $\rho^0(0) = 0$*, and that, moreover,*

$$\|u_x^0\|^2 + \kappa\|\rho^0\|^2 \geq 0. \tag{2.6}$$

Then at the origin $x = 0, u_x(t, 0)$ *blows up in finite time* T_0 *(time of break-down at the origin). The blow-up rate of* $u_x(t, 0)$ *is*

$$\lim_{t \to T_0} \{(T_0 - t)\, u_x(t, 0)\} = -2.$$

Proof. Due to the algebraic structure of the equations in (2.1), we note that (2.1) is invariant under the transformations

$$u(x) \to -u(-x) \quad \text{and} \quad \rho(x) \to \rho(-x).$$

Then under our assumption on the initial data, we see that $u(t, \cdot), \rho(t, \cdot)$ remain odd or even, respectively. Observe next that

$$\rho(t, 0) = 0 \tag{2.7}$$

for all times of existence. Indeed, one has

$$\frac{\partial}{\partial t}\, \rho(t, 0) = -(u\rho_x)(t, 0) - (u_x\rho)(t, 0).$$

Note that the first term on the right-hand side vanishes since both u and ρ_x are odd. Together with the assumption that $\rho^0(0) = 0$, this proves (2.7).

Let us now set

$$\zeta(t) := u_x(t, 0).$$

The resulting ordinary differential equation for the evolution of ζ reads as follows:

$$\begin{cases} \frac{d}{dt}\zeta(t) = -\frac{1}{2}\zeta(t)^2 + \frac{\kappa}{2}\ \rho(t, 0)^2 + a(t) \overset{(2.7)}{=} -\frac{1}{2}\zeta(t)^2 + a(t), \\ \zeta(0) = \zeta^0 = u_x^0(0). \end{cases} \tag{2.8}$$

When $\alpha = -1$, due to Lemma 2.4, $a(t)$ is a constant. Then by (2.6) we know that $a \leq 0$. Thus, we have

$$\frac{d}{dt}\, \zeta(t) \leq -\frac{1}{2}\zeta(t)^2,$$

which implies

$$\zeta(t) \leq \frac{2\zeta^0}{2 + \zeta^0\, t}$$

As a consequence, for $T_0 = -\frac{2}{\zeta^0}$,

$$\lim_{t \to T_0} \zeta(t) = -\infty. \tag{2.9}$$

For $a(0) = -\frac{\kappa}{2}\|\rho^0\|^2 - \frac{1}{2}\|u_x^0\|^2 \le 0$, the following chain of inequalities holds

$$-|a(0)| - 1 < \frac{d}{dt}\,\zeta(t) + \frac{1}{2}\zeta(t)^2 < |a(0)| + 1. \tag{2.10}$$

Because of (2.9), there exists a number $\varepsilon \in (0, \frac{1}{2})$ and a time t_ε such that

$$\zeta(t)^2 \ge \frac{|a(0)| + 1}{\varepsilon} > 0, \quad \forall t \in (t_\varepsilon, T_0).$$

Hence

$$-\frac{1}{2} - \varepsilon < \frac{1}{\zeta(t)^2}\,\frac{d}{dt}\zeta(t) < -\frac{1}{2} + \varepsilon, \quad \forall t \in (t_\varepsilon, T_0),$$

from which we glean, upon integrating from $t > t_\varepsilon$ to T_0, that

$$-\frac{1}{2} - \varepsilon < \frac{1}{(T_0 - t)\;\zeta(t)} < -\frac{1}{2} + \varepsilon.$$

We may thus conclude the assertion of the proposition, since ε was chosen arbitrarily. $\square$

Remark 2.6. It remains an open problem to determine the first time of break-down, since the ODE describing the evolution of $m(t) := \inf_x u(t,x)$ is more involved than (2.8), and double-sided estimates of $\frac{d}{dt}m(t)$—as in (2.10)—would require uniform bounds on $\|\rho(t,.)\|_{L^\infty}$ (cf. [48]). We observe that the rate of break-down we obtained is in accordance with the one computed for the Camassa–Holm system [20].

Remark 2.7. For some special cases, the exact blow-up time T_0 can be computed. For instance,
(i) $\|u_x^0\|^2 = -\kappa\|\rho^0\|^2$, then $a(0) = 0$,
(ii) $\|u_x^0\|^2 = -\kappa\|\rho^0\|^2 + 1$, then $a(0) = -\frac{1}{2}$.
In case (i), the explicit solution to (2.8) with $a(0) = 0$ reads

$$\zeta(t) = \frac{2\zeta^0}{2 + \zeta^0\, t} < 0.$$

Then the blow-up time is given by

$$T_0 = -2/\zeta^0 > 0.$$

In case (ii), the explicit solution to (2.8) with $a(0) = -\frac{1}{2}$ reads

$$\zeta(t) = \tan\left(\arctan(\zeta^0) - \frac{t}{2}\right) < 0. \tag{2.11}$$

Thus the blow-up time of $\zeta(t) = u_x(t,0)$ can be given exactly as

$$T_0 = \pi + 2\arctan(\zeta^0) \;\in\; (0, \pi).$$

We note that a similar conclusion was obtained in [29] for the (one-component) Hunter–Saxton equation by using geometric arguments.

Even if the condition (2.6) does not hold, we can still construct some solutions that break down at the origin.

Corollary 2.8. *Let $z(t,x)$ be a solution to (2.1) with parameters $(\alpha, \kappa) \in \{-1\} \times \mathbb{R}_-$ and initial datum $z^0 \in H^s \times H^{s-1}, s \ge 2$. In addition, we assume that u^0 is odd with $u_x^0(0) < 0$ and ρ^0 is even with $\rho^0(0) = 0$. Moreover, if, instead of (2.6), we assume that*

$$u_x^0(0) < -\sqrt{2\left|-\frac{1}{2}\|u_x^0\|^2 - \frac{\kappa}{2}\|\rho^0\|^2\right|}, \tag{2.12}$$

then $u_x(t,x)$ blows up at the origin $x = 0$ in finite time. The blow-up rate of $u_x(t,0)$ is

$$\lim_{t \to T_0}\{(T_0 - t)\; u_x(t,0)\} = -2.$$

Proof. Similarly to the proof of Proposition 2.5, we have

$$\frac{d}{dt}\,\zeta(t) = -\frac{1}{2}\zeta(t)^2 + a(0) \le -\frac{1}{2}\zeta(t)^2 + |a(0)|. \tag{2.13}$$

Thus, if (2.12) holds, namely, $\zeta(0) < -(2|a(0)|)^{\frac{1}{2}}$, then $\zeta(t) < -(2|a(0)|)^{\frac{1}{2}}$ for all $t \in [0, T_0)$, where $T_0 > 0$ is the existence time (ensured by Theorem 2.2). By solving the standard Riccati type inequality (2.13), it follows that (cf. e.g., [20])

$$\lim_{t\to T_0} \zeta(t) = -\infty, \quad \text{with} \quad 0 < T_0 < (2|a(0)|)^{-\frac{1}{2}} \ln \frac{\zeta^0 - (2|a(0)|)^{\frac{1}{2}}}{\zeta^0 + (2|a(0)|)^{\frac{1}{2}}}.$$

The computation of the blow-up rate is now exactly the same as for Proposition 2.5. □

3. Persistence of Solutions for $(\alpha, \kappa) \in \mathbb{R} \times \mathbb{R}$

In this section, we consider the question of looking for a suitable bound on the solutions to (2.1), which will ensure that the local solutions obtained in Theorem 2.2 can be extended to be global ones.

We first introduce some lemmata that are useful in the subsequent estimates:

Lemma 3.1. (Kato–Ponce commutator estimate) *Denote $\Lambda = (1 - \partial_x^2)^{1/2}$. For $s > 0,\ p \in (1, \infty)$,*

$$\|[\Lambda^s, f]\, v\|_{L^p} \lesssim \|f_x\|_{L^\infty} \|\Lambda^{s-1} v\|_{L^p} + \|\Lambda^s f\|_{L^p} \|v\|_{L^\infty}.$$

Lemma 3.2. *If $s > 0$, then $H^s \cap L^\infty$ is an algebra. Moreover,*

$$\|fg\|_{H^s} \lesssim \|f\|_{L^\infty} \|g\|_{H^s} + \|f\|_{H^s} \|g\|_{L^\infty}.$$

The main result of this section is as follows

Theorem 3.3. *Suppose $(\alpha, \kappa) \in \mathbb{R} \times \mathbb{R}$. For any $z^0 = (u^0, \rho^0)^{tr} \in H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S}), s \ge 2$, let T be the existence time of the solution $z = (u, \rho)^{tr}$ to system (2.1) corresponding to z^0. If there exists a constant $M > 0$ such that*

$$\|u_x(t, \cdot)\|_{L^\infty} + \|\rho(t, \cdot)\|_{L^\infty} + \|\rho_x(t, \cdot)\|_{L^\infty} \le M, \quad \forall t \in [0, T), \tag{3.1}$$

then $\|z(t, \cdot)\|_{H^s \times H^{s-1}}$ is bounded on $[0, T)$.

Proof. In the proof, we perform only formal calculations which can, however, be justified rigorously using Friedrichs' mollifiers and passing to the limit (cf. [33,43]). The proof is similar to [18, Theorem 3.1] for the 2-component Camassa–Holm equations but requires some modifications. For the sake of completeness, we sketch it here.

First, we notice that for any function $f \in H^s$ satisfying $\int_{\mathbb{S}} f dx = 0$ (with zero mean), there holds

$$\|\partial_x^{-1} f\|_{H^{s+1}} \lesssim \|f\|_{H^s}.$$

Step 1. Estimates for the first component u.

For $s \ge 2$, we calculate that (using (2.4), (2.3) and taking $h = 0$)

$$\begin{aligned} \frac{d}{dt}\|u\|_{H^s}^2 &= 2\langle \Lambda^s u_t, \Lambda^s u \rangle \\ &= -2\langle \Lambda^s (u u_x), \Lambda^s u \rangle + (\alpha + 2) \left\langle \Lambda^s \partial_x^{-1} \left(u_x^2 - \int_{\mathbb{S}} u_x^2 dx \right), \Lambda^s u \right\rangle \\ &\quad + \kappa \left\langle \Lambda^s \partial_x^{-1} \left(\rho^2 - \int_{\mathbb{S}} \rho^2 dx \right), \Lambda^s u \right\rangle \\ &:= I_1 + I_2 + I_3. \end{aligned}$$

The first term can be estimated as in [43] by using the Kato–Ponce estimate:

$$
\begin{aligned}
|I_1| &= 2|\langle [\Lambda^s, u]u_x, \Lambda^s u\rangle + \langle u\Lambda^s u_x, \Lambda^s u\rangle| \\
&\leq 2|\langle [\Lambda^s, u]u_x, \Lambda^s u\rangle| + |\langle u_x\Lambda^s u, \Lambda^s u\rangle| \\
&\lesssim \|[\Lambda^s, u]u_x\|\|\Lambda^s u\| + \|u_x\|_{L^\infty}\|\Lambda^s u\|^2 \\
&\lesssim \|u_x\|_{L^\infty}\|u\|^2_{H^s}.
\end{aligned}
$$

For the second term, by the Kato–Ponce commutator estimate (Lemma 3.1), the fact $s-1\geq 1$ and the continuous embedding $H^{s-1}(\mathbb{S}) \hookrightarrow L^\infty(\mathbb{S})$ to obtain

$$
\begin{aligned}
|I_2| &\leq |\alpha+2| \left\| u_x^2 - \int_{\mathbb{S}} u_x^2 dx \right\|_{H^{s-1}} \|u\|_{H^s} \\
&\leq |\alpha+2| \left(\|u_x^2\|_{H^{s-1}} + \left\| \int_{\mathbb{S}} u_x^2 dx \right\|_{H^{s-1}} \right) \|u\|_{H^s} \\
&\lesssim |\alpha+2| \left(\|u_x\|_{L^\infty}\|u_x\|_{H^{s-1}} + \left| \int_{\mathbb{S}} u_x^2 dx \right| \right) \|u\|_{H^s} \\
&\lesssim |\alpha+2|(\|u_x\|_{L^\infty}\|u\|_{H^{s-1}}\|u\|_{H^s} + \|u_x\|^2_{L^\infty}\|u\|_{H^s}) \\
&\lesssim |\alpha+2|\|u_x\|_{L^\infty}\|u\|^2_{H^s}.
\end{aligned}
$$

Similarly, we can bound the third term involving the density ρ by

$$
\begin{aligned}
|I_3| &\lesssim |\kappa| \left\| \rho^2 - \int_{\mathbb{S}} \rho^2 dx \right\|_{H^{s-1}} \|u\|_{H^s} \\
&\lesssim |\kappa|(\|\rho\|_{L^\infty}\|\rho\|_{H^{s-1}} + \|\rho\|^2_{L^\infty})\|u\|_{H^s} \\
&\lesssim |\kappa|\|\rho\|_{L^\infty}\|\rho\|^2_{H^{s-1}} + |\kappa|\|\rho\|_{L^\infty}\|u\|^2_{H^s}.
\end{aligned}
$$

We infer from the estimates for I_1, I_2, I_3 that

$$
\begin{aligned}
\frac{d}{dt}\|u\|^2_{H^s} \lesssim{}& [(|\alpha+2|+1)\|u_x\|_{L^\infty} + |\kappa|\|\rho\|_{L^\infty}]\|u\|^2_{H^s} \\
&+ |\kappa|\|\rho\|_{L^\infty}\|\rho\|^2_{H^{s-1}}.
\end{aligned} \tag{3.2}
$$

Step 2. Estimates for the second component ρ.

We calculate that

$$
\begin{aligned}
\frac{d}{dt}\|\rho\|^2_{H^{s-1}} &= 2\alpha\langle \Lambda^{s-1}(u_x\rho), \Lambda^{s-1}\rho\rangle - 2\langle \Lambda^{s-1}(\rho_x u), \Lambda^{s-1}\rho\rangle \\
&:= J_1 + J_2.
\end{aligned} \tag{3.3}
$$

The first term J_1 can be estimated like I_2 by Lemma 3.2

$$
\begin{aligned}
|J_1| &\lesssim |\alpha|\|u_x\rho\|_{H^{s-1}}\|\rho\|_{H^{s-1}} \\
&\lesssim |\alpha|(\|u_x\|_{L^\infty}\|\rho\|_{H^{s-1}} + \|\rho\|_{L^\infty}\|u_x\|_{H^{s-1}})\|\rho\|_{H^{s-1}} \\
&\lesssim |\alpha|(\|u_x\|_{L^\infty} + \|\rho\|_{L^\infty})(\|u\|^2_{H^s} + \|\rho\|^2_{H^{s-1}}).
\end{aligned} \tag{3.4}
$$

Then we apply by the Kato–Ponce estimate (Lemma 3.1) to J_2:

$$\begin{aligned} |J_2| &= 2|\langle[\Lambda^{s-1},u]\rho_x,\Lambda^{s-1}\rho\rangle + \langle u\Lambda^{s-1}\rho_x,\Lambda^{s-1}\rho\rangle| \\ &\le 2|\langle[\Lambda^{s-1},u]\rho_x,\Lambda^{s-1}\rho\rangle| + |\langle u_x\Lambda^{s-1}\rho,\Lambda^{s-1}\rho\rangle| \\ &\lesssim \|[\Lambda^{s-1},u]\rho_x\|\|\Lambda^{s-1}\rho\| + \|u_x\|_{L^\infty}\|\Lambda^{s-1}\rho\|^2 \\ &\lesssim \|u_x\|_{L^\infty}\|\Lambda^{s-2}\rho_x\|\|\Lambda^{s-1}\rho\| + \|\rho_x\|_{L^\infty}\|\Lambda^{s-1}u\|\|\Lambda^{s-1}\rho\| \\ &\quad + \|u_x\|_{L^\infty}\|\Lambda^{s-1}\rho\|^2 \\ &\lesssim \|u_x\|_{L^\infty}\|\Lambda^{s-1}\rho\|^2 + \|\rho_x\|_{L^\infty}\|\Lambda^{s-1}u\|\|\Lambda^{s-1}\rho\|. \end{aligned} \tag{3.5}$$

It follows from (3.3)–(3.5) and the Hölder inequality that

$$\begin{aligned} \frac{d}{dt}\|\rho\|^2_{H^{s-1}} &\lesssim [(1+|\alpha|)\|u_x\|_{L^\infty} + |\alpha|\|\rho\|_{L^\infty} + \|\rho_x\|_{L^\infty}] \\ &\quad \times(\|u\|^2_{H^s} + \|\rho\|^2_{H^{s-1}}). \end{aligned} \tag{3.6}$$

Combining (3.2) and (3.6), we can see that

$$\begin{aligned} \frac{d}{dt}(\|u\|^2_{H^s} + \|\rho\|^2_{H^{s-1}}) &\lesssim [(1+|\alpha|+|\alpha+2|)\|u_x\|_{L^\infty} + (|\alpha|+|\kappa|)\|\rho\|_{L^\infty} + \|\rho_x\|_{L^\infty}] \\ &\quad \times(\|u\|^2_{H^s} + \|\rho\|^2_{H^{s-1}}). \end{aligned}$$

Under the assumption (3.1), for $t \in [0,T)$, it holds

$$\frac{d}{dt}(\|u\|^2_{H^s} + \|\rho\|^2_{H^{s-1}}) \lesssim (1+|\alpha|+|\alpha+2|+|\kappa|)M(\|u\|^2_{H^s} + \|\rho\|^2_{H^{s-1}}).$$

By the Gronwall inequality, we see that $\|(u,\rho)^{tr}\|_{H^s\times H^{s-1}}$ is bounded for $t \in [0,T)$. The proof is complete. ☐

4. Blow-up Scenarios for $(\alpha,\kappa) \in \{-1\} \times \mathbb{R}_+$

In Sect. 3, we have shown a persistence result for all $(\alpha,\kappa) \in \mathbb{R}\times\mathbb{R}$. Concerning the interesting case of $(\alpha,\kappa) \in \{-1\}\times\mathbb{R}_+$, we consider the precise blow-up scenarios for regular solutions.

Lemma 4.1. *Suppose that $(\alpha,\kappa) \in \{-1\}\times\mathbb{R}_+$. Given any $z^0 = (u^0,\rho^0)^{tr} \in H^s(\mathbb{S})\times H^{s-1}(\mathbb{S})$, $s \ge 2$. For the solution $z = (u,\rho)^{tr}$ of system (2.1) corresponding to z^0, we have*

$$\|\rho(t)\|^2 + \|u_x(t)\|^2 \le C(\|\rho^0\|, \|u^0_x\|), \quad \forall\ t \in [0,T). \tag{4.1}$$

Moreover,

$$\|u(t)\| \le C(T, \|u^0\|_{H^1}, \|\rho^0\|), \quad \forall\ t \in [0,T). \tag{4.2}$$

Proof. Estimate (4.1) follows from Lemma 2.4 and the facts that $\alpha = -1, \kappa > 0$. For the proof of (4.2), we refer to [47, pp. 653]. ☐

Let $u(t,x)$ be the solution of (2.1). We consider the initial value problem for the Lagrangian flow map:

$$\partial_t\, \varphi(t,x) = u(t,\varphi(t,x)), \quad \varphi(0,x) = x. \tag{4.3}$$

We note that this local flow is a geodesic flow and refer to [17] for details about geometric aspects of two-component systems similar to (1.1). It is well-known that (cf. e.g., [33]) the following lemma is valid.

Lemma 4.2. *Let $u \in C([0,T);H^s)\cap C^1([0,T);H^{s-1})$, $s \ge 2$. Then problem (4.3) admits a unique solution $\varphi \in C^1([0,T)\times\mathbb{S};\mathbb{S})$. Moreover, $\{\varphi(t,\cdot)\}_{t\in[0,T)}$ is a family of orientation-preserving diffeomorphisms on the circle $\mathbb{S}$ and*

$$\varphi_x(t,x) = e^{\int_0^t u_x(s,\varphi(s,x))ds} > 0, \quad \forall\ (t,x) \in [0,T)\times\mathbb{S}. \tag{4.4}$$

Moreover, if $\alpha = -1$, in analogy to [18, Lemma 3.4], we can show that

Lemma 4.3. *Suppose that* $(\alpha, \kappa) \in \{-1\} \times \mathbb{R}_+$. *Given any* $z^0 = (u^0, \rho^0)^{tr} \in H^2(\mathbb{S}) \times H^1(\mathbb{S})$. *Let* $z = (u, \rho)^{tr}$ *be the solution to system* (2.1) *corresponding to* z^0 *on* $[0, T)$. *We have*

$$\rho(t, \varphi(t, x))\varphi_x(t, x) = \rho^0(x), \quad \forall\ (t, x) \in [0, T) \times \mathbb{S}. \tag{4.5}$$

Moreover, if there exists $M_1 > 0$ *such that* $u_x(t, x) \geq -M_1$ *for all* $(t, x) \in [0, T) \times \mathbb{S}$, *then*

$$\|\rho(t, \cdot)\|_{L^\infty} = \|\rho(t, \varphi(t, \cdot))\|_{L^\infty} \leq e^{M_1 T}\|\rho^0(\cdot)\|_{L^\infty}, \quad \forall\ t \in [0, T). \tag{4.6}$$

Theorem 4.4. *Suppose that* $(\alpha, \kappa) \in \{-1\} \times \mathbb{R}_+$. *For any* $z^0 = (u^0, \rho^0)^{tr} \in H^2(\mathbb{S}) \times H^1(\mathbb{S})$, *let* T *be the maximal existence time of the solution* $z = (u, \rho)^{tr}$ *to* (2.1) *corresponding to the initial datum* z^0. *Then the solution blows up in finite time if and only if*

$$\liminf_{t \to T^-} \left\{ \inf_{x \in \mathbb{S}} u_x(t, x) \right\} = -\infty. \tag{4.7}$$

Proof. Consider the equation describing the dynamics of $\rho(t, x)$ in (2.1), and differentiate it once in space

$$\rho_{xt} + u\rho_{xx} + 2u_x\rho_x + u_{xx}\rho = 0. \tag{4.8}$$

Multiplying the first equation in (2.1) by u_{xx}, and (4.8) by ρ_x, upon adding the resultants together, we deduce that

$$\frac{1}{2}\partial_t(u_{xx}^2 + \rho_x^2) + 2u_x(u_{xx}^2 + \rho_x^2) + \frac{1}{2}u\partial_x(u_{xx}^2 + \rho_x^2) + (1 - \kappa)\rho\rho_x u_{xx} = 0. \tag{4.9}$$

Integrating in space and using the periodic boundary conditions for u, ρ, we have

$$\begin{aligned} \frac{d}{dt}(\|u_{xx}\|^2 + \|\rho_x\|^2) &= -3\int_{\mathbb{S}} u_x(u_{xx}^2 + \rho_x^2)dx + (1 - \kappa)\int_{\mathbb{S}} \rho\rho_x u_{xx}dx \\ &\leq -3\int_{\mathbb{S}} u_x(u_{xx}^2 + \rho_x^2)dx + |1 - \kappa|\|\rho\|_{L^\infty}\int_{\mathbb{S}}(u_{xx}^2 + \rho_x^2)dx. \end{aligned} \tag{4.10}$$

Assume that there exists $M_1 > 0$ such that

$$u_x(t, x) \geq -M_1, \quad \forall\ (t, x) \in [0, T) \times \mathbb{S}. \tag{4.11}$$

Then it follows from (4.6) and (4.10) that

$$\frac{d}{dt}(\|u_{xx}\|^2 + \|\rho_x\|^2) \leq (3M_1 + |1 - \kappa|e^{M_1 T}\|\rho^0\|_{L^\infty})(\|u_{xx}\|^2 + \|\rho_x\|^2).$$

By Gronwall's inequality we have

$$\|u_{xx}(t)\|^2 + \|\rho_x(t)\|^2 \leq (\|u_{xx}^0\|^2 + \|\rho_x^0\|^2)e^{(3M_1 + |1-\kappa|e^{M_1 T}\|\rho^0\|_{L^\infty})t}, \quad \forall\ t \in [0, T).$$

This and Lemma 4.1 ensure that the solution z does not blow up in finite time.

On the other hand, by Theorem 3.3, we see that if (4.7) holds, then the solution will blow up in finite time. The proof is complete. □

Lemma 4.5. *Suppose that* $(\alpha, \kappa) \in \{-1\} \times \mathbb{R}_+$. *Let* $z^0 = (u^0, \rho^0)^{tr} \in H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S}), s > 2$, *and let* T *be the maximal existence time of the solution* $z = (u, \rho)^{tr}$ *to* (2.1) *with the initial datum* z^0. *If there exist two constants* $M_1, M_2 > 0$ *such that*

$$u_x(t, x) \geq -M_1, \quad \|\rho_x(t, \cdot)\|_{L^\infty} \leq M_2, \quad \forall\ (t, x) \in [0, T) \times \mathbb{S}, \tag{4.12}$$

then $\|z(t, \cdot)\|_{H^s \times H^{s-1}}$ *will not blow up in finite time.*

Proof. Under our current assumption (4.12), it follows from the argument in Theorem 4.4 that $\|z\|_{H^2 \times H^1}$ is bounded for all $t \in [0, T)$. By Sobolev's embedding theorem, we can see that $\|u_x\|_{L^\infty}$ and $\|\rho\|_{L^\infty}$ are also bounded. Then our conclusion easily follows from Theorem 3.3. □

Now we discuss a first precise blow-up scenario for sufficiently regular solutions:

Theorem 4.6. *Suppose that* $(\alpha, \kappa) \in \{-1\} \times \mathbb{R}_+$*. For any* $z^0 = (u^0, \rho^0)^{tr} \in H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S}), s > \frac{5}{2}$*, let* T *be the maximal existence time of the solution* $z = (u, \rho)^{tr}$ *to* (2.1) *with initial datum* z^0*. Then the corresponding solution blows up in finite time if and only if*

$$\liminf_{t\to T^-} \left\{ \inf_{x\in\mathbb{S}} u_x(t,x) \right\} = -\infty, \quad or \quad \limsup_{t\to T^-} \|\rho_x(t,\cdot)\|_{L^\infty} = +\infty. \tag{4.13}$$

Proof. By Lemma 4.5, we can see that if there exist $M_1, M_2 > 0$ such that assumption (4.12) are satisfied, then $\|z\|_{H^s\times H^{s-1}}$ will not blow up in finite time. On the other hand, by Sobolev's embedding theorem, we see that if (4.13) holds, then the solution will blow up in finite time. The proof is complete. □

We note that a blow-up scenario similar to Theorem 4.6 that involves the condition (4.13) was obtained for regular solutions to a two-component Camassa–Holm equations (cf. e.g., [18]). Later in [52], the authors obtained an improved blow-up scenario that only needs the condition on one of the component (i.e., (4.7) for u). In what follows, we derive an improved blow-up scenario for our two-component Hunter–Saxton system, which shows that the assumption (4.7), is actually enough to determine wave breaking of the regular solutions ($s > \frac{5}{2}$) in finite time. The key observation is that the quantity $\|\rho_x\|_{L^\infty}$ can be controlled by the lower-bound of u_x.

Theorem 4.7. *Suppose that* $(\alpha, \kappa) \in \{-1\} \times \mathbb{R}_+$*. For any* $z^0 = (u^0, \rho^0)^{tr} \in H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S}), s > \frac{5}{2}$*, let* T *be the maximal existence time of the solution* $z = (u, \rho)^{tr}$ *to* (2.1) *with initial datum* z^0*. Then the corresponding solution blows up in finite time if and only if* (4.7) *holds.*

Proof. Using the Lagrangian flow map, we set

$$\begin{aligned} M(t,x) &= u_x(t,\varphi(t,x)),\\ \gamma(t,x) &= \rho(t,\varphi(t,x)),\\ N(t,x) &= u_{xx}(t,\varphi(t,x)),\\ \varpi(t,x) &= \rho_x(t,\varphi(t,x)). \end{aligned}$$

It follows from (4.9) that

$$\partial_t(N^2 + \varpi^2) + 4M(N^2 + \varpi^2) + 2(1-\kappa)\gamma\varpi N = 0. \tag{4.14}$$

Then we have

$$\partial_t(N^2 + \varpi^2) \le (-4M + |1-\kappa|\|\gamma\|_{L^\infty})(N^2 + \varpi^2). \tag{4.15}$$

Assume that there exists $M_1 > 0$ such that (4.11) holds. Then it follows from (4.6) and (4.15) that

$$\partial_t(N^2 + \varpi^2) \le (4M_1 + |1-\kappa| e^{M_1 T}\|\rho^0\|_{L^\infty})(N^2 + \varpi^2).$$

By Gronwall's inequality and the Sobolev embedding theorem ($s > \frac{5}{2}$), for all $(t, x) \in [0, T) \times \mathbb{S}$, we have

$$\begin{aligned} N(t,x)^2 + \varpi(t,x)^2 &\le (\|u^0_{xx}\|^2_{L^\infty} + \|\rho^0_x\|^2_{L^\infty}) e^{(4M_1 + |1-\kappa| e^{M_1 T}\|\rho^0\|_{L^\infty})t}\\ &\le C(\|u^0\|^2_{H^s} + \|\rho^0\|^2_{H^{s-1}}) e^{(4M_1 + |1-\kappa| e^{M_1 T}\|\rho^0\|_{L^\infty})t}. \end{aligned}$$

In particular, this implies that there exists a constant $M_2 > 0$ such that

$$\|\rho_x(t,\cdot)\|_{L^\infty} \le M_2, \quad \forall\, t \in [0, T). \tag{4.16}$$

Thus, the condition (4.12) made in Lemma 4.5 is now satisfied, and as a result, we conclude that $\|z(t,\cdot)\|_{H^s\times H^{s-1}}$ will not blow up in finite time.

On the other hand, by Theorem 3.3, we see that if (4.7) holds, then the solution will blow up in finite time. The proof is complete. □

5. Global Existence for $(\alpha, \kappa) \in \{-1, 0\} \times \mathbb{R}_+$

We remark here that the assumption (3.1) is somewhat artificial: There is no reason to assume that $\|\rho_x(t,\cdot)\|_{L^\infty}$ actually stays bounded in time (but note that this assumption was also made in [18] for the 2-component Camassa–Holm equations). It turns out, however, that we can dispense with (3.1) if we impose some sign condition on the initial datum ρ^0. Our results show that if $\rho^0(x)$ keeps its sign for all $x \in \mathbb{S}$, then existence of global solutions to system (2.1) will be guaranteed for $(\alpha, \kappa) \in \{-1\} \times \mathbb{R}_+$. Besides, in the previous work [48], a smallness condition on the quantity $\|u_x^0\|^2 + \kappa\|\rho^0\|^2$ was required to obtain the (global-in-time) lower-order estimate of the solutions. In what follows, we improve the former results by showing that only the sign condition of the initial data can ensure the existence of regular solutions of our system.

Theorem 5.1. *Suppose that $(\alpha, \kappa) \in \{-1\} \times \mathbb{R}_+$. Given any $z^0 = (u^0, \rho^0)^{tr} \in H^2(\mathbb{S}) \times H^1(\mathbb{S})$, if we further assume that*

$$\rho^0(x) > 0, \quad or \quad \rho^0(x) < 0, \tag{5.1}$$

then the solution $z = (u, \rho)^{tr}$ to system (2.1) corresponding to z^0 is global.

Proof. It suffices to get some uniform *a priori* estimates for the solution $(u, \rho)^{tr}$.

Step 1. Estimates for $\|\rho\|_{L^\infty}$ and $\|u_x\|_{L^\infty}$.

Using the Lagrangian flow map, we set

$$M(t,x) = u_x(t, \varphi(t,x)), \quad \gamma(t,x) = \rho(t, \varphi(t,x)).$$

Then we have (cf. [10,47,48])

$$\begin{aligned} M_t(t,x) &= -\frac{1}{2}M(t,x)^2 + \frac{\kappa}{2}\gamma(t,x)^2 + a, \\ \gamma_t(t,x) &= -M(t,x)\gamma(t,x). \end{aligned}$$

By the assumptions (5.1), we infer from (4.4) and (4.5) that if $\gamma(0,x) > 0$ (or $\gamma(0,x) < 0$) then $\gamma(t,x) > 0$ (or $\gamma(t,x) < 0$) for $t \in [0,T)$. Thus, we can construct the following strictly positive auxiliary function (cf. [10,47])

$$w(t,x) := \kappa\gamma(0,x)\gamma(t,x) + \frac{\gamma(0,x)}{\gamma(t,x)}(1 + M(t,x)^2).$$

Computing the evolution of w, we get

$$\begin{aligned} \partial_t w(t,x) &= \kappa\gamma(0,x)\, \partial_t\gamma(t,x) - \frac{\gamma(0,x)}{\gamma(t,x)^2}\, \partial_t\gamma(t,x)(1 + M(t,x)^2) \\ &\quad + 2\frac{\gamma(0,x)}{\gamma(t,x)}M(t,x)\, \partial_t M(t,x) \\ &= -\kappa\gamma(0,x)\gamma(t,x)M(t,x) + \frac{\gamma(0,x)}{\gamma(t,x)^2}\gamma(t,x)M(t,x)(1 + M(t,x)^2) \\ &\quad - \frac{\gamma(0,x)}{\gamma(t,x)}M(t,x)^3 + \kappa\frac{\gamma(0,x)}{\gamma(t,x)}M(t,x)\gamma(t,x)^2 + 2a\frac{\gamma(0,x)}{\gamma(t,x)}M(t,x) \\ &= (1 + 2a)\frac{\gamma(0,x)}{\gamma(t,x)}M(t,x). \end{aligned}$$

The last quantity can be estimated by

$$\begin{aligned} (1 + 2|a|)\frac{\gamma(0,x)}{\gamma(t,x)}|M(t,x)| &\le (1 + 2|a|)\frac{\gamma(0,x)}{\gamma(t,x)}(1 + M(t,x)^2) \\ &\le (1 + 2|a|)w(t,x). \end{aligned}$$

By Gronwall's inequality, we obtain

$$w(t,x) \le w(0,x)\, e^{(1+2|a|)t}, \quad \forall\, t \in [0,T), \tag{5.2}$$

which together with (2.5) implies the following estimate

$$\begin{aligned}\|\rho(t)\|_{L^\infty} + \|u_x(t)\|_{L^\infty} &\le C(T, \|\rho^0\|_{L^\infty}, \|u_x^0\|_{L^\infty}) \\ &\le C(T, \|\rho^0\|_{H^1}, \|u^0\|_{H^2}).\end{aligned} \tag{5.3}$$

Step 2. Estimates for $\|u\|_{H^2}$ and $\|\rho\|_{H^1}$.

It follows from (4.10) that

$$\frac{d}{dt}(\|u_{xx}\|^2 + \|\rho_x\|^2) \le (3\|u_x\|_{L^\infty} + |1-\kappa|\|\rho\|_{L^\infty})(\|u_{xx}\|^2 + \|\rho_x\|^2).$$

By Gronwall's inequality we have

$$\|u_{xx}(t)\|^2 + \|\rho_x(t)\|^2 \le e^{(3\|u_x\|_{L^\infty} + |1-\kappa|\|\rho\|_{L^\infty})t}(\|u_{xx}^0\|^2 + \|\rho_x^0\|^2),$$

which together with (5.3) and Lemma 4.1 yields that

$$\|u(t)\|_{H^2}^2 + \|\rho(t)\|_{H^1}^2 \le C(T, \kappa, \|\rho^0\|_{H^1}, \|u^0\|_{H^2}), \quad \forall\, t \in [0,T). \tag{5.4}$$

The proof is complete. □

Remark 5.2. We notice that, for $\kappa = 1$, one only needs a bound on $\|u_x(t,.)\|_{L^\infty}$ to get the uniform estimate (5.4). Besides, in contrast with [47, Proposition 6.1], we have shown that in order to have global existence in $H^2 \times H^1$, one does not need to impose certain smallness assumptions on the initial data, and it only requires that ρ^0 is strictly nonzero (cf. (5.1)). This follows from an idea of [20].

Theorem 5.3. *Suppose that $(\alpha, \kappa) \in \{-1\} \times \mathbb{R}_+$. Let $z^0 = (u^0, \rho^0)^{tr} \in H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S}), s > \frac{5}{2}$ satisfying (5.1) be given. Then system (2.1) admits a unique global solution $z = (u, \rho)^{tr}$ such that for any $T > 0$,*

$$\|z(t,\cdot)\|_{H^s \times H^{s-1}} \le C, \quad \forall\, t \in [0,T),$$

where C is a constant depending on $\|u_0\|_{H^s}, \|\rho_0\|_{H^{s-1}}$ and T.

Proof. We know that if $(u^0, \rho^0) \in H^2 \times H^1$, there exists a constant $K > 0$ such that (cf. (5.3))

$$\|u_x(t)\|_{L^\infty} + \|\rho(t)\|_{L^\infty} \le K, \quad \forall\, t \in [0,T),$$

which implies (4.11). Then as in the proof of Theorem 4.7, we obtain the estimate of $\|\rho_x\|$ on $[0,T)$ (cf. (4.16)). This yields that the condition (4.12) in Lemma 4.5 is satisfied, and thus leads to the conclusion. □

From Lemma 2.4, we see that a very important property for the case $(\alpha, \kappa) \in \{-1\} \times \mathbb{R}_+$ is the conservation law for the quantity $a(t)$, which can be bounded by $\|u_x^0\|, \|\rho^0\|$. However, this nice property may be lost for other choices of α (cf. Lemma 2.4). This fact leads to a different procedure to prove that solutions exist globally.

Theorem 5.4. *Suppose that $(\alpha, \kappa) \in \{0\} \times \mathbb{R}_+$. Given any $z^0 = (u^0, \rho^0)^{tr} \in H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S}), s \ge 3$, we assume that ρ^0 satisfies the sign condition (5.1), and T is the existence time of the solution $z = (u, \rho)^{tr}$ to system (2.1) corresponding to z^0. Then $\|z(t,\cdot)\|_{H^s \times H^{s-1}}$ is bounded on $[0,T)$.*

Proof. Now for $\alpha = 0$, we no longer have the conservation law for $a(t)$ (cf. Lemma 2.4). As a result, we lose the control of $\|\rho\|, \|u_x\|$, in contrast with the case $\alpha = -1$ (see Lemma 4.1). Fortunately, however, the equation for ρ now is just a transport equation, which implies that

$$\|\rho(t,\cdot)\|_{L^\infty} = \|\rho^0\|_{L^\infty} \le C(\|\rho^0\|_{H^2}), \quad \forall\, t \in [0,T). \tag{5.5}$$

Besides, it follows from [48, Proposition 4.1] that for $t \in [0,T)$,

$$\sup_{x \in \mathbb{S}} u_x(t,x) \le C(\|u^0\|_{H^3}, \|\rho^0\|_{H^2}, \kappa, T), \tag{5.6}$$

$$\|u_x(t)\| \le C(\|u^0\|_{H^3}, \|\rho^0\|_{H^2}, \kappa, T). \tag{5.7}$$

Multiplying (2.4) by u, integrating over the circle, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u\|^2 &= \int_{\mathbb{S}} u\partial_x^{-1}\left(\frac{1}{2}\rho^2+u_x^2+a(t)\right)dx + h(t)\int_{\mathbb{S}} udx \\ &\le \frac{1}{2}|h(t)| + \frac{1}{2}(1+|h(t)|)\int_{\mathbb{S}} u^2dx + \frac{1}{2}\left[\int_{\mathbb{S}}\left(\frac{1}{2}\rho^2+u_x^2+|a(t)|\right)dx\right]^2.\end{aligned}$$

It follows from (5.5), (5.7) that

$$\begin{aligned}|a(t)| &\le \frac{\kappa}{2}\|\rho(t)\|_{L^\infty}^2 + \|u_x(t)\|^2 \\ &\le C(\|u^0\|_{H^3}, \|\rho^0\|_{H^2}, \kappa, T), \quad \forall t\in[0,T),\end{aligned} \tag{5.8}$$

which yields

$$\frac{d}{dt}\|u\|^2 \le C(1+\|u\|^2) + C',$$

where C is a constant depending on $h(t)$ and C' is a constant depending on $\|u^0\|_{H^3}, \|\rho^0\|_{H^2}, T$ and κ. By the Gronwall inequality and (5.7), we see that

$$\|u(t)\|_{H^1} \le C(\|u^0\|_{H^3}, \|\rho^0\|_{H^2}, \kappa, T), \quad t\in[0,T). \tag{5.9}$$

Recalling the functions M, γ introduced in the proof of Theorem 5.1, in our present case we have

$$\partial_t M(t,x) = \frac{\kappa}{2}\gamma(t,x)^2 + a(t), \quad \partial_t\gamma(t,x) = 0.$$

Then we compute the time derivative of

$$\tilde{w}(t,x) = \kappa\gamma(0,x)^2 + (1+M(t,x)^2)$$

such that

$$\begin{aligned}\partial_t\tilde{w} &= 2M\partial_t M = (\kappa\gamma^2+2a)M \\ &\le \frac{\kappa}{2}\gamma^2(1+M^2) + |a|(1+M^2) \\ &\le \left(\frac{\kappa}{2}\gamma^2+|a|\right)\tilde{w}.\end{aligned}$$

It follows from the Gronwall inequality, (5.5), (5.8), and the definition of $\tilde{w}$ that

$$\|u_x(t,\cdot)\|_{L^\infty} \le C(\|u^0\|_{H^3}, \|\rho^0\|_{H^2}, \kappa, T), \quad \forall\ t\in[0,T). \tag{5.10}$$

Using the estimate (5.10) and a similar argument for Theorem 5.3, we can easily see that

$$\|u(t)\|_{H^s}^2 + \|\rho(t)\|_{H^{s-1}}^2 \le C(\|u^0\|_{H^s}, \|\rho^0\|_{H^{s-1}}, \kappa, T), \quad \forall\ t\in[0,T),$$

which completes the proof. □

Remark 5.5. In Theorem 5.4, we assumed that $s \ge 3$. This is because in order to obtain the estimate (5.6), one has to make use of an abstract lemma due to Constantin and Escher [9] which requires that $u_x \in C^1([0,T];H^1)$, i.e., $z^0 \in H^3\times H^2$ (cf. e.g., [48]).

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