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On the Distribution of Rotation Angles

How great is the mean rotation angle of a random rotation?

If you choose a random rotation in 3 dimensions, its angle is far from being uniformly distributed. And the $[n/2]$ angles of a rotation in n dimensions are strongly correlated. I shall study these phenomena, making some concrete calculations involving the Haar measure of the rotation groups.

The Angle of a Random Rotation in 3 Dimensions

Any rotation of the oriented euclidean 3-space $\mathbb{R}^3$ has a well-defined rotation angle $\alpha \in [0, \pi]$, and, in the case $0 < \alpha < \pi$, also a well-defined axis, which may be represented by a unit vector $\xi \in S^2$. For the identity, only the angle $\alpha = 0$ is well-defined, whereas any $\xi \in S^2$ can be considered as axis; if $\alpha = \pi$, there are two axis vectors $\pm \xi$. By a *random rotation* we understand a random variable in SO(3), which is uniformly distributed with respect to Haar measure. It is clear that the axis of such a random rotation must be uniformly distributed on the sphere S^2 with respect to the natural area measure, but what about the rotation angle?

It is certainly not uniformly distributed: The rotations by a small angle α, let's say with $0 \leq \alpha < 1°$, form a small neighbourhood U of the identity $\mathbb{1} \in$ SO(3), whereas the rotations with $179° < \alpha \leq 180°$ constitute a neighbourhood V of the set of all rotations by 180°, which make up a surface (a projective plane) in SO(3). It is plausible that V has a greater volume than U, i.e., the distribution of rotation angles should give more weight to large angles than to small ones. In order to calculate the distribution of the rotation angle, I first express the Haar measure of SO(3) in appropriate coordinates.

The Haar measure of SO(3)

Proposition 1: *If one describes* SO(3) *by the parametrization*

$$\rho : [0, \pi] \times S^2 \to \mathrm{SO}(3),$$
$$\rho(\alpha, \xi) := \text{rotation by the angle } \alpha \text{ about } \xi,$$

the Haar measure of SO(3) *satisfies*

$$\rho^* d\mu_{\mathrm{SO}(3)}(\alpha, \xi) = \frac{1}{2\pi^2} \sin^2\left(\frac{\alpha}{2}\right) d\alpha\, d\lambda(\xi)$$
$$= \frac{1}{4\pi^2} (1 - \cos \alpha) d\alpha\, d\lambda(\xi),$$

where $d\lambda$ is the area element of the unit sphere S^2.

Proof: To begin with, observe that the restriction of ρ to $]0, \pi[\times S^2$ is a diffeomorphism onto an open set U in SO(3), and that the null set $\{0, \pi\} \times S^2$ is mapped by ρ onto $\mathrm{SO}(3) \setminus U$, which is a null set with respect to Haar measure. We can therefore use α and ξ to describe the Haar measure of SO(3), even if they are not coordinates in the strong sense.

The mapping ρ is related to the adjoint representation of the group Q of unit quaternions, and it is easy to calculate the Haar measure of Q. Decomposing a quaternion into its real and imaginary parts, we may describe this group as follows:

$$Q = \{(t, \xi) \in \mathbb{R} \times \mathbb{R}^3;\ t^2 + \|\xi\|^2 = 1\}$$

with multiplication

$$(t, \xi) \cdot (s, \eta) = (ts - \langle \xi, \eta \rangle,\ t\eta + s\xi + \xi \times \eta).$$

The natural riemannian metric on $Q = S^3 \subset \mathbb{R}^4$ is invariant, and therefore the Haar measure of Q is just a multiple of the riemannian volume element. Using the parametrization

$$\varphi : [0, \pi] \times S^2 \to Q, \quad \varphi(\gamma, \xi) := (\cos \gamma, \xi \sin \gamma)$$

and taking into account the total volume of S^3, we get for the Haar measure of Q

$$\varphi^* \, d\mu_Q(\gamma, \xi) = \frac{1}{2\pi^2} \sin^2\gamma \, d\gamma d\lambda(\xi),$$

where $d\lambda$ denotes the area element of the unit sphere S^2.

To get from this the Haar measure of SO(3), we use the adjoint representation $\tau = \mathrm{Ad} : Q \to \mathrm{SO}(3)$, defined by $\tau_q(\zeta) = q\zeta\bar{q}$ for $q \in Q$ and $\zeta \in \mathbb{R}^3$. This is a twofold covering and

$$\tau^* d\mu_{\mathrm{SO}(3)} = 2 d\mu_Q.$$

In the parametrization $\psi := \tau \circ \varphi : [0, \pi] \times S^2 \to \mathrm{SO}(3)$ we have therefore

$$\psi^* d\mu_{\mathrm{SO}(3)} = \frac{1}{\pi^2} \sin^2\gamma \, d\gamma d\lambda(\xi).$$

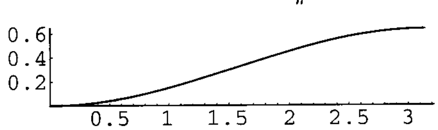
To finish the proof, we observe that $\psi(\gamma, \xi)$ is just the rotation by 2γ about the axis ξ, i.e., $\rho(\alpha, \xi) = \psi(\frac{\alpha}{2}, \xi)$. $\square$

See also [1], pp. 327–329, and [6].

The distribution of the rotation angle

The parametrization ρ is well adapted to our problem, because the subset of rotations by a fixed angle α is just the image of the sphere $\{\alpha\} \times S^2$. If we integrate our expression for the Haar measure of SO(3) over these spheres, we obtain the following result:

Proposition 2: *The angle $\alpha \in [0, \pi]$ of a random rotation is distributed with density $f(\alpha) = \frac{1}{\pi}(1 - \cos \alpha)$:*



See also [7], pp. 89–93.

Generating random rotations

Integrating our expression of the Haar measure of SO(3) over the segment $[0, \pi] \times \{\xi\}$ for any $\xi \in S^2$ confirms that the axis ξ of a random rotation is uniformly distributed with respect to the natural area measure $d\lambda$ on S^2. Using this fact and knowing the distribution of the rotation angle, we can generate random rotations by choosing axis and angle as follows:

The horizontal projection of the unit sphere S^2 onto the tangent cylinder along the equator is an area-preserving map; thus we may choose a point on the cylinder and take the corresponding point on the sphere as axis. This means choosing a random point (λ, h) in the rectangle $[-\pi, \pi] \times [-1, 1]$ and taking the rotation axis $\xi = (\sqrt{1 - h^2} \cos \lambda, \sqrt{1 - h^2} \sin \lambda, h)$.

For the rotation angle α, we choose a random number $a \in [0, 1]$ and take $\alpha := F^{-1}(a)$, where

$$F(\alpha) = \int_0^\alpha f(t)dt = \frac{1}{\pi}(\alpha - \sin \alpha)$$

is the distribution function. Linear algebra tells us how to calculate from ξ and α the matrix $g \in \mathrm{SO}(3)$.

To test this generator of random rotations, I fixed $x \in S^2$ together with a tangent vector $\xi \in T_xS^2$ and calculated with *Mathematica* the tangent vectors $dg(x; \xi)$ for 600 random rotations $g \in \mathrm{SO}(3)$. The mapping $g \to (g(x), dg(x; \xi))$ is a diffeomorphism from SO(3) onto the unit tangent bundle of S^2 and thus makes the rotations g visible by the "flags" $(g(x), dg(x; \xi))$ (Fig. 1).

For the sake of curiosity, I calculated the mean rotation angle for 5,000 random rotations: The result $\mathrm{E}_{5{,}000}(\alpha) = 126°13'55''$ matches the theory, because an easy calculation gives the answer to the question of the subtitle as a consequence of proposition 2:

Corollary: *The expectation of the rotation angle of a random rotation is*

$$\frac{\pi}{2} + \frac{2}{\pi} \approx 126°\,28'\,32''.$$

Random Rotations in 4 Dimensions

The Haar measure of SO(4)

If we identify the euclidean $\mathbb{R}^4$ with the skew field of quaternions $\mathbb{H}$, the group $Q \simeq S^3$ of unit quaternions acts on $\mathbb{R}^4$ by left and right multiplication with $q \in Q$, $L_q : \mathbb{H} \to \mathbb{H}$ and $R_q : \mathbb{H} \to \mathbb{H}$, which are linear isometries, i.e., elements of SO(4). These special rotations generate the whole group SO(4):

$$\Phi : Q \times Q \to \mathrm{SO}(4), \quad \Phi(p, q) := L_p \circ R_{\bar{q}}$$

is a group epimorphism with kernel $\{(1, 1), (-1, -1)\}$. (See also [1], pp. 329–330.)

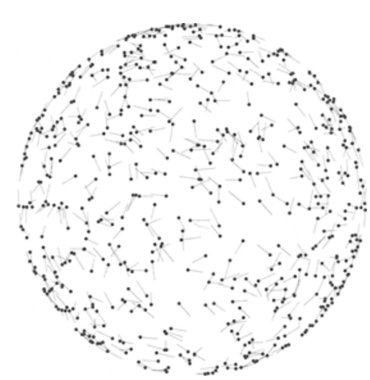


FIGURE 1

Using the parametrization φ for either factor of the product $Q \times Q$, we obtain a parametrization of SO(4):

$$\Psi : [0, \pi] \times [0, \pi] \times S^2 \times S^2 \to \mathrm{SO}(4),$$
$$\Psi(s, t, \xi, \eta) := \Phi(\varphi(s, \xi), \varphi(t, \eta)).$$

If we admit $s, t \in [0, 2\pi]$ and calculate modulo 2π, Ψ becomes a fourfold covering $\Psi : T^2 \times S^2 \times S^2 \to \mathrm{SO}(4)$ with branching locus $(\{(0, 0)\} \cup (\pi, \pi)\}) \times S^2 \times S^2$:

$$\begin{aligned}\Psi(s, t, \xi, \eta) &= \Psi(\pi - s, \pi - t, -\xi, -\eta)\\ &= \Psi(\pi + s, \pi + t, \xi, \eta)\\ &= \Psi(2\pi - s, 2\pi - t, -\xi, -\eta)\end{aligned}$$

for $0 \le s, t \le \pi$, and even $\Psi(0, 0, \xi, \eta) = \Psi(\pi, \pi, \xi, \eta) = \mathbb{1}$ for all $\xi, \eta \in S^2$. The Haar measure of SO(4) therefore satisfies

$$\Psi^* d\mu_{\mathrm{SO}(4)} = c \sin^2 s \sin^2 t \, ds \, dt \, d\lambda(\xi) \, d\lambda(\eta),$$

with a constant c.

Pairs of rotation angles

Any rotation $g \in \mathrm{SO}(4)$ is conjugate to a standard rotation

$$\begin{pmatrix}\mathrm{Rot}_{\vartheta_1} & 0\\ 0 & \mathrm{Rot}_{\vartheta_2}\end{pmatrix} \text{ with } \mathrm{Rot}_\vartheta := \begin{pmatrix}\cos\vartheta & -\sin\vartheta\\ \sin\vartheta & \cos\vartheta\end{pmatrix}.$$

Choosing the rotation angles ϑ_1, ϑ_2 in the interval $[0, 2\pi]$, the following pairs are equivalent, i.e., the corresponding rotations are conjugate:

$$(\vartheta_1, \vartheta_2) \sim (\vartheta_2, \vartheta_1) \sim (2\pi - \vartheta_1, 2\pi - \vartheta_2) \sim (2\pi - \vartheta_2, 2\pi - \vartheta_1).$$

The class of these equivalent pairs will be called **the pair of rotation angles** $[\vartheta_1, \vartheta_2]$. This is an element of $T^2/\sim$, where the equivalence relation $\sim$ is considered on the torus $T^2 = S^1 \times S^1$. Two rotations in SO(4) are conjugate if and only if they have the same pair of rotation angles $[\vartheta_1, \vartheta_2]$.

The following lemmas are needed to determine the pair of rotation angles for an element $\Phi(p, q) \in \mathrm{SO}(4)$.

Lemma 1: *For $p, q, p', q' \in Q$, the rotations $\Phi(p, q)$ and $\Phi(p', q')$ in SO(4) are conjugate if and only if p is conjugate to $\pm p'$ and q is conjugate to $\pm q'$ in Q, with the same sign in either case.*

Proof: $\Phi(p, q)$ is conjugate to $\Phi(p', q')$ if and only if there exists a $T \in \mathrm{SO}(4)$ with $\Phi(p, q) = T \circ \Phi(p', q') \circ T^{-1}$. As $T = \Phi(u, v)$ for some $u, v \in Q$, we have:

$\Phi(p, q)$ is conjugate to $\Phi(p', q')$ if and only if there exist $u, v \in Q$ with

$$\begin{aligned}L_p \circ R_{\bar q} &= L_u \circ R_{\bar v} \circ L_{p'} \circ R_{\bar q'} \circ L_{\bar u} \circ R_v\\ &= L_u \circ L_{p'} \circ L_{\bar u} \circ R_{\bar v} \circ R_{\bar q'} \circ R_v\\ &= L_{up'\bar u} \circ R_{(vq'\bar v)^-} = \Phi(up'\bar u, vq'\bar v).\end{aligned}$$

The kernel of Φ contains only the two elements $(1, 1)$ and $(-1, -1)$; therefore we have shown that $\Phi(p, q)$ is conjugate to $\Phi(p', q')$ if and only if there exist $u, v \in Q$ such that $p = \pm up'\bar u$ and $q = \pm vq'\bar v$, with the same sign in either case. $\square$

Lemma 2: Let $\xi \in S^2$ *be a purely imaginary quaternion with norm* 1. *Then the quaternion $p = \cos t + \xi \sin t$ is conjugate to $p' = \cos t + \mathrm{i} \sin t$.*

Proof: We must find a $u \in Q$ with $\tau_u(\xi) := u\xi\bar u = \mathrm{i}$. But as SO(3) acts transitively on S^2, there exists a rotation which sends ξ to i, and as the adjoint representation $\tau : Q \to \mathrm{SO}(3)$ is onto, there exists $u \in Q$ such that this rotation is τ_u, i.e. $\tau_u(\xi) = \mathrm{i}$. $\square$

These lemmas allow us to show:

Proposition 3: *Let $p = \cos s + \xi \sin s$, $q = \cos t + \eta \sin t$, where ξ and η are purely imaginary unit quaternions. Then the rotation $\Phi(p, q) \in \mathrm{SO}(4)$ has the pair of rotation angles $[s - t, s + t]$.*

Proof: By the two lemmas, the rotation $\Phi(p, q)$ is conjugate to $\Phi(\cos s + \mathrm{i} \sin s, \cos t + \mathrm{i} \sin t)$ and has therefore the same pair of rotation angles. Let us calculate the matrix of the latter rotation with respect to the canonical base $(1, \mathrm{i}, \mathrm{j}, \mathrm{k})$ of $\mathbb{R}^4 = \mathbb{H}$:

$$L_{\cos s + \mathrm{i} \sin s} = \begin{pmatrix}\mathrm{Rot}_s & 0\\ 0 & \mathrm{Rot}_s\end{pmatrix},$$

$$R_{\cos t - \mathrm{i} \sin t} = \begin{pmatrix}\mathrm{Rot}_{-t} & 0\\ 0 & \mathrm{Rot}_t\end{pmatrix},$$

whence

$$\Phi(\cos s + \mathrm{i} \sin s, \cos t + \mathrm{i} \sin t) = \begin{pmatrix}\mathrm{Rot}_{s-t} & 0\\ 0 & \mathrm{Rot}_{s+t}\end{pmatrix},$$

which finishes the proof. $\square$

Corollary (Fig. 2): *The pair of rotation angles is distributed with density*

$$\begin{aligned}f([\vartheta_1, \vartheta_2]) &= \frac{1}{\pi^2} \sin^2\left(\frac{\vartheta_1 + \vartheta_2}{2}\right) \sin^2\left(\frac{\vartheta_1 - \vartheta_2}{2}\right)\\ &= \frac{1}{4\pi^2} (\cos\vartheta_1 - \cos\vartheta_2)^2.\end{aligned}$$

Here f is considered as a function on $[0, 2\pi] \times [0, 2\pi]$, i.e., it is normalized so that integrating it over $[0, 2\pi] \times [0, 2\pi]$ gives 1.

Proof: Starting with the parametrization

$$\Psi : [0, \pi] \times [0, \pi] \times S^2 \times S^2 \to \mathrm{SO}(4)$$

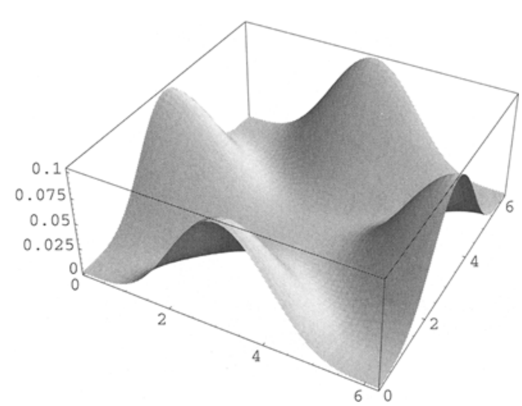


FIGURE 2

and using the relation $[\vartheta_1, \vartheta_2] = [s - t, s + t]$, we obtain a new parametrization:

$$\psi(\vartheta_1, \vartheta_2, \xi, \eta) := \Psi\left(\frac{\vartheta_1 + \vartheta_2}{2}, \frac{\vartheta_1 - \vartheta_2}{2}, \xi, \eta\right).$$

With respect to these parameters the Haar measure satisfies

$$\psi^* d\mu_{SO(4)} = C \sin^2\left(\frac{\vartheta_1 + \vartheta_2}{2}\right) \sin^2\left(\frac{\vartheta_1 - \vartheta_2}{2}\right) d\vartheta_1\, d\vartheta_2\, d\lambda(\xi)\, d\lambda(\eta).$$

Integrating over $\{(\vartheta_1, \vartheta_2)\} \times S^2 \times S^2$ for fixed ϑ_1, ϑ_2 gives us the density

$$\begin{aligned} f(\vartheta_1, \vartheta_2) &= C' \sin^2((\vartheta_1 + \vartheta_2)/2) \sin^2((\vartheta_1 - \vartheta_2)/2) \\ &= \frac{C'}{4} (\cos \vartheta_1 - \cos \vartheta_2)^2. \end{aligned}$$

The constant $C' = 1/\pi^2$ is obtained by integrating this function over $[0, 2\pi] \times [0, 2\pi]$. □

Rotations in Dimension $n \geq 4$

The results obtained in dimensions 3 and 4 can be generalized to dimension $n \geq 4$ using Hermann Weyl's method of integration of central functions on a compact Lie group. A central function is one which is constant on conjugacy classes. In the case of SO(3) this is simply a function of the rotation angle, and in the case of SO(4) of the pair of rotation angles. In dimension $n > 4$ we can introduce the notion of a **multiangle** characterizing the conjugacy classes.

Multiangles of rotation

Let us begin with the case of a rotation $g \in SO(n)$ for even $n = 2m$: as in the case $n = 4$, there are m rotation angles $\vartheta_1 \ldots, \vartheta_m$ corresponding to the decomposition of g as direct sum of m plane rotations:

$g = \mathrm{Rot}_{\vartheta_1} \oplus \ldots \oplus \mathrm{Rot}_{\vartheta_m}$. For odd $n = 2m + 1$, there are also m angles $\vartheta_1, \ldots, \vartheta_m$. Calculating modulo 2π, the list $(\vartheta_1, \ldots \vartheta_m)$ is an element of the m-torus T^m and is unique up to the following symmetries, which define an equivalence relation $\sim$ on T^m:

the ϑ_i may be permuted;

ϑ_i may be replaced by $-\vartheta_i$, but only for an even number of indices i if n is even; for odd n there is no such restriction.

Let us call the class $\mathrm{ma}(g) := [\vartheta_1, \ldots \vartheta_m] \in T^m/\sim$ the **multiangle** of the rotation $g \in SO(n)$.

Two rotations in $SO(n)$ are conjugate if and only if they have the same multiangle. To determine the multiangle of a rotation $x \in SO(n)$, we fix an orthonormal base of $\mathbb{R}^n$ and consider a conjugate of x in the maximal torus $T \subset SO(n)$ the elements of which have, with respect to the chosen base, the form

$$\vartheta = \begin{pmatrix} \mathrm{Rot}_{\vartheta_1} & \ldots & 0 \\ & \ddots & \\ 0 & \ldots & \mathrm{Rot}_{\vartheta_m} \end{pmatrix}$$

in the case $n = 2m$; in the case $n = 2m + 1$, one has to add a first column and a first row with first element 1 and zeroes elsewhere. In either case we identify T with the standard torus T^m. Obviously, $\vartheta \in T^m$ has the multiangle $\mathrm{ma}(\vartheta) = [\vartheta]$, and this is the same for the whole conjugacy class:

$$\mathrm{ma}(g\vartheta g^{-1}) = [\vartheta] \text{ for all } \vartheta \in T^m \text{ and } g \in SO(n).$$

The Haar measure of a compact Lie group

Let G be a compact and connected Lie group and $T \subset G$ a maximal torus. There exists a natural mapping $\psi : G/T \times T \to G$ such that the diagram

$$\begin{tikzcd} G \times T \arrow[r, "\varphi"] \arrow[d] & G \\ G/T \times T \arrow[ur, "\psi"'] & \end{tikzcd}$$

commutes, where $\varphi(g, \vartheta) := \tau_g(\vartheta) = g\vartheta g^{-1}$ and the vertical arrow is the natural projection.

The Lie algebra $\mathfrak{g}$ is endowed with an Ad-invariant scalar product, and if $\mathfrak{t} \subset \mathfrak{g}$ is the Lie algebra of the maximal torus T, its orthogonal complement $\mathfrak{t}^\perp$ is stable under the mappings $\mathrm{Ad}\, g : \mathfrak{g} \to \mathfrak{g}$ for $g \in G$. The restriction of $\mathrm{Ad}\, g$ to $\mathfrak{t}^\perp$ is denoted by $\mathrm{Ad}^\perp g$.

With these notations, the Haar measure of G can be expressed in terms of that of T together with the invariant measure of G/T:

Proposition 4: *$\psi : G/T \times T \to G$ is a finite branched covering. Let $d\mu_G$ and $d\mu_T$ denote the Haar measures of G and T, and let $d\mu_{G/T}$ be the G-invariant normalized measure of the homogeneous space G/T. Then*

$$\psi^* d\mu_G = d\mu_{G/T} \times J\, d\mu_T,$$

where $J : T \to \mathbb{R}$ is the function

$$J(\vartheta) := \det(\mathbb{1} - \mathrm{Ad}^\perp \vartheta).$$

For a proof of this formula, see [2], pp. 87–95.

The distribution of the multiangle

Proposition 4 may be applied in our case, with $G = SO(n)$ and $T = T^m$. Now $\psi([g], \vartheta) = g\vartheta g^{-1}$ has for every $[g] \in SO(n)/T^m$ the same multiangle $[\vartheta]$, i.e.,

$$\mathrm{ma}(\psi([g], \vartheta)) = [\vartheta] \in T/\sim.$$

Therefore the density of the multiangle $[\vartheta]$, considered as a symmetric function on the torus T^m, has the form

$$f(\vartheta) = c \int_{G/T} J(\vartheta)\, d\mu_{G/T} = c J(\vartheta)$$

with a normalizing constant c.

To calculate $J(\vartheta) = \det(\mathbb{1} - \mathrm{Ad}^\perp \vartheta)$, we observe that in the case $n = 2m$ the elements of $\mathfrak{t}^\perp$ are the symmetric matrices of the form

$$A = \begin{pmatrix} 0 & A_{12} & A_{13} \ldots A_{1m} \\ A_{12}^* & 0 & A_{23} \ldots A_{2m} \\ \ldots & & \ddots \quad \ldots \\ A_{1m}^* & & \ldots \quad 0 \end{pmatrix},$$

where the A_{ij} are 2×2-blocks.

A direct calculation shows that $\mathrm{Ad}^{\perp}\vartheta$ transforms this matrix by replacing every block A_{ij} by the block

$$R_{\vartheta_i,\vartheta_j}(A_{ij}) := \mathrm{Rot}_{\vartheta_i} A_{ij} \mathrm{Rot}_{\vartheta_j}^{-1}.$$

If we identify $\mathbb{R}^{2\times 2}$ with $\mathbb{R}^2 \otimes \mathbb{R}^2$, $R_{\vartheta_i,\vartheta_j}$ becomes the tensor product of the two rotations $\mathrm{Rot}_{\vartheta_i}$ and $\mathrm{Rot}_{\vartheta_j}$. The eigenvalues are therefore $e^{\pm i\vartheta_i} e^{\pm i\vartheta_j} = e^{\pm i(\vartheta_i \pm \vartheta_j)}$, and we obtain

$$\det(\mathbb{1} - R_{\vartheta_i,\vartheta_j}) = \left(2 \sin \frac{\vartheta_i + \vartheta_j}{2}\right)^2 \left(2 \sin \frac{\vartheta_i - \vartheta_j}{2}\right)^2 = 4(\cos \vartheta_i - \cos \vartheta_j)^2.$$

Now $\mathrm{Ad}^{\perp}\vartheta$ is the direct sum of the $R_{\vartheta_i,\vartheta_j}$. Combining these results:

$$J(\vartheta) = 2^{m(m-1)} \prod_{1\le i<j\le m} (\cos \vartheta_i - \cos \vartheta_j)^2$$

and

$$f_{2m}(\vartheta) = C' \prod_{1\le i<j\le m} (\cos \vartheta_i - \cos \vartheta_j)^2.$$

These formulae apply to the case of even $n = 2m$. In the case of odd $n = 2m + 1$ one has

$$J(\vartheta) = 2^{m^2} \prod_{i=1}^{m} (1 - \cos \vartheta_i) \prod_{1\le i<j\le m} (\cos \vartheta_i - \cos \vartheta_j)^2$$

and

$$f_{2m+1}(\vartheta) = C \prod_{i=1}^{m} (1 - \cos \vartheta_i) \prod_{1\le i<j\le m} (\cos \vartheta_i - \cos \vartheta_j)^2.$$

Figure 3 illustrates the function $f_5(\vartheta)$ for SO(5), where the normalizing factor is $C = 1/(2\pi^2)$:

You see a "sharper" correlation between the two angles than in the case SO(4). The rotations with the pair of angles $[\arccos(1/3), \pi] = [70°31'44'', 180°]$ are the "most frequent" ones. We shall see that the cases SO(4) and SO(5) are representative of a general phenomenon: The density of the multiangle has always a well-defined maximum with $0 \le \vartheta_1 < \ldots < \vartheta_m \le \pi$, and for this maximum $\vartheta_m = \pi$, whereas $\vartheta_1 = 0$ for even n and $\vartheta_1 > 0$ for odd n.

To study the density functions $f_n(\vartheta)$, observe that they may be written as

$$f_n(\vartheta) = C g_n(\cos \vartheta_1, \ldots, \cos \vartheta_m), \quad m = [n/2],$$

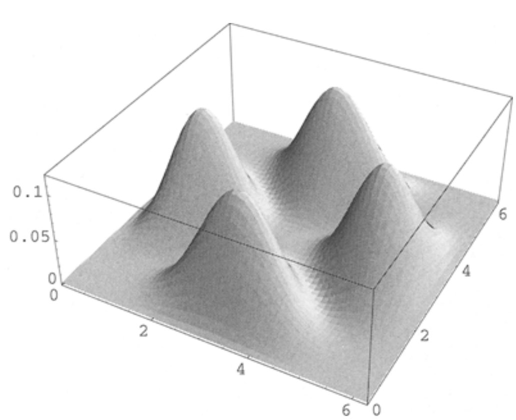


FIGURE 3

with

$$g_{2m}(x_1, \ldots, x_m) = \prod_{1\le i<j\le m} (x_i - x_j)^2$$

and

$$g_{2m+1}(x_1, \ldots, x_m) = g_{2m}(x_1, \ldots, x_m) \prod_{i=1}^{m} (1 - x_i).$$

g_{2m} is a well-known function, namely the discriminant of the polynomial $(x - x_1) \cdot \ldots \cdot (x - x_m)$. Here we consider the functions g_{2m} and g_{2m+1} on the compact simplex $D := \{x \in \mathbb{R}^m;\ 1 \ge x_1 \ge \ldots \ge x_m \ge -1\}$ where they are not negative and must have a maximum.

Proposition 5: *The global maximum of g_n in D is also the only local maximum in D.*

For the maximum of g_{2m}, $1 = x_1 > \ldots > x_m = -1$; for that of g_{2m+1}, $1 > x_1 > \ldots > x_m = -1$.

Proof: Let us consider the even case, i.e., the function g_{2m}: Obviously, one has $x_1 = 1$ and $x_m = -1$ for any local maximum x. Fix these two coordinates and define

$$h(x_2, \ldots, x_{m-1}) := \ln g_{2m}(1, x_2, \ldots, x_{m-1}, -1).$$

On the boundary of $D' := \{1 \ge x_2 \ge \ldots \ge x_{m-1} \ge -1\}$, h has the value $-\infty$, and this function is strictly concave in the interior: its Hessian is the matrix $H_h(x) = (h_{ij}(x))$ with

$$h_{ij}(x) = \begin{cases} -\displaystyle\sum_{\substack{k=1\\k\ne i}}^{m} \frac{2}{(x_i - x_k)^2} & \text{for } i = j \\ \dfrac{2}{(x_i - x_j)^2} & \text{for } i \ne j. \end{cases}$$

The diagonal elements are strictly negative, the other elements are strictly positive but still sufficiently small to make the sum of the elements of any row negative. Therefore, the Hessian is negative definite and h is a strictly concave function and has a unique local maximum in the interior of D'. As the natural logarithm is strictly increasing, the function $g_{2m}(1, x_2, \ldots, x_{m-1}, -1)$ has also a unique local maximum.

For g_{2m+1} the reasoning is similar. □

As a consequence of this proposition, the density $f_n(\vartheta)$ of the multiangle of SO(n) has always one and only one maximum in $\{0 \le \vartheta_1 < \ldots < \vartheta_{[n/2]} \le \pi\}$; for this maximum, $\vartheta_1 = 0$ and $\vartheta_m = \pi$ if $n = 2m$, whereas $\vartheta_1 > 0$ and $\vartheta_m = \pi$ in the case $n = 2m + 1$.

Here is a list of the most frequent multiangles, i.e., the $[\vartheta]$ with maximal density $f_n(\vartheta)$, for $n \le 10$:

SO(3):	[180°],
SO(4):	[0°, 180°],
SO(5):	[70°31′44″, 180°],
SO(6):	[0°, 90°, 180°],
SO(7):	[46°22′41″, 106°51′07″, 180°],
SO(8):	[0°, 63°26′06″, 116°33′54″, 180°],
SO(9):	[34°37′55″, 79°33′46″, 125°07′13″, 180°],
SO(10):	[0°, 64°37′23″, 90°, 115°22′37″, 180°]

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It seems that there is almost no literature on the subject; however, [3], [4], and [5] treat related topics.

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