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Arbitrarily large families of spaces of the same volume

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Abstract In any connected non-compact semi-simple Lie group without factors locally isomorphic to $SL_2(\mathbb{R})$, there can be only finitely many lattices (up to isomorphism) of a given covolume. We show that there exist arbitrarily large families of pairwise non-isomorphic arithmetic lattices of the same covolume. We construct these lattices with the help of Bruhat-Tits theory, using Prasad's volume formula to control their covolumes.

Keywords Locally symmetric spaces · Arithmetic lattices · Volume

Mathematics Subject Classification (1991) 22E40 · 11E57 · 20G30 · 51M25

1 Introduction

Let $\mathscr{G}$ be a connected semi-simple real Lie group without compact factors. For simplicity we will suppose that $\mathscr{G}$ is adjoint (i.e., with trivial center), though this is not a major restriction in this article. Any choice of a Haar measure μ on $\mathscr{G}$ assigns a covolume $\mu(\Gamma\backslash\mathscr{G}) \in \mathbb{R}_{>0}$ to each lattice Γ in $\mathscr{G}$. Wang's theorem [1] asserts that there exist only finitely many irreducible lattices (up to conjugation) of bounded covolumes in $\mathscr{G}$ unless $\mathscr{G}$ is isomorphic to $PSL_2(\mathbb{R})$ or $PSL_2(\mathbb{C})$. In particular, there exist only finitely many irreducible lattices in $\mathscr{G}$ of a given covolume. For $\mathscr{G}$ isomorphic to $PSL_2(\mathbb{C})$ this property is still true, as follows from the work of Thurston and Jørgensen [2, Ch. 6]. In this paper we prove that the number of lattices in $\mathscr{G}$ of the same covolume can be arbitrarily large. In most cases, arbitrarily large families of lattices of equal covolume appear in the commensurability class of any arithmetic lattice of

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Table 1 Simple lie groups not covered in Theorem 1

Type A_1:	$PSL_2(\mathbb{R})$ and $PSL_2(\mathbb{C})$;
Type A_2:	$PSL_3(\mathbb{R})$, $PSL_3(\mathbb{C})$ and $PU(2, 1)$;
Type A_3:	$PSL_4(\mathbb{R})$, $PSL_4(\mathbb{C})$, $PSO(5, 1)$, $PU(3, 1)$ and $PU(2, 2)$.

$\mathscr{G}$. This is the content of the following theorem. The symbol $\mathfrak{g}_{\mathbb{C}}$ denotes the complexification of the Lie algebra of $\mathscr{G}$.

Theorem 1 *Let $\mathscr{G}$ be a connected adjoint semi-simple real Lie group without compact factors. We suppose that $\mathfrak{g}_{\mathbb{C}}$ has a simple factor that is not of type* A_1, A_2 *or* A_3. *Let Γ be an arithmetic lattice in $\mathscr{G}$. Then, for every $m \in \mathbb{N}$, there exist a family of m lattices commensurable to Γ that are pairwise non-isomorphic and have the same covolume in $\mathscr{G}$. These lattices can be chosen torsion-free.*

Every arithmetic lattice $\Gamma \subset \mathscr{G}$ is constructed with the help of some algebraic group G defined over a number field k (see Sect. 2.1). To prove Theorem 1, we use Bruhat-Tits theory to construct families of arithmetic subgroups in $G(k)$ that are non-conjugate, and have equal covolume. By strong (Mostow) rigidity one obtains the analogous result with "pairwise non-conjugate" replaced with "pairwise non-isomorphic". To control the covolume we use some computations that appear in Prasad's volume formula [3]. To ensure that the subgroups constructed are not conjugate we need to exhibit parahoric subgroups in $G(k_v)$ (where k_v is a non-archimedean completion of k) that are not conjugate but of the same volume. This can be easily achieved when G is not of type A_n and is split over k_v. When G is of type A_n the Bruhat-Tits building of a split $G(k_v)$ has more symmetries, and the argument must be slightly adapted. In particular, there we need the assumption $n \geq 4$, which explains the excluded cases in the statement of Theorem 1. The simple Lie groups excluded are listed in Table 1.

For the Lie groups of type A_2 and A_3 we can use algebraic groups that are outer forms (type ${}^2A_2$ and ${}^2A_3$) to show the existence of arbitrarily large families of arithmetic lattices of the same covolume. In contrast with Theorem 1, now each family corresponds to a different commensurability class.

Theorem 2 *Let $\mathscr{G}$ be a connected adjoint semi-simple Lie group without compact factors. We suppose that $\mathfrak{g}_{\mathbb{C}}$ contains only factors of type* A_2 *(resp. only factors of type* A_3*). Let $m \in \mathbb{N}$. Then there exists a family $\{\Gamma_1, \ldots, \Gamma_m\}$ of irreducible arithmetic lattices in $\mathscr{G}$ such that for $i, j \in \{1, \ldots, m\}$:*

1. *Γ_i is commensurable to Γ_j;*
2. *Γ_i and Γ_j have the same covolume in $\mathscr{G}$;*
3. *if $i \neq j$, then Γ_i and Γ_j are not isomorphic.*

The lattices $\{\Gamma_i\}$ can be chosen torsion-free. Moreover, they can be chosen cocompact. They can be chosen non-cocompact unless there are no such lattices in $\mathscr{G}$.

It follows from Margulis' arithmeticity theorem that irreducible lattices can only exist in a Lie group $\mathscr{G}$ that is isotypic (i.e., for which all the simple factors of $\mathfrak{g}_{\mathbb{C}}$ have the same type), so that the assumptions in Theorem 2 are minimal. The existence of irreducible cocompact lattices in any isotypic $\mathscr{G}$ was proved by Borel and Harder [4]. Non-compact irreducible quotients of $\mathscr{G}$ do not always exist. For example there is no such quotient of $PU(3, 1) \times PSO(5, 1)$ (this example is detailed in [5, Prop. (15.31)]). A general criterion for the existence of non-cocompact arithmetic lattices appears in the work of Prasad-Rapinchuk [6], where the authors extend the results of [4]. The proof of Theorem 2 uses these existence results.

By Wang's theorem, it is clear that the covolume common to the lattices of a family grows with the size of the family. Even though in this article we focus on qualitative results, we note that the proofs of Theorems 1–2 could be used to obtain quantitative results on the growth of the covolume with the size of the family.

We now discuss the geometric significance of our results. Let X be the symmetric space associated with $\mathscr{G}$, that is $X = \mathscr{G}/K$ for a maximal compact subgroup $K \subset \mathscr{G}$. This class of spaces includes the *hyperbolic n-space* $\mathscr{H}^n$; we have that $\mathscr{H}^2$ is associated with $\mathscr{G} = \mathrm{PSL}_2(\mathbb{R})$, and $\mathscr{H}^3$ with $\mathscr{G} = \mathrm{PSL}_2(\mathbb{C})$. For a torsion-free irreducible lattice $\Gamma \subset \mathscr{G}$, the locally symmetric space $\Gamma\backslash X$ will be called *an X-manifold* (in particular it is irreducible and has finite volume). The following result follows directly from Theorems 1–2 and the existence of cocompact arithmetic lattices in $\mathscr{G}$ (see for instance [6, Theorem 1]).

Corollary 3 *Let X be a Riemannian symmetric space of non-compact type that contains no factor isometric to $\mathscr{H}^2$ or $\mathscr{H}^3$, and suppose that irreducible quotients of X do exist. Then there exist arbitrarily large families of pairwise non-isometric commensurable compact X-manifolds having the same volume. The analogue statement with non-compact X-manifolds is true unless all X-manifolds are compact.*

The result for compact X-manifolds associated with non-compact simple Lie groups (including $\mathrm{PSL}_2(\mathbb{R})$ and $\mathrm{PSL}_2(\mathbb{C})$) already follows from a recent paper of McReynolds [7], who constructed families of manifolds with the stronger property of being isospectral. His construction uses arithmetic lattices except for the case $X = \mathscr{H}^n$, where he proved the result by considering the non-arithmetic lattices constructed by Gromov and Piatetski-Shapiro.

The result for $X = \mathscr{H}^3$ was proved by Wielenberg for the case of non-compact manifolds [8], and later by Apasanov-Gutsul for compact manifolds [9]. For $X = \mathscr{H}^4$ the result with non-compact manifolds was proved by Ivanšić in his thesis [10]. All these results are obtained by geometric methods. In [11] Zimmerman gave a new proof for $X = \mathscr{H}^3$ by exhibiting examples of $\mathscr{H}^3$-manifolds M with first Betti number β_1 at least 2, and showing that this property implies the existence of arbitrarily large families of covering spaces of M of same degree. In [12] Lubotzky showed that there exist (many) hyperbolic manifolds with $\beta_1 \geq 2$ in every dimension. Thus for all $X = \mathscr{H}^n$ we have a proof of Corollary 3 by Zimmerman's method. Since super-rigidity implies that $H^1(\Gamma\backslash X, \mathbb{R}) = 0$ for irreducible lattices Γ in $\mathscr{G}$ with $\mathbb{R}\text{−rank}(\mathscr{G}) \geq 2$, the same approach cannot be used to prove the result in this situation. Conversely, it does not seem that our method can be adapted to include the case of $\mathscr{H}^2$ and $\mathscr{H}^3$.

Very recently, Aka constructed non-isomorphic arithmetic lattices that have isomorphic profinite completions [13]. In particular, his construction gives arbitrarily large families of lattices of equal covolume in the Lie group $\mathrm{SL}_n(\mathbb{C})$, for any $n \geq 3$.

2 Arithmetic lattices

We can obviously reduce the proof of Theorem 1 to the case of an irreducible Γ. Then, like in Theorem 2, $\mathscr{G}$ is supposed to be isotypic.

2.1

For generalities on arithmetic groups we refer the reader to [14] and [15]. We briefly explain here how irreducible arithmetic lattices in $\mathscr{G}$ are obtained. Let k be a number field with ring of integers $\mathscr{O}$. Let G be an absolutely simple simply connected algebraic group defined over k.

We denote by $\overline{\mathrm{G}}$ the adjoint group of G, i.e., the k-group defined as G modulo its center, and by $\pi : \mathrm{G} \to \overline{\mathrm{G}}$ the natural isogeny. Let $\mathscr{S}$ be the set of archimedean places v of k such that $\mathrm{G}(k_v)$ is non-compact. We denote by $\mathrm{G}_{\mathscr{S}}$ the product $\prod_{v\in\mathscr{S}} \mathrm{G}(k_v)$, and similarly for $\overline{\mathrm{G}}_{\mathscr{S}}$. Note that $\mathrm{G}_{\mathscr{S}}$ is connected. For any matrix realization of G, the group $\mathrm{G}(\mathscr{O})$ is an irreducible lattice in $\mathrm{G}_{\mathscr{S}}$. Suppose that the connected component $(\overline{\mathrm{G}}_{\mathscr{S}})^{\circ}$ of $\overline{\mathrm{G}}_{\mathscr{S}}$ is isomorphic to $\mathscr{G}$. Then π extends to a surjective map $\pi_{\mathscr{S}} : \mathrm{G}_{\mathscr{S}} \to \mathscr{G}$. An irreducible lattice in $\mathscr{G}$ is called *an arithmetic lattice* if it is commensurable with a subgroup of the form $\pi_{\mathscr{S}}(\mathrm{G}(\mathscr{O}))$ for some k-group G as above.

In the following G will always be a k-group as above, which determines a commensurability class of arithmetic lattices in $\mathscr{G}$.

2.2

We denote by V_{f} the set of finite places of k, and by $\mathbb{A}_{\mathrm{f}}$ the ring of finite adèles of k. For each $v \in V_{\mathrm{f}}$ we consider k_v the completion of k with respect to v, and $\mathscr{O}_v \subset k_v$ its associated valuation ring. A collection $P = (P_v)_{v\in V_{\mathrm{f}}}$ of compact subgroups $P_v \subset \mathrm{G}(k_v)$ is called *coherent* if the product $\mathscr{K}_P = \prod_{v\in V_{\mathrm{f}}} P_v$ is open in the adelic group $\mathrm{G}(\mathbb{A}_{\mathrm{f}})$ (see [15, Ch. 6] for information on adelic groups). For example, for any matrix realization of G, the collection $(\mathrm{G}(\mathscr{O}_v))_{v\in V_{\mathrm{f}}}$ is coherent. For a coherent collection $P = (P_v)$, the group

$$\Lambda_P = \mathrm{G}(k) \cap \prod_{v\in V_{\mathrm{f}}} P_v, \tag{1}$$

where $\mathrm{G}(k)$ is seen diagonally embedded into $\mathrm{G}(\mathbb{A}_{\mathrm{f}})$, is an arithmetic subgroup of $\mathrm{G}(k)$ (and thus an arithmetic lattice in $\mathrm{G}_{\mathscr{S}}$). This follows from the equality $\mathrm{G}(\mathscr{O}) = \mathrm{G}(k) \cap \prod_v \mathrm{G}(\mathscr{O}_v)$ together with the inequality

$$[\Lambda_P : \Lambda_{P'}] \le [\mathscr{K}_P : \mathscr{K}_{P'}], \tag{2}$$

valid for any two coherent collections P and P' with $P'_v \subset P_v$ for each $v \in V_{\mathrm{f}}$. Since G is simply connected, strong approximation holds [15, Theorem 7.12] and it follows that (2) is in fact an equality. We put this (known) result in the following lemma.

Lemma 4 *Let $P = (P_v)_{v\in V_{\mathrm{f}}}$ and $P' = (P'_v)_{v\in V_{\mathrm{f}}}$ be two coherent collections of compact subgroups such that $P'_v \subset P_v \subset \mathrm{G}(k_v)$ for all $v \in V_{\mathrm{f}}$. Then*

$$[\Lambda_P : \Lambda_{P'}] = \prod_{v\in V_{\mathrm{f}}} [P_v : P'_v].$$

2.3

For every field extension $L|k$ with algebraic closure $\overline{L}$, the group of L-points given by $\overline{\mathrm{G}}(L)$ is identified with the inner automorphisms of G that are defined over L. Note that in general $\overline{\mathrm{G}}(L)$ is larger than the image of $\mathrm{G}(L)$ in $\overline{\mathrm{G}}(\overline{L})$.

Lemma 5 *Let P and P' be two coherent collections of compact subgroups P_v, $P'_v \subset \mathrm{G}(k_v)$. Suppose that there exist a place $w \in V_{\mathrm{f}}$ such that P_w and P'_w are not conjugate by the action of $\overline{\mathrm{G}}(k_w)$. Moreover, we suppose that P_w and P'_w contain the center of $\mathrm{G}(k_w)$. Then $\pi_{\mathscr{S}}(\Lambda_P)$ and $\pi_{\mathscr{S}}(\Lambda_{P'})$ are not conjugate in $\mathscr{G}$.*

Proof Let C be the center of G. We may assume that each P_v (resp. P'_v) contains the center $\mathrm{C}(k_v)$. If not replace P_v by $\mathrm{C}(k_v) \cdot P_v$; the image $\pi_{\mathscr{S}}(\Lambda_P)$ does not change with this modification, and the hypothesis at w is kept.

Suppose that $\pi_{\mathscr{S}}(\Lambda_P)$ and $\pi_{\mathscr{S}}(\Lambda_{P'})$ are conjugate in $\mathscr{G}$. Then Λ_P and $\Lambda_{P'}$ are conjugate under the action of $\mathscr{G} \cong (\overline{\mathrm{G}}_{\mathscr{S}})^\circ$. Since arithmetic subgroups of G are Zariski-dense, we have more precisely that Λ_P and $\Lambda_{P'}$ are conjugate by an element $g \in \overline{\mathrm{G}}(k)$. By strong approximation the closure of Λ_P (resp. $\Lambda_{P'}$) in $\mathrm{G}(k_w)$ is P_w (resp. P'_w), and it follows that g conjugates P_w and P'_w. □

3 Parahoric subgroups and volume

In the following we assume that the reader has some knowledge of Bruhat-Tits theory. All the facts we need can be found in Tits' survey [16]. See [15, §3.4] for a more elementary introduction.

3.1

Let $v \in V_{\mathrm{f}}$. A *parahoric subgroup* of $\mathrm{G}(k_v)$, a certain kind of compact open subgroup of $\mathrm{G}(k_v)$, is by definition the stabilizer of a simplex in the Bruhat-Tits building attached to $\mathrm{G}(k_v)$. There are a finite number of conjugacy classes of parahoric subgroups in $\mathrm{G}(k_v)$; these conjugacy classes in $\mathrm{G}(k_v)$ correspond canonically to proper subsets of the local Dynkin diagram Δ_v of $\mathrm{G}(k_v)$. If $P_v \subset \mathrm{G}(k_v)$ is a parahoric subgroup, we denote by $\tau(P_v) \subset \Delta_v$ its associated subset, and we call it the *type* of P_v. Two parahoric subgroups P_v and P'_v can be conjugate by an element of $\overline{\mathrm{G}}(k_v)$ only if there is an automorphism of Δ_v that sends $\tau(P_v)$ to $\tau(P'_v)$.

3.2

Let us denote by $\mathfrak{f}_v$ the residual field of k_v. To each parahoric subgroup $P_v \subset \mathrm{G}(k_v)$, a smooth affine group scheme over $\mathscr{O}_v$ is associated in a canonical way [16, §3.4.1]. By reduction modulo v, this determines in turn an algebraic group over $\mathfrak{f}_v$. Its maximal reductive quotient is a $\mathfrak{f}_v$-group that will be denoted by the symbol $\overline{M}_v$. The structure of $\overline{M}_v$ can be determined from $\tau(P_v)$ and the local index of $\mathrm{G}(k_v)$ by the procedure described in [16, §3.5].

3.3

Let $(\overline{M}_v, \overline{M}_v)$ be the commutator group of $\overline{M}_v$, and let $R(\overline{M}_v)$ be the radical of $\overline{M}_v$. Both are defined over $\mathfrak{f}_v$, and we have (see [17, 8.1.6])

$$\overline{M}_v = (\overline{M}_v, \overline{M}_v) \cdot R(\overline{M}_v).$$

The radical $R(\overline{M}_v)$ is a central torus in $\overline{M}_v$, whose intersection with $(\overline{M}_v, \overline{M}_v)$ is finite [17, 7.3.1]. It follows that the product map

$$(\overline{M}_v, \overline{M}_v) \times R(\overline{M}_v) \to \overline{M}_v$$

is an isogeny. By applying Lang's isogeny theorem [15, Prop. 6.3], we obtain that the order of $\overline{M}_v(\mathfrak{f}_v)$ is given by the following:

$$|\overline{M}_v(\mathfrak{f}_v)| = |(\overline{M}_v, \overline{M}_v)(\mathfrak{f}_v)| \cdot |R(\overline{M}_v)(\mathfrak{f}_v)|. \tag{3}$$

Theorem 6 (Prasad) *Let μ be a Haar measure on $\mathrm{G}_{\mathscr{S}}$. Then there exists a constant c_{G} (depending on the algebraic group* G*) such that for any coherent collection P of parahoric* *subgroups* $P_v \subset \mathrm{G}(k_v)$*, we have*

$$\mu(\Lambda_P\backslash \mathrm{G}_{\mathscr{S}}) = c_{\mathrm{G}} \prod_{v\in V_{\mathrm{f}}} \frac{|\mathfrak{f}_v|^{(t_v+\dim \overline{M}_v)/2}}{|\overline{M}_v(\mathfrak{f}_v)|},$$

where for each $v \in V_{\mathrm{f}}$ *the integer* t_v *depends only on the* k_v*-structure of* G.

This theorem is a much weaker form of Prasad's volume formula, given in [3, Theorem 3.7]. In fact, Prasad's result explicitly gives the value of c_{G} for a natural normalization of the Haar measure μ. Moreover, the integers t_v are explicitly known. Since we want to prove qualitative results, we will not need more than the statement of Theorem 6.

4 Proof of Theorem 1

We now prove Theorem 1, assuming that the group $\mathscr{G}$ is isotypic. Let $\Gamma \subset \mathscr{G}$ be an irreducible arithmetic lattice, with G and $\overline{\mathrm{G}}$ the associated k-groups as in Sect. 2.1. We retain all notation introduced above.

4.1

The group G is quasi-split over k_v for almost all places v [15, Theorem 6.7]. Let us denote by T the set of the places $v \in V_{\mathrm{f}}$ such G is not quasi-split over k_v. Let $\ell|k$ be the smallest Galois extension such that G is an inner form over ℓ (see for instance [17, Ch. 17], where this field is denoted by E_τ). If $v \notin T$ is totally split in $\ell|k$, i.e., if $\ell \subset k_v$, then G is split over k_v. It follows from the Chebotarev density theorem that the set of places $v \notin T$ that are totally split in $\ell|k$ is infinite. Let us denote this infinite subset of V_{f} by S.

4.2

Let $v \in S$. The local Dynkin diagram Δ_v of $\mathrm{G}(k_v)$ can be found in [16, §4.2]. Let n be the absolute rank of $\mathscr{G}$ (and of G). We suppose first that $\mathscr{G}$ (and consequently G as well) is not of absolute type A_n. Then there exist two vertices $\alpha_1, \alpha_2 \in \Delta_v$ such that α_1 is hyperspecial and α_2 is not. Let $P_v^{(1)}$ (resp. $P_v^{(2)}$) be a parahoric subgroup in $\mathrm{G}(k_v)$ of type $\tau(P_v^{(1)}) = \{\alpha_1\}$ (resp. $\tau(P_v^{(2)}) = \{\alpha_2\}$). Then $P_v^{(1)}$ and $P_v^{(2)}$ are not conjugate by the action of $\overline{\mathrm{G}}(k_v)$ (see Sect. 3.1). Note also that these two groups, being parahoric subgroups, contain the center of $\mathrm{G}(k_v)$. We consider the subgroup $\overline{M}_v$ associated with $P_v^{(1)}$ (resp. associated with $P_v^{(2)}$). In both cases $i = 1, 2$ the radical $R(\overline{M}_v)$ is a split torus of rank $n-1$ and the semi-simple part $(\overline{M}_v, \overline{M}_v)$ is of type A_1. From (3) we see that the order of $\overline{M}_v(\mathfrak{f}_v)$ is the same for $P_v^{(1)}$ and $P_v^{(2)}$.

If G is of type A_n then Δ_v is a cycle of $n+1$ vertices, all hyperspecial. The group $\overline{\mathrm{G}}(k_v)$ acts simply transitively by rotations on Δ_v. Let us choose a labelling $\alpha_0, \ldots, \alpha_n$ of the vertices that follows an orientation of Δ_v. We now consider $P_v^{(1)}$ with $\tau(P_v^{(1)}) = \{\alpha_0, \alpha_2\}$, and $P_v^{(2)}$ with $\tau(P_v^{(2)}) = \{\alpha_0, \alpha_3\}$. If $n \geq 4$ then no rotation of Δ_v sends $\tau(P_v^{(1)})$ to $\tau(P_v^{(2)})$, so that $P_v^{(1)}$ and $P_v^{(2)}$ are not conjugate by $\overline{\mathrm{G}}(k_v)$. Moreover, we can check as above that the order of $\overline{M}_v$ is the same for $P_v^{(1)}$ and $P_v^{(2)}$.

4.3

We consider a coherent collection P of parahoric subgroups $P_v \subset \mathrm{G}(k_v)$. Let $m \in \mathbb{N}$ and choose a finite subset $S_m \subset S$ of length m. For each $v \in S_m$ we replace P_v by either $P_v^{(1)}$ or $P_v^{(2)}$, and consider the arithmetic subgroup in $\mathrm{G}(k)$ associated with this modified coherent collection. Thus we obtain 2^m different arithmetic subgroups in $\mathrm{G}(k)$, and by Lemma 5 their images in $\mathscr{G}$ are pairwise non-conjugate. But by Theorem 6 they all have the same covolume.

To obtain families of torsion-free lattices we make the following change. Let us choose two distinct places $v_1, v_2 \in S \setminus S_m$, and for $i = 1, 2$ replace P_{v_i} by its subgroup K_i defined as the kernel of the reduction modulo v_i. We denote this modified coherent collection by P'. Let p_i be the characteristic of $\mathfrak{f}_{v_i}$. Then K_i is a pro-p_i-group [15, Lemma 3.8], and since $p_1 \neq p_2$ we have that $K_1 \cap K_2$ is torsion-free. Thus $\Lambda_{P'}$ is torsion-free. The above construction with the coherent collection P' instead of P now gives non-conjugate lattices in $\mathscr{G}$ that are torsion-free. Using Lemma 4 we see that these sublattices also share the same covolume.

4.4

Let $\mathrm{Aut}(\mathscr{G})$ be the automorphism group of $\mathscr{G}$. Then $\mathrm{Aut}(\mathscr{G})/\mathscr{G}$ (where $\mathscr{G}$ acts on itself as inner automorphisms) is a group whose order is bounded by the symmetries of the Dynkin diagram of $\mathscr{G}$. In particular, it is a finite group. By letting m tends to infinity, we have constructed arbitrarily large families of non-conjugate lattices in $\mathscr{G}$ of the same covolume. By considering each family modulo the equivalence induced by the action of $\mathrm{Aut}(\mathscr{G})/\mathscr{G}$, we see that there exist arbitrarily large families of lattices that are not conjugate by $\mathrm{Aut}(\mathscr{G})$. Since strong rigidity holds for all the lattices under consideration (see [14, §5.1] and the references given there), we get that these families consist of non-isomorphic lattices.

5 Proof of Theorem 2

We now give the proof of Theorem 2. Thus we suppose that $\mathfrak{g}_{\mathbb{C}}$ has only factors of type A_n (with $n = 2$ or $n = 3$). Let $m \in \mathbb{N}$.

5.1

Let k be a number field that has as many complex places as there are simple factor of $\mathscr{G}$ isomorphic to $\mathrm{PSL}_{n+1}(\mathbb{C})$. Let $\ell | k$ be a quadratic extension having one complex place for each factor of $\mathscr{G}$ that is projective unitary (i.e., of the form $\mathrm{PU}(p, q)$) or isomorphic to $\mathrm{PSL}_{n+1}(\mathbb{C})$. Using approximation for k (see [18, Theorem (3.4)]) it is possible to choose $\alpha \in k$ such that $\ell = k(\sqrt{\alpha})$ is as above with the additional property that for the set $R \subset V_{\mathrm{f}}$ of ramified places in $\ell | k$ we have $2^{\# R} \geq m$.

5.2

Let G_0 be the quasi-split simply connected k-group of type A_n with splitting field ℓ. By [6, Theorem 1], there exists an inner form G of G_0 such that $\mathrm{G}|k_v$ is quasi-split for all $v \in R$ and such that $(\overline{\mathrm{G}}_{\mathscr{S}})^\circ \cong \mathscr{G}$. The group G can be chosen to be k-isotropic unless the condition (1) in [6] is not satisfied at infinite places, in which case there is no isotropic k-group G with $(\overline{\mathrm{G}}_{\mathscr{S}})^\circ \cong \mathscr{G}$. We can always choose G to be anisotropic, by specifying in [6, Theorem 1] that G is k_v-anisotropic at some $v \in V_{\mathrm{f}} \setminus R$.

5.3

The local Dynkin diagram Δ_v of $\mathrm{G}(k_v)$ for $v \in R$ is shown in [16, §4.2]; it is named C–BC_1 for the type A_2, and C–B_2 for A_3 ($= \mathrm{D}_3$). With this diagram at hand we can easily construct (similarly to Sect. 4.2) a pair of non-conjugate parahoric subgroups of $\mathrm{G}(k_v)$ ($v \in R$) that have equal volume. Taking them as part of coherent collection we produce m pairwise non-conjugate arithmetic subgroups that, by Theorem 6, are of the same covolume in $\mathscr{G}$. By Godement's compactness criterion, these lattices are cocompact exactly when G is anisotropic. The last steps of the proof are verified exactly as in Sects. 4.3–4.4.

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