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Prices are macro-observables! Stylized facts from evolutionary finance

S. Reimann · A. Tupak

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Abstract Prices are macro-observables of a financial market that result from the trading actions of a huge number of individual investors. Major stylized facts of empirical asset returns concern (i) non-Gaussian distribution of empirical asset returns and (ii) volatility clustering, i.e., the slow decay of auto-correlations of absolute returns. We propose a model for the aggregate dynamics of the market which is generated by the coupling of a 'slow' and a 'fast' dynamical component, where the 'fast' component can be seen as a perturbation of the 'slow' one. Statistical properties of price changes in this model are estimated by simulation; sample size is 4×10^6. It is shown that increasing the decoupling of these two dynamical levels generates a crossover in the distribution of log returns from a concave Gaussian-like distribution to a convex, truncated Levy-like one. For a sufficiently large degree of dynamic decoupling, the return trails exhibit pronounced volatility clustering.

1 Introduction

Today, a huge proportion of the economic welfare of individuals and of whole countries is tightly bound to the functioning of international networks of economic institutions and financial markets. It is of vital interest to understand their organization and particularly their dynamics. Markets exhibit structures on various levels of organization. These structures put the agents' actions into interdependencies. In a connected network, interdependencies do not remain local in general (see Foellmer, 1974). For understanding dynamic properties, such as *price evolution*, it is therefore important to consider the dynamics of the

S. Reimann (✉) · A. Tupak
Swiss Banking Institute, University of Zurich, Zurich, Switzerland
e-mail: sreimann@isb.unizh.ch

individual states of the agents (their wealth dynamics, for example), as well as the dynamics of network structure itself.

Typically, an agent lives in a situation where both the number and diversity of his interactions is huge, while interactions might be long-range instead of localized. On the other hand, agents realistically only have finite computational power, and in general can only recognize their neighborhoods instead of the entire network. How, then, can they deal with this situation? The assumption of rationality would help solve this dilemma, but is quite unrealistic. This was already mentioned some time ago by Simon (1957). It is therefore natural to consider a financial market as an evolving network of interacting agents which are 'boundedly rational'. For a detailed survey and discussion of Heterogeneous Agents' Models and their properties, see the survey by Hommes (2006) for deep discussions and an extended overview of literature on various aspects and models.

Weakening the assumption of strict rationality opens the door for a variety of ad hoc 'behavioral' assumptions in such models and hence for some arbitrariness. Typically, such models exhibit strong non-linearities and have many degrees of freedom, allowing for a wide range of qualitatively different results. It may not be surprising that such models can generate time series that look realistic. The natural question then, of course, is whether related properties are generic or left to particular parameter settings only. While single parameters might not be measurable with sufficient precision, the question is which model in this 'zoo' is reasonable. Our standpoint is that this judgement has to be left to data, while the simplicity of a model is another criterion for its explanatory value! What are the relevant data? Given that the outcome of an experiment is merely stochastic, these data can only be *typical* statistical properties, i.e., properties that are common to almost all realizations. For a financial market, this means that relevant data are those statistical properties that are common to all empirical observations. These invariant properties were termed '*Stylized Facts*'. Therefore, *a particular model is judged in terms of its capability to reproduce stylized facts.*

The number of agents in a financial market is huge. As a result of their aggregated actions, prices are formed on this market; prices thus have to be regarded as 'macro-observables'. A strict bottom-up modeling approach therefore would consist in writing down the 'equations of motion' of these individual agents. For a realistic number of traders on the market, this is not only practicably impossible, but also unnecessary, because the quantity being dealt with—prices—is an 'average property' of the ensemble. Individual dynamics are therefore largely irrelevant. This is precisely analogous to the situation in *Thermodynamics*, which considers macroscopic properties of systems consisting of 10^{23} particles, in the order of magnitude. Thermodynamics is one of the most successful disciplines in physics, and it has survived fundamental paradigmatic changes in physics *because* it considers typical properties of the system, i.e., properties that are, by definition, independent from a particular microscopic realization, such as the temperature of a gas or its pressure.

Serious consideration of prices as typical properties of a financial market has an important consequence: the macroscopic level cannot be seen as a simple ‘up-stream’ analog of the microscopic level. As an example, consider a Bernoulli random graph with a fixed and large number of nodes, in which edges exist with some probability independently from each other. Nodes are identical in the sense that all nodes have the same probability to have a given edge degree. On the macro level the distribution of degrees in this random graph is not uniform but binomial. This binomial distribution does not have any analog on the microscopic level. In the context of economic reasoning, the lesson to be drawn may be the following: while microscopic equations of motions of the agents might be derived from some ‘micro-founded, optimization-based’ framework, this does not necessarily carry over to typical properties. Therefore, there is no need to assume that the behavior of a market on the aggregate level has to be directed by principles that exist on the micro level. In other words, even if the behavior of all traders were consistent with rational behavior, there is no reason that the dynamics of prices should be generated by something like a *‘rational’* agent (see Kirman, 1992).

From these considerations, the main topic of our note follows: *the data we are concerned with are so-called stylized facts of empirical asset returns. We want to understand better the origin of these data by means of an economically reasonable model for the dynamics of a financial market on an aggregate level.*

The next section gives a brief overview of the empirical stylized facts we are concerned with: the non-Gaussian distribution of empirical returns and ‘volatility clustering’ in return trails. Section 2 defines our interacting agents model, while in Sect. 3 we consider the ‘adiabatic’[1] market model whose simulations are displayed in Sect. 4. We show that stylized facts mentioned above are reproduced by this financial markets model. In particular, the distribution of log returns exhibits a crossover from a concave Gaussian like distribution, as observed on long time-scales such as months or years, to a convex (truncated) Levy-like distribution, as observed in high-frequency data. Simulated return trails also exhibit volatility clustering, which becomes more pronounced the further both components are decoupled from each other.

2 Stylized facts to be considered

Asset prices are macro-observables of a financial market, while index prices are weighted averages of the prices of their constituents. A typical trail of daily prices and log-returns of the NIKKEI from 1966 to 1989 is shown in Fig. 1.

Some properties of empirical asset returns have turned out to be remarkably similar or even identical for different assets on different markets in different time periods. These properties, invariant under the choice of a particular asset in a market at some point, are usually called ‘stylized facts’. Cont (2001) listed

[1] The term ‘adiabatic’ is actually a misuse of the physical term. It is meant to indicate that we consider two dynamic levels, a slow one and a fast one; see below.

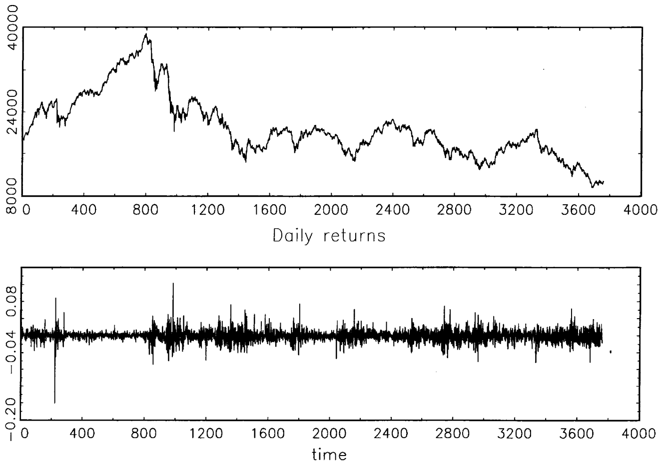

Fig. 1 The upper picture shows a price trail of daily data from the NIKKEI from 1966 to 1989, while the lower picture displays corresponding log-returns.

a number of stylized facts, including those we consider in this paper: the non-Gaussian distribution of asset return, and the slow decay of auto-correlations of squared asset returns, which can be regarded as a measure of volatility clustering.

For comparability, we consider *normalized* returns in the following. Let s_t denote the price of an index today and s_{t-1} its price yesterday, then the relative price change can be approximated by the log-return defined by $z_t = \ln \frac{s_t}{s_{t_1}}$, provided that $|s_t - s_{t-1}|$ is small.

The normalized return Z_t is then defined as $Z_t = \frac{z_t - \langle z \rangle_t}{\sigma(z)}$, where $\langle \cdots \rangle_t$ denotes the time average and $\sigma(\cdot)$ the standard deviation of its argument. If the tails decay sufficiently fast, the second moment is defined. $P(z) = \mathbb{P}[Z = z]$ denotes the probability (relative frequency) that a return of size z occures. Recall that, in a semi-logarithmic plot, a Gaussian distribution looks like a parabola standing on its head (inner cure in Fig. 2), a Laplacian looks like a tent, while the outer curve in this figure represents a Levy distribution. In our pictures, we also use semi-logarithmic plots for better visualization.

There is empirical evidence that, from longer to shorter time-scales, respective distributions show a crossover from a concave Gaussian-like distribution to a convex one. Typically, return distributions of high-frequency data are convex shaped (see Fig. 2) while distributions of daily returns are tent-shaped (see Fig. 3, for example). For longer time-scales, such as monthly ones, return distributions are close to a Gaussian distribution (see Fig. 4); hence a crossover from convex to concave, depending on the time-scale being considered.

Cont and Bouchaud (2000) suggested a model of a large number of traders who imitate their neighbors. Each trader can be in one of three states: if he is active, he can either buy or sell or he can be inactive and do nothing. The time scale is defined by the probability of being active. As the probability of

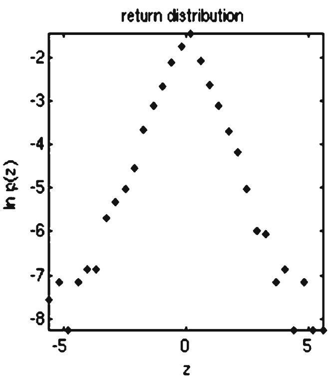

Fig. 2 Return distribution for daily data for the NIKKEI 500 (1979 – 2004) shows that the distribution is close to a Laplacian

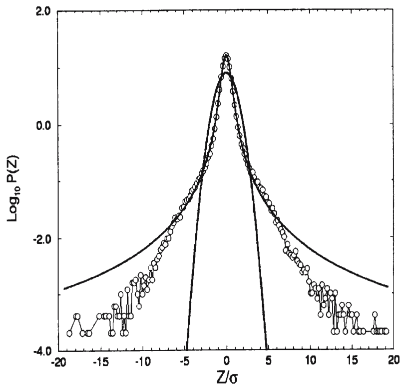

Fig. 3 Return distribution for high-frequency data for the S&P 500 (1 min from 1984 to 1989), adapted from Mantegna and Stanley (1995)

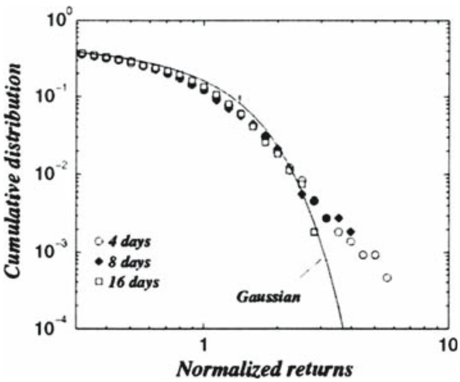

Fig. 4 The cumulative return distribution for the S&P 500 shows a slow crossover to a Gaussian for longer time-scales

being active increases from low (isolated traders) to high (clustered traders), the distribution changes from a convex to a concave function. This model points to imitative behavior (herding behavior) as a possible source of intermittency in financial time series. For generalizations of this model, see the literature in Stauffer (1999).

Before defining our model of a financial market as an interacting agents' system, let us fix some notation here: *A vector* $\boldsymbol{x} = (x_k) = (a, \ldots, a)$ *is denoted*

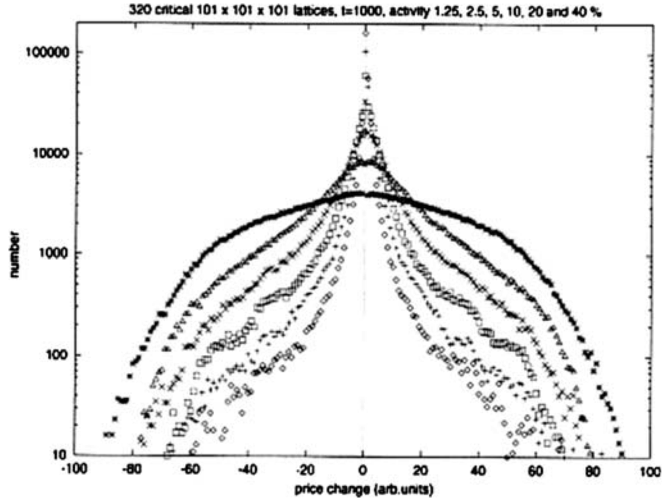

Fig. 5 The crossover from a Gaussian distribution of log-returns to a convex power law-like distribution, from Stauffer and Penna (1998)

by $\boldsymbol{a}$*. Its norm* $\|\boldsymbol{x}\| := \sum_k |x_k|$ *is simply its 1-norm. Component-wise division is defined as* $\frac{\boldsymbol{x}}{\boldsymbol{y}} = \left(\frac{x_k}{y_k}\right)$ *for vectors* $\boldsymbol{x} = (x_k)$ *and* $\boldsymbol{y} = (y_k)$*. Component-wise multiplication is denoted by* $\boldsymbol{x} \star \boldsymbol{y} = (x_n\ y_n)$*, while* $\boldsymbol{x}\ \boldsymbol{y} = \sum_k x_k y_k$ *denotes the standard scalar product (*Fig. 5*).*

3 The financial market as an interacting agent's system

Since we consider the period $[t, t+1)$, we omit the 'time-index' t for the moment. There is a constant number of assets $k = 1 \ldots K$ on the market, where q^k is the price of asset k and $\boldsymbol{q} = (q^k)$ is the vector of prices at time t. We assume that assets are infinitely divisible. Each asset has an uncertain value D^k in the future. We define the set of agents as the set of all investment strategies $\boldsymbol{\lambda}^a$ on the market, where λ^a_k is the portion of wealth to be invested in asset k. If short-selling is excluded, as we will assume in the following, we have $\lambda^a_k \geq 0$ and $\|\boldsymbol{\lambda}^a\| = 1$. The weight of agent a is the amount of wealth r^a which is invested according to $\boldsymbol{\lambda}^a$. If Δ is the simplex spanned by $\{\boldsymbol{\lambda}^a\}$ and if $\{\boldsymbol{\phi}^\kappa\}$ its basis,[2] then the state of each agent is characterized by a tuple $(r^a, \boldsymbol{\ell}^a)$, where $\boldsymbol{\ell}^a = (\ell^a_\kappa)$ is the coordinate of agent a in Δ given by

$$\boldsymbol{\lambda}^a = \sum_\kappa \ell^a_\kappa\ \boldsymbol{\phi}^\kappa \tag{1}$$

[2] Here we assume that the basis is fixed for all times.

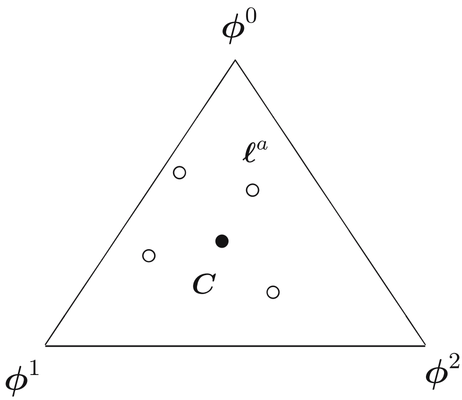

Fig. 6 The population of agents $\{a\}$ in a financial market is a cloud of points $\boldsymbol{\ell}^a$ with masses r^a in the simplex spanned by the fundamental investment styles $\{\boldsymbol{\phi}^\kappa\}$, $\kappa = 0, 1, 2$. The center of mass has coordinate $\boldsymbol{C}$

3.1 Wealth dynamics

According to its state $(r^a, \boldsymbol{\ell}^a)$, agent a builds his portfolio $\boldsymbol{\theta} = \frac{r^a \boldsymbol{\lambda}^a}{\boldsymbol{q}}$, where θ_k is the number of units of asset k the agent purchased at their prices q^k. Given that each unit of asset k has some uncertain future value D^k tomorrow, the wealth of agent a tomorrow reads $(r^a)' = \boldsymbol{D}\,\boldsymbol{\theta}^a$. Therefore the stochastic growth rate of his wealth is given by $\frac{\boldsymbol{D}}{\boldsymbol{q}}\boldsymbol{\lambda}^a$. Note that today the future growth rate is uncertain because $\boldsymbol{D}$ is uncertain. The growth rate, or wealth relative in this period, i.e., $R^a = \frac{(r')^a}{r^a} = \frac{\boldsymbol{D}}{\boldsymbol{q}}\,\boldsymbol{\lambda}^a$, which in the basis of the simplex Δ therefore reads (Fig. 6)

$$R^a = \boldsymbol{Y}\,\boldsymbol{\ell}^a,$$

where $Y^\kappa = Y^\kappa(\boldsymbol{q}) = \frac{\boldsymbol{D}}{\boldsymbol{q}}\,\boldsymbol{\phi}^\kappa$ is the projection of $\frac{\boldsymbol{D}}{\boldsymbol{q}}$ on the fundamental style $\boldsymbol{\phi}^\kappa$. Therefore, the wealth evolution of agent a is described by a multiplicative random process

$$r^a_{t+1} = R^a_{t+1}\,r^a_t, \tag{2}$$

whose stochastic growth rate $R^a_{t+1} = \boldsymbol{Y}_{t+1}\,\boldsymbol{\ell}^a_t$ depends on the current price $\boldsymbol{q}_t$ and his current investment strategy $\boldsymbol{\ell}^a_t$.

The following remark might be in place here. One could regard the above-mentioned model as a model for short-lived assets in which, according to Blume and Easley (1992), the random variable $\boldsymbol{D}$ is interpreted as the dividend payoff of the assets so that $\boldsymbol{Y}$ is, up to a projection, the dividend yield. This interpretation restricts the model to sufficiently long time-scales in which dividends are paid off; that is, to say, the time-scale should be quarterly or even longer. This limitation is due to interpretation rather than to the formal structure of the model. Our interpretation of the factor R^a_{t+1} is that it represents the *expectation* agent a has at time t about the future growth rate of his wealth. In other words, at time t, agent a decides to follow strategy $\boldsymbol{\ell}^t$, whose expected growth rate yields $\boldsymbol{Y}_{t+1}\,\boldsymbol{\ell}^a_t$, where $\boldsymbol{Y}_{t+1}$ is a stochastic variable with a time-varying support which depends on the current price $\boldsymbol{q}_t$. In this sense wealth evolution in this model depends on the heterogeneous expectations of the agents.

3.2 Prices

In this financial market are assumed to be entirely determined by the additional requirement that the amount of available assets is conserved over time. Note that, because of the two requirements that assets are infinitely divisible and that the number of available units of assets is conserved, prices are so-called *market clearing prices* $\boldsymbol{q}_t = \sum_a r_t^a \boldsymbol{\lambda}_t^a$, In our formulation, there is a close relation between an interacting agent's system and a cloud of point-masses. What are prices in this picture? In fact prices are related to the center of mass of the agent's cloud by

$$\boldsymbol{q}_t = \sum_\kappa C_{\kappa,t}\, \boldsymbol{\phi}^\kappa, \tag{3}$$

where $\boldsymbol{C}_t = \frac{1}{r_t} \sum_a r_t^a \boldsymbol{\ell}^a$ is the center of mass of the agents' cloud in the simplex Δ. In other words, the evolution of prices can be viewed as the motion of the center of mass of the particle system.

3.3 Dynamics of investment strategies

At any time t, agent a follows his investment strategy $\boldsymbol{\lambda}_t^a = \sum_\kappa \ell_{\kappa,t}^a \boldsymbol{\phi}^\kappa$, while some time later he might follow another strategy $\boldsymbol{\lambda}_{t+1}^a = \sum_\kappa \ell_{\kappa,t+1}^a \boldsymbol{\phi}^\kappa$. Thus, over time, the mixture of fundamental investment styles $\boldsymbol{\phi}^\kappa$ may change. This obviously corresponds to a 'motion' in Δ. Changes in investment strategies, i.e., the choice of $\boldsymbol{\ell}_t^a$, may depend on some 'signal' $\boldsymbol{x}_t^a$, which agent a observes and uses to rebuild his investment strategy. To explicitly mention the effect of the signal $\boldsymbol{x}_t^a$, one may also write $\boldsymbol{\ell}_t^a = \boldsymbol{\ell}^a(\boldsymbol{x}_t^a)$. Thus, if $\boldsymbol{\ell}_t^a$ is constant in time, the financial agent neglects this signal and stays with his initially chosen strategy for all time. On the other hand, the signal may suggest that the agent should give more weight to some $\boldsymbol{\phi}^\kappa$ in his strategy. The nature of the signal $\boldsymbol{x}_t^a$, that agent a receives and uses for updating his investment rule is left open. Signals may include economic observables like prices, or entire charts of assets, performances of other strategies, interest rates, 'news' about macro-economic entities, as well as expectations about growth rates of companies acording to analysts, rumours, and other sources. In this model, we avoid modeling the level of individual decisions, because price evolution is essentially an average property of the ensemble of financial agents.

Two agents can be distinguished by their *susceptibility* with respect to a signal. The susceptibility of agent a may be defined as $\chi^a(\boldsymbol{x}^a) := \|J\boldsymbol{\ell}^a(\boldsymbol{x}^a)\|$, where $J\boldsymbol{\ell}^a(\boldsymbol{x}^a)$ is the Jacobian of $\boldsymbol{\ell}^a$ in $\boldsymbol{x}^a$. Therefore, when giving the same signal $\boldsymbol{x}$ to both agents, we say that agent a is 'slower' than agent b, if his susceptibility is less than that of the other agent, $\chi^a(\boldsymbol{x}) < \chi^b(\boldsymbol{x})$.

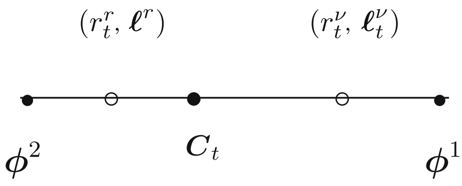

Fig. 7 The agents $\{r, \nu\}$ in a financial market are a couple of points $\boldsymbol{\ell}^a$ with masses r^a, $a = r, \nu$ in the simplex spanned by the fundamental investment styles $\{\boldsymbol{\phi}^{1,2}\}$. The center of mass has coordinate $\boldsymbol{C}_t$

4 The 'adiabatic' market model

Recall that we are interested in prices, which are *ensemble properties* of the aggregate actions of agents on the market. To the variety of agents on the financial market corresponds a whole spectrum of time-scales, defined by the agents' susceptibilities. As an approximation, we assume that the aggregate dynamics of the financial market can be described by the coupling of two dynamical components, a slow one and a fast one. Both components can be thought to be represented by agents with different susceptibilities. There is a 'rigid' financial agent r who does not change his investment strategy, while the other one ν is 'flexible,' in that he changes his investment strategy over time with respect to some signal, $0 \approx \chi^r < \chi^\nu$.

Moreover, we assume that there are only two fixed fundamental investment styles $\boldsymbol{\phi}^{1,2}$, such that each agent is characterized by the coordinates $\boldsymbol{\ell}^a = \binom{\ell^a}{1-\ell^a}$ (see Fig. 7) according to $\boldsymbol{\lambda}^a_t = \ell^a_t \boldsymbol{\phi}^1 + (1 - \ell^a_t)\boldsymbol{\phi}^2$. Without loss of generality, we take these fundamental investment styles to be the Euclidian ones, i.e., $\boldsymbol{\phi}^1 = \binom{1}{0}$ and $\boldsymbol{\phi}^2 = \binom{0}{1}$. The two agents r, ν are then characterized by the following investment strategies

$$\boldsymbol{\lambda}^r := \begin{pmatrix} r \\ 1 - r \end{pmatrix}, \qquad 0 \le r \le 1, \tag{4}$$

i.e., agent r follows a constant investment strategy, while agent ν is flexible in that he builds his investment strategy on some signal x_t that he receives. His reception is modeled by a sigmoid function $\ell(x_t) = \frac{e^{\nu x_t}}{1+e^{\nu x_t}}$ parameterized by some real ν. His investment strategy is given by

$$\boldsymbol{\lambda}^\nu_t := \begin{pmatrix} \ell(x_t) \\ 1 - \ell(x_t) \end{pmatrix}. \tag{5}$$

For the sake of simplicity, we denote a rigid strategy $\boldsymbol{\lambda}^r$ with parameter r by $[r]$, while the flexible strategy $\boldsymbol{\lambda}^\nu_t$ is denoted by (ν), i.e.

$$[r] := \boldsymbol{\lambda}^r, \qquad (\nu) := \boldsymbol{\lambda}^\nu_t.$$

We briefly denote this aggregate financial market with the two agents r, ν by the pair

$$\Big([r], (\nu)\Big),$$

where parameter ranges are $0 < r < 1, -\infty < \nu < \infty$.

If $x_t = 0$, then $\ell(0) = 1/2$, so that the agent gives the same weight to both styles. On the other hand, if $|x_t| = \infty$, he invests in only one style. Susceptibilities are therefore

$$\chi^r = 0, \qquad \chi^\nu(0) = \frac{|\nu|}{2} \tag{6}$$

The wealth evolution of agent $a \in \{r, \nu\}$ is given by $r^a_{t+1} = \boldsymbol{Y}^a_{t+1}\ r^a_t$ according to Eq. 2. In the following section, we present simulations of return trails in this financial market for different $\chi^\nu(0)$.

4.1 Excess performance as the signal

Many different signals can serve as 'news', according to which an agent can update his investment strategy. In the following, we consider the situation in which the agent weights the styles $\boldsymbol{\phi}^{1,2}$ in his investment strategy according to their mutual performance, i.e., the signal he receives is the excess performance of the two styles in the previous period $[t-1, t)$, i.e.,

$$x_t = R^1_t - R^2_t.$$

Thus, if $\nu > 0$ and $x_t > 0$, then the agent prefers style $\boldsymbol{\phi}^1$ over $\boldsymbol{\phi}^2$. That is to say, he prefers the style that performed better in the previous period. On the other hand, if his parameter $\nu < 0$, he prefers the worse one. Therefore, an agent with positive parameter ν might be called a *trend follower*, whereas if his parameter is negative, he might be called a *mean reverter*. In any case the update is a deterministic function of the signal.

5 Results

Parameter values of the market $([r], (\nu))$ considered in the following are listed in the table below. Results were obtained from numerical simulations of the stylized market. The length of each trail is 4.000 to keep the results comparable with our data, comprising daily returns over a period of 15 years from January 1, 1990 to December 31, 2004 of 3914 data points. The total amount of data points used for the histograms is thus 4×10^6. Another important reason for taking a trail length of 4.000 is that, in the case of $([r], (0))$, $0 < r < 1$, the convergence of the return trails to zero is so fast that, for longer trails numerical results for the distribution would collapse into a numerical Dirac function, which is a single

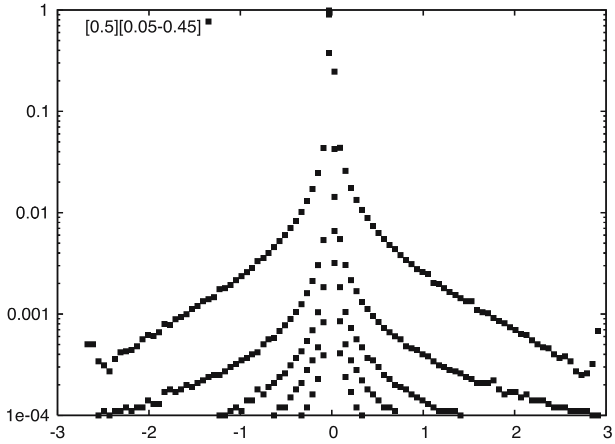

Fig. 8 Distribution on the market ([r] (0)), trail length 20.000, 1.000 trails

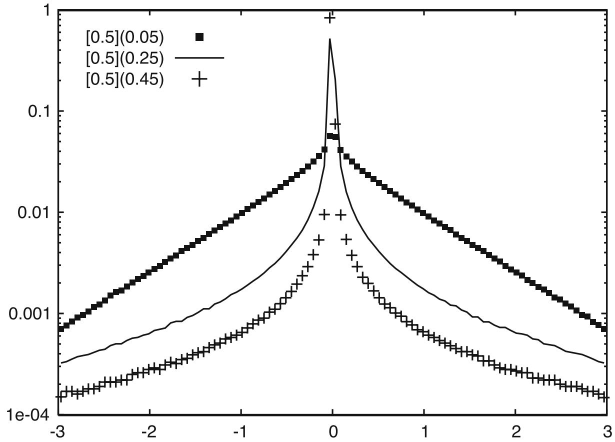

Fig. 9 Distribution on the market ([1/2] (ν)), trail length 20.000, 1.000 trails

point. In Figs. 8–10, the length of each trail is 20.000, while the number of runs is 1.000. The total amount of data points used for the histograms thus is 10^7.

([r](0))	([1/2] (ν))	([0.45] (ν))
Fig. 8	Fig. 9	Fig. 10

We consider the case:

$$\big([r],(0)\big) = \big([r],[1/2]\big). \tag{7}$$

First, notice the symmetry of the problem. Two parameters, r and r' with $|1/2 - r| = |1/2 - r'|$ leads to the same distributions. Therefore, we can restrict our simulation to some interval $r \in [0, 1/2]$.

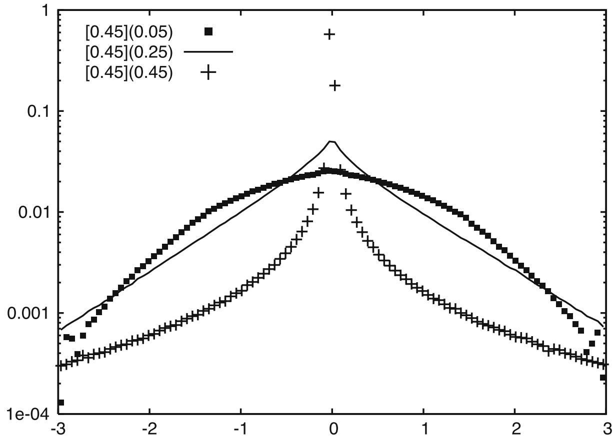

Fig. 10 Distribution on the market ([0.45](ν)), trail length 20.000, 1.000 trails

The distribution of the log returns is essentially determined by the fact that the return sequence rapidly converges to 0.[3] In fact, for infinitely long trails, the distribution would be a Dirac function, with all its mass on 0. As seen in Fig. 8, for all ν the distributions are convex functions, whith the highest one being $\nu = 0.05$, for which the flexible strategy is close to [1/2].

Next, we consider the case

$$([1/2], (\nu)), \quad \nu = 0.05, 0.25, 0.45. \tag{8}$$

For $\nu = 0.05$, the distribution is almost exponential, i.e., a tent in the semi-log plot. With a larger ν the convergence generated by the strategy [1/2] forces the distribution to become more and more convex, i.e., it becomes more peaked with a larger ν. This behavior follows essentially from the dominating convergence of prices due to the existence of $\boldsymbol{\lambda}^* = [1/2]$.

Finally, we consider the empirically interesting case

$$([0.45], (\nu)), \qquad \nu = 0.05, 0.25, 0.45, \tag{9}$$

$[r], r \neq 1/2$.

None of the agents is clearly dominating. In Fig. 10, we see a clear crossover from a concave Gaussian-like distribution of small χ^ν to a convex distribution for large χ^ν. Particularly in case 8, with $\nu = 0.05$, the distribution of log returns is exponential (see Fig. 9); it is Gaussian-like in case 9 (see Fig. 10).

[3] Convergence for the strategies [1/2] and (0): Random value factors D_1, D_2 are assumed to be independently and uniformly distributed in some interval. In this case, there is a strategy $\lambda^* = \mathbf{\frac{1}{2}}$ which will overtake the market asymptotically. Since $r_t^* \to r_t$ asymptotically, prices become asymptotically constant $\boldsymbol{q}_\infty$, as shown in Amir, Evstigneev, Schenk, Hens & Schenk-Hopp (2005). Thus, there are two exceptional parameter settings. One is the rigid strategy $\frac{1}{2}$, while the other is the 'flexible' strategy (0).

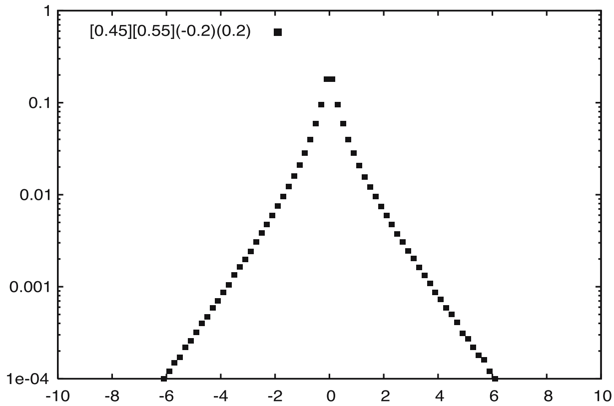

Fig. 11 A 4-component aggregates model $([r_1]\ [r_2](\nu)(-\nu))$, where $[r_1], [r_2] \neq \boldsymbol{\lambda}^*$.

The stylized market considered so far consisted of two components, a rigid one and a flexible one. We denoted this market by $\big([r], (\nu)\big)$. By varying the time-scale parameter and the susceptibility of the agent (ν), we obtained quantitatively different system behaviors. One might suspect that this approximation of a financial market is artificial and exhibits singular features. In the following, we show a more complex model (see Fig. 11). We divided the market into four components, two rigid components and two flexible components

$$\big([r_1]\ [r_2]\ (\nu)\ (-\nu)\big).$$

We chose r_1, r_2, symmetric in the sense that $r_1 = 1/2 - \epsilon$ while $r_2 = 1/2 + \epsilon$. In the simulation, $\epsilon = 0.05$ so that $[r_1] = [0.45]$ prefers style $\boldsymbol{\phi}^2$ over $\boldsymbol{\phi}^1$, while $[r_2] = [0.55]$ prefers $\boldsymbol{\phi}^1$ over $\boldsymbol{\phi}^2$. While both flexible agents have the same susceptibility with respect to the signal, one can be regarded as a trend-follower, while the other might be called a mean reverter. The choice of the parameters was essentially arbitrary, except that above symmetries had to respected. Therefore, in fact, only two parameters in this model are free. Extended simulations showed that the space of the resulting distribution is qualitatively insensitive to changes in the parameters in a wide range. In this sense, the result is quite representative. The following simulation used a trail length of 20.000; while 500 trails were considered. The simulated return distribution should be compared with the one in Fig. 3 for high-frequency returns of the S&P 500.

Although even the lowest-order approximation, with only two time-scales, already creates important system behavior, higher-order approximations with the inclusion of more agents do not lead to qualitatively new results. Comparison with empirical data suggest that his model is capable of reproducing important dynamical patterns of real financial markets.

5.1 Volatility clustering

The phenomenon of high-volatility events, tending to cluster together in time, is denoted by the term '*volatility clustering*.' It is a model-free property of

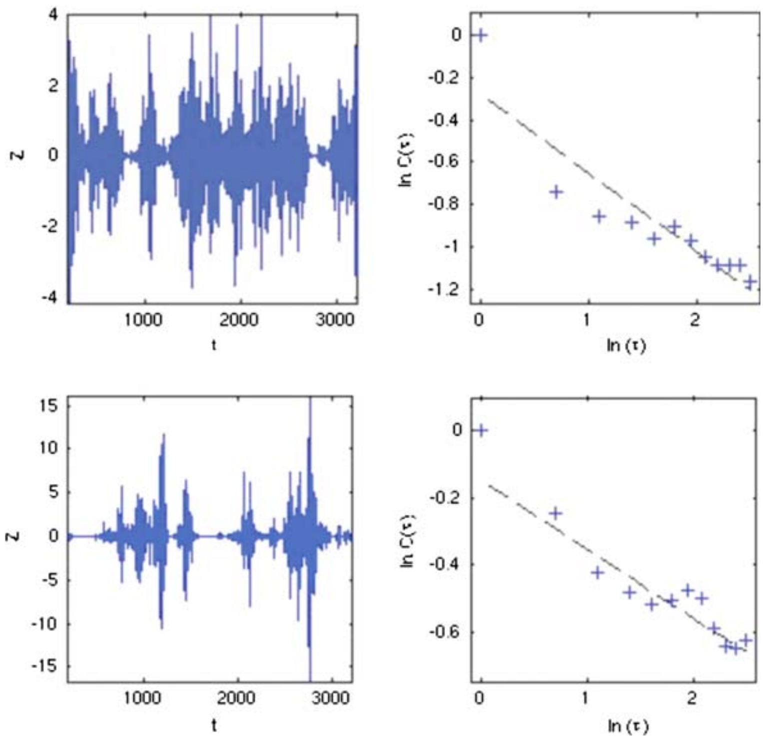

Fig. 12 The slow decay of the auto correlation of squared returns for the model ([0.4], (0.32)) (above) and ([0.4], (0.36)) (below)

empirical returns which does not rely on some GARCH hypothesis. Our model generates volatility clustering, while its extent is directly related to the degree of decoupling of the two components (Fig. 12). In other words, the greater the susceptibility of χ^{ν}, the more pronounced the clustering of high-volatility events. This is easily seen in the following sequence of typical return trails, obtained from simulations on the market ([0.45] (ν)) (see the Figs. 13–18).

The auto correlation $C(\tau)$ of squared returns Z_t is often taken as a measure for volatility clustering

$$C_2(\tau) \;=\; \text{corr}\,\left(Z_t^2, Z_{t+\tau}^2\right), \quad \tau \geq 0.$$

Figure 12 contrasts $C_2(\tau)$ with the corresponding trails. The inscribed line indicates that $C_2(\tau)$ is strictly polynomial in τ, i.e., $C_2(\tau) \propto \tau^{-\gamma}$.

6 Concluding remarks

Prices are macro-observables of a financial market because they are the result of the aggregate trading action of a huge number of individual investors. Research

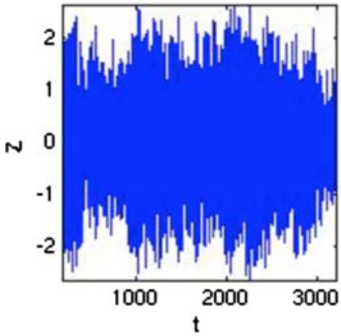

Fig. 13 [0.45], (0.05)

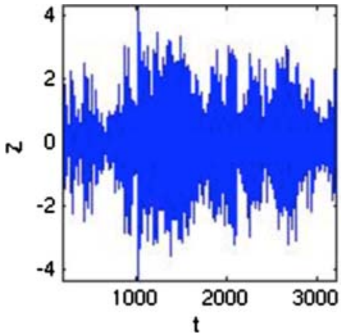

Fig. 14 [0.45], (0.2)

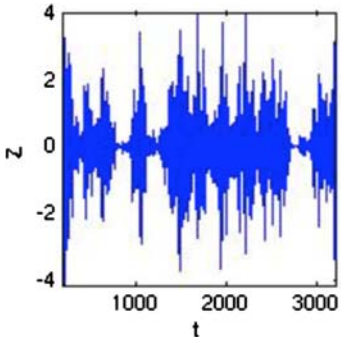

Fig. 15 [0.45], (0.25)

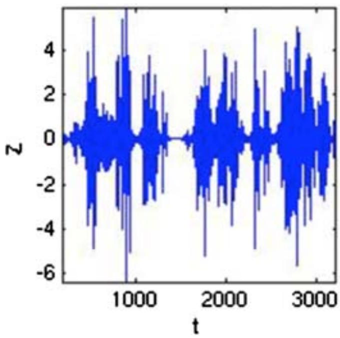

Fig. 16 [0.45], (0.3)

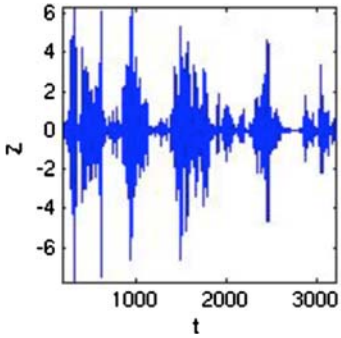

Fig. 17 [0.45], (0.4)

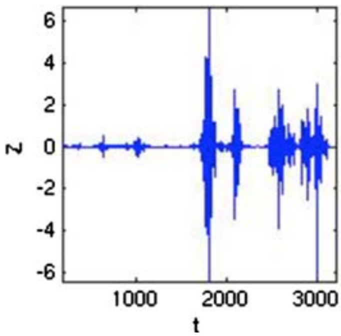

Fig. 18 [0.45], (0.45)

on price fluctuations has revealed that different markets show qualitatively similar or even almost identical stochastic properties, called stylized facts. These stylized facts may serve as fingerprints for a more basic mechanism driving the dynamics on financial markets. Considering seriously that prices are macro-observables, we proposed an elementary model for the *dynamics* of ‘a financial market’ on the aggregate level. For modeling a financial market, we chose to represent it as a network of interacting boundedly rational agents.[4] Agents are non-rational in that their decision rules are not micro-founded and derived from some optimization principles, while decisions are based on *signals* or news. Agents may differ in the extent to which they react to incoming signals. Some may ignore signals, while others respond more or less strongly to a signal by updating their investment strategies. In other words, agents show different *reactivity* with respect to incoming signals. A financial market thus can be seen as being constituted by a number of agents with different ‘time-scales’ due to their respective reactivity.

The aim is to describe the dynamics of the market, not to model individual agents’ dynamics. Our approach to this problem is inspired by what is known as ‘mean field approximation’ in complex systems, where dynamics are divided into a constant part representing the mean of the dynamics, and a second part representing fluctuations around this mean. The idea is thus to approximate the market’s *dynamics* by a ‘slow’ and a ‘fast’ dynamical component, where the ’fast’ component can be seen as a perturbation of the ‘slow’ one. A ‘slow’ component has lesser reactivity to a signal than a fast one. The corresponding model exhibits two components, one describing the slow component $[r]$, and the other describing the ‘fast’ one (ν), i.e., representing the effect of incoming signals. The ‘market’ was represented by the pair

$$\big([r],\ (\nu)\big).$$

We simulated the market for various parameter settings. Unless otherwise stated, we did 1.000 trails, each of length 4.000. Therefore, the histograms use samples of size 5×10^6 points. The most important parameter in this model turned out to be the susceptibility of the fast agent, which in some sense also determines the degree of dynamic decoupling between these two levels. Varying the dynamic decoupling between the two levels by increasing the susceptibility from 0 changes dynamics, and changes the return distribution significantly. In particular, the shape of the return distribution crosses over from a concave Gaussian-like shape to a convex one for increasing susceptibility. This relates

[4] Here, a financial agent here is an investment strategy which is equipped with some wealth and whose state therefore is described by his wealth and his investment strategy. This naturally leads to representing the set of agents as a system of point-masses characterized by their masses and coordinates. We stayed with traditional assumptions: agents interact only via prices, which, moreover, are market-clearing prices. Under the assumption that short-selling is excluded, dynamics are such that, at any time, the market is in equilibrium. Since prices are macro-variables, price evolution has to be described on the aggregate level. In this picture, a market-clearing price is the center-of-gravity of the mass-point cloud.

to an increase in volatility clustering. Since our model has essentially only two free parameters, the question concerning fitting real data to the model is natural. Fitting would be of explanatory value if the fitted parameters had a clear economic meaning and are measurable.

Comparing simulated return trails with empirical ones, one sees that the model generates return trials whose statistical properties are qualitatively similar to those seen in empirical data in a wide range of parameters. Therefore, this behavior is generic rather than restricted to some special parameter settings. One can narrow the range of empirically reasonable parameters in our model as follows. We analyzed nine major indices, such as the american indices DJIA, the NASDAQ, and the S&P 500; european indices, including the english FTSE, the french CAC, the german DAX and the swiss SPI; and also the asian indices NIKKEI and the Hang Seng. Their distribution of daily asset returns is close to a Laplacian. From inspection of Fig. 19, it is therefore evident that, for a given parameter $r = 0.4$, parameter values ν therefore should be around 0.25, since this is the value where the crossover from the concave to the convex shape in the distribution is observed. For the same reason, $\nu > 0.25$ if high-frequency data are considered, since their distribution is more convex.

As an example, we stay with the NIKKEI. Figure 19 summarizes considered stylized facts of the NIKKEI, using daily data from January 1, 1990 to December

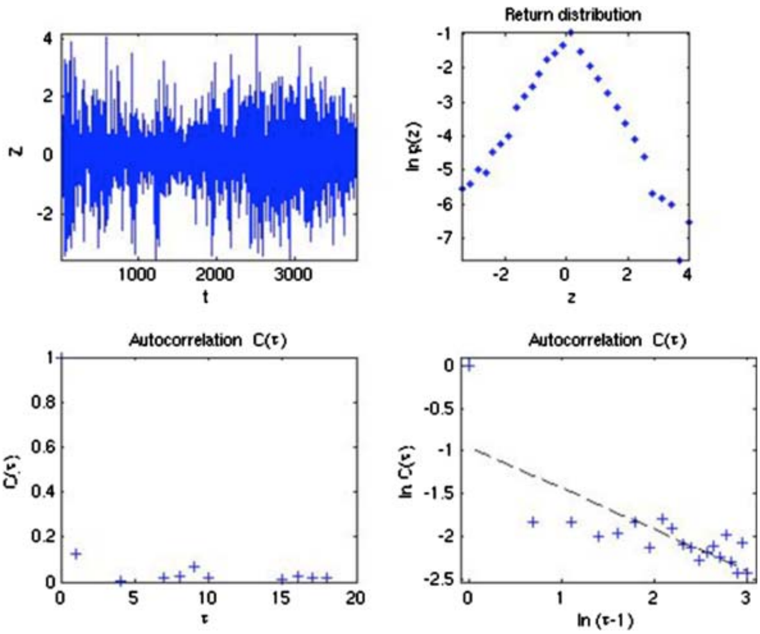

Fig. 19 Summary of stylized facts of an index showing NIKKEI daily data from 1990 to 2004. The lower row displays the auto-correlations of returns (left) and of absolute returns (right)

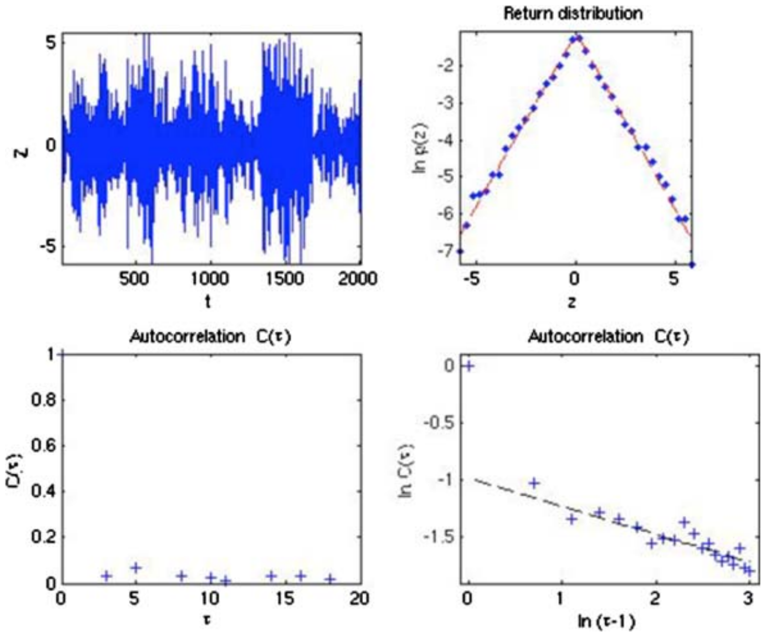

Fig. 20 Summary of stylized fact obtained from our model ([0.4], (0.25)). The lower row displays the auto-correlations of returns (left) and of absolute returns (right)

31, 2004. For comparison, Figure 20 displays the same stylized properties for our model

$$\big([0.4],\ (0.25)\big).$$

The model is not about economic agents and their behavior, nor does it represent sophisticated economic structures and interdependencies. What then does this model tell us? Stylized facts can be obtained by a large class of models. One may conclude that stylized facts do not provide a sufficiently rich fundamental basis for sound economic modeling. Another conclusion may be that stylized facts are typical properties of a far more general system, while financial markets are only special realizations of it.

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