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Blocking Visibility for Points in General Position

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Abstract For a finite set P in the plane, let $b(P)$ be the smallest possible size of a set Q, $Q \cap P = \emptyset$, such that every segment with both endpoints in P contains at least one point of Q. We raise the problem of estimating $b(n)$, the minimum of $b(P)$ over all n-point sets P with no three points collinear. We review results providing bounds on $b(n)$ and mention some additional observations.

Keywords Visibility · Visibility-blocking set · Behrend's construction

Let P be an n-point set in the plane (or, more generally, in $\mathbf{R}^d$). We define a *visibility-blocking set for* P as a set Q that is *disjoint* from P and such that every segment with endpoints in P contains at least one point of Q.

If the points of P are all collinear, then there is a visibility blocking set with $n-1$ points. The question raised in this note is what is the smallest possible size of a visibility-blocking set for P having no three points collinear? That is, we let

$$b(P) := \min\{|Q| : Q \text{ a visibility-blocking set for } P\},$$

$$b(n) := \min\{b(P) : P \subset \mathbf{R}^2 \text{ with no three points collinear, } |P| = n\},$$

and we would like to estimate the asymptotics of $b(n)$ for large n.

I arrived at this question when (unsuccessfully) attempting to prove the following nice conjecture of Kára, Pór, and Wood [7, Conjecture 2]: *For all integers $k, \ell \geq 2$,*

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there is an integer n such that every n-point set in the plane contains ℓ collinear points or k pairwise visible points.

I obtained Theorems 1 and 3 below, but after this note was accepted for publication in the Klee Festschrift, it turned out that both of these results had been known earlier in somewhat different contexts. Moreover, I learned that other people considered the asymptotics of $b(n)$ independently (a group of researchers at the Courant Institute in New York including Andreas Holmsen, János Pach, Radoš Radoičić, and Gábor Tardos [6]) and also rediscovered some of these results. Thus, the present note mostly reviews known results and adds some observations, which have not appeared in print as far as I know and which illustrate some of the difficulties inherent in the problem.

I still believe that the inclusion of this note in the Festschrift is warranted by the beauty of the problem—I think that Vic Klee would like it.

Midpoints and an Upper Bound For a point set P, let $\mu(P)$ be the cardinality of the set $\{\frac{1}{2}(p+q) : p, q \in P, p \neq q\}$ of *midpoints* of all pairs of points of P. Clearly, $b(P) \leq \mu(P)$. The problem of estimating $\mu(n) := \min \mu(P)$, where the minimum is over all n-point planar sets P in general position, was raised, according to Pach [10], by F. Hurtado. Earlier Erdős, Fishburn, and Füredi [4] studied the problem of estimating $\min \mu(P)$ over all n-point planar sets P *in convex position* and established the (surprising) lower and upper bounds of $0.40n^2$ and $0.45n^2$, respectively.

Now we recall an upper bound on $\mu(n)$ (and $b(n)$) due to Pach [10] (based on Erdős et al. [5]). For an integer N, let $\nu(N)$ be the maximum number of elements of a set $A \subseteq \{1, 2, \ldots, N\}$ that contains no three-term arithmetic progression. Behrend [3], improving on a construction by Salem and Spencer, proved that $\nu(N) \geq N^{1-O(1/\sqrt{\log N})}$. The following theorem, whose proof we recall, is based on Behrend's construction.

Theorem 1 (Pach [10]) *There is a constant C such that*

$$b(n) \leq \mu(n) \leq n\mathrm{e}^{C\sqrt{\log n}}$$

for all sufficiently large n.

Proof Let n be given and large, and let m and s be integer parameters to be specified later. Let $G := \{0, 1, \ldots, s-1\}^m \subset \mathbf{R}^m$, and let $S_k := \{x \in G : \|x\|^2 = k\}$, where $\|\cdot\|$ is the Euclidean norm.

As in Behrend's argument, $G = \bigcup_{k=0}^{m(s-1)^2} S_k$, and thus by the pigeonhole principle there exists k with $|S_k| \geq s^{m-2}/m$. We let $P := S_k$ for some such k.

Since the points of P lie on a sphere, no three of them are collinear. The midpoint of every two points $p, q \in G$ lies in $G' := \{0, \frac{1}{2}, 1, \frac{3}{2}, \ldots, s-1\}^m$, and thus $Q := G' \setminus P$ is a visibility-blocking set for P.

We let $m := \lfloor \ln n \rfloor$, and let s be the smallest integer with $s^{m-2}/m \geq n$. Then $|P| \geq n$, $(s-1)^{m-2} < mn$, and

$$|Q| \leq |G'| = (2s-1)^m \leq \big(3(s-1)\big)^m \leq 3^m(s-1)^2mn \leq n\mathrm{e}^{O(\sqrt{\log n})},$$

as can easily be calculated (using $(s-1)^2 \le (mn)^{2/(m-2)} \le (n^2)^{4/m}$, say). The implicit constant in the exponent can be improved by a more careful choice of m and by more precise calculations.

The bound in the theorem follows by projecting P and Q to a generic two-dimensional subspace of $\mathbf{R}^m$. □

Lower Bounds I am aware only of the following rather trivial lower bound for $b(n)$:

Observation 2 *If P is an n-point planar set with no three points collinear and with* $\operatorname{conv}(P)$ *having p vertices, then $b(P) \ge 3n - p - 3$. In particular, $b(n) \ge 2n - 3$.*

Proof A triangulation of P has $3n - p - 3$ edges, and each point of a visibility-blocking set covers at most one of these. □

For point sets in convex position, a slightly superlinear lower bound can be given. The result is implicitly contained in Araujo et al. [1], and the argument goes back (at least) to Kostochka and Kratochvíl [8].

Theorem 3 *For every n-point planar set P in (strictly) convex position, we have*

$$b(P) \ge \begin{cases} n\sum_{k=1}^{m} 1/k & \text{for } n = 2m+1 \text{ odd,} \\ 1 + n\sum_{k=1}^{m-1} 1/k & \text{for } n = 2m \text{ even.} \end{cases}$$

Thus, $b(P) = \Omega(n \log n)$.

Proof Let $p_1, p_2, \dots, p_n$ be the points of P numbered along the circumference of $\operatorname{conv}(P)$. We define the *length* $\ell(p_i p_j)$ of the segment $p_i p_j$, $i < j$, as the number of convex hull edges between p_i and p_j, i.e., $\min(j-i, n+i-j)$.

Let us suppose that Q is a visibility-blocking set for P, and let $q \in Q$. The key observation is that if ℓ is the smallest of the lengths of the segments $p_i p_j$ incident to q, then q is incident to at most ℓ segments.

Thus, if we give the segment $p_i p_j$ weight $1/\ell(p_i p_j)$, no $q \in Q$ is incident to segments of total weight more than 1. The theorem follows by summing the weights of all segments. □

For the case where P is the vertex set of a *regular* convex n-gon, Poonen and Rubinstein [13] show that, apart from the center, no point is the intersection of eight or more of the diagonals of the n-gon, and thus $b(P) = \Omega(n^2)$ in this case.

Small Cases Now we return to arbitrary sets (not necessarily in convex position). The first few values of $b(n)$ are $b(2) = 1$, $b(3) = 3$, $b(4) = 5$, $b(5) = 8$, $b(6) = 10$. The upper bounds are witnessed by Fig. 1. The lower bounds for $n \le 4$ are trivial (or follow from Observation 2). For $n = 5, 6$, we distinguish two cases: If all vertices of P are on the convex hull, then the lower bound follows from Theorem 3, and otherwise, it is given by Observation 2. Determining the exact value gets more complicated for larger n, and I am not aware of a reasonably clean argument for any

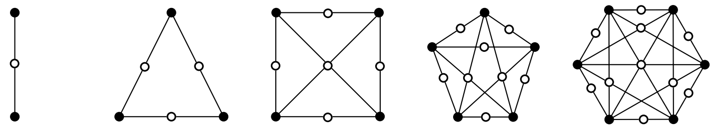

Fig. 1 Upper bound examples for small n

$n \geq 7$. Interesting upper bound constructions for several small cases, to be reported elsewhere, were given by Snoeyink and Speckmann (private communication).

Additional Remarks

1. A possibly easier version of the problem deals with pseudosegments instead of straight segments. That is, for a given point set P, we want to construct an arrangement $\mathcal{A}$ of pseudolines and a subset Q of its vertices such that $P \cap Q = \emptyset$, each $p \in P$ is a vertex of $\mathcal{A}$, no three points of P lie on a common pseudoline, and every two points of P lie on a common pseudoline ℓ and have a point of Q on the segment of ℓ between them.

   The lower bound of Theorem 3 still applies in this setting (with "convex position" interpreted appropriately in terms of the arrangement $\mathcal{A}$), and here it is easy to provide an $O(n \log n)$ upper bound.

   Is there a linear upper bound for some P, not in "convex position", in the pseudoline setting?

   Another variant of the problem, asked by Pach, is partitioning all the $\binom{n}{2}$ straight segments defined by the points into a small number of *crossing families*, i.e., families in which every two segments cross. Is a superlinear number of such families always necessary? (Also see Pach et al. [11] for some related blocking-type questions.)
2. Here is another variation of the problem: Instead of requiring that each *segment* determined by two points of P contains a point of Q, it now suffices that each *line* determined by two points of P contains a point of Q. Perhaps surprisingly, there are examples showing that $O(n)$ points suffice to block all lines.

   One such example can be constructed using integer points on the curve $y = x^3$ (a similar construction was used, e.g., for the *orchard problem*; see [9]). A simpler example, inspired by a relation of the problem to Ungar's theorem (see [9] again), was communicated to me by Pinchasi [12]: Take the vertex set of a regular $(2n)$-gon centered at the origin and apply a projective transform that sends the line at infinity to the x-axis, obtaining a $(2n)$-point set P (the points of P lie on a hyperbola). Since the vertices of the regular $(2n)$-gon determine only $2n$ distinct directions, there is a $(2n)$-point set Q on the x-axis that intersects all the lines determined by P.
3. The set P in the example just mentioned can also be partitioned into two n-point subsets P_1 and P_2 (namely, the points above and below the x-axis) in such a way that every segment p_1p_2, $p_1 \in P_1$, $p_2 \in P_2$, contains a point of Q. This shows that a natural bipartite version of the original visibility-blocking problem has a linear upper bound.

Pach [10] proves a superlinear lower bound for $\mu(n)$, the minimum cardinality of the set of all midpoints for n points in general position, using Freiman's theorem on set addition. This argument can be adapted to give a superlinear lower bound for the midpoints in the bipartite setting as well. Indeed, let P_1, P_2 be disjoint n-point sets with $P := P_1 \cup P_2$ in general position, and suppose that the set $P_1 + P_2 = \{p_1 + p_2 : p_1 \in P_1, p_2 \in P_2\}$ has cardinality $O(n)$. Then P has $\Omega(n^3)$ *additive four-tuples*, i.e., four-tuples (p_1, p_2, p_3, p_4) with $p_1 + p_2 = p_3 + p_4$, and by the Balog–Szemerédi theorem [2] there is a subset $P' \subseteq P$ of size $\Omega(n)$ with $|P' + P'| = O(|P'|)$. Then, as in Pach's argument, Freiman's theorem implies that for a sufficiently large n, the set P' contains three collinear points, a contradiction. This shows that Pach's lower bound idea for $\mu(n)$ is not directly applicable to $b(n)$.

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