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Existence of stationary solutions of the Navier–Stokes equations in the presence of a wall

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Abstract. We consider the problem of a body moving within an incompressible fluid at constant speed parallel to a wall, in an otherwise unbounded domain. This situation is modeled by the incompressible Navier–Stokes equations in an exterior domain in a half space, with appropriate boundary conditions on the wall, the body, and at infinity. Here, we prove existence of stationary solutions for this problem for the simplified situation where the body is replaced by a source term of compact support.

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Keywords. Navier–Stokes equations · Stationary solutions · Fourier transform · Fluid structure interaction.

Contents

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1. Introduction

In this paper, we consider the three-dimensional stationary Navier–Stokes equations

$$-\partial_x \mathbf{u} + \Delta \mathbf{u} = \mathbf{F} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p \ , \tag{1}$$

in the domain $\Omega_+ = \{(x, y, z) \in \mathbb{R}^3 \mid z > 1\}$, subjected to the incompressibility condition

$$\nabla \cdot \mathbf{u} = 0 \ , \tag{2}$$

and the boundary conditions

$$\mathbf{u}(x, y, 1) = 0 \ , \qquad (x, y) \in \mathbb{R}^2 \ , \tag{3}$$

$$\lim_{\mathbf{x} \to \infty} \mathbf{u}(\mathbf{x}) = 0 \ , \tag{4}$$

and with $\mathbf{F}$ a smooth vector field with compact support in Ω_+, i.e., $\mathbf{F} \in C_c^{\infty}(\Omega_+)$.

This model can be used to describe the motion of a body moving within an incompressible fluid at constant speed parallel to a wall, in an otherwise unbounded domain. A very important practical application of such a situation is the description of the motion of bubbles rising in a liquid parallel to a nearby wall. Interesting recent experimental work is described in [20,21]. Numerical studies can be found in [4,6,12,18].

In what follows, we consider the situation of a single bubble of fixed shape which rises with constant velocity in a regime of Reynolds numbers less than about fifty. The resulting fluid flow is then laminar. The Stokes equations provide a good quantitative description (forces determined within an error of one percent) only for Reynolds numbers less than one. For the larger Reynolds numbers under consideration, the Navier–Stokes equations need to be solved in order to obtain precise results. The vertical speed of the bubble depends on the drag, and the distance from the wall at which the bubble rises requires one to find the position relative to the wall where the transverse force is zero. Since at low Reynolds numbers the transverse forces are orders of magnitude smaller than the forces along the flow, this turns out to be a very delicate problem which needs to be solved numerically with the help of high precision computations. But, if done by brute force, such computations are excessively costly even with today's computers. In [1,2,14,15], the third author and his collaborators have developed techniques that lead for similar problems to an overall gain of computational efficiency of typically several orders of magnitude. These techniques use as an input a precise asymptotic description of the flow. The present work is an important step toward the extension of this technique to the case of motions close to a wall.

We explain now in more detail the background of our problem. For convenience later on, we have placed the position of the wall at $z = 1$. Let $\boldsymbol{B} \subset \Omega_+$ be a compact set with smooth boundary $\partial \boldsymbol{B}$, and $e_1 = (1, 0, 0)$. Then, as described in [17], in a frame co-moving with the body, the Navier–Stokes equations which model laminar flow around this body are

$$-\partial_x \mathbf{u} - \mathbf{u} \cdot \nabla \mathbf{u} + \Delta \mathbf{u} - \nabla p = 0, \tag{5}$$

which have to be solved together with divergence-free condition (2) in the domain $\Omega = \Omega_+ \backslash \boldsymbol{B}$, subjected to the boundary conditions (3), (4), and

$$\mathbf{u}|_{\partial \boldsymbol{B}} = -e_1. \tag{6}$$

A standard technique to solve this problem is to prove the existence of weak solutions. Such solutions are constructed by considering a nested sequence of finite domains that converges to Ω_+. Existence then follows by a compactness argument. See for example [9,10,19], for the case of $\Omega = \mathbb{R}^3 \backslash \boldsymbol{B}$, and [11,13] for the case of a half space in two dimensions. The weak solutions constructed in this way are smooth; the only shortcoming of the method is that only little information is obtained about the behavior of solutions at infinity.

In order to obtain such information, a classic way is to consider the problem in an appropriately chosen weighted Sobolev space. Such methods are well developed for the case of isotropic weights, but become very technical if, as in the present case, anisotropic weights are needed. See for example [3,7].

In the present paper, we follow the strategy that we have proposed in [16] for the two-dimensional case: we take advantage of the anisotropy of the problem to obtain information at infinity by constructing a classical solution in a function space which is motivated by the theory of dynamical systems. Namely, we choose the coordinate z to play the role of time and rewrite our equation as a system of evolution equations with respect to this variable. Information on the large time behavior of the dynamical system then naturally provides detailed information at infinity. In order to get a system of ordinary differential equations, we use the Fourier transform in the x and y coordinates. We then choose the function spaces which are well adapted to the problem. These spaces come up naturally once the problem is formulated in this form.

However, to use our techniques based on the Fourier transform, we need that the problem is formulated on all of Ω_+. This is achieved as follows, see [5]. Let $(\tilde{\mathbf{u}}, \tilde{p})$ be a smooth solution to the problem (2)–(6), let D_1 and D_2 be two disks such that $\boldsymbol{B} \subset D_1 \subset D_2 \subset \Omega_+$. We also consider a stream function $\tilde{\psi}$ which is divergence free such that $\tilde{\mathbf{u}} = \nabla \times \tilde{\psi}$. We then use a smooth cutoff function χ which interpolates between zero in the interior of D_1 and one in the exterior of D_2, define $\mathbf{u}$ and p to be zero in the interior of D_1 and by the equations

$$\begin{aligned} \mathbf{u} &= \nabla \times \left(\chi \tilde{\psi}\right) = \nabla \times (\chi\ \nabla \times (G * \tilde{\mathbf{u}})), \\ p &= \chi \tilde{p}, \end{aligned}$$

in the exterior of D_1, where

$$(G * \tilde{\mathbf{u}})\,(\mathbf{x}) = \int_{\mathbb{R}^3} \frac{\tilde{\mathbf{u}}(\mathbf{y})}{4\pi |\mathbf{x} - \mathbf{y}|} d^3\mathbf{y}.$$

By construction, $\mathbf{u}$ and p are smooth and satisfy (1), (2) for a certain function $\mathbf{F}$ which is smooth and of compact support in D_2. Motivated by these remarks we consider the problem proposed at the beginning of this section.

The following theorem is our main result (see Sect. 3, Theorem 8, for a precise formulation):

Theorem 1. *For all $\mathbf{F} \in C_c^\infty(\Omega_+)$ with $\mathbf{F}$ sufficiently small in a sense to be defined below, there exists a vector field $\mathbf{u} = (u_1, u_2, u_3) \in H^1(\Omega_+)$ and a function p satisfying the Navier–Stokes equations (1) and (2) in Ω_+ subjected to the boundary conditions (3), (4). Moreover, there exists a constant C such that, uniformly in $(x, y, z) \in \Omega_+$, $|u_i(x, y, z)| \leq C/z^2$, for $i = 1, 2, 3$.*

This theorem provides basic information on the decay of solutions at infinity. Using the present result as a starting point, the detailed asymptotic behavior of the velocity field has been explored in a recent publication [8], with results analogous to what has been proved for the two-dimensional case [22]. In an upcoming paper, we will show that the vorticity of the fluid decays algebraically at infinity, not only in the wake region but also in directions transverse to the fluid flow and in particular also in the direction transverse to the wall. This is in sharp contrast to the behavior of the vorticity for exterior problems in the whole space, where the decay is exponential outside the wake region. The presence of the wall therefore leads to a very basic and important modification of the flow field, which is not limited to the mere appearance of a boundary layer on the wall. Hence, the importance of analyzing the flow field in the direction transversal to the wall.

The smallness condition on the source term is imposed here since, on one hand, the contraction mapping principle is used to prove the existence of a solution, and, on the other hand, the only goal here is to obtain precise decay estimates for strong solutions for small data. These results are then used to prove a weak–strong uniqueness result which shows that for small data weak solutions decay at infinity exactly like the strong solutions constructed here.

The rest of this paper is organized as follows. In Sect. 2, we reduce the Eqs. (1) and (2) to a set of integral equations for an evolution equation for which the coordinate z plays the role of time. In Sect. 3, we formulate the problem as a functional equation. Existence of solutions is proved in Sect. 4.

2. Reduction to an evolution equation

Let $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{F} = (F_1, F_2, F_3)$. Then, the Navier–Stokes equations (1) are equivalent to

$$-\partial_x \boldsymbol{\omega} + \nabla \times (\mathbf{u} \times \boldsymbol{\omega}) + \Delta \boldsymbol{\omega} - \nabla \times \mathbf{F} = 0, \tag{7}$$

$$\nabla \cdot \boldsymbol{\omega} = 0, \tag{8}$$

where $\boldsymbol{\omega} = \nabla \times \mathbf{u} = (\omega_1, \omega_2, \omega_3)$ is the vorticity vector. Let $\mathbf{q} = \mathbf{u} \times \boldsymbol{\omega} = (q_1, q_2, q_3)$ and $\mathbf{Q} = \mathbf{q} - \mathbf{F} = (Q_1, Q_2, Q_3)$. Then we have, in component form,

$$\omega_1 = \partial_y u_3 - \partial_z u_2, \tag{9}$$

$$\omega_2 = \partial_z u_1 - \partial_x u_3, \tag{10}$$

$$\omega_3 = \partial_x u_2 - \partial_y u_1, \tag{11}$$

and

$$\partial_x^2 \omega_1 + \partial_y^2 \omega_1 + \partial_z^2 \omega_1 - \partial_x \omega_1 + \partial_y Q_3 - \partial_z Q_2 = 0, \tag{12}$$

$$\partial_x^2 \omega_2 + \partial_y^2 \omega_2 + \partial_z^2 \omega_2 - \partial_x \omega_2 - \partial_x Q_3 + \partial_z Q_1 = 0, \tag{13}$$

$$\partial_x^2 \omega_3 + \partial_y^2 \omega_3 + \partial_z^2 \omega_3 - \partial_x \omega_3 + \partial_x Q_2 - \partial_y Q_1 = 0. \tag{14}$$

Once the Eqs. (7) and (8) are solved, the pressure p can be obtained by using standard techniques, our by also solving the equation

$$\Delta p = -\nabla \cdot (\mathbf{F} + \mathbf{u} \cdot \nabla \mathbf{u}).$$

also in our function spaces.

We now rewrite (2), (9), (10), (12) and (13) as evolution equations with z playing the role of time. Namely, as is easily verified, these equations are equivalent to

$$\partial_z\omega_1 = \partial_x\eta_{1,1} + \partial_y\eta_{1,2} + Q_2, \tag{15}$$
$$\partial_z\omega_2 = \partial_x\eta_{2,1} + \partial_y\eta_{2,2} - Q_1, \tag{16}$$
$$\partial_z\eta_{1,1} = -\partial_x\omega_1 + \omega_1, \tag{17}$$
$$\partial_z\eta_{1,2} = -\partial_y\omega_1 - Q_3, \tag{18}$$
$$\partial_z\eta_{2,1} = -\partial_x\omega_2 + \omega_2 + Q_3, \tag{19}$$
$$\partial_z\eta_{2,2} = -\partial_y\omega_2, \tag{20}$$
$$\partial_z u_1 = \partial_x u_3 + \omega_2, \tag{21}$$
$$\partial_z u_2 = \partial_y u_3 - \omega_1, \tag{22}$$
$$\partial_z u_3 = -\partial_x u_1 - \partial_y u_2. \tag{23}$$

More precisely, the Eqs. (15), (17), (18) are equivalent to (12), the Eqs. (16), (19), (20) are equivalent to (13), the Eqs. (21), (22) are equivalent to (10) and (9) and (23) is equivalent to (2). Equation (11) defines ω_3 as a function of u_1 and u_2 and (14) then follows using (12), (13) and the boundary conditions. We now convert (15)–(23) into a system of ordinary differential equations by taking the Fourier transform in the x and y directions.

Definition 2. *Let $\hat{f}$, $\hat{g}$ be complex valued functions defined almost everywhere on Ω_+. Then, we define the inverse Fourier transform $f = \mathcal{F}^{-1}[\hat{f}]$ by*

$$f(x,y,z) = \mathcal{F}^{-1}[\hat{f}](x,y,z) = \frac{1}{4\pi^2}\int_{\mathbb{R}^2} e^{-ik_1x}e^{-ik_2y}\hat{f}(k_1,k_2,z)\ dk_1dk_2, \tag{24}$$

and $\hat{\pi} = \hat{f} * \hat{g}$ by

$$\hat{\pi}(\mathbf{k},z) = (\hat{f} * \hat{g})(\mathbf{k},z) = \int_{\mathbb{R}^2} \hat{f}(\mathbf{k}-\mathbf{k}',z)\hat{g}(\mathbf{k}',z)\ d^2\mathbf{k}',$$

whenever the integrals make sense.

We note that for functions f, g which are smooth and of compact support in Ω_+ we have that $f = \mathcal{F}^{-1}[\hat{f}]$, and that $fg = \mathcal{F}^{-1}[\hat{f} * \hat{g}]$, where

$$\hat{f}(k_1,k_2,z) = \mathcal{F}[f](k_1,k_2,z) = \int_{\mathbb{R}^2} e^{ik_1x}e^{ik_2y}f(x,y,z)\ dxdy,$$

and similarly for $\hat{g} = \mathcal{F}[g]$. With this definition, we formally have in Fourier space, instead of (15)–(23), the equations

$$\partial_z\hat{\omega}_1 = -ik_1\hat{\eta}_{1,1} - ik_2\hat{\eta}_{1,2} + \hat{Q}_2, \tag{25}$$
$$\partial_z\hat{\omega}_2 = -ik_1\hat{\eta}_{2,1} - ik_2\hat{\eta}_{2,2} - \hat{Q}_1, \tag{26}$$
$$\partial_z\hat{\eta}_{1,1} = ik_1\hat{\omega}_1 + \hat{\omega}_1, \tag{27}$$
$$\partial_z\hat{\eta}_{1,2} = ik_2\hat{\omega}_1 - \hat{Q}_3, \tag{28}$$
$$\partial_z\hat{\eta}_{2,1} = ik_1\hat{\omega}_2 + \hat{\omega}_2 + \hat{Q}_3, \tag{29}$$
$$\partial_z\hat{\eta}_{2,2} = ik_2\hat{\omega}_2, \tag{30}$$
$$\partial_z\hat{u}_1 = -ik_1\hat{u}_3 + \hat{\omega}_2, \tag{31}$$

$$\partial_z \hat{u}_2 = -ik_2 \hat{u}_3 - \hat{\omega}_1, \tag{32}$$

$$\partial_z \hat{u}_3 = ik_1 \hat{u}_1 + ik_2 \hat{u}_2, \tag{33}$$

with $\hat{Q}_i = \hat{q}_i - \hat{F}_i$, $i = 1, 2, 3$, and

$$\hat{q}_1 = \frac{1}{4\pi^2} (\hat{\omega}_3 * \hat{u}_2 - \hat{\omega}_2 * \hat{u}_3), \tag{34}$$

$$\hat{q}_2 = \frac{1}{4\pi^2} (\hat{\omega}_1 * \hat{u}_3 - \hat{\omega}_3 * \hat{u}_1), \tag{35}$$

$$\hat{q}_3 = \frac{1}{4\pi^2} (\hat{\omega}_2 * \hat{u}_1 - \hat{\omega}_1 * \hat{u}_2). \tag{36}$$

It is (25)–(36) that we solve in Sect. 3 in appropriate function spaces. We also show that the constructed solution corresponds via inverse Fourier transform to a strong solution of (1)–(4) and that the solution has a finite Dirichlet integral.

We now rewrite (25), (26) and (31)–(33) as a system of integral equations (see "Appendix A" for a detailed derivation). Note that the integral equation for $\hat{\omega}_3$ can be obtained from the integral equations of $\hat{u}_1$ and $\hat{u}_2$ using that $\hat{\omega}_3 = -ik_1\hat{u}_2 + ik_2\hat{u}_1$. Note also that the integral equations for $\eta_{i,j}$, $i = 1, 2$, $j = 1, 2$ do not need to be considered since they do not appear in the nonlinearities q_1, q_2 and q_3. The functions $\eta_{i,j}$ are only used in an intermediate formal step in order to derive the integral equations. To insist on the dynamical system point of view, we will use from now on s, $t \geq 1$ instead of z for the "time" variable, and σ, $\tau \geq 0$ for "time" differences. We set

$$k = \sqrt{k_1^2 + k_2^2}, \quad \kappa = \sqrt{k^2 - ik_1}, \tag{37}$$

and define, for $\mathbf{k} = (k_1, k_2) \in \mathbb{R}^2 \setminus \{0\}$ and $\tau \geq 0$, the functions K_n by

$$K_n(\mathbf{k}, \tau) = \frac{1}{2} e^{-\kappa\tau}, \quad \text{for } n = 1, 2, \tag{38}$$

$$K_3(\mathbf{k}, \tau) = \frac{1+k}{2\kappa} \left(e^{\kappa\tau} - e^{-\kappa\tau}\right), \tag{39}$$

the functions G_n by

$$G_n(\mathbf{k}, \tau) = \frac{1}{2} e^{-k\tau}, \quad \text{for } n = 1, 2, \tag{40}$$

$$G_3(\mathbf{k}, \tau) = \frac{1+k}{2k} \left(e^{k\tau} - e^{-k\tau}\right), \tag{41}$$

and the functions H_n by

$$H_n(\mathbf{k}, \tau) = \frac{\kappa + k}{k_1} (K_n - G_n), \quad \text{for } n = 1, 2, \tag{42}$$

$$H_3(\mathbf{k}, \tau) = \frac{k}{k_1} (K_3 - G_3). \tag{43}$$

We furthermore define, for $t \geq 1$, and $n = 1, 2, 3$, the intervals I_n by, $I_1 = [1, t]$, and $I_n = [t, \infty)$, otherwise. Using this notation and given Q_1, Q_2, Q_3, a representation in Fourier space of a classical solution of (2), (9)–(14), which satisfies the boundary conditions (3), (4), is

$$\hat{\omega}_i = \sum_{m=1,2,3} \sum_{n=1,2,3} \hat{\omega}_{i,n,m}, \quad i = 1, 2, 3, \tag{44}$$

$$\hat{u}_i = \sum_{m=1,2,3} \sum_{n=1,2,3} \hat{u}_{i,n,m}, \quad i = 1, 2, 3, \tag{45}$$

where

$$\hat{\omega}_{i,n,m}(\mathbf{k},t) = K_n(\mathbf{k},t-1)\int_{I_n} \alpha_{i,n,m}(\mathbf{k},s-1)\hat{Q}_m(\mathbf{k},s)\ ds, \quad i=1,2, \tag{46}$$

$$\begin{aligned}\hat{\omega}_{3,n,m}(\mathbf{k},t) &= K_n(\mathbf{k},t-1)\int_{I_n} \alpha_{3,n,m}(\mathbf{k},s-1)\hat{Q}_m(\mathbf{k},s)\ ds + G_n(\mathbf{k},t-1)\int_{I_n} \beta_{3,n,m}(\mathbf{k},s-1)\hat{Q}_m(\mathbf{k},s)\ ds\\ &\quad + H_n(\mathbf{k},t-1)\int_{I_n} \gamma_{3,n,m}(\mathbf{k},s-1)\hat{Q}_m(\mathbf{k},s)\ ds,\end{aligned} \tag{47}$$

$$\hat{u}_{1,n,m}(\mathbf{k},t) = K_n(\mathbf{k},t-1)\int_{I_n} f_{1,n,m}(\mathbf{k},s-1)\hat{Q}_m(\mathbf{k},s)\ ds + G_n(\mathbf{k},t-1)\int_{I_n} g_{1,n,m}(k,s-1)\hat{Q}_m(\mathbf{k},s)\ ds, \tag{48}$$

$$\begin{aligned}\hat{u}_{i,n,m}(\mathbf{k},t) &= K_n(\mathbf{k},t-1)\int_{I_n} f_{i,n,m}(\mathbf{k},s-1)\hat{Q}_m(\mathbf{k},s)\ ds + G_n(\mathbf{k},t-1)\int_{I_n} g_{i,n,m}(\mathbf{k},s-1)\hat{Q}_m(\mathbf{k},s)\ ds\\ &\quad + H_n(\mathbf{k},t-1)\int_{I_n} h_{i,n,m}(\mathbf{k},s-1)\hat{Q}_m(\mathbf{k},s)\ ds, \quad i=2,3,\end{aligned} \tag{49}$$

with K_n, G_n, H_n, and I_n defined as above. The expressions for the functions $\alpha_{i,n,m}$, $\beta_{i,n,m}$, $\gamma_{i,n,m}$, $f_{i,n,m}$, $g_{i,n,m}$, and $h_{i,n,m}$ are given in "Appendix A".

3. Proof of main result

3.1. Functional framework

We now define the function spaces that will be used.

Let $\alpha, r \geq 0$ and $\mathbf{k} = (k_1, k_2) \in \mathbb{R}^2$, and let

$$\mu_{\alpha,r}(\mathbf{k},t) = \frac{1}{1 + (|\mathbf{k}|\, t^r)^\alpha}.$$

Let furthermore

$$\bar{\mu}_\alpha(\mathbf{k},t) = \mu_{\alpha,1}(\mathbf{k},t).$$

Definition 3. *Let $\mathbb{R}_0^2 = \mathbb{R}^2\backslash\{0\}$. We define, for fixed $\alpha \geq 0$ and $p \geq 0$, $\mathcal{B}_{\alpha,p}$ to be the Banach space of functions $f \in C(\mathbb{R}_0^2 \times [1,\infty), \mathbb{C})$, for which the norm*

$$||f;\mathcal{B}_{\alpha,p}|| = \sup_{t\geq 1}\sup_{\mathbf{k}\in\mathbb{R}_0^2} \frac{|f(\mathbf{k},t)|}{\frac{1}{t^p}\bar{\mu}_\alpha(\mathbf{k},t)}$$

is finite. Furthermore, we set

$$\mathcal{B}_{\alpha,p}^n = \underbrace{\mathcal{B}_{\alpha,p} \times \cdots \times \mathcal{B}_{\alpha,p}}_{n\ times},$$

and

$$\mathcal{W}_\alpha = \mathcal{B}_{\alpha,3}^3, \quad \mathcal{V}_\alpha = \mathcal{B}_{\alpha,1}^3 \times \mathcal{B}_{\alpha,0}^3.$$

The following properties of the spaces $\mathcal{B}_{\alpha,p}$ will be important and will be routinely used without mention.

Proposition 4. 1. *If α, $\alpha' \geq 0$, and p, $p' \geq 0$, then*

$$\mathcal{B}_{\alpha,p} \cap \mathcal{B}_{\alpha',p'} \subset \mathcal{B}_{\min\{\alpha',\alpha,\},\min\{p',p\}}.$$

2. *For $p \geq 0$, if $f \in \mathcal{B}_{\alpha,p}$, then $|\mathbf{k}|f \in \mathcal{B}_{\alpha-1,p+1}$. For $p > 0$, if $\partial_z f \in \mathcal{B}_{\alpha,p}$, then $|\mathbf{k}|f \in \mathcal{B}_{\alpha-1,p}$.*

3. *if* $\alpha > \frac{1}{2}$, $p \geq 0$, *then*

$$(\mathbf{k}, t) \mapsto \frac{1}{t^p} \bar{\mu}_\alpha(\mathbf{k}, t) \in L^2([1, \infty) \times \mathbb{R}^2).$$

Therefore, and because the Fourier transform is an isometry of $L^2(\mathbb{R}^2)$, *we have that* $f = \mathcal{F}^{-1}[\hat{f}] \in L^2(\Omega_+)$, *whenever* $\hat{f} \in \mathcal{B}_{\alpha,p}$ *for some* $\alpha > \frac{1}{2}$, $p \geq 0$.

4. *If* $\alpha > 2$, $p \geq 0$, *then* $\hat{f} \in \mathcal{B}_{\alpha,p}$ *is bounded by* $\|\hat{f}; \mathcal{B}_{\alpha,p}\|(1 + |\mathbf{k}|^\alpha)^{-1}$, *uniformly in* t. *Therefore, the function* $\mathbf{k} \mapsto \sup_{t \geq 1} |\hat{f}(., t)|$ *is in* $L^1(\mathbb{R}^2)$.

Proof. We only give the proof of the second point, the others are direct consequences of the definitions. Suppose that $f \in \mathcal{B}_{\alpha,p}$, then

$$|f(k, t)| \leq \text{Const.} \frac{1}{t^p} \frac{1}{1 + (|\mathbf{k}|t)^\alpha}.$$

Therefore

$$\begin{aligned} |\mathbf{k}||f(k, t)| &\leq \text{Const.} \frac{1}{t^{p+1}} \frac{|\mathbf{k}|t}{1 + (|\mathbf{k}|t)^\alpha} \\ &\leq \text{Const.} \frac{1}{t^{p+1}} \frac{1}{1 + (|\mathbf{k}|t)^{\alpha-1}}, \end{aligned}$$

this implies $|\mathbf{k}| f \in \mathcal{B}_{\alpha-1,p+1}$. If $\partial_t f \in \mathcal{B}_{\alpha,p}$, then, for $z \geq 1$,

$$\begin{aligned} |\mathbf{k}||f(k, t)| &= \left| \int_z^\infty |\mathbf{k}| \partial_t f(k, t)\, dt \right| \leq \int_z^\infty |\mathbf{k}|\, |\partial_t f|\, dt \\ &\leq \int_z^\infty \frac{1}{t^{p+1}} \frac{|\mathbf{k}|t}{1 + (|\mathbf{k}|t)^\alpha} dt \\ &\leq \text{Const.} \int_z^\infty \frac{1}{t^{p+1}} \frac{1}{1 + (|\mathbf{k}|t)^{\alpha-1}} dt \\ &\leq \text{Const.} \frac{1}{1 + (|\mathbf{k}|z)^{\alpha-1}} \int_z^\infty \frac{1}{t^{p+1}} dt \\ &\leq \text{Const.} \frac{1}{t^p} \frac{1}{1 + (|\mathbf{k}|z)^{\alpha-1}}, \end{aligned}$$

and therefore, $|\mathbf{k}| f \in \mathcal{B}_{\alpha-1,p}$. □

Next, we rewrite the problem of solving (25)–(36) as a functional equation.

Lemma 5. *Let* $\alpha > 2$. *Then,*

$$\begin{array}{rccl} \mathcal{C} : & \mathcal{V}_\alpha \times \mathcal{V}_\alpha & \to & \mathcal{W}_\alpha \\ & ((\hat{\boldsymbol{\omega}}_1, \hat{\mathbf{u}}_1), (\hat{\boldsymbol{\omega}}_2, \hat{\mathbf{u}}_2)) & \longmapsto & \hat{\mathbf{q}}, \end{array} \tag{50}$$

where

$$\hat{\boldsymbol{\omega}}_i = (\hat{\omega}_{i1}, \hat{\omega}_{i2}, \hat{\omega}_{i3}), \ \hat{\mathbf{u}}_i = (\hat{u}_{i1}, \hat{u}_{i2}, \hat{u}_{i3}), \quad i = 1, 2,$$

and $\hat{\mathbf{q}} = (\hat{q}_1, \hat{q}_2, \hat{q}_3)$ *with*

$$\begin{aligned} \hat{q}_1 &= \frac{1}{4\pi^2} \left(\hat{\omega}_{23} * \hat{u}_{12} - \hat{\omega}_{22} * \hat{u}_{13} \right), \\ \hat{q}_2 &= \frac{1}{4\pi^2} \left(\hat{\omega}_{21} * \hat{u}_{13} - \hat{\omega}_{23} * \hat{u}_{11} \right), \\ \hat{q}_3 &= \frac{1}{4\pi^2} \left(\hat{\omega}_{22} * \hat{u}_{11} - \hat{\omega}_{21} * \hat{u}_{12} \right), \end{aligned}$$

defines a continuous bilinear map.

Lemma 6. *Let $\alpha > 2$. Then,*

$$\begin{array}{rcl} \mathcal{L} : \mathcal{W}_\alpha & \to & \mathcal{V}_\alpha \\ \hat{\mathbf{Q}} & \longmapsto & (\hat{\boldsymbol{\omega}}, \hat{\mathbf{u}}), \end{array} \tag{51}$$

defines a continuous linear map.

The maps $\mathcal{C}$ and $\mathcal{L}$ are studied in Sects. 4.1 and 4.2, respectively. Now let $\mathbf{F} = (F_1, F_2, F_3) \in C_c^\infty(\Omega_+)$, and let $\hat{\mathbf{F}} = (\mathcal{F}[F_1], \mathcal{F}[F_2], \mathcal{F}[F_3])$ be the Fourier transform of $\mathbf{F}$. Note that $\hat{\mathbf{F}} \in \mathcal{W}_\alpha$ for all $\alpha > 2$.

Definition 7. *Let $\alpha > 2$. A pair $(\hat{\boldsymbol{\omega}}, \hat{\mathbf{u}})$ is called an α-solution for $\hat{\mathbf{F}}$ if:*

(i) $(\hat{\boldsymbol{\omega}}, \hat{\mathbf{u}}) \in \mathcal{V}_\alpha$,

(ii) $(\hat{\boldsymbol{\omega}}, \hat{\mathbf{u}}) = \mathcal{L}[\mathcal{C}[(\hat{\boldsymbol{\omega}}, \hat{\mathbf{u}}), (\hat{\boldsymbol{\omega}}, \hat{\mathbf{u}})] - \hat{\mathbf{F}}]$.

With this definition at hand, we can now give a precise formulation of Theorem 1:

Theorem 8. (Existence) *Let $\alpha > 2$, $\mathbf{F} = (F_1, F_2, F_3) \in C_c^\infty(\Omega_+)$, and let $\hat{\mathbf{F}}$ be the Fourier transform of $\mathbf{F}$. If $\|\hat{\mathbf{F}}; \mathcal{W}_\alpha\|$ is sufficiently small, then there exists an α-solution $(\hat{\boldsymbol{\omega}}, \hat{\mathbf{u}})$ for $\hat{\mathbf{F}}$ in $\mathcal{V}_\alpha$, with $\|(\hat{\boldsymbol{\omega}}, \hat{\mathbf{u}}); \mathcal{V}_\alpha\| \leq C_\alpha \|\hat{\mathbf{F}}; \mathcal{W}_\alpha\|$, for some constant C_α depending only on the choice of α.*

Proof. Let $\varepsilon_\alpha := \|\hat{\mathbf{F}}; \mathcal{W}_\alpha\|$. Since $\alpha > 2$, we have by Lemmas 5 and 6 that the map $\mathcal{N} \colon \mathcal{V}_\alpha \to \mathcal{V}_\alpha$, $\mathcal{N}[x] = \mathcal{L}[\mathcal{C}[x, x] - \hat{\mathbf{F}}]$ is continuous. We now show that for ε_α small enough there is a constant ρ_α such that $\mathcal{N}$ is a contraction on the ball $\mathcal{U}_\alpha = \{x \in \mathcal{V}_\alpha \mid \|x; \mathcal{V}_\alpha\| < \rho_\alpha\}$. Namely, let $x \in \mathcal{U}_\alpha$, then, by Lemma 5, there exists a constant C_1 such that $\|\mathcal{C}[x, x]; \mathcal{W}_\alpha\| \leq C_1(\rho_\alpha)^2$, and therefore $\|\mathcal{C}[x, x] - \hat{\mathbf{F}}; \mathcal{W}_\alpha\| \leq C_1(\rho_\alpha)^2 + \varepsilon_\alpha$. Using now Lemma 6 it follows that there exists a constant C_2 such that $\|\mathcal{N}[x]; \mathcal{V}_\alpha\| \leq C_2(C_1(\rho_\alpha)^2 + \varepsilon_\alpha)$. Now, we assume that

$$\varepsilon_\alpha < \frac{1}{8C_1C_2^2} =: \varepsilon_\alpha^0, \tag{52}$$

and let

$$\rho_\alpha = 2C_2\varepsilon_\alpha. \tag{53}$$

Then, we find that

$$\begin{aligned} \|\mathcal{N}[x]; \mathcal{V}_\alpha\| &\leq C_2(C_1(2C_2\varepsilon_\alpha)^2 + \varepsilon_\alpha) < \left(4C_1C_2^2\varepsilon_\alpha^0 + 1\right) C_2\varepsilon_\alpha \\ &< 2C_2\varepsilon_\alpha = \rho_\alpha, \end{aligned}$$

which shows that for ρ_α as defined in (53) and with ε_α satisfying (52), we have that $\mathcal{N}[\mathcal{U}] \subset \mathcal{U}$. Now let $x, y \in \mathcal{U}$. By the linearity of $\mathcal{L}$, we have that $\mathcal{N}[x] - \mathcal{N}[y] = \mathcal{L}[\mathcal{C}[x, x] - \mathcal{C}[y, y]]$, and therefore by the bilinearity of $\mathcal{C}$ that $\mathcal{N}[x] - \mathcal{N}[y] = \mathcal{L}[\mathcal{C}[x - y, x] + \mathcal{C}[y, x - y]]$. With the same constants C_1 and C_2 as before, and using (52), (53), we therefore find that

$$\begin{aligned} \|\mathcal{N}[x] - \mathcal{N}[y]; \mathcal{V}_\alpha\| &\leq 2C_2C_1\rho_\alpha\|x - y; \mathcal{V}_\alpha\| \leq 4C_2^2C_1\varepsilon_\alpha^0\|x - y; \mathcal{V}_\alpha\| \\ &= \frac{1}{2}\|x - y; \mathcal{V}_\alpha\|. \end{aligned}$$

This shows that $\mathcal{N}$ is a contraction of $\mathcal{U}$ into $\mathcal{U}$. Theorem 8 now follows by the contraction mapping principle. ☐

3.2. Proof of Theorem 1

The definition of α-solutions has been obtained from (1), (2), (3) on a formal level. We now prove that for $\alpha > 3$ any α-solution provides a classical solution $(\mathbf{u}, p)$ to (1), (2), (3). In what follows $\mathbf{F}$ is a smooth source term of compact support. So assume $(\hat{\boldsymbol{\omega}}, \hat{\mathbf{u}})$ is an α-solution for given $\mathbf{F}$ (not necessarily small). By definition, we have that

$$\hat{\boldsymbol{\omega}} \in \mathcal{B}^3_{\alpha,1}, \quad \hat{\mathbf{u}} \in \mathcal{B}^3_{\alpha,0}. \tag{54}$$

Applying Lemma 5, we obtain that the function $\hat{\mathbf{q}} = (\hat{q}_1, \hat{q}_2, \hat{q}_3)$ satisfies

$$\hat{\mathbf{q}} \in \mathcal{B}^3_{\alpha,3}, \tag{55}$$

and therefore, $\hat{\mathbf{Q}} = \hat{\mathbf{q}} - \mathcal{F}[\mathbf{F}]$ belongs to the same space. Finally, by definition of α-solution, we have that $(\hat{\boldsymbol{\omega}}, \hat{\mathbf{u}}) = \mathcal{L}[\hat{\mathbf{Q}}]$. By construction the functions $(\hat{\boldsymbol{\omega}}, \hat{\mathbf{u}})$ are components of a solution of the system of ordinary differential equations (25)–(33) with continuous coefficients, and the functions $(\hat{\boldsymbol{\omega}}, \hat{\mathbf{u}})$ therefore admit partial derivatives with respect to the z variable. Using (25)–(33) and the above information about $(\hat{\boldsymbol{\omega}}, \hat{\mathbf{u}})$, we obtain, in view of Proposition 4,

$$\partial_z \hat{\mathbf{u}} \in \mathcal{B}^3_{\alpha-1,1}, \text{ and } \partial_z \hat{\omega}_i \in \mathcal{B}_{\alpha-2,1},\ i = 1,2;\ \partial_z \hat{\omega}_3 \in \mathcal{B}_{\alpha-2,2}. \tag{56}$$

In order to get information on the second-order derivatives of $\hat{\mathbf{u}}$, we need to differentiate (25)–(33) with respect to z. For this purpose, we note that standard techniques for integrals depending on a parameter imply that $\hat{\mathbf{q}}$ admits partial derivative with respect to the variable z and that

$$\partial_z \hat{q}_1 = \frac{1}{4\pi^2}(\partial_z \hat{\omega}_3 * \hat{u}_2 + \hat{\omega}_3 * \partial_z \hat{u}_2 - \partial_z \hat{\omega}_2 * \hat{u}_3 - \hat{\omega}_2 * \partial_z \hat{u}_3).$$

The functions $\partial_z \hat{q}_2$ and $\partial_z \hat{q}_3$ are similar. Using Corollary 10, we find from (54) and (56) that

$$\partial_z \hat{\mathbf{q}} \in \mathcal{B}_{\alpha-2,3}\ . \tag{57}$$

Since $\partial_z^{\mathbf{n}} \hat{\mathbf{F}} \in \mathcal{W}_{\alpha'}$ for all $\mathbf{n} = (n_1, n_2, n_3)$, where $n_i \in \mathbb{N} \cup \{0\}$, $\alpha' \geq 0$, we find that $\partial_z \hat{\mathbf{Q}}$ exists and is the same space as $\partial_z \hat{\mathbf{q}}$. Therefore, we can differentiate the equations in (25)–(33) with respect to z, and using the above information on the first-order derivatives, and Proposition 4, it is straightforward to verify that

$$\partial_{zz} \hat{u}_i \in \mathcal{B}_{\alpha-2,1}, \quad i = 1,2;\ \partial_{zz} \hat{u}_3 \in \mathcal{B}_{\alpha-2,2}\ ,$$

and

$$\partial_{zz} \hat{\omega}_i \in \mathcal{B}_{\alpha-2,2}, \quad i = 1,2;\ \partial_{zz} \hat{\omega}_3 \in \mathcal{B}_{\alpha-2,3}\ .$$

One then sets $\omega = \mathcal{F}^{-1}[\hat{\omega}]$ and $\mathbf{u} = \mathcal{F}^{-1}[\hat{\mathbf{u}}]$. Using the properties of the spaces $\mathcal{B}_{\alpha,p}$ and standard techniques for integrals depending on a parameter, it follows that the functions $(\boldsymbol{\omega}, \mathbf{u})$ are well defined and are in $C^2(\Omega_+)$ (remember that we assume that $\alpha > 3$). Also, since $\mathcal{F}$ is an isometry in $L^2(\mathbb{R}^2)$, it follows from Proposition 4 that $(\mathbf{u}, \nabla \mathbf{u}) \in L^2(\Omega_+)$, and therefore, $\mathbf{u}$ has a finite Dirichlet integral, and $\mathbf{u} \in H^1_0(\Omega_+)$, with zero boundary values by construction of the integral equations. Next, since $(\hat{\boldsymbol{\omega}}, \hat{\mathbf{u}})$ satisfy (25)–(33), we find that $(\boldsymbol{\omega}, \mathbf{u})$ satisfy (3), (4) and (7), (8). Finally, by standard arguments, there exists a function p, such that $(\mathbf{u}, p)$ is a solution to (1), (2), (3) and (4) in the sense of distributions. By slight abuse of terminology, we refer in what follows to solutions $\mathbf{u}$ constructed this way as α-solutions.

In the remainder of this section, we discuss the behavior of the solution $\mathbf{u}$ at infinity. By Theorem 8, there exists an α-solution $(\hat{\boldsymbol{\omega}}, \hat{\mathbf{u}}) \in \mathcal{V}_\alpha$ satisfying

$$\|(\hat{\boldsymbol{\omega}}, \hat{\mathbf{u}}); \mathcal{V}_\alpha\| \leq 2C_2 \varepsilon_\alpha,$$

with C_2 as in Theorem 8 and with $\varepsilon_\alpha = \|\hat{\mathbf{F}}; \mathcal{W}_\alpha\|$, and furthermore, for $\alpha > 2$, $\mathbf{u} = \mathcal{F}^{-1}([\hat{\mathbf{u}}]) \in H^1_0(\Omega_+)$. Since, for $\alpha > 2$ and $z \geq 1$,

$$\int_{\mathbb{R}^2} \left(\frac{1}{1 + (|\mathbf{k}|\, z)^\alpha} \right)\, d\mathbf{k} \leq \frac{\text{const.}}{z^2},$$

we find for $(x, y, z) \in \Omega_+$ the pointwise bounds

$$|u_i(x,y,z)| \leq \frac{C_\alpha \varepsilon_\alpha}{z^2}, \quad i = 1,2,3. \tag{58}$$

This completes the proof of our main theorem.

4. Proof of main lemmas

In what follows, we give a proof of Lemmas 5 and 6.

4.1. Proof of Lemma 5

Proposition 9. *Let $\alpha > 2$, and let a_1, a_2 be continuous functions from $\mathbb{R}_0^2 \times [1,\infty)$ to $\mathbb{C}$ satisfying the bounds,*

$$|a_i(\mathbf{k},t)| \leq \bar{\mu}_\alpha(\mathbf{k},t), \quad i = 1,2.$$

Then, the convolution product $a_1 * a_2$ is a continuous function from $\mathbb{R}^2 \times [1,\infty)$ to $\mathbb{C}$ and we have the bound

$$|(a_1 * a_2)(\mathbf{k},t)| \leq \text{const.}\frac{1}{t^2}\bar{\mu}_\alpha(\mathbf{k},t), \tag{59}$$

uniformly in $t \geq 1$, $\mathbf{k} \in \mathbb{R}^2$.

Proof. Continuity is elementary. We now prove (59). Let

$$D(\mathbf{k}) = \{\mathbf{k}' \in \mathbb{R}^2 | \ |\mathbf{k}-\mathbf{k}'| \leq k/2\},$$

where $k = |\mathbf{k}|$. For $\mathbf{k}' \in D(\mathbf{k})$ and $k' = |\mathbf{k}'|$ we have that

$$k' \geq k - |\mathbf{k}-\mathbf{k}'| \geq \frac{1}{2}k.$$

Therefore, we have for the convolution $a_1 * a_2$,

$$\begin{aligned}
|(a_1 * a_2)(\mathbf{k},t)| &\leq \int_{\mathbb{R}^2\setminus D(\mathbf{k})} \bar{\mu}_\alpha(\mathbf{k}',t)\bar{\mu}_\alpha(\mathbf{k}-\mathbf{k}',t)\ d\mathbf{k}' + \int_{D(\mathbf{k})} \bar{\mu}_\alpha(\mathbf{k}',t)\bar{\mu}_\alpha(\mathbf{k}-\mathbf{k}',t)\ d\mathbf{k}' \\
&\leq \left(\sup_{\mathbf{k}'\in\mathbb{R}^2\setminus D(\mathbf{k})} \bar{\mu}_\alpha(\mathbf{k}-\mathbf{k}',t)\right)\int_{\mathbb{R}^2\setminus D(\mathbf{k})} \bar{\mu}_\alpha(\mathbf{k}',t)\ d\mathbf{k}' \\
&\quad + \left(\sup_{\mathbf{k}'\in D(\mathbf{k})} \bar{\mu}_\alpha(\mathbf{k}',t)\right)\int_{D(\mathbf{k})} \bar{\mu}_\alpha(\mathbf{k}-\mathbf{k}',t)\ d\mathbf{k}' \\
&\leq \text{const.}\bar{\mu}_\alpha(\mathbf{k}/2,t)\left(\int_{\mathbb{R}^2} \bar{\mu}_\alpha(\mathbf{k}',t)\ d\mathbf{k}' + \int_{\mathbb{R}^2} \bar{\mu}_\alpha(\mathbf{k}-\mathbf{k}',t)\ d\mathbf{k}'\right) \\
&\leq \text{const.}\frac{1}{t^2}\bar{\mu}_\alpha(\mathbf{k}/2,t) \leq \text{const.}\frac{1}{t^2}\bar{\mu}_\alpha(\mathbf{k},t),
\end{aligned}$$

and (59) follows. □

Corollary 10. *Let, for $i = 1, 2$, $\alpha_i > 2$, and $p_i \geq 0$. Let $f_i \in \mathcal{B}_{\alpha_i,p_i}$, then $f_1 * f_2 \in \mathcal{B}_{\alpha,p}$ and there exists a constant C, depending only on α_i, such that*

$$\|f_1 * f_2; \mathcal{B}_{\alpha,p}\| \leq C \, \|f_1; \mathcal{B}_{\alpha_1,p_1}\| \cdot \|f_2; \mathcal{B}_{\alpha_2,p_2}\| \,, \tag{60}$$

where $\alpha = \min\{\alpha_1, \alpha_2\}$ and $p = p_1 + p_2 + 2$.

Proof. We have $|f_i(\mathbf{k}, t)| \leq \|f_i; \mathcal{B}_{\alpha_i,p_i}\| \cdot \bar{\mu}_{\alpha_i}(\mathbf{k}, t)$ and by Proposition 9 we have

$$\frac{1}{t^{p_1}} \bar{\mu}_{\alpha_1} * \frac{1}{t^{p_2}} \bar{\mu}_{\alpha_2} \leq C \frac{1}{t^{p_1+p_2+2}} \bar{\mu}_{\min\{\alpha_1,\alpha_2\}},$$

with C depending only on α_1, α_2, and therefore (60) follows. □

Now let $(\hat{\boldsymbol{\omega}}_1, \hat{\mathbf{u}}_1), (\hat{\boldsymbol{\omega}}_2, \hat{\mathbf{u}}_2) \in \mathcal{V}_\alpha$. Using Corollary 10, we find that $\hat{\omega}_{23} * \hat{u}_{12} - \hat{\omega}_{22} * \hat{u}_{13} \in \mathcal{B}_{\alpha,3}$ with

$$\begin{aligned}\|\hat{\omega}_{23} * \hat{u}_{12} - \hat{\omega}_{22} * \hat{u}_{13}; \mathcal{B}_{\alpha,3}\| &\leq \text{const.} \left(\|\hat{u}_{12}; \mathcal{B}_{\alpha,0}\| \cdot \|\hat{\omega}_{23}; \mathcal{B}_{\alpha,1}\| + \|\hat{u}_{13}; \mathcal{B}_{\alpha,0}\| \cdot \|\hat{\omega}_{22}; \mathcal{B}_{\alpha,1}\|\right) \\ &\leq \text{const.} \|(\hat{\boldsymbol{\omega}}_1, \hat{\mathbf{u}}_1); \mathcal{V}_\alpha\| \cdot \|(\hat{\boldsymbol{\omega}}_2, \hat{\mathbf{u}}_2); \mathcal{V}_\alpha\|,\end{aligned}$$

and we conclude that $\hat{\mathbf{q}} \in \mathcal{W}_\alpha = \mathcal{B}^3_{\alpha,3}$ and that

$$\|\hat{\mathbf{q}}; \mathcal{W}_\alpha\| \leq \text{const.} \|(\hat{\boldsymbol{\omega}}_1, \hat{\mathbf{u}}_1); \mathcal{V}_\alpha\| \cdot \|(\hat{\boldsymbol{\omega}}_2, \hat{\mathbf{u}}_2); \mathcal{V}_\alpha\|.$$

This completes the proof of Lemma 5.

4.2. Proof of Lemma 6

Let k, κ be as defined in (37), and define Λ_- by

$$\Lambda_- = -\operatorname{Re}(\kappa) = -\frac{1}{2}\sqrt{2\sqrt{k_1^2 + k^4} + 2k^2}. \tag{61}$$

We have that

$$|\kappa| = (k_1^2 + k^4)^{1/4} \leq |k_1|^{1/2} + k \leq 2^{3/4} |\kappa| \leq 2^{3/4} \left(1 + k\right), \tag{62}$$

and that

$$k \leq |\Lambda_-| \leq |\kappa| \leq \sqrt{2} |\Lambda_-|. \tag{63}$$

Therefore, we have in particular that for $\sigma \geq 0$ and $\mathbf{k} \in \mathbb{R}^2$,

$$e^{\Lambda_- \sigma} \leq e^{-k\sigma}. \tag{64}$$

We will also need the following inequalities. For all $N \in \mathbb{N}_0$, we have for $z \in \mathbb{C}$ with $\operatorname{Re}(z) \leq 0$,

$$\left| \frac{e^z - \sum_{n=0}^N \frac{1}{n!} z^n}{z^{N+1}} \right| \leq \text{const.}, \tag{65}$$

and for all $z \in \mathbb{C}$ with $\operatorname{Re}(z) > 0$,

$$\left| \frac{e^z - \sum_{n=0}^N \frac{1}{n!} z^n}{z^{N+1}} \right| \leq \text{const.} e^{\operatorname{Re}(z)}. \tag{66}$$

In the following, we will routinely use (65) and (66) without mention. In what follows we prove Lemma 6 by providing bounds for the norms of $\hat{\omega}_i$ and $\hat{u}_i$ in terms of the norms of $\hat{Q}_i$. We systematically use the notation introduced above, but, for simplicity, we set

$$\mu(\mathbf{k}, s) = \frac{1}{s^3} \bar{\mu}_\alpha(\mathbf{k}, s), \tag{67}$$

and $\|Q\| = C \, \|\hat{\mathbf{Q}}; \mathcal{W}_\alpha\|$ with C a constant independent of $\mathbf{k}$ and t. This constant may be different from instance to instance changing even within the same line.

4.2.1. Bounds for $\hat{\omega}_1$. For the integral kernels of $\hat{\omega}_1$ we have:

Proposition 11. *Let $\alpha_{1,i,j}$ be as given in "Appendix A". Then we have the bounds*

$$|\alpha_{1,1,1}(\mathbf{k},\sigma)| \leq \text{const.} e^{-k\sigma} \min\{k^{\frac{1}{2}}(1+k^{\frac{1}{2}}), k\sigma\}, \tag{68}$$
$$|\alpha_{1,1,2}(\mathbf{k},\sigma)| \leq \text{const.} |\Lambda_-|\sigma e^{|\Lambda_-|\sigma}, \tag{69}$$
$$|\alpha_{1,1,3}(\mathbf{k},\sigma)| \leq \text{const.} \left(|\Lambda_-|^{\frac{3}{2}} + |\Lambda_-|^2\right)\sigma^2 e^{|\Lambda_-|\sigma}, \tag{70}$$
$$|\alpha_{1,2,1}(\mathbf{k},\sigma)| \leq \text{const.} e^{-k\sigma} \min\{k^{\frac{1}{2}}(1+k^{\frac{1}{2}}), k\sigma\}, \tag{71}$$
$$|\alpha_{1,2,2}(\mathbf{k},\sigma)| \leq \text{const.} (1+k\sigma)e^{-k\sigma}, \tag{72}$$
$$|\alpha_{1,2,3}(\mathbf{k},\sigma)| \leq \text{const.} k\sigma e^{-k\sigma}, \tag{73}$$
$$|\alpha_{1,3,2}(\mathbf{k},\sigma)| \leq |\Lambda_-| e^{\Lambda_-\sigma}, \tag{74}$$
$$|\alpha_{1,3,3}(\mathbf{k},\sigma)| \leq |\Lambda_-| e^{\Lambda_-\sigma}, \tag{75}$$

uniformly in $\sigma \geq 0$ and $\mathbf{k} \in \mathbb{R}^2$.

Proof. We first prove (68) and (71). From (185), we immediately get that

$$|\alpha_{1,1,1}(\mathbf{k},\sigma)| \leq \text{const.} k^{\frac{1}{2}}(1+k^{\frac{1}{2}})e^{-k\sigma}.$$

On the other hand, we have

$$\alpha_{1,1,1}(\mathbf{k},\sigma) = -\frac{2ik_2(k+\kappa)}{k} e^{-k\sigma} (k-\kappa)\sigma \frac{\left(e^{(k-\kappa)\sigma}-1\right)}{(k-\kappa)\sigma}, \tag{76}$$

and therefore, we find from (76) using (62) that

$$|\alpha_{1,1,1}(\mathbf{k},\sigma)| \leq \text{const.} k\sigma e^{-k\sigma}.$$

This completes the proof of (68). The proof of (71) is the same as for (68). We now prove (69). We have

$$\begin{aligned} |\alpha_{1,1,2}(\mathbf{k},\sigma)| &= \left| \left(e^{\kappa\sigma} - e^{-\kappa\sigma}\right) + \frac{2ik_2^2(k+\kappa)}{kk_1}\left(e^{-k\sigma} - e^{-\kappa\sigma}\right) \right| \\ &\leq \text{const.} |\Lambda_-|\sigma e^{|\Lambda_-|\sigma}. \end{aligned}$$

In order to prove (70), we use that

$$\begin{aligned} \alpha_{1,1,3}(\mathbf{k},\sigma) &= \frac{ik_2}{\kappa}\left[e^{-k\sigma} - e^{\kappa\sigma}\right] + \frac{k_2(\kappa+k)^2}{\kappa k_1}\left[e^{-\kappa\sigma} - e^{-k\sigma}\right] \\ &= -\frac{2k_2(k+\kappa)}{k_1}\left(e^{-k\sigma} - e^{-\kappa\sigma}\right) + \frac{ik_2}{\kappa}\left(e^{-\kappa\sigma} - e^{\kappa\sigma}\right) \\ &= -2ik_2\sigma\left[e^{-k\sigma}\frac{\left(1-e^{(k-\kappa)\sigma}\right)}{(k-\kappa)\sigma} + e^{\kappa\sigma}\left(\frac{e^{-2\kappa\sigma}-1}{-2\kappa\sigma}\right)\right] \\ &= 2ik_2(k-\kappa)\sigma^2 e^{-k\sigma}\frac{\left(e^{(k-\kappa)\sigma} - 1 - (k-\kappa)\sigma\right)}{(k-\kappa)^2\sigma^2} + 2ik_2\sigma e^{-k\sigma} \\ &\quad + 4i\kappa k_2\sigma^2 e^{\kappa\sigma}\left(\frac{e^{-2\kappa\sigma} - 1 + 2\kappa\sigma}{(-2\kappa\sigma)^2}\right) - 2ik_2\sigma e^{\kappa\sigma} \end{aligned}$$

$$= 2ik_2\,(k-\kappa)\,\sigma^2 e^{-k\sigma}\frac{\left(e^{(k-\kappa)\sigma}-1-(k-\kappa)\,\sigma\right)}{(k-\kappa)^2\,\sigma^2}$$
$$+\,4i\kappa k_2\sigma^2 e^{\kappa\sigma}\left(\frac{e^{-2\kappa\sigma}-1+2\kappa\sigma}{(-2\kappa\sigma)^2}\right)-2i\,(k+\kappa)\,k_2\sigma^2 e^{\kappa\sigma}\frac{\left(e^{-(k+\kappa)\sigma}-1\right)}{-\,(k+\kappa)\,\sigma}. \tag{77}$$

It is easy to get from (77) that

$$|\alpha_{1,1,3}(\mathbf{k},\sigma)|\le \text{const.}\left(|\Lambda_-|^{\frac{3}{2}}+|\Lambda_-|^2\right)\sigma^2 e^{|\Lambda_-|\sigma},$$

which yields the bound (70). For (72), we have

$$|\alpha_{1,2,2}(\mathbf{k},\sigma)| = \left|-2e^{-\kappa\sigma}-\frac{2k_2^2}{k}e^{-k\sigma}\sigma\frac{\left(1-e^{(k-\kappa)\sigma}\right)}{(k-\kappa)\,\sigma}\right|$$
$$\le \text{const.}(1+k\sigma)e^{-k\sigma}.$$

For (73), we have

$$|\alpha_{1,2,3}(\mathbf{k},\sigma)| = \left|\frac{2k_2(\kappa+k)}{k_1}\left(e^{-\kappa\sigma}-e^{-k\sigma}\right)\right|$$
$$= \left|2ik_2e^{-k\sigma}\sigma\frac{\left(e^{(k-\kappa)\sigma}-1\right)}{(k-\kappa)\,\sigma}\right|$$
$$\le \text{const.}k\sigma e^{-k\sigma}.$$

The bounds (74) and (75) are immediate. □

As a consequence of Proposition 11, we have:

Proposition 12. *Let $\alpha > 2$. Then, $\hat{\mathbf{Q}} \mapsto \hat{\omega}_1$ defines a continuous linear map from $\mathcal{W}_\alpha$ to $\mathcal{B}_{\alpha,1}$. More precisely, $\hat{\mathbf{Q}} \mapsto \hat{\omega}_{1,i,j}$, with $\hat{\omega}_{1,i,j}$ as given in (44), define continuous linear maps on $\mathcal{W}_\alpha$, with $\hat{\omega}_{1,1,i}\in\mathcal{B}_{\alpha,1}$, $i=1,2$, $\hat{\omega}_{1,1,3}\in\mathcal{B}_{\alpha,\frac{3}{2}-\varepsilon}$, $\hat{\omega}_{1,2,1}\in\mathcal{B}_{\alpha,\frac{5}{2}}$, $\hat{\omega}_{1,2,i}\in\mathcal{B}_{\alpha,2}$, $i=2,3$ and $\hat{\omega}_{1,3,i}\in\mathcal{B}_{\alpha,2}$, $i=2,3$, where ε is positive and sufficiently small.*

Proof. Let μ as defined in (67). From (68), we find with Propositions 31 and 32 that

$$|\hat{\omega}_{1,1,1}(\mathbf{k},t)| \le \|Q\|\ e^{\Lambda_-(t-1)}\int_1^t e^{|\Lambda_-|(s-1)}\min\{|\Lambda_-|^{\frac{1}{2}}(1+|\Lambda_-|^{\frac{1}{2}}),|\Lambda_-|\,(s-1)\}\ \mu(\mathbf{k},s)\ ds$$
$$= \|Q\|\ e^{\Lambda_-(t-1)}\int_1^t e^{|\Lambda_-|(s-1)}\min\{|\Lambda_-|^{\frac{1}{2}}(1+|\Lambda_-|^{\frac{1}{2}}),|\Lambda_-|\,(s-1)\}\ \left(\frac{1}{s^3}\bar{\mu}_\alpha(\mathbf{k},s)\right)\ ds$$
$$\le \|Q\|\left(\frac{1}{t}\bar{\mu}_\alpha(\mathbf{k},t)+\frac{1}{t^2}\bar{\mu}_\alpha(\mathbf{k},t)\right),$$

and therefore, $\hat{\omega}_{1,1,1}\in\mathcal{B}_{\alpha,1}$. From (69), we find with Propositions 31 and 32 that

$$|\hat{\omega}_{1,1,2}(\mathbf{k},t)| \le \|Q\|\ e^{\Lambda_-(t-1)}\int_1^t |\Lambda_-|(s-1)e^{|\Lambda_-|(s-1)}\ \mu(\mathbf{k},s)\ ds$$
$$\le \|Q\|\left(\frac{1}{t}\bar{\mu}_\alpha(\mathbf{k},t)+\frac{1}{t^2}\bar{\mu}_\alpha(\mathbf{k},t)\right),$$

and therefore, $\hat{\omega}_{1,1,2} \in \mathcal{B}_{\alpha,1}$. Using (70) with $\min\{1, |\Lambda_-|(s-1)\} \leq |\Lambda_-|(s-1)$, and Propositions 31, 32, we find that

$$|\hat{\omega}_{1,1,3}(\mathbf{k},t)| \leq \|Q\|\ e^{\Lambda_-(t-1)} \int_1^t e^{|\Lambda_-|(s-1)} \left(|\Lambda_-|^{\frac{3}{2}} + |\Lambda_-|^2\right)(s-1)^2\ \mu(\mathbf{k},s)\ ds$$
$$\leq \|Q\| \left(\frac{1}{t^{\frac{3}{2}-\varepsilon}}\bar{\mu}_\alpha(\mathbf{k},t) + \frac{1}{t^{\frac{3}{2}}}\bar{\mu}_\alpha(\mathbf{k},t)\right), \quad \text{for } \varepsilon > 0 \text{ sufficiently small.}$$

and therefore, $\hat{\omega}_{1,1,3} \in \mathcal{B}_{\alpha,\frac{3}{2}-\varepsilon}$. Using (71) and Proposition 37, we find

$$|\hat{\omega}_{1,2,1}(\mathbf{k},t)| \leq \|Q\|\ e^{\Lambda_-(t-1)} \int_t^\infty e^{-k(s-1)} \min\{k^{\frac{1}{2}}(1+k^{\frac{1}{2}}), k\,(s-1)\}\ \mu(\mathbf{k},s)\ ds$$
$$\leq \|Q\| \left(\frac{1}{t^{\frac{5}{2}}}\bar{\mu}_\alpha(\mathbf{k},t)\right),$$

and therefore, $\hat{\omega}_{1,2,1} \in \mathcal{B}_{\alpha,\frac{5}{2}}$. Using (72) and Proposition 37, we find that

$$|\hat{\omega}_{1,2,2}(\mathbf{k},t)| \leq \|Q\|\ e^{\Lambda_-(t-1)} \int_t^\infty (1+k(s-1))e^{-k(s-1)}\ \mu(\mathbf{k},s)\ ds$$
$$\leq \|Q\| \left(\frac{1}{t^2}\bar{\mu}_\alpha(\mathbf{k},t)\right),$$

and therefore, $\hat{\omega}_{1,2,2} \in \mathcal{B}_{\alpha,2}$. Using (73) and Proposition 37, we find that

$$|\hat{\omega}_{1,2,3}(\mathbf{k},t)| \leq \|Q\|\ e^{\Lambda_-(t-1)} \int_t^\infty k(s-1)e^{-k(s-1)}\ \mu(\mathbf{k},s)\ ds$$
$$\leq \|Q\| \left(\frac{1}{t^2}\bar{\mu}_\alpha(\mathbf{k},t)\right),$$

and therefore, $\hat{\omega}_{1,2,3} \in \mathcal{B}_{\alpha,2}$. Similarly, we find from (74) and Proposition 33 that

$$|\hat{\omega}_{1,3,2}(\mathbf{k},t)| \leq \|Q\|\ \left|\frac{1+k}{2\kappa}\left(e^{\kappa(t-1)} - e^{-\kappa(t-1)}\right)\right| \int_t^\infty |\Lambda_-|e^{\Lambda_-(s-1)}\ \mu(\mathbf{k},s)\ ds$$
$$\leq \|Q\| \left(\frac{1}{t^2}\bar{\mu}_\alpha(\mathbf{k},t)\right),$$

and therefore, $\hat{\omega}_{1,3,2} \in \mathcal{B}_{\alpha,2}$. Finally, we find from (75) and Proposition 33 that

$$|\hat{\omega}_{1,3,3}(\mathbf{k},t)| \leq \|Q\|\ \left|\frac{1+k}{2\kappa}\left(e^{\kappa(t-1)} - e^{-\kappa(t-1)}\right)\right| \int_t^\infty |\Lambda_-|e^{\Lambda_-(s-1)}\ \mu(\mathbf{k},s)\ ds$$
$$\leq \|Q\| \left(\frac{1}{t^2}\bar{\mu}_\alpha(\mathbf{k},t)\right),$$

and therefore $\hat{\omega}_{1,3,3} \in \mathcal{B}_{\alpha,2}$. □

4.2.2. Bounds for $\hat{\omega}_2$. For the integral kernels of $\hat{\omega}_2$ we have:

Proposition 13. *Let $\alpha_{2,i,j}$ be as given in "Appendix A". Then we have the bounds*

$$|\alpha_{2,1,1}(\mathbf{k},\sigma)| \leq \text{const.}|\Lambda_-|\sigma e^{|\Lambda_-|\sigma}, \tag{78}$$

$$|\alpha_{2,1,2}(\mathbf{k},\sigma)| \leq \text{const.}e^{-k\sigma}\min\{k^{\frac{1}{2}}(1+k^{\frac{1}{2}}), k\sigma\}, \tag{79}$$

$$|\alpha_{2,1,3}(\mathbf{k},\sigma)| \leq \text{const.}(1+|\Lambda_-|)e^{|\Lambda_-|\sigma}\min\{1, |\Lambda_-|^2\sigma^2\}, \tag{80}$$

$$|\alpha_{2,2,1}(\mathbf{k},\sigma)| \leq \text{const.}(1+k\sigma)e^{-k\sigma}, \tag{81}$$

$$|\alpha_{2,2,2}(\mathbf{k},\sigma)| \leq \text{const.}e^{-k\sigma}\min\{k^{\frac{1}{2}}(1+k^{\frac{1}{2}}), k\sigma\}, \tag{82}$$

$$|\alpha_{2,2,3}(\mathbf{k},\sigma)| \leq \text{const.}k^{\frac{1}{2}}(1+k^{\frac{1}{2}})e^{-k\sigma}\ , \tag{83}$$

$$|\alpha_{2,3,1}(\mathbf{k},\sigma)| \leq |\Lambda_-|e^{\Lambda_-\sigma}, \tag{84}$$

$$|\alpha_{2,3,3}(\mathbf{k},\sigma)| \leq |\Lambda_-|e^{\Lambda_-\sigma}, \tag{85}$$

uniformly in $\sigma \geq 0$ and $\mathbf{k} \in \mathbb{R}^2$.

Proof. From (193), we get that

$$\begin{aligned}|\alpha_{2,1,1}(\mathbf{k},\sigma)| &= \left|\left(e^{-\kappa\sigma}-e^{\kappa\sigma}\right)+\frac{2ik_1(\kappa+k)}{k}\left(e^{-\kappa\sigma}-e^{-k\sigma}\right)\right| \\ &= \left|-e^{\kappa\sigma}2\kappa\sigma\frac{\left(e^{-2\kappa\sigma}-1\right)}{(-2\kappa\sigma)}-\frac{2k_1^2}{k}\sigma e^{-k\sigma}\frac{\left(e^{(k-\kappa)\sigma}-1\right)}{(k-\kappa)\,\sigma}\right| \\ &= \text{const.}|\Lambda_-|\sigma e^{|\Lambda_-|\sigma},\end{aligned}$$

which shows (78). From (195), we get that

$$|\alpha_{2,1,3}(\mathbf{k},\sigma)| \leq \text{const.}(1+|\Lambda_-|)e^{|\Lambda_-|\sigma}.$$

When expanding the exponential functions in (195), the first two terms cancel, so that

$$\begin{aligned}\alpha_{2,1,3}(\mathbf{k},\sigma) = \frac{ik_1}{\kappa}\left(e^{\kappa\sigma}-1-\kappa\sigma\right)-\frac{(\kappa+k)^2}{\kappa}\left(e^{-\kappa\sigma}-1+\kappa\sigma\right) \\ +2\left(k+\kappa\right)\left(e^{-k\sigma}-1+k\sigma\right),\end{aligned}$$

and we find that

$$|\alpha_{2,1,3}(\mathbf{k},\sigma)| \leq \text{const.}|\Lambda_-|^2\sigma^2(1+|\Lambda_-|)e^{|\Lambda_-|\sigma}.$$

Finally, in order to prove (81), we note that

$$\begin{aligned}|\alpha_{2,2,1}(\mathbf{k},\sigma)| &= \left|2e^{-\kappa\sigma}+\frac{2ik_1(\kappa+k)}{k}\left(e^{-\kappa\sigma}-e^{-k\sigma}\right)\right| \\ &= \left|2e^{-\kappa\sigma}-\frac{2k_1^2}{k}e^{-k\sigma}\sigma\frac{\left(e^{(k-\kappa)\sigma}-1\right)}{(k-\kappa)\,\sigma}\right| \\ &\leq \text{const.}(1+k\sigma)e^{-k\sigma},\end{aligned}$$

and (81) follows. The bounds (79) and (82) are the same as (68), and the bounds (83), (84) and (85) are trivial. □

As a consequence of Proposition 13, we have:

Proposition 14. *Let $\alpha > 2$. Then, $\hat{\mathbf{Q}} \mapsto \hat{\omega}_2$ defines a continuous linear map from $\mathcal{W}_\alpha$ to $\mathcal{B}_{\alpha,1}$. More precisely, $\hat{\mathbf{Q}} \mapsto \hat{\omega}_{2,i,j}$, with $\hat{\omega}_{2,i,j}$ as given in (44), define continuous linear maps on $\mathcal{W}_\alpha$, with $\hat{\omega}_{2,1,i} \in \mathcal{B}_{\alpha,1}$, $i = 1,2$, $\hat{\omega}_{2,1,3} \in \mathcal{B}_{\alpha,2-\varepsilon}$, $\hat{\omega}_{2,2,1} \in \mathcal{B}_{\alpha,2}$, $\hat{\omega}_{2,2,i} \in \mathcal{B}_{\alpha,\frac{5}{2}}$, $i = 2,3$ and $\hat{\omega}_{2,3,i} \in \mathcal{B}_{\alpha,2}$, $i = 1,3$.*

Proof. Using (78), Propositions 31, and 32, we find that

$$|\hat{\omega}_{2,1,1}(\mathbf{k},t)| \leq \|Q\| \; e^{\Lambda_-(t-1)} \int_1^t e^{|\Lambda_-|(s-1)} |\Lambda_-|(s-1) \; \mu(\mathbf{k},s) \; ds$$
$$\leq \|Q\| \left(\frac{1}{t} \bar{\mu}_\alpha(\mathbf{k},t) + \frac{1}{t^2} \bar{\mu}_\alpha(\mathbf{k},t) \right),$$

and therefore, $\hat{\omega}_{2,1,1} \in \mathcal{B}_{\alpha,1}$. Using (79), Propositions 35, and 36, we find that

$$|\hat{\omega}_{2,1,2}(\mathbf{k},t)| \leq \|Q\| \; e^{\Lambda_-(t-1)} \int_1^t e^{-k(s-1)} \min\{k^{\frac{1}{2}}(1+k^{\frac{1}{2}}), k(s-1)\} \; \mu(\mathbf{k},s) \; ds$$
$$\leq \|Q\| \left(\frac{1}{t} \bar{\mu}_\alpha(\mathbf{k},t) + \frac{1}{t^2} \bar{\mu}_\alpha(\mathbf{k},t) \right),$$

and therefore, $\hat{\omega}_{2,1,2} \in \mathcal{B}_{\alpha,1}$. From (80), Propositions 31, and 32, we get that

$$|\hat{\omega}_{2,1,3}(\mathbf{k},t)| \leq \|Q\| \; e^{\Lambda_-(t-1)} \int_1^t (1+|\Lambda_-|) e^{|\Lambda_-|(s-1)} \min\{1, |\Lambda_-|^2(s-1)^2\} \; \mu(\mathbf{k},s) \; ds$$
$$\leq \|Q\| \left(\frac{1}{t^{2-\varepsilon}} \bar{\mu}_\alpha(\mathbf{k},t) \right), \quad \text{for } \forall \varepsilon > 0.$$

and therefore, $\hat{\omega}_{2,1,3} \in \mathcal{B}_{\alpha,2-\varepsilon}$. Using (81) and Proposition 37, we find that

$$|\hat{\omega}_{2,2,1}(\mathbf{k},t)| \leq \|Q\| \; e^{\Lambda_-(t-1)} \int_t^\infty (1+k(s-1)) e^{-k(s-1)} \; \mu(\mathbf{k},s) \; ds$$
$$\leq \|Q\| \left(\frac{1}{t^2} \bar{\mu}_\alpha(\mathbf{k},t) \right),$$

and therefore, $\hat{\omega}_{2,2,1} \in \mathcal{B}_{\alpha,2}$. Using (82) and Proposition 37, we find that

$$|\hat{\omega}_{2,2,2}(\mathbf{k},t)| \leq \|Q\| \; e^{\Lambda_-(t-1)} \int_t^\infty e^{-k(s-1)} \min\{k^{\frac{1}{2}}(1+k^{\frac{1}{2}}), k(s-1)\} \; \mu(\mathbf{k},s) \; ds$$
$$\leq \|Q\| \; \left(\frac{1}{t^{\frac{5}{2}}} \bar{\mu}_\alpha(\mathbf{k},t) \right),$$

and therefore, $\hat{\omega}_{2,2,2} \in \mathcal{B}_{\alpha,\frac{5}{2}}$. Using (83) and Proposition 37, we find that

$$|\hat{\omega}_{2,2,3}(\mathbf{k},t)| \leq \|Q\| \; e^{\Lambda_-(t-1)} \int_t^\infty k^{\frac{1}{2}}(1+k^{\frac{1}{2}}) e^{-k(s-1)} \; \mu(\mathbf{k},s) \; ds$$
$$\leq \|Q\| \left(\frac{1}{t^{\frac{5}{2}}} \bar{\mu}_\alpha(\mathbf{k},t) \right),$$

and therefore, $\hat{\omega}_{2,2,3} \in \mathcal{B}_{\alpha,\frac{5}{2}}$. From (84) and Proposition 33, we find that

$$|\hat{\omega}_{2,3,1}(\mathbf{k},t)| \leq \|Q\| \; \left| \frac{1+k}{2\kappa} \left(e^{\kappa(t-1)} - e^{-\kappa(t-1)} \right) \right| \int_t^\infty |\Lambda_-| \, e^{\Lambda_-(s-1)} \; \mu(\mathbf{k},s) \; ds$$
$$\leq \|Q\| \left(\frac{1}{t^2} \bar{\mu}_\alpha(\mathbf{k},t) \right),$$

and therefore, $\hat{\omega}_{2,3,1} \in \mathcal{B}_{\alpha,2}$. Finally, we find from (85) and Proposition 33 that

$$\begin{aligned}|\hat{\omega}_{2,3,3}(\mathbf{k},t)| &\leq \|Q\| \left|\frac{1+k}{2\kappa}\left(e^{\kappa(t-1)} - e^{-\kappa(t-1)}\right)\right| \int_t^\infty |\Lambda_-| \, e^{\Lambda_-(s-1)} \, \mu(\mathbf{k},s) \, ds \\ &\leq \|Q\| \left(\frac{1}{t^2}\bar{\mu}_\alpha(\mathbf{k},t)\right),\end{aligned}$$

and therefore, $\hat{\omega}_{2,3,3} \in \mathcal{B}_{\alpha,2}$. □

4.2.3. Bounds for $\hat{\boldsymbol{\omega}}_3$.

For the integral kernels of $\hat{\omega}_3$, we have:

Proposition 15. *Let $\alpha_{3,i,j}$ be as given in "Appendix A". Then we have the bounds*

$$|\alpha_{3,1,1}(\mathbf{k},\sigma)| \leq \text{const.} e^{|\Lambda_-|\sigma} \min\{1, |\Lambda_-|\,\sigma\}, \tag{86}$$

$$|\alpha_{3,1,2}(\mathbf{k},\sigma)| \leq \begin{cases} \text{const.}\,(|\Lambda_-| + |\Lambda_-|\,\sigma)\, e^{|\Lambda_-|\sigma} & \text{for } |k| \leq 1 \\ \text{const.} e^{|\Lambda_-|\sigma}|\Lambda_-| & \text{for } |k| > 1 \end{cases}, \tag{87}$$

$$|\alpha_{3,1,3}(\mathbf{k},\sigma)| \leq \text{const.} \min\{1, k\sigma\} k e^{k\sigma}, \tag{88}$$

$$|\alpha_{3,2,2}(\mathbf{k},\sigma)| \leq \text{const.}(k + k^{\frac{1}{2}}) e^{-\frac{k}{2}\sigma}, \tag{89}$$

$$|\alpha_{3,2,3}(\mathbf{k},\sigma)| \leq \text{const.} k e^{-\frac{k}{2}\sigma}, \tag{90}$$

$$|\alpha_{3,3,1}(\mathbf{k},\sigma)| \leq |\Lambda_-|\, e^{\Lambda_-\sigma}, \tag{91}$$

$$|\alpha_{3,3,2}(\mathbf{k},\sigma)| \leq |\Lambda_-|\, e^{\Lambda_-\sigma}, \tag{92}$$

$$|\alpha_{3,3,3}(\mathbf{k},\sigma)| \leq \text{const.}\, |\Lambda_-|\, e^{\Lambda_-\sigma}, \tag{93}$$

uniformly in $\sigma \geq 0$ and $\mathbf{k} \in \mathbb{R}^2$.

Proof. From (201), we immediately get that $|\alpha_{3,1,1}(\mathbf{k},\sigma)| \leq \text{const.} e^{|\Lambda_-|\sigma}$. We also have

$$\begin{aligned}|\alpha_{3,1,1}(\mathbf{k},\sigma)| &= \left|2ik_2 e^{\kappa\sigma}\sigma \frac{\left(1 - e^{-2\kappa\sigma}\right)}{-2\kappa\sigma}\right| \\ &\leq \text{const.}\, |\Lambda_-|\, \sigma e^{|\Lambda_-|\sigma}.\end{aligned}$$

The bound (86) thus follows. Next, we note that

$$\alpha_{3,1,2}(\mathbf{k},\sigma) = \frac{ik_1}{\kappa}\left(e^{-\kappa\sigma} - e^{\kappa\sigma}\right) + \frac{2i\kappa k_2^2(\kappa + k)}{kk_1}\left(e^{-\kappa\sigma} - e^{-k\sigma}\right) + \frac{k_2^2}{k}\left(e^{-k\sigma} - e^{k\sigma}\right). \tag{94}$$

From (94) we get for $k > 1$ that

$$|\alpha_{3,1,2}(\mathbf{k},\sigma)| \leq \text{const.}\, |\Lambda_-|\, e^{|\Lambda_-|\sigma},$$

and for $k \leq 1$ that

$$\begin{aligned}|\alpha_{3,1,2}(\mathbf{k},\sigma)| &= \left|-2ik_1 e^{\kappa\sigma}\sigma \frac{\left(e^{-2\kappa\sigma} - 1\right)}{-2\kappa\sigma} - \frac{2\kappa k_2^2}{k}\sigma e^{-k\sigma}\frac{\left(e^{(k-\kappa)\sigma} - 1\right)}{(k-\kappa)\,\sigma} - 2k_2^2\sigma e^{k\sigma}\frac{\left(e^{-2k\sigma} - 1\right)}{-2k\sigma}\right| \\ &\leq \text{const.}(ke^{|\Lambda_-|\sigma}\sigma + |\kappa|) \leq \text{const.}\,(|\Lambda_-| + |\Lambda_-|\,\sigma)\, e^{|\Lambda_-|\sigma}.\end{aligned}$$

This shows (87). From (203), we find

$$\begin{aligned}|\alpha_{3,1,3}(\mathbf{k},\sigma)| &= \left| ik_2\left(e^{k\sigma}-e^{-k\sigma}\right) - \frac{2kk_2(k+\kappa)}{k_1}\left(e^{-\kappa\sigma}-e^{-k\sigma}\right)\right| \\ &= \left| ik_2\left(e^{k\sigma}-e^{-k\sigma}\right) - 2ikk_2\sigma e^{-k\sigma}\frac{\left(e^{(k-\kappa)\sigma}-1\right)}{(k-\kappa)\,\sigma}\right| \\ &\le \text{const.}\min\{1,k\sigma\}ke^{k\sigma}.\end{aligned}$$

This shows the bound (88). From (204), we get

$$\begin{aligned}|\alpha_{3,2,2}(\mathbf{k},\sigma)| &= \left|\frac{2k_2^2\kappa}{k}\sigma e^{-k\sigma}\frac{\left(e^{(k-\kappa)\sigma}-1\right)}{(k-\kappa)\,\sigma}\right| \\ &\le \text{const.}k(k+k^{\frac{1}{2}})\sigma e^{-k\sigma} \\ &\le \text{const.}(k+k^{\frac{1}{2}})e^{-\frac{k}{2}\sigma},\end{aligned}$$

where we have used the fact that for all $\sigma \ge 0$, $k \ge 0$,

$$k\sigma e^{-k\sigma} \le \text{const.}e^{-\frac{k}{2}\sigma}. \tag{95}$$

For (205), we find, using again (95), that

$$\begin{aligned}|\alpha_{3,2,3}(\mathbf{k},\sigma)| &= \left|-2ik_2e^{-k\sigma} - \frac{2k_2k(k+\kappa)}{k_1}\left(e^{-\kappa\sigma}-e^{-k\sigma}\right)\right| \\ &\le \text{const.}k(1+k\sigma)e^{-k\sigma} \le \text{const.}ke^{-\frac{k}{2}\sigma}.\end{aligned}$$

The bounds (91), (92), and (93) are obvious. □

Proposition 16. *Let $\beta_{3,i,j}$ be as given in "Appendix A". Then we have the bounds*

$$|\beta_{3,1,2}(\mathbf{k},\sigma)| \le \text{const.}\min\{1,k\sigma\}(k^{\frac{1}{2}}+k)e^{k\sigma}, \tag{96}$$

$$|\beta_{3,1,3}(\mathbf{k},\sigma)| \le \text{const.}\min\{1,k\sigma\}ke^{k\sigma}, \tag{97}$$

$$|\beta_{3,2,2}(\mathbf{k},\sigma)| \le \text{const.}(k^{\frac{1}{2}}+k)e^{-\frac{k}{2}\sigma}, \tag{98}$$

$$|\beta_{3,2,3}(\mathbf{k},\sigma)| \le \text{const.}ke^{-\frac{k}{2}\sigma}, \tag{99}$$

$$|\beta_{3,3,2}(\mathbf{k},\sigma)| \le |\Lambda_-|\,e^{\Lambda_-\sigma}, \tag{100}$$

$$|\beta_{3,3,3}(\mathbf{k},\sigma)| \le |\Lambda_-|\,e^{\Lambda_-\sigma}, \tag{101}$$

uniformly in $\sigma \ge 0$ and $\mathbf{k} \in \mathbb{R}^2$.

Proof. The bounds (98) and (99) follow immediately from (89) and (90). The bounds (100) and (101) are trivial. We only need to prove (96) and (97). From (209) and (210), we get

$$\begin{aligned}|\beta_{3,1,2}(\mathbf{k},\sigma)| &= \left|\frac{k_2^2}{k}\left(e^{k\sigma}-e^{-k\sigma}\right) + \frac{2\kappa k_2^2}{k^2}k\sigma e^{-k\sigma}\frac{\left(e^{(k-\kappa)\sigma}-1\right)}{(k-\kappa)\,\sigma}\right| \\ &\le \text{const.}(k^{\frac{1}{2}}+k)e^{k\sigma},\end{aligned}$$

on the other hand,

$$\begin{aligned}|\beta_{3,1,2}(\mathbf{k},\sigma)| &= \left|\frac{k_2^2}{k}\left(e^{k\sigma}-e^{-k\sigma}\right) + \frac{2\kappa k_2^2}{k^2}k\sigma e^{-k\sigma}\frac{\left(e^{(k-\kappa)\sigma}-1\right)}{(k-\kappa)\,\sigma}\right| \\ &\le \text{const.}(k^{\frac{1}{2}}+k)k\sigma e^{k\sigma},\end{aligned}$$

which show (96), and similarly,

$$\begin{aligned}|\beta_{3,1,3}(\mathbf{k},\sigma)| &= \left| ik_2\left(e^{-k\sigma}-e^{k\sigma}\right)+2ik_2k\sigma e^{-k\sigma}\frac{\left(e^{(k-\kappa)\sigma}-1\right)}{(k-\kappa)\,\sigma}\right| \\ &\leq \text{const.}\min\{1,k\sigma\}ke^{k\sigma},\end{aligned}$$

which shows (97). □

Proposition 17. *Let $\gamma_{3,i,j}$ be as given in "Appendix A". Then we have the bounds*

$$|\gamma_{3,1,2}(\mathbf{k},\sigma)| \leq \text{const.}e^{k\sigma}\min\{k^2\sigma,k\}, \tag{102}$$

$$|\gamma_{3,1,3}(\mathbf{k},\sigma)| \leq \text{const.}e^{k\sigma}\min\{k^2\sigma,k\}, \tag{103}$$

$$|\gamma_{3,2,2}(\mathbf{k},\sigma)| \leq \text{const.}(k+k^{\frac{1}{2}})e^{-\frac{k}{2}\sigma}, \tag{104}$$

$$|\gamma_{3,2,3}(\mathbf{k},\sigma)| \leq \text{const.}ke^{-\frac{k}{2}\sigma}, \tag{105}$$

$$|\gamma_{3,3,2}(\mathbf{k},\sigma)| \leq |\Lambda_-|\,e^{\Lambda_-\sigma}, \tag{106}$$

$$|\gamma_{3,3,3}(\mathbf{k},\sigma)| \leq |\Lambda_-|\,e^{\Lambda_-\sigma}, \tag{107}$$

uniformly in $\sigma \geq 0$ and $\mathbf{k} \in \mathbb{R}^2$.

Proof. The same techniques that we have used to prove the bounds on $\beta_{3,i,j}$ can be applied to prove the bounds for $\gamma_{3,i,j}$. From (215), we immediately get that

$$\begin{aligned}|\gamma_{3,1,2}(\mathbf{k},\sigma)| &= \left|\frac{k_1k_2^2}{k\,(\kappa+k)}\left(e^{k\sigma}-e^{-\kappa\sigma}\right)-\frac{k_2^2k_1}{k^2}k\sigma e^{-k\sigma}\frac{\left(1-e^{(k-\kappa)\sigma}\right)}{(k-\kappa)\,\sigma}\right| \\ &\leq \text{const.}ke^{k\sigma}.\end{aligned}$$

We also have

$$\begin{aligned}|\gamma_{3,1,2}(\mathbf{k},\sigma)| &= \left|-\frac{k_1k_2^2}{k}\sigma e^{k\sigma}\frac{\left(1-e^{-(k+\kappa)\sigma}\right)}{-(k+\kappa)\,\sigma}-\frac{k_2^2k_1}{k}\sigma e^{-k\sigma}\frac{\left(1-e^{(k-\kappa)\sigma}\right)}{(k-\kappa)\,\sigma}\right| \\ &\leq \text{const.}k^2\sigma e^{k\sigma}.\end{aligned}$$

This completes the proof of (102). Similarly, we get from (216) the bound (103). From the expression (217), we get

$$\begin{aligned}|\gamma_{3,2,2}(\mathbf{k},\sigma)| &= \left|\frac{2\kappa k_2^2k_1}{k\,(k+\kappa)}\sigma e^{-k\sigma}\frac{\left(e^{(k-\kappa)\sigma}-1\right)}{(k-\kappa)\,\sigma}\right| \\ &\leq \text{const.}(k+k^{\frac{1}{2}})e^{-\frac{k}{2}\sigma},\end{aligned}$$

which implies (104), and (218) leads to

$$\begin{aligned}|\gamma_{3,2,3}(\mathbf{k},\sigma)| &= \left|\frac{2ik_2k_1}{(k+\kappa)}k\sigma e^{-k\sigma}\frac{\left(e^{(k-\kappa)\sigma}-1\right)}{(k-\kappa)\,\sigma}+\frac{2ik_2k_1}{(k+\kappa)}e^{-k\sigma}\right| \\ &\leq \text{const.}ke^{-\frac{k}{2}\sigma},\end{aligned}$$

which implies (105). The bounds (106) and (107) are trivial. □

As a consequence of Propositions 15–17, we have

Proposition 18. *Let $\alpha > 2$. Then, $\hat{\mathbf{Q}} \mapsto \hat{\omega}_3$ defines a continuous linear map from $\mathcal{W}_\alpha$ to $\mathcal{B}_{\alpha,1}$. More precisely, $\hat{\mathbf{Q}} \mapsto \hat{\omega}_{3,i,j}$, with $\hat{\omega}_{3,i,j}$ as given in (44), define continuous linear maps on $\mathcal{W}_\alpha$, with $\hat{\omega}_{3,1,i} \in \mathcal{B}_{\alpha,1}$, $i = 1,3$, $\hat{\omega}_{3,1,2} \in \mathcal{B}_{\alpha,1}$, $\hat{\omega}_{3,2,2} \in \mathcal{B}_{\alpha,\frac{3}{2}}$, $\hat{\omega}_{3,2,3} \in \mathcal{B}_{\alpha,2}$, and $\hat{\omega}_{3,3,1} \in \mathcal{B}_{\alpha,2}$, $\hat{\omega}_{3,3,i} \in \mathcal{B}_{\alpha,1}$, $i = 2,3$.*

Proof. From (86), Propositions 31 and 32, we find

$$|\hat{\omega}_{3,1,1}(\mathbf{k},t)| \leq \|Q\| \left[e^{\Lambda_-(t-1)} \int_1^t e^{|\Lambda_-|(s-1)} \min\{1, |\Lambda_-|(s-1)\} \, \mu(\mathbf{k},s) \, ds \right] \leq \|Q\| \left(\frac{1}{t} \bar{\mu}_\alpha(\mathbf{k},t) + \frac{1}{t^2} \bar{\mu}_\alpha(\mathbf{k},t) \right),$$

and therefore, $\hat{\omega}_{3,1,1} \in \mathcal{B}_{\alpha,1}$. From Propositions 31, 32, 35, and 36, we get for $k \leq 1$,

$$|\hat{\omega}_{3,1,2}(\mathbf{k},t)| \leq \|Q\| \left[e^{\Lambda_-(t-1)} \int_1^t (|\Lambda_-| + |\Lambda_-|(s-1)) \, e^{|\Lambda_-|(s-1)} \, \mu(\mathbf{k},s) \, ds + e^{-k(t-1)} \int_1^t (k^{\frac{1}{2}} + k) k (s-1) e^{k(s-1)} \, \mu(\mathbf{k},s) \, ds + |H_1| \int_1^t \min\{k^2(s-1), k\} e^{k(s-1)} \, \mu(\mathbf{k},s) \, ds \right] \leq \|Q\| \left(\frac{1}{t} \bar{\mu}_\alpha(\mathbf{k},t) \right),$$

and for $k > 1$ we have

$$|\hat{\omega}_{3,1,2}(\mathbf{k},t)| \leq \|Q\| \left(\frac{1}{t} \bar{\mu}_\alpha(\mathbf{k},t) \right),$$

and therefore, $\hat{\omega}_{3,1,2} \in \mathcal{B}_{\alpha,1}$. Similarly, we get from Propositions 31, 32, 35 and 36 that

$$|\hat{\omega}_{3,1,3}(\mathbf{k},t)| \leq \|Q\| \left[e^{\Lambda_-(t-1)} \int_1^t \min\{1, k\sigma\} k e^{k(s-1)} \, \mu(\mathbf{k},s) \, ds + e^{-k(t-1)} \int_1^t \min\{1, k\sigma\} k e^{k(s-1)} \, \mu(\mathbf{k},s) \, ds + |H_1| \int_1^t \min\{k^2(s-1), k\} e^{k(s-1)} \, \mu(\mathbf{k},s) \, ds \right] \leq \|Q\| \left(\frac{1}{t} \bar{\mu}_\alpha(\mathbf{k},t) \right),$$

and therefore, $\hat{\omega}_{3,1,3} \in \mathcal{B}_{\alpha,1}$. Using Proposition 37 we find from (89), (98) and (104) that

$$|\hat{\omega}_{3,2,2}(\mathbf{k},t)| \leq \|Q\| \left[e^{\Lambda_-(t-1)} \int_t^\infty (k + k^{\frac{1}{2}}) e^{-\frac{k}{2}(s-1)} \, \mu(\mathbf{k},s) \, ds + e^{-k(t-1)} \int_t^\infty (k^{\frac{1}{2}} + k) e^{-\frac{k}{2}(s-1)} \mu(\mathbf{k},s) \, ds + |H_2| \int_t^\infty (k + k^{\frac{1}{2}}) e^{-\frac{k}{2}(s-1)} \mu(\mathbf{k},s) \, ds \right] \leq \|Q\| \left(\frac{1}{t^{\frac{3}{2}}} \bar{\mu}_\alpha(\mathbf{k},t) \right),$$

and therefore, $\hat{\omega}_{3,2,2} \in \mathcal{B}_{\alpha,\frac{3}{2}}$. Using again Proposition 37, again we find from (90), (99) and (105) that

$$|\hat{\omega}_{3,2,3}(\mathbf{k},t)| \leq \|Q\| \left[e^{\Lambda_-(t-1)} \int_t^\infty k e^{-\frac{k}{2}(s-1)} \, \mu(\mathbf{k},s) \, ds + e^{-k(t-1)} \int_t^\infty k e^{-\frac{k}{2}(s-1)} \, \mu(\mathbf{k},s) \, ds + |H_2| \int_t^\infty k e^{-\frac{k}{2}(s-1)} \, \mu(\mathbf{k},s) \, ds \right] \leq \|Q\| \left(\frac{1}{t^2} \bar{\mu}_\alpha(\mathbf{k},t) \right),$$

and therefore, $\hat{\omega}_{3,2,3} \in \mathcal{B}_{\alpha,2}$. From (91) and Proposition 33, we get

$$\begin{aligned} |\hat{\omega}_{3,3,1}(\mathbf{k},t)| &\leq \|Q\| \ |K_3| \int_t^\infty |\Lambda_-| \, e^{\Lambda_-(s-1)} \ \mu(\mathbf{k},s) \ ds \\ &\leq \|Q\| \ \left(\frac{1}{t^2}\bar{\mu}_\alpha(\mathbf{k},t)\right), \end{aligned}$$

and therefore, $\hat{\omega}_{3,3,1} \in \mathcal{B}_{\alpha,2}$. For $\hat{\omega}_{3,3,2}$, we have

$$\begin{aligned} |\hat{\omega}_{3,3,2}(\mathbf{k},t)| \leq \|Q\| \ &\left[|K_3| \int_t^\infty |\Lambda_-| \, e^{\Lambda_-(s-1)} \ \mu(\mathbf{k},s) \ ds \right. \\ &\left. + \ |G_3| \int_t^\infty |\Lambda_-| \, e^{\Lambda_-(s-1)} \ \mu(\mathbf{k},s) \ ds + |H_3| \int_t^\infty |\Lambda_-| \, e^{\Lambda_-(s-1)} \ \mu(\mathbf{k},s) \ ds \right] \\ &\leq \|Q\| \ \left(\frac{1}{t}\bar{\mu}_\alpha(\mathbf{k},t)\right). \end{aligned} \tag{108}$$

For the first term in (108), we use (92) and Proposition 33; for the second term, we use (100) and Proposition 33; for the third term, we use (106) and Proposition 34; and we get that $\hat{\omega}_{3,3,2} \in \mathcal{B}_{\alpha,1}$. Similarly,

$$\begin{aligned} |\hat{\omega}_{3,3,3}(\mathbf{k},t)| \leq \|Q\| \ &\left[|K_3| \int_t^\infty |\Lambda_-| \, e^{\Lambda_-(s-1)} \ \mu(\mathbf{k},s) \ ds \right. \\ &\left. + \ |G_3| \int_t^\infty |\Lambda_-| \, e^{\Lambda_-(s-1)} \ \mu(\mathbf{k},s) \ ds + |H_3| \int_t^\infty |\Lambda_-| \, e^{\Lambda_-(s-1)} \ \mu(\mathbf{k},s) \ ds \right] \\ &\leq \|Q\| \ \left(\frac{1}{t}\bar{\mu}_\alpha(\mathbf{k},t)\right), \end{aligned}$$

and therefore, $\hat{\omega}_{3,3,3} \in \mathcal{B}_{\alpha,1}$. □

4.2.4. Bounds for $\hat{u}_1$. For the integral kernels of $\hat{u}_1$ we have:

Proposition 19. *Let $f_{1,i,j}$ be as given in "Appendix A". Then we have the bounds*

$$|f_{1,1,1}(\mathbf{k},\sigma)| \leq \text{const.}(1+|\Lambda_-|)\sigma e^{|\Lambda_-|\sigma}, \tag{109}$$

$$|f_{1,1,2}(\mathbf{k},\sigma)| \leq \text{const.}|\Lambda_-|\sigma e^{|\Lambda_-|\sigma}, \tag{110}$$

$$|f_{1,1,3}(\mathbf{k},\sigma)| \leq \text{const.} \, |\Lambda_-| \, \sigma e^{|\Lambda_-|\sigma}, \tag{111}$$

$$|f_{1,2,1}(\mathbf{k},\sigma)| \leq \text{const.}(1+k)e^{-k\sigma}, \tag{112}$$

$$|f_{1,2,2}(\mathbf{k},\sigma)| \leq \text{const.}(k^{\frac{1}{2}}+k)\sigma e^{-k\sigma}, \tag{113}$$

$$|f_{1,2,3}(\mathbf{k},\sigma)| \leq \text{const.}(1+k\sigma)e^{-k\sigma}, \tag{114}$$

$$|f_{1,3,1}(\mathbf{k},\sigma)| \leq (1+|\Lambda_-|)e^{\Lambda_-\sigma}, \tag{115}$$

$$|f_{1,3,2}(\mathbf{k},\sigma)| \leq |\Lambda_-|e^{\Lambda_-\sigma}, \tag{116}$$

$$|f_{1,3,3}(\mathbf{k},\sigma)| \leq |\Lambda_-|e^{\Lambda_-\sigma}, \tag{117}$$

uniformly in $\sigma \geq 0$ and $\mathbf{k} \in \mathbb{R}^2$.

Proof. We rewrite (221) as follows,

$$\begin{aligned} f_{1,1,1}(\mathbf{k},\sigma) &= \frac{ik_1+1}{\kappa}\left(e^{\kappa\sigma}-e^{-\kappa\sigma}\right) + \frac{2\kappa(\kappa+k)}{k}\left(e^{-\kappa\sigma}-e^{-k\sigma}\right) \\ &= -2\left(ik_1+1\right)e^{\kappa\sigma}\sigma\frac{\left(1-e^{-2\kappa\sigma}\right)}{-2\kappa\sigma} + \frac{2i\kappa k_1}{k}\sigma e^{-k\sigma}\frac{\left(e^{(k-\kappa)\sigma}-1\right)}{(k-\kappa)\,\sigma}, \end{aligned}$$

from which we easily get that

$$|f_{1,1,1}(\mathbf{k},\sigma)| \leq \text{const.}(1+|\Lambda_-|)\sigma e^{|\Lambda_-|\sigma}.$$

This shows (109). From (222), we have

$$\begin{aligned} f_{1,1,2}(\mathbf{k},\sigma) &= \frac{ik_2}{\kappa}\left(e^{\kappa\sigma}-e^{-\kappa\sigma}\right)+\frac{2\kappa k_2(\kappa+k)}{kk_1}\left(e^{-\kappa\sigma}-e^{-k\sigma}\right) \\ &= -2ik_2\sigma e^{\kappa\sigma}\frac{\left(1-e^{-2\kappa\sigma}\right)}{-2\kappa\sigma}+\frac{2i\kappa k_2}{k}\sigma e^{-k\sigma}\frac{\left(e^{(k-\kappa)\sigma}-1\right)}{(k-\kappa)\,\sigma}, \end{aligned}$$

and therefore,

$$|f_{1,1,2}(\mathbf{k},\sigma)| \leq \text{const.}|\Lambda_-|\sigma e^{|\Lambda_-|\sigma},$$

which shows (110). From (223), we get

$$\begin{aligned} f_{1,1,3}(\mathbf{k},\sigma) &= \left(e^{\kappa\sigma}-e^{-\kappa\sigma}\right)-\frac{2i\kappa\,(k+\kappa)}{k_1}\left(e^{-k\sigma}-e^{-\kappa\sigma}\right) \\ &= -2\kappa\sigma e^{\kappa\sigma}\frac{\left(1-e^{-2\kappa\sigma}\right)}{-2\kappa\sigma}+2\kappa\sigma e^{-k\sigma}\frac{\left(1-e^{(k-\kappa)\sigma}\right)}{(k-\kappa)\,\sigma}, \end{aligned}$$

and therefore,

$$|f_{1,1,3}(\mathbf{k},\sigma)| \leq \text{const.}\,|\Lambda_-|\,\sigma e^{|\Lambda_-|\sigma}.$$

The same technique can be applied to (225) and (226). We get

$$\begin{aligned} f_{1,2,2}(\mathbf{k},\sigma) &= \frac{2\kappa k_2\,(\kappa+k)}{kk_1}\left(e^{-\kappa\sigma}-e^{-k\sigma}\right) \\ &= \frac{2i\kappa k_2}{k}\sigma e^{-k\sigma}\frac{\left(e^{(k-\kappa)\sigma}-1\right)}{(k-\kappa)\,\sigma}, \end{aligned}$$

and

$$\begin{aligned} f_{1,2,3}(\mathbf{k},\sigma) &= \frac{2ik(\kappa+k)}{k_1}e^{-\kappa\sigma}-\frac{2i\kappa(k+\kappa)}{k_1}e^{-k\sigma} \\ &= -2k\sigma e^{-k\sigma}\frac{\left(e^{(k-\kappa)\sigma}-1\right)}{(k-\kappa)\,\sigma}-2e^{-k\sigma}, \end{aligned}$$

and the bounds (113) and (114) follow. The remaining bounds (112) and (115)–(117) are trivial. □

Proposition 20. *Let $g_{1,i,j}$ be as given in "Appendix A". Then, we have the bounds*

$$|g_{1,1,1}(\mathbf{k},\sigma)| \leq \text{const.}(1+k)e^{k\sigma}\min\{1,(k^{\frac{1}{2}}+k)\sigma\}, \tag{118}$$

$$|g_{1,1,2}(\mathbf{k},\sigma)| \leq \text{const.}(k^{\frac{1}{2}}+k)\sigma e^{k\sigma}, \tag{119}$$

$$|g_{1,1,3}(\mathbf{k},\sigma)| \leq \text{const.}k\sigma e^{k\sigma}, \tag{120}$$

$$|g_{1,2,1}(\mathbf{k},\sigma)| \leq \text{const.}(1+k)e^{-k\sigma}, \tag{121}$$

$$|g_{1,2,2}(\mathbf{k},\sigma)| \leq \text{const.}(k^{\frac{1}{2}}+k)\sigma e^{-k\sigma}, \tag{122}$$

$$|g_{1,2,3}(\mathbf{k},\sigma)| \leq \text{const.}(1+k\sigma)e^{-k\sigma}, \tag{123}$$

$$|g_{1,3,1}(\mathbf{k},\sigma)| \leq ke^{-k\sigma}, \tag{124}$$

$$|g_{1,3,2}(\mathbf{k},\sigma)| \leq ke^{-k\sigma}, \tag{125}$$

$$|g_{1,3,3}(\mathbf{k},\sigma)| \leq ke^{-k\sigma}, \tag{126}$$

uniformly in $\sigma \geq 0$ and $\mathbf{k} \in \mathbb{R}^2$.

Proof. From (230), we get that

$$|g_{1,1,1}(\mathbf{k},\sigma)| = \left| -\frac{ik_1}{k}e^{k\sigma} + \frac{(\kappa+k)^2}{k}e^{-k\sigma} - \frac{2\kappa(\kappa+k)}{k}e^{-\kappa\sigma} \right| \\ \leq \text{const.}(1+k)e^{k\sigma}.$$

When one expands the exponential functions in (230), the first term cancels, so that

$$g_{1,1,1}(\mathbf{k},\sigma) = -\frac{ik_1}{k}\left(e^{k\sigma}-1\right) + \frac{(\kappa+k)^2}{k}\left(e^{-k\sigma}-1\right) - \frac{2\kappa(\kappa+k)}{k}\left(e^{-\kappa\sigma}-1\right),$$

and therefore,

$$|g_{1,1,1}(\mathbf{k},\sigma)| \leq (1+k)(k^{\frac{1}{2}}+k)\sigma e^{k\sigma}.$$

From (231) and (232), we get

$$g_{1,1,2}(\mathbf{k},\sigma) = \frac{k_2(\kappa+k)^2}{k_1 k}\left(e^{-k\sigma}-e^{-\kappa\sigma}\right) + \frac{ik_2}{k}\left(e^{-\kappa\sigma}-e^{k\sigma}\right) \\ = \frac{ik_2(\kappa+k)}{k}\sigma e^{-k\sigma}\frac{\left(1-e^{(k-\kappa)\sigma}\right)}{(k-\kappa)\,\sigma} - \frac{ik_2\,(k+\kappa)}{k}\sigma e^{k\sigma}\frac{\left(e^{-(k+\kappa)\sigma}-1\right)}{-\,(k+\kappa)\,\sigma},$$

and

$$g_{1,1,3}(\mathbf{k},\sigma) = -e^{k\sigma} + \frac{i(\kappa+k)^2}{k_1}e^{-k\sigma} - \frac{2ik(\kappa+k)}{k_1}e^{-\kappa\sigma} \\ = -2k\sigma e^{k\sigma}\frac{\left(e^{-2k\sigma}-1\right)}{-2k\sigma} + 2k\sigma e^{-k\sigma}\frac{\left(e^{(k-\kappa)\sigma}-1\right)}{(k-\kappa)\,\sigma},$$

and the bounds (119) and (120) follow. The remaining bounds (121)–(126) are similar to the bounds (112)–(117). □

As a consequence of Propositions 19 and 20, we have

Proposition 21. *Let $\alpha > 2$. Then, $\hat{\mathbf{Q}} \mapsto \hat{u}_1$ defines a continuous linear map from $\mathcal{W}_\alpha$ to $\mathcal{B}_{\alpha,0}$. More precisely, $\hat{\mathbf{Q}} \mapsto \hat{u}_{1,i,j}$, with $\hat{u}_{1,i,j}$ as given in (45), define continuous linear maps on $\mathcal{W}_\alpha$, with $\hat{u}_{1,1,1} \in \mathcal{B}_{\alpha,0}$, $\hat{u}_{1,1,2} \in \mathcal{B}_{\alpha,\frac{1}{2}}$, $\hat{u}_{1,1,3} \in \mathcal{B}_{\alpha,1}$, $\hat{u}_{1,2,i} \in \mathcal{B}_{\alpha,2}$, $i = 1,3$, $\hat{u}_{1,2,2} \in \mathcal{B}_{\alpha,\frac{3}{2}}$ and $\hat{u}_{1,3,1} \in \mathcal{B}_{\alpha,1}$, $\hat{u}_{1,3,i} \in \mathcal{B}_{\alpha,2}$, $i = 2,3$.*

Proof. From (109) and (118), We find with Propositions 31, 32, 35 and 36

$$|\hat{u}_{1,1,1}(\mathbf{k},t)| \leq \|Q\| \left[e^{\Lambda_-(t-1)}\int_1^t e^{|\Lambda_-|(s-1)}(1+|\Lambda_-|)(s-1)\;\mu(\mathbf{k},s)\;ds \right. \\ \left. + \; e^{-k(t-1)}\int_1^t (1+k)e^{k(s-1)}\min\{1,(k^{\frac{1}{2}}+k)(s-1)\}\;\mu(\mathbf{k},s)\;ds \right] \\ \leq \|Q\|\left(\bar{\mu}_\alpha(\mathbf{k},t)\right),$$

and therefore, $\hat{u}_{1,1,1} \in \mathcal{B}_{\alpha,0}$. From (110) and (119), Propositions 31, 32, 35 and 36 we get

$$|\hat{u}_{1,1,2}(\mathbf{k},t)| \leq \|Q\| \left[e^{\Lambda_-(t-1)}\int_1^t |\Lambda_-|\,(s-1)e^{|\Lambda_-|(s-1)}\;\mu(\mathbf{k},s)\;ds \right. \\ \left. + \; e^{-k(t-1)}\int_1^t (k^{\frac{1}{2}}+k)(s-1)e^{k(s-1)}\;\mu(\mathbf{k},s)\;ds \right] \\ \leq \|Q\|\left(\frac{1}{t^{\frac{1}{2}}}\bar{\mu}_\alpha(\mathbf{k},t)\right),$$

and therefore, $\hat{u}_{1,1,2} \in \mathcal{B}_{\alpha,\frac{1}{2}}$. Using (111), (120), Propositions 31, 32, 35, and 36 we get

$$\begin{aligned} |\hat{u}_{1,1,3}(\mathbf{k},t)| &\leq \|Q\| \left[e^{\Lambda_-(t-1)} \int_1^t |\Lambda_-|\,(s-1) e^{|\Lambda_-|(s-1)}\, \mu(\mathbf{k},s)\, ds \right. \\ &\quad \left. + e^{-k(t-1)} \int_1^t k(s-1) e^{k(s-1)}\, \mu(\mathbf{k},s)\, ds \right] \\ &\leq \|Q\| \left(\frac{1}{t} \bar{\mu}_\alpha(\mathbf{k},t) \right), \end{aligned}$$

and therefore, $\hat{u}_{1,1,3} \in \mathcal{B}_{\alpha,1}$. From (112), (121) and Proposition 37, we find that

$$\begin{aligned} |\hat{u}_{1,2,1}(\mathbf{k},t)| &\leq \|Q\| \left[e^{\Lambda_-(t-1)} \int_t^\infty (1+k) e^{-k(s-1)}\, \mu(\mathbf{k},s)\, ds \right. \\ &\quad \left. + e^{-k(t-1)} \int_t^\infty (1+k) e^{-k(s-1)}\, \mu(\mathbf{k},s)\, ds \right] \\ &\leq \|Q\| \left(\frac{1}{t^2} \bar{\mu}_\alpha(\mathbf{k},t) \right), \end{aligned}$$

and therefore, $\hat{u}_{1,2,1} \in \mathcal{B}_{\alpha,2}$. Similarly, we get from (113) and (122) with Proposition 37 that

$$\begin{aligned} |\hat{u}_{1,2,2}(\mathbf{k},t)| &\leq \|Q\| \left[e^{\Lambda_-(t-1)} \int_t^\infty (k^{\frac{1}{2}}+k)(s-1) e^{-k(s-1)}\, \mu(\mathbf{k},s)\, ds \right. \\ &\quad \left. + e^{-k(t-1)} \int_t^\infty (k^{\frac{1}{2}}+k)(s-1) e^{-k(s-1)}\, \mu(\mathbf{k},s)\, ds \right] \\ &\leq \|Q\| \left(\frac{1}{t^{\frac{3}{2}}} \bar{\mu}_\alpha(\mathbf{k},t) \right), \end{aligned}$$

and therefore, $\hat{u}_{1,2,2} \in \mathcal{B}_{\alpha,\frac{3}{2}}$. From (114), (123) and Proposition 37, we get

$$\begin{aligned} |\hat{u}_{1,2,3}(\mathbf{k},t)| &\leq \|Q\| \left[e^{\Lambda_-(t-1)} \int_t^\infty (1+k(s-1)) e^{-k(s-1)}\, \mu(\mathbf{k},s)\, ds \right. \\ &\quad \left. + e^{-k(t-1)} \int_t^\infty (1+k(s-1)) e^{-k(s-1)}\, \mu(\mathbf{k},s)\, ds \right] \\ &\leq \|Q\| \left(\frac{1}{t^2} \bar{\mu}_\alpha(\mathbf{k},t) \right), \end{aligned}$$

and therefore, $\hat{u}_{1,2,3} \in \mathcal{B}_{\alpha,2}$. Finally, using (115)–(117), (124)–(126), Propositions 33 and 37, we find that

$$\begin{aligned} |\hat{u}_{1,3,1}(\mathbf{k},t)| &\leq \|Q\| \left[|K_3| \int_t^\infty (1+|\Lambda_-|) e^{\Lambda_-(s-1)}\, \mu(\mathbf{k},s)\, ds + |G_3| \int_t^\infty k e^{-k(s-1)}\, \mu(\mathbf{k},s)\, ds \right] \\ &\leq \|Q\| \left(\frac{1}{t} \bar{\mu}_\alpha(\mathbf{k},t) \right), \end{aligned}$$

and therefore, $\hat{u}_{1,3,1} \in \mathcal{B}_{\alpha,1}$, and

$$\begin{aligned} |\hat{u}_{1,3,i}(\mathbf{k},t)| &\leq \|Q\| \left[|K_3| \int_t^\infty |\Lambda_-| e^{\Lambda_-(s-1)}\, \mu(\mathbf{k},s)\, ds + |G_3| \int_t^\infty k e^{-k(s-1)}\, \mu(\mathbf{k},s)\, ds \right] \\ &\leq \|Q\| \left(\frac{1}{t^2} \bar{\mu}_\alpha(\mathbf{k},t) \right), \quad i = 2, 3. \end{aligned}$$

and therefore, $\hat{u}_{1,3,i} \in \mathcal{B}_{\alpha,2}$, $i = 2, 3$. □

4.2.5. Bounds for $\hat{\boldsymbol{u}}_2$. For the integral kernels of $\hat{u}_2$, we have:

Proposition 22. *Let $f_{2,i,j}$ be as given in "Appendix A". Then we have the bounds*

$$|f_{2,1,1}(\mathbf{k},\sigma)| \leq \text{const.}(k^{\frac{1}{2}}+k)\sigma e^{|\Lambda_-|\sigma}, \tag{127}$$

$$|f_{2,1,2}(\mathbf{k},\sigma)| \leq \text{const.}(1+k\sigma)\sigma e^{|\Lambda_-|\sigma}, \tag{128}$$

$$|f_{2,1,3}(\mathbf{k},\sigma)| \leq \text{const.}\sigma e^{|\Lambda_-|\sigma}\min\{1,|\Lambda_-|\sigma\}, \tag{129}$$

$$|f_{2,2,1}(\mathbf{k},\sigma)| \leq \text{const.}(k^{\frac{1}{2}}+k)\sigma e^{-k\sigma}, \tag{130}$$

$$|f_{2,3,1}(\mathbf{k},\sigma)| \leq |\Lambda_-|e^{\Lambda_-\sigma}, \tag{131}$$

uniformly in $\sigma \geq 0$ and $\mathbf{k} \in \mathbb{R}^2$.

Proof. The bound (127) is simple, and similar to (119) and (130) is similar to (122). We now prove (128) and (129). From (240), we get

$$\begin{aligned}
f_{2,1,2}(\mathbf{k},\sigma) &= \frac{k_1+ik_2^2}{\kappa k_1}\left(e^{\kappa\sigma}-e^{-\kappa\sigma}\right)+\frac{ik_2^2}{kk_1}\left(e^{-k\sigma}-e^{k\sigma}\right)\\
&= \frac{1}{\kappa}\left(e^{\kappa\sigma}-e^{-\kappa\sigma}\right)+\frac{ik_2^2}{k_1}\left[\frac{1}{\kappa}\left(e^{\kappa\sigma}-e^{-\kappa\sigma}\right)-\frac{1}{k}\left(e^{k\sigma}-e^{-k\sigma}\right)\right]\\
&= \frac{1}{\kappa}\left(e^{\kappa\sigma}-e^{-\kappa\sigma}\right)+\frac{ik_2^2}{k_1}\int_{-\sigma}^{\sigma}\left(e^{\kappa s}-e^{ks}\right)ds\\
&= \frac{1}{\kappa}\left(e^{\kappa\sigma}-e^{-\kappa\sigma}\right)-\frac{k_2^2}{(k+\kappa)}\int_{-\sigma}^{\sigma}se^{\kappa s}\frac{\left(1-e^{(k-\kappa)s}\right)}{(k-\kappa)s}ds.
\end{aligned}$$

Therefore,

$$|f_{2,1,2}(\mathbf{k},\sigma)| \leq \text{const.}(1+k\sigma)\sigma e^{|\Lambda_-|\sigma}.$$

Similarly, we obtain from (241)

$$\begin{aligned}
|f_{2,1,3}(\mathbf{k},\sigma)| &= \left|\frac{k_2}{k_1}\left(e^{\kappa\sigma}-e^{k\sigma}\right)+\frac{k_2}{k_1}\left(e^{-\kappa\sigma}-e^{-k\sigma}\right)\right|\\
&= \left|\frac{ik_2}{k+\kappa}\sigma e^{\kappa\sigma}\frac{\left(1-e^{(k-\kappa)\sigma}\right)}{(k-\kappa)\sigma}+\frac{ik_2}{k+\kappa}\sigma e^{-k\sigma}\frac{\left(e^{(k-\kappa)\sigma}-1\right)}{(k-\kappa)\sigma}\right|\\
&\leq \text{const.}\sigma e^{|\Lambda_-|\sigma}.
\end{aligned}$$

We also have,

$$\begin{aligned}
f_{2,1,3}(\mathbf{k},\sigma) &= \frac{k_2}{k_1}\left(e^{\kappa\sigma}-e^{k\sigma}+e^{-\kappa\sigma}-e^{-k\sigma}\right)\\
&= \frac{k_2}{k_1}\left(e^{\frac{\kappa+k}{2}\sigma}-e^{\frac{-\kappa-k}{2}\sigma}\right)\left(e^{\frac{\kappa-k}{2}\sigma}-e^{\frac{k-\kappa}{2}\sigma}\right)\\
&= -ik_2\sigma^2 e^{\kappa\sigma}\frac{\left(1-e^{-(\kappa+k)\sigma}\right)}{-(\kappa+k)\sigma}\frac{\left(1-e^{(k-\kappa)\sigma}\right)}{(k-\kappa)\sigma},
\end{aligned}$$

and therefore,

$$\begin{aligned}
|f_{2,1,3}(\mathbf{k},\sigma)| &\leq \left|-ik_2\sigma^2 e^{\kappa\sigma}\frac{\left(1-e^{-(\kappa+k)\sigma}\right)}{-(\kappa+k)\sigma}\frac{\left(1-e^{(k-\kappa)\sigma}\right)}{(k-\kappa)\sigma}\right|\\
&\leq \text{const.}|\Lambda_-|\sigma^2 e^{|\Lambda_-|\sigma}.
\end{aligned}$$

This shows (129). The bound (131) is obvious. □

Proposition 23. *Let $g_{2,i,j}$ be as given in "Appendix A". Then we have the bounds*

$$|g_{2,1,1}(\mathbf{k},\sigma)| \leq \text{const.}(k^{\frac{1}{2}}+k)\sigma e^{k\sigma}, \tag{132}$$

$$|g_{2,2,1}(\mathbf{k},\sigma)| \leq \text{const.}(k^{\frac{1}{2}}+k)\sigma e^{-k\sigma}, \tag{133}$$

$$|g_{2,3,1}(\mathbf{k},\sigma)| \leq ke^{-k\sigma}, \tag{134}$$

$$|g_{2,3,2}(\mathbf{k},\sigma)| \leq \text{const.}(1+k\sigma)e^{-k\sigma}, \tag{135}$$

$$|g_{2,3,3}(\mathbf{k},\sigma)| \leq \text{const.}(1+k\sigma)e^{-k\sigma}, \tag{136}$$

uniformly in $\sigma \geq 0$ and $\mathbf{k} \in \mathbb{R}^2$.

Proof. The proof of (132) is identical to the one of (119). From (245), we have

$$\begin{aligned} |g_{2,2,1}(\mathbf{k},\sigma)| &= \left| \frac{2i\kappa k_2}{k} \sigma e^{-k\sigma} \frac{\left(e^{(k-\kappa)\sigma}-1\right)}{(k-\kappa)\,\sigma} \right| \\ &\leq \text{const.}(k^{\frac{1}{2}}+k)\sigma e^{-k\sigma}. \end{aligned}$$

This shows (133). For (135), we have

$$\begin{aligned} |g_{2,3,2}(\mathbf{k},\sigma)| &= \left| \frac{1}{1+k} \left[\frac{-k_2^2}{(k+\kappa)} \sigma e^{-k\sigma} \frac{\left(e^{(k-\kappa)\sigma}-1\right)}{(k-\kappa)\,\sigma} + e^{-\kappa\sigma} \right] \right| \\ &\leq \text{const.}(1+k\sigma)e^{-k\sigma}, \end{aligned}$$

which shows (135). From (248), we get

$$\begin{aligned} |g_{2,3,3}(\mathbf{k},\sigma)| &= \left| \frac{1}{1+k} \left[\frac{ikk_2}{(k+\kappa)} \sigma e^{-k\sigma} \frac{\left(1-e^{(k-\kappa)\sigma}\right)}{(k-\kappa)\,\sigma} + \frac{ik_2}{(k+\kappa)} e^{-\kappa\sigma} \right] \right| \\ &\leq \text{const.}(1+k\sigma)e^{-k\sigma}, \end{aligned}$$

which is the bound (136). The bound (134) is obvious. □

Proposition 24. *Let $h_{2,i,j}$ be as given in "Appendix A". Then we have the bounds*

$$|h_{2,1,2}(\mathbf{k},\sigma)| \leq \text{const.}k\sigma e^{k\sigma}, \tag{137}$$

$$|h_{2,1,3}(\mathbf{k},\sigma)| \leq \text{const.}\min\{1,(k^{\frac{1}{2}}+k)\sigma\}k\sigma e^{k\sigma}, \tag{138}$$

$$|h_{2,2,2}(\mathbf{k},\sigma)| \leq \text{const.}k\sigma e^{-k\sigma}, \tag{139}$$

$$|h_{2,2,3}(\mathbf{k},\sigma)| \leq \text{const.}(1+k\sigma)e^{-k\sigma}, \tag{140}$$

$$|h_{2,3,2}(\mathbf{k},\sigma)| \leq \text{const.}\left(\frac{|k_1|}{k\,(1+k)} + |\Lambda_-|\right) e^{\Lambda_-\sigma}, \tag{141}$$

$$|h_{2,3,3}(\mathbf{k},\sigma)| \leq \text{const.}\min\{1,|\Lambda_-|\}e^{\Lambda_-\sigma}, \tag{142}$$

uniformly in $\sigma \geq 0$ and $\mathbf{k} \in \mathbb{R}^2$.

Proof. The bounds (139) and (140) are similar to (133) and (135). We now prove (137) and (138). From (249), we have

$$\begin{aligned} h_{2,1,2}(\mathbf{k},\sigma) &= \frac{ik_2^2}{k\,(\kappa+k)}\left(e^{k\sigma}-e^{-\kappa\sigma}\right) - \frac{k_2^2(\kappa+k)}{kk_1}\left(e^{-k\sigma}-e^{-\kappa\sigma}\right) \\ &= -\frac{ik_2^2}{k}\sigma e^{k\sigma}\frac{\left(1-e^{-(k+\kappa)\sigma}\right)}{-(k+\kappa)\,\sigma} - \frac{ik_2^2}{k}\sigma e^{-k\sigma}\frac{\left(1-e^{(k-\kappa)\sigma}\right)}{(k-\kappa)\,\sigma}\ , \end{aligned}$$

and therefore,

$$|h_{2,1,2}(\mathbf{k},\sigma)| \leq \text{const.} k\sigma e^{k\sigma}.$$

A similar argument applied to (250) yields the bound

$$|h_{2,1,3}(\mathbf{k},\sigma)| \leq \text{const.} k\sigma e^{k\sigma}. \tag{143}$$

On the other hand, we have for (250) that

$$\begin{aligned}
h_{2,1,3}(\mathbf{k},\sigma) &= \frac{k_2}{\kappa + k}\left(e^{k\sigma} - e^{-\kappa\sigma}\right) + \frac{ik_2(k+\kappa)}{k_1}\left(e^{-\kappa\sigma} - e^{-k\sigma}\right) \\
&= -\sigma k_2 e^{k\sigma}\frac{\left(1 - e^{-(k+\kappa)\sigma}\right)}{-(k+\kappa)\sigma} - k_2\sigma e^{-k\sigma}\frac{\left(e^{(k-\kappa)\sigma} - 1\right)}{(k-\kappa)\,\sigma} \\
&= \sigma k_2 e^{k\sigma}\left[-(k+\kappa)\sigma\frac{e^{-(k+\kappa)\sigma} - 1 + (k+\kappa)\sigma}{(k+\kappa)^2\sigma^2}\right] \\
&\quad - k_2\sigma e^{-k\sigma}\left[(k-\kappa)\,\sigma\frac{e^{(k-\kappa)\sigma} - 1 - (k-\kappa)\,\sigma}{(k-\kappa)^2\sigma^2}\right] \\
&\quad + k_2 k\sigma^2\frac{\left(e^{k\sigma} - e^{-k\sigma}\right)}{k\sigma}.
\end{aligned}$$

Therefore,

$$|h_{2,1,3}(\mathbf{k},\sigma)| \leq \text{const.} k(k^{\frac{1}{2}} + k)\sigma^2 e^{k\sigma}. \tag{144}$$

The combination of (143) and (144) gives (138). The bounds (141) and (142) are obvious. □

As a consequence of Propositions 22–24, we have

Proposition 25. *Let $\alpha > 2$. Then, $\hat{\mathbf{Q}} \mapsto \hat{u}_2$ defines a continuous linear map from $\mathcal{W}_\alpha$ to $\mathcal{B}_{\alpha,0}$. More precisely, $\hat{\mathbf{Q}} \mapsto \hat{u}_{2,i,j}$, with $\hat{u}_{2,i,j}$ as given in (45), define continuous linear maps on $\mathcal{W}_\alpha$, with $\hat{u}_{2,1,1} \in \mathcal{B}_{\alpha,\frac{1}{2}}$, $\hat{u}_{2,1,2} \in \mathcal{B}_{\alpha,0}$, $\hat{u}_{2,1,3} \in \mathcal{B}_{\alpha,\frac{1}{2}-\varepsilon}$, $\hat{u}_{2,2,1} \in \mathcal{B}_{\alpha,\frac{3}{2}}$, $\hat{u}_{2,2,2} \in \mathcal{B}_{\alpha,1}$, $\hat{u}_{2,2,3} \in \mathcal{B}_{\alpha,1}$, $\hat{u}_{2,3,1} \in \mathcal{B}_{\alpha,2}$ and $\hat{u}_{2,3,i} \in \mathcal{B}_{\alpha,1}$, $i = 2, 3$.*

Proof. Using (127), (132), Propositions 31, 32, 35, and 36, we find that

$$\begin{aligned}
|\hat{u}_{2,1,1}(\mathbf{k},t)| &\leq \|Q\| \left[e^{\Lambda_-(t-1)}\int_1^t (k^{\frac{1}{2}} + k)(s-1)e^{|\Lambda_-|(s-1)}\,\mu(\mathbf{k},s)\,ds\right. \\
&\quad \left. + e^{-k(t-1)}\int_1^t (k^{\frac{1}{2}} + k)(s-1)e^{k(s-1)}\,\mu(\mathbf{k},s)\,ds\right] \\
&\leq \|Q\|\left(\frac{1}{t^{\frac{1}{2}}}\bar{\mu}_\alpha(\mathbf{k},t)\right),
\end{aligned}$$

and therefore, $\hat{u}_{2,1,1} \in \mathcal{B}_{\alpha,\frac{1}{2}}$. Similarly, we get from (128), (137), Propositions 31, 32, 35, and 36 that

$$\begin{aligned}
|\hat{u}_{2,1,2}(\mathbf{k},t)| &\leq \|Q\| \left[e^{\Lambda_-(t-1)}\int_1^t (1 + k(s-1))(s-1)e^{|\Lambda_-|(s-1)}\,\mu(\mathbf{k},s)\,ds\right. \\
&\quad \left. + |H_1|\int_1^t k(s-1)e^{k(s-1)}\,\mu(\mathbf{k},s)\,ds\right] \\
&\leq \|Q\|\left(\bar{\mu}_\alpha(\mathbf{k},t)\right),
\end{aligned}$$

and therefore, $\hat{u}_{2,1,2} \in \mathcal{B}_{\alpha,0}$. From (129), (138), Propositions 31, 32, 35, and 36, we get

$$\begin{aligned} |\hat{u}_{2,1,3}(\mathbf{k},t)| &\leq \|Q\| \left[e^{\Lambda_-(t-1)} \int_1^t (s-1) e^{|\Lambda_-|(s-1)} \min\{1, |\Lambda_-|(s-1)\}\; \mu(\mathbf{k},s)\; ds \right. \\ &\quad \left. +\; |H_1| \int_1^t \min\{1, (k^{\frac{1}{2}}+k)(s-1)\} k(s-1) e^{k(s-1)} \mu(\mathbf{k},s)\; ds \right] \\ &\leq \|Q\| \left(\frac{1}{t^{\frac{1}{2}-\varepsilon}} \bar{\mu}_\alpha(\mathbf{k},t) \right), \quad \text{for } \forall \varepsilon > 0. \end{aligned}$$

and therefore, $\hat{u}_{2,1,3} \in \mathcal{B}_{\alpha,\frac{1}{2}-\varepsilon}$. From (130), (133) and Proposition 37, we have

$$\begin{aligned} |\hat{u}_{2,2,1}(\mathbf{k},t)| &\leq \|Q\| \left[e^{\Lambda_-(t-1)} \int_t^\infty (k^{\frac{1}{2}}+k)(s-1) e^{-k(s-1)}\; \mu(\mathbf{k},s)\; ds \right. \\ &\quad \left. +\; e^{-k(t-1)} \int_t^\infty (k^{\frac{1}{2}}+k)(s-1) e^{-k(s-1)}\; \mu(\mathbf{k},s)\; ds \right] \\ &\leq \|Q\| \left(\frac{1}{t^{\frac{3}{2}}} \bar{\mu}_\alpha(\mathbf{k},t) \right), \end{aligned}$$

and therefore, $\hat{u}_{2,2,1} \in \mathcal{B}_{\alpha,\frac{3}{2}}$. Using (139) and Proposition 37, we find that

$$\begin{aligned} |\hat{u}_{2,2,2}(\mathbf{k},t)| &\leq \|Q\| \left[|H_2| \int_t^\infty k(s-1) e^{-k(s-1)}\; \mu(\mathbf{k},s)\; ds \right] \\ &\leq \|Q\| \left(\frac{1}{t} \bar{\mu}_\alpha(\mathbf{k},t) \right), \end{aligned}$$

and therefore, $\hat{u}_{2,2,2} \in \mathcal{B}_{\alpha,1}$. Using (140) and Proposition 37, we find that

$$\begin{aligned} |\hat{u}_{2,2,3}(\mathbf{k},t)| &\leq \|Q\| \left[|H_2| \int_t^\infty (1+k(s-1)) e^{-k(s-1)}\; \mu(\mathbf{k},s)\; ds \right] \\ &\leq \|Q\| \left(\frac{1}{t} \bar{\mu}_\alpha(\mathbf{k},t) \right), \end{aligned}$$

and therefore $\hat{u}_{2,2,3} \in \mathcal{B}_{\alpha,1}$. Finally, (131), (134), Propositions 33, and 37 give

$$\begin{aligned} |\hat{u}_{2,3,1}(\mathbf{k},t)| &\leq \|Q\| \left[|K_3| \int_t^\infty |\Lambda_-|\, e^{\Lambda_-(s-1)}\; \mu(\mathbf{k},s)\; ds + |G_3| \int_t^\infty k e^{-k(s-1)}\; \mu(\mathbf{k},s)\; ds \right] \\ &\leq \|Q\| \left(\frac{1}{t^2} \bar{\mu}_\alpha(\mathbf{k},t) \right), \end{aligned}$$

and therefore, $\hat{u}_{2,3,1} \in \mathcal{B}_{\alpha,2}$. Similar arguments using (135) and (141) in combination with Propositions 34 and 37 yield

$$\begin{aligned} |\hat{u}_{2,3,2}(\mathbf{k},t)| &\leq \|Q\| \left[|G_3| \int_t^\infty (1+k(s-1)) e^{-k(s-1)}\; \mu(\mathbf{k},s)\; ds \right. \\ &\quad \left. + \left| \frac{1}{1+k}(K_3 - G_3) \right| \int_t^\infty e^{\Lambda_-(s-1)}\; \mu(\mathbf{k},s)\; ds + |H_3| \int_t^\infty |\Lambda_-|\, e^{\Lambda_-(s-1)}\; \mu(\mathbf{k},s)\; ds \right] \\ &\leq \|Q\| \left(\frac{1}{t} \bar{\mu}_\alpha(\mathbf{k},t) \right), \end{aligned}$$

and therefore, $\hat{u}_{2,3,2} \in \mathcal{B}_{\alpha,1}$. Finally, we get from (136), (142), Propositions 34 and 37 that

$$|\hat{u}_{2,3,3}(\mathbf{k},t)| \leq \|Q\| \left[|G_3| \int_t^\infty (1+k(s-1))e^{-k(s-1)}\ \mu(\mathbf{k},s)\ ds + |H_3| \int_t^\infty \min\{1,|\Lambda_-|\} e^{\Lambda_-(s-1)}\ \mu(\mathbf{k},s)\ ds\right]$$
$$\leq \|Q\| \left(\frac{1}{t}\bar{\mu}_\alpha(\mathbf{k},t)\right),$$

and therefore, $\hat{u}_{2,3,3} \in \mathcal{B}_{\alpha,1}$. □

4.2.6. Bounds for $\hat{u}_3$. For the integral kernels of $\hat{u}_3$, we have:

Proposition 26. *Let $f_{3,i,j}$ be as given in "Appendix A". Then we have the bounds*

$$|f_{3,1,1}(\mathbf{k},\sigma)| \leq \text{const.}\,|\Lambda_-|\,\sigma e^{|\Lambda_-|\sigma}, \tag{145}$$
$$|f_{3,1,2}(\mathbf{k},\sigma)| \leq \text{const.}\sigma e^{|\Lambda_-|\sigma} \min\{1,|\Lambda_-|\sigma\}, \tag{146}$$
$$|f_{3,1,3}(\mathbf{k},\sigma)| \leq \text{const.}k\sigma^2 e^{|\Lambda_-|\sigma}, \tag{147}$$
$$|f_{3,2,1}(\mathbf{k},\sigma)| \leq \text{const.}(1+k\sigma)e^{-k\sigma}, \tag{148}$$
$$|f_{3,3,1}(\mathbf{k},\sigma)| \leq |\Lambda_-|e^{\Lambda_-\sigma}, \tag{149}$$

uniformly in $\sigma \geq 0$ and $\mathbf{k} \in \mathbb{R}^2$.

Proof. From (255), we obtain

$$|f_{3,1,1}(\mathbf{k},\sigma)| = \left|-2\kappa\sigma e^{\kappa\sigma}\frac{\left(1-e^{-2\kappa\sigma}\right)}{-2\kappa\sigma} - 2k\sigma e^{-k\sigma}\frac{\left(1-e^{(k-\kappa)\sigma}\right)}{(k-\kappa)\,\sigma}\right|$$
$$\leq \text{const.}\,|\Lambda_-|\,\sigma e^{|\Lambda_-|\sigma}.$$

This shows (145). From (257), we get

$$f_{3,1,3}(\mathbf{k},\sigma) = \frac{ik}{k_1}\left(e^{k\sigma}-e^{-k\sigma}\right) + \frac{ik^2}{\kappa k_1}\left(e^{-\kappa\sigma}-e^{\kappa\sigma}\right)$$
$$= \frac{ik^2}{k_1}\left[\int_{-\sigma}^{\sigma}\left(e^{ks}-e^{\kappa s}\right)ds\right]$$
$$= -\frac{k^2}{(k+\kappa)}\int_{-\sigma}^{\sigma} se^{\kappa s}\frac{\left(e^{(k-\kappa)s}-1\right)}{(k-\kappa)\,s}ds,$$

which gives

$$|f_{3,1,3}| \leq \text{const.}k\sigma^2 e^{|\Lambda_-|\sigma},$$

which shows (147). The bounds (146) and (148) are similar to (129) and (123), respectively. The bound (149) is obvious. □

Proposition 27. *Let $g_{3,i,j}$ be as given in "Appendix A". Then we have the bounds*

$$|g_{3,1,1}(\mathbf{k},\sigma)| \leq \text{const.}(k^{\frac{1}{2}}+k)\sigma e^{k\sigma}, \tag{150}$$
$$|g_{3,2,1}(\mathbf{k},\sigma)| \leq \text{const.}\,(1+k\sigma)\,e^{-k\sigma}, \tag{151}$$
$$|g_{3,3,1}(\mathbf{k},\sigma)| \leq ke^{-k\sigma}, \tag{152}$$
$$|g_{3,3,2}(\mathbf{k},\sigma)| \leq \text{const.}(1+k\sigma)e^{-k\sigma}, \tag{153}$$
$$|g_{3,3,3}(\mathbf{k},\sigma)| \leq \text{const.}k\sigma e^{-k\sigma}, \tag{154}$$

uniformly in $\sigma \geq 0$ and $\mathbf{k} \in \mathbb{R}^2$.

Proof. First, we prove the bound (150). From (260), we have

$$|g_{3,1,1}(\mathbf{k},\sigma)| = \left| \frac{2i\kappa(\kappa+k)}{k_1} \left(e^{-\kappa\sigma} - e^{-k\sigma}\right) + \left(e^{-k\sigma} - e^{k\sigma}\right) \right|$$
$$= \left| 2\kappa\sigma e^{-k\sigma} \frac{\left(e^{(k-\kappa)\sigma} - 1\right)}{(k-\kappa)\,\sigma} - 2k\sigma e^{k\sigma} \frac{\left(e^{-2k\sigma} - 1\right)}{-2k\sigma} \right|$$
$$\leq \text{const.}(k^{\frac{1}{2}} + k)\sigma e^{k\sigma} \ ,$$

and therefore, we get the bound (150). The bounds (151), (153) and (154) are similar to (123), (136) and (133), respectively. The bound (152) is obvious. □

Proposition 28. *Let $h_{3,i,j}$ be as given in "Appendix A". Then we have the bounds*

$$|h_{3,1,2}(\mathbf{k},\sigma)| \leq \text{const.} k\sigma e^{k\sigma}, \tag{155}$$
$$|h_{3,1,3}(\mathbf{k},\sigma)| \leq \text{const.} \min\{1, (k^{\frac{1}{2}} + k)\sigma\} k\sigma e^{k\sigma}, \tag{156}$$
$$|h_{3,2,2}(\mathbf{k},\sigma)| \leq \text{const.}(1 + k\sigma)e^{-k\sigma}, \tag{157}$$
$$|h_{3,2,3}(\mathbf{k},\sigma)| \leq \text{const.} k\sigma e^{-k\sigma}, \tag{158}$$
$$|h_{3,3,2}(\mathbf{k},\sigma)| \leq \text{const.} \min\{1, |\Lambda_-|\} e^{\Lambda_- \sigma} \ , \tag{159}$$
$$|h_{3,3,3}(\mathbf{k},\sigma)| \leq \text{const.} \min\{1, |\Lambda_-|\} e^{\Lambda_- \sigma}, \tag{160}$$

uniformly in $\sigma \geq 0$ and $\mathbf{k} \in \mathbb{R}^2$.

Proof. The bounds (157) and (158) are similar to (140) and (133). The bound (156) is similar to (138). The bounds (159) and (160) are obvious. We now prove the bound (155). From (265) we get

$$|h_{3,1,2}(\mathbf{k},\sigma)| = \left| -k_2\sigma e^{k\sigma} \frac{\left(1 - e^{-(k+\kappa)\sigma}\right)}{-(k+\kappa)\,\sigma} - k_2\sigma e^{-k\sigma} \frac{\left(1 - e^{(k-\kappa)\sigma}\right)}{(k-\kappa)\,\sigma} \right|$$
$$\leq \text{const.} k\sigma e^{k\sigma},$$

which shows (155). □

As a consequence of Propositions 26–28, we have

Proposition 29. *Let $\alpha > 2$. Then, $\hat{\mathbf{Q}} \mapsto \hat{u}_3$ defines a continuous linear map from $\mathcal{W}_\alpha$ to $\mathcal{B}_{\alpha,0}$. More precisely, $\hat{\mathbf{Q}} \mapsto \hat{u}_{3,i,j}$, with $\hat{u}_{3,i,j}$ as given in (45), defines continuous linear maps on $\mathcal{W}_\alpha$, with $\hat{u}_{3,1,1} \in \mathcal{B}_{\alpha,\frac{1}{2}}$, $\hat{u}_{3,1,2} \in \mathcal{B}_{\alpha,0}$, $\hat{u}_{3,1,3} \in \mathcal{B}_{\alpha,\frac{1}{2}-\varepsilon}$, $hatu_{3,2,1} \in \mathcal{B}_{\alpha,2}$, $\hat{u}_{3,2,i} \in \mathcal{B}_{\alpha,1}$, $i = 2, 3$, and $\hat{u}_{3,3,1} \in \mathcal{B}_{\alpha,2}$, $\hat{u}_{3,3,i} \in \mathcal{B}_{\alpha,1}$, $i = 2, 3$.*

Proof. From (145), (150), Propositions 31, 32, 35 and 36, we get that

$$|\hat{u}_{3,1,1}(\mathbf{k},t)| \leq \|Q\| \ \left[e^{\Lambda_-(t-1)} \int_1^t |\Lambda_-|(s-1)e^{|\Lambda_-|(s-1)} \ \mu(\mathbf{k},s) \ ds \right.$$
$$\left. + \ e^{-k(t-1)} \int_1^t (k^{\frac{1}{2}} + k)(s-1)e^{k(s-1)} \ \mu(\mathbf{k},s) \ ds \right]$$
$$\leq \|Q\| \left(\frac{1}{t^{\frac{1}{2}}} \bar{\mu}_\alpha(\mathbf{k},t) \right),$$

and therefore, $\hat{u}_{3,1,1} \in \mathcal{B}_{\alpha,\frac{1}{2}}$. From (146), (155), Propositions 31, 32, 35 and 36, we get that

$$\begin{aligned}|\hat{u}_{3,1,2}(\mathbf{k},t)| \leq \|Q\| \; &\left[e^{\Lambda_-(t-1)}\int_1^t \min\{1,|\Lambda_-|(s-1)\}(s-1)e^{|\Lambda_-|(s-1)}\;\mu(\mathbf{k},s)\;ds\right.\\&\left.+\;|H_1|\int_1^t k(s-1)e^{k(s-1)}\;\mu(\mathbf{k},s)\;ds\right]\\&\leq \|Q\|\left(\bar{\mu}_\alpha(\mathbf{k},t)\right),\end{aligned}$$

and therefore, $\hat{u}_{3,1,2} \in \mathcal{B}_{\alpha,0}$. Similarly, we get from (147) and (156) with Propositions 31, 32, 35 and 36 that

$$\begin{aligned}|\hat{u}_{3,1,3}(\mathbf{k},t)| \leq \|Q\| \; &\left[e^{\Lambda_-(t-1)}\int_1^t k(s-1)^2e^{|\Lambda_-|(s-1)}\;\mu(\mathbf{k},s)\;ds\right.\\&\left.+\;|H_1|\int_1^t \min\{1,(k^{\frac{1}{2}}+k)(s-1)\}k(s-1)e^{k(s-1)}\;\mu(\mathbf{k},s)\;ds\right]\\&\leq \|Q\|\left(\frac{1}{t^{\frac{1}{2}-\varepsilon}}\bar{\mu}_\alpha(\mathbf{k},t)\right),\end{aligned}$$

and therefore, $\hat{u}_{3,1,3} \in \mathcal{B}_{\alpha,\frac{1}{2}-\varepsilon}$. From (148), (151) and Proposition 37, we find that

$$\begin{aligned}|\hat{u}_{3,2,1}(\mathbf{k},t)| \leq \|Q\| \; &\left[e^{\Lambda_-(t-1)}\int_t^\infty (1+k(s-1))e^{-k(s-1)}\;\mu(\mathbf{k},s)\;ds\right.\\&\left.+\;e^{-k(t-1)}\int_t^\infty (1+k(s-1))e^{-k(s-1)}\;\mu(\mathbf{k},s)\;ds\right]\\&\leq \|Q\|\left(\frac{1}{t^2}\bar{\mu}_\alpha(\mathbf{k},t)\right),\end{aligned}$$

and therefore, $\hat{u}_{3,2,1} \in \mathcal{B}_{\alpha,2}$. Using (157) and Proposition 37, we get

$$\begin{aligned}|\hat{u}_{3,2,2}(\mathbf{k},t)| \leq \|Q\| \; &\left[|H_2|\int_t^\infty (1+k(s-1))e^{-k(s-1)}\;\mu(\mathbf{k},s)\;ds\right]\\&\leq \|Q\|\left(\frac{1}{t}\bar{\mu}_\alpha(\mathbf{k},t)\right),\end{aligned}$$

and therefore, $\hat{u}_{3,2,2} \in \mathcal{B}_{\alpha,1}$. Using (158) and Proposition 37, we get

$$\begin{aligned}|\hat{u}_{3,2,3}(\mathbf{k},t)| \leq \|Q\| \; &\left[\;|H_2|\int_t^\infty k(s-1)e^{-k(s-1)}\;\mu(\mathbf{k},s)\;ds\right]\\&\leq \|Q\|\left(\frac{1}{t}\bar{\mu}_\alpha(\mathbf{k},t)\right),\end{aligned}$$

and therefore, $\hat{u}_{3,2,3} \in \mathcal{B}_{\alpha,1}$. Next, we find from (149), (152), Propositions 33 and 37,

$$\begin{aligned}|\hat{u}_{3,3,1}(\mathbf{k},t)| \leq \|Q\| \; &\left[|K_3|\int_t^\infty |\Lambda_-|\,e^{\Lambda_-(s-1)}\;\mu(\mathbf{k},s)\;ds+|G_3|\int_t^\infty ke^{-k(s-1)}\;\mu(\mathbf{k},s)\;ds\right]\\&\leq \|Q\|\left(\frac{1}{t^2}\bar{\mu}_\alpha(\mathbf{k},t)\right),\end{aligned}$$

and therefore, $\hat{u}_{3,3,1} \in \mathcal{B}_{\alpha,2}$. Similar arguments show using (153), (159), Propositions 34 and 37 that

$$|\hat{u}_{3,3,2}(\mathbf{k},t)| \leq \|Q\| \left[|G_3| \int_t^{\infty} (1+k(s-1))e^{-k(s-1)}\mu(\mathbf{k},s)\, ds \right.$$
$$\left. + |H_3| \int_t^{\infty} \min\{1,|\Lambda_-|\}e^{\Lambda_-(s-1)}\, \mu(\mathbf{k},s)\, ds \right] \leq \|Q\| \left(\frac{1}{t}\bar{\mu}_\alpha(\mathbf{k},t)\right),$$

and therefore, $\hat{u}_{3,3,2} \in \mathcal{B}_{\alpha,1}$. Finally, we get from (154), (160), Propositions 34 and 37 that

$$|\hat{u}_{3,3,3}(\mathbf{k},t)| \leq \|Q\| \left[|G_3| \int_t^{\infty} k(s-1))e^{-k(s-1)}\mu(\mathbf{k},s)\, ds + |H_3| \int_t^{\infty} \min\{1,|\Lambda_-|\}e^{\Lambda_-(s-1)}\mu(\mathbf{k},s)\, ds\right]$$
$$\leq \|Q\| \left(\frac{1}{t}\bar{\mu}_\alpha(\mathbf{k},t)\right),$$

and therefore, $\hat{u}_{3,3,3} \in \mathcal{B}_{\alpha,1}$. □

This completes the proof of Lemma 6.

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Appendix A: Derivation of the integral equations

Let $\mathbf{k} = (k_1, k_2) \in \mathbb{R}^2$. We have, see (37),

$$k = \sqrt{k_1^2 + k_2^2}, \quad \kappa = \sqrt{k^2 - ik_1}.$$

In order to derive the integral equations (44) and (45), we note that the Eqs. (25)–(33) are of the form $\partial_z \mathbf{U} = L\mathbf{U} + \mathbf{\Gamma}$, with $\mathbf{U} = (\hat{\omega}_1, \hat{\omega}_2, \hat{\eta}_{1,1}, \hat{\eta}_{1,2}, \hat{\eta}_{2,1}, \hat{\eta}_{2,2}, \hat{u}_1, \hat{u}_2, \hat{u}_3)^T$, $\mathbf{\Gamma} = \left(\hat{Q}_2, -\hat{Q}_1, 0, -\hat{Q}_3, \hat{Q}_3, 0, 0, 0, 0\right)^T$, with

$$L = \begin{pmatrix} L_1 & 0 \\ L_3 & L_2 \end{pmatrix},$$

with L_1 a 6×6 matrix, L_2 a 3×3 matrix, L_3 a 3×6 matrix and 0 the 6×3 zero matrix. Explicitly, we have

$$L_1 = \begin{pmatrix} 0 & 0 & -ik_1 & -ik_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -ik_1 & -ik_2 \\ ik_1+1 & 0 & 0 & 0 & 0 & 0 \\ ik_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & ik_1+1 & 0 & 0 & 0 & 0 \\ 0 & ik_2 & 0 & 0 & 0 & 0 \end{pmatrix},$$

and

$$L_2 = \begin{pmatrix} 0 & 0 & -ik_1 \\ 0 & 0 & -ik_2 \\ ik_1 & ik_2 & 0 \end{pmatrix}, \quad L_3 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The matrix L can be diagonalized (see the "Appendix C" for details). One gets that $SLS^{-1} = D$, with S and D matrices which have the same block structure as L,

$$S = \begin{pmatrix} S_1 & 0 \\ S_3 & S_2 \end{pmatrix}, \ D = \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix},$$

with $\mathrm{diag}(D_1) = (0, 0, \kappa, \kappa, -\kappa, -\kappa)$, $\mathrm{diag}(D_2) = (0, k, -k)$, with

$$S_1 = \begin{pmatrix} 0 & 0 & -\frac{i\kappa}{k_2} & 0 & \frac{i\kappa}{k_2} & 0 \\ 0 & 0 & 0 & -\frac{i\kappa}{k_2} & 0 & \frac{i\kappa}{k_2} \\ -\frac{k_2}{k_1} & 0 & \frac{k_1-i}{k_2} & 0 & \frac{k_1-i}{k_2} & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & -\frac{k_2}{k_1} & 0 & \frac{k_1-i}{k_2} & 0 & \frac{k_1-i}{k_2} \\ 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}, \tag{161}$$

and with

$$S_2 = \begin{pmatrix} -\frac{k_2}{k_1} & -\frac{ik_1}{k} & \frac{ik_1}{k} \\ 1 & -\frac{ik_2}{k} & \frac{ik_2}{k} \\ 0 & 1 & 1 \end{pmatrix}, \ S_3 = \begin{pmatrix} 0 & 0 & -1 & \frac{k_1-i}{k_2} & -1 & \frac{k_1-i}{k_2} \\ 0 & 0 & \frac{ik_1-k_2^2}{k_1k_2} & 1 & \frac{ik_1-k_2^2}{k_1k_2} & 1 \\ 0 & 0 & -\frac{i\kappa}{k_1} & \frac{i\kappa}{k_2} & \frac{i\kappa}{k_1} & -\frac{i\kappa}{k_2} \end{pmatrix}. \tag{162}$$

Now let $\mathbf{U} = S\mathbf{Y}$, where $\mathbf{Y} = (\hat{\omega}_{0,1}, \hat{\omega}_{0,2}, \hat{\omega}_{+,1}, \hat{\omega}_{+,2}, \hat{\omega}_{-,1}, \hat{\omega}_{-,2}, \hat{u}_0, \hat{u}_+, \hat{u}_-)$. Then, we obtain the equation $\partial_z \mathbf{Y} = D\mathbf{Y} + S^{-1}\mathbf{\Gamma}$ with (see "Appendix C" for details),

$$S^{-1} = \begin{pmatrix} S_1^{-1} & 0 \\ (S^{-1})_3 & S_2^{-1} \end{pmatrix}$$

again a matrix with the same block structure as L, with

$$S_1^{-1} = \begin{pmatrix} 0 & 0 & -\frac{k_1k_2}{\kappa^2} & \frac{k_1^2-ik_1}{\kappa^2} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{k_1k_2}{\kappa^2} & \frac{k_1^2-ik_1}{\kappa^2} \\ \frac{ik_2}{2\kappa} & 0 & \frac{k_1k_2}{2\kappa^2} & \frac{k_2^2}{2\kappa^2} & 0 & 0 \\ 0 & \frac{ik_2}{2\kappa} & 0 & 0 & \frac{k_1k_2}{2\kappa^2} & \frac{k_2^2}{2\kappa^2} \\ -\frac{ik_2}{2\kappa} & 0 & \frac{k_1k_2}{2\kappa^2} & \frac{k_2^2}{2\kappa^2} & 0 & 0 \\ 0 & -\frac{ik_2}{2\kappa} & 0 & 0 & \frac{k_1k_2}{2\kappa^2} & \frac{k_2^2}{2\kappa^2} \end{pmatrix}, \quad S_2^{-1} = \begin{pmatrix} -\frac{k_1k_2}{k^2} & \frac{k_1^2}{k^2} & 0 \\ \frac{ik_1}{2k} & \frac{ik_2}{2k} & \frac{1}{2} \\ -\frac{ik_1}{2k} & -\frac{ik_2}{2k} & \frac{1}{2} \end{pmatrix},$$

and

$$(S^{-1})_3 = \begin{pmatrix} 0 & 0 & -\frac{ik_1^3}{k^2\kappa^2} & -\frac{ik_1^2k_2}{k^2\kappa^2} & -\frac{ik_1^2k_2}{k^2\kappa^2} & -\frac{ik_1k_2^2}{k^2\kappa^2} \\ -\frac{k_2}{2k_1} & \frac{1}{2} & \frac{ik_2}{2k} & \frac{ik_2^2}{2k_1k} & -\frac{ik_1}{2k} & -\frac{ik_2}{2k} \\ -\frac{k_2}{2k_1} & \frac{1}{2} & -\frac{ik_2}{2k} & -\frac{ik_2^2}{2k_1k} & \frac{ik_1}{2k} & \frac{ik_2}{2k} \end{pmatrix}. \tag{163}$$

Using the definitions, we find that $\partial_z \mathbf{Y} = D\mathbf{Y} + \mathbf{T}$ with $\mathbf{T} = S^{-1}\mathbf{\Gamma}$, where

$$\mathbf{T} = \begin{pmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \\ T_7 \\ T_8 \\ T_9 \end{pmatrix} = \begin{pmatrix} \frac{\left(k_2^2-\kappa^2\right)}{\kappa^2}\hat{Q}_3 \\ -\frac{k_1k_2}{\kappa^2}\hat{Q}_3 \\ \frac{ik_2}{2\kappa}\hat{Q}_2 - \frac{k_2^2}{2\kappa^2}\hat{Q}_3 \\ -\frac{ik_2}{2\kappa}\hat{Q}_1 + \frac{k_1k_2}{2\kappa^2}\hat{Q}_3 \\ -\frac{ik_2}{2\kappa}\hat{Q}_2 - \frac{k_2^2}{2\kappa^2}\hat{Q}_3 \\ \frac{ik_2}{2\kappa}\hat{Q}_1 + \frac{k_1k_2}{2\kappa^2}\hat{Q}_3 \\ 0 \\ -\frac{1}{2}\hat{Q}_1 - \frac{k_2}{2k_1}\hat{Q}_2 - \frac{ik}{2k_1}\hat{Q}_3 \\ -\frac{1}{2}\hat{Q}_1 - \frac{k_2}{2k_1}\hat{Q}_2 + \frac{ik}{2k_1}\hat{Q}_3 \end{pmatrix}.$$

In component form, we have for $\mathbf{U} = S\mathbf{Y}$, using from now on the letter t instead of z for the "time variable",

$$\hat{\omega}_1(\mathbf{k},t) = -\frac{i\kappa}{k_2}\hat{\omega}_{+,1} + \frac{i\kappa}{k_2}\hat{\omega}_{-,1}, \tag{164}$$

$$\hat{\omega}_2(\mathbf{k},t) = -\frac{i\kappa}{k_2}\hat{\omega}_{+,2} + \frac{i\kappa}{k_2}\hat{\omega}_{-,2}, \tag{165}$$

$$\hat{\eta}_{1,1}(\mathbf{k},t) = -\frac{k_2}{k_1}\hat{\omega}_{0,1} + \frac{k_1 - i}{k_2}\hat{\omega}_{+,1} + \frac{k_1 - i}{k_2}\hat{\omega}_{-,1}, \tag{166}$$

$$\hat{\eta}_{1,2}(\mathbf{k},t) = \hat{\omega}_{0,1} + \hat{\omega}_{+,1} + \hat{\omega}_{-,1}, \tag{167}$$

$$\hat{\eta}_{2,1}(\mathbf{k},t) = -\frac{k_2}{k_1}\hat{\omega}_{0,2} + \frac{k_1 - i}{k_2}\hat{\omega}_{+,2} + \frac{k_1 - i}{k_2}\hat{\omega}_{-,2}, \tag{168}$$

$$\hat{\eta}_{2,2}(\mathbf{k},t) = \hat{\omega}_{0,2} + \hat{\omega}_{+,2} + \hat{\omega}_{-,2}, \tag{169}$$

$$\hat{u}_1(\mathbf{k},t) = -\hat{\omega}_{+,1} + \frac{k_1 - i}{k_2}\hat{\omega}_{+,2} - \hat{\omega}_{-,1} + \frac{k_1 - i}{k_2}\hat{\omega}_{-,2} - \frac{k_2}{k_1}\hat{u}_0 - \frac{ik_1}{k}\hat{u}_+ + \frac{ik_1}{k}\hat{u}_-, \tag{170}$$

$$\hat{u}_2(\mathbf{k},t) = \frac{ik_1 - k_2^2}{k_1k_2}\hat{\omega}_{+,1} + \hat{\omega}_{+,2} + \frac{ik_1 - k_2^2}{k_1k_2}\hat{\omega}_{-,1} + \hat{\omega}_{-,2} + \hat{u}_0 - \frac{ik_2}{k}\hat{u}_+ + \frac{ik_2}{k}\hat{u}_-, \tag{171}$$

$$\hat{u}_3(\mathbf{k},t) = -\frac{i\kappa}{k_1}\hat{\omega}_{+,1} + \frac{i\kappa}{k_2}\hat{\omega}_{+,2} + \frac{i\kappa}{k_1}\hat{\omega}_{-,1} - \frac{i\kappa}{k_2}\hat{\omega}_{-,2} + \hat{u}_+ + \hat{u}_-. \tag{172}$$

Given $\hat{\mathbf{Q}}$, the equation $\partial_t \mathbf{Y} = D\mathbf{Y} + \mathbf{T}$ can be integrated. We integrate forward in "time" for negative eigenvalues, and backward in "time" for positive and zero eigenvalues, and use the boundary condition at infinity which requires that $\hat{u}(\mathbf{k}, \infty) \to 0$. We get:

$$\hat{\omega}_{0,1}(\mathbf{k}, t) = -\int_t^\infty T_1(\mathbf{k},s)ds, \tag{173}$$

$$\hat{\omega}_{0,2}(\mathbf{k}, t) = -\int_t^\infty T_2(\mathbf{k},s)ds, \tag{174}$$

$$\hat{\omega}_{+,1}(\mathbf{k}, t) = -\int_t^\infty e^{\kappa(t-s)} T_3(\mathbf{k},s)ds, \tag{175}$$

$$\hat{\omega}_{+,2}(\mathbf{k}, t) = -\int_t^\infty e^{\kappa(t-s)} T_4(\mathbf{k},s)ds, \tag{176}$$

$$\hat{\omega}_{-,1}(\mathbf{k}, t) = \hat{\omega}^*_{-,1}(\mathbf{k}) e^{-\kappa(t-1)} + \int_1^t e^{-\kappa(t-s)} T_5(\mathbf{k},s)ds, \tag{177}$$

$$\hat{\omega}_{-,2}(\mathbf{k}, t) = \hat{\omega}^*_{-,2}(\mathbf{k}) e^{-\kappa(t-1)} + \int_1^t e^{-\kappa(t-s)} T_6(\mathbf{k},s)ds, \tag{178}$$

$$\hat{u}_0(\mathbf{k}, t) = 0, \tag{179}$$

$$\hat{u}_+(\mathbf{k}, t) = -\int_t^\infty e^{k(t-s)} T_8(\mathbf{k},s)ds, \tag{180}$$

$$\hat{u}_-(\mathbf{k}, t) = \hat{u}^*_-(\mathbf{k}) e^{-k(t-1)} + \int_1^t e^{-k(t-s)} T_9(\mathbf{k},s)ds. \tag{181}$$

The functions $\hat{\omega}^*_{-,1}$, $\hat{\omega}^*_{-,2}$, and $\hat{u}^*_-$ can be determined from the boundary condition at $t = 1$. We have

$$\begin{aligned}
\hat{\omega}_{0,1}(\mathbf{k}, 1) &= -\int_1^\infty T_1(\mathbf{k}, s)ds,\\
\hat{\omega}_{0,2}(\mathbf{k}, 1) &= -\int_1^\infty T_2(\mathbf{k}, s)ds,\\
\hat{\omega}_{+,1}(\mathbf{k}, 1) &= -\int_1^\infty e^{\kappa(1-s)} T_3(\mathbf{k}, s)ds,\\
\hat{\omega}_{+,2}(\mathbf{k}, 1) &= -\int_1^\infty e^{\kappa(1-s)} T_4(\mathbf{k}, s)ds,\\
\hat{\omega}_{-,1}(\mathbf{k}, 1) &= \hat{\omega}^*_{-,1}(\mathbf{k}),\\
\hat{\omega}_{-,2}(\mathbf{k}, 1) &= \hat{\omega}^*_{-,2}(\mathbf{k}),\\
\hat{u}_0(\mathbf{k}, 1) &= 0,\\
\hat{u}_+(\mathbf{k}, 1) &= -\int_1^\infty e^{k(1-s)} T_8(\mathbf{k}, s)ds,\\
\hat{u}_-(\mathbf{k}, 1) &= \hat{u}^*_-(\mathbf{k}).
\end{aligned}$$

Substituting (173)–(181) into (164)–(172) and using that $\hat{\mathbf{u}}(\mathbf{k}, 1) = 0$, we get

$$\begin{aligned}
-\omega^*_{-,1}(\mathbf{k}) + \frac{k_1 - i}{k_2}\omega^*_{-,2}(\mathbf{k}) + \frac{ik_1}{k} u^*_-(\mathbf{k}) &= \Phi_1(\mathbf{k}),\\
\frac{ik_1 - k_2^2}{k_1 k_2}\omega^*_{-,1}(\mathbf{k}) + \omega^*_{-,2}(\mathbf{k}) + \frac{ik_2}{k} u^*_-(\mathbf{k}) &= \Phi_2(\mathbf{k}),\\
\frac{i\kappa}{k_1}\omega^*_{-,1}(\mathbf{k}) - \frac{i\kappa}{k_2}\omega^*_{-,2}(\mathbf{k}) + u^*_-(\mathbf{k}) &= \Phi_3(\mathbf{k}),
\end{aligned}$$

where

$$\Phi_1(\mathbf{k}) = -\int_1^\infty e^{\kappa(1-s)}T_3(\mathbf{k},s)ds + \frac{k_1-i}{k_2}\int_1^\infty e^{\kappa(1-s)}T_4(\mathbf{k},s)ds - \frac{ik_1}{k}\int_1^\infty e^{k(1-s)}T_8(\mathbf{k},s)ds,$$
$$\Phi_2(\mathbf{k}) = \frac{ik_1-k_2^2}{k_1k_2}\int_1^\infty e^{\kappa(1-s)}T_3(\mathbf{k},s)ds + \int_1^\infty e^{\kappa(1-s)}T_4(\mathbf{k},s)ds - \frac{ik_2}{k}\int_1^\infty e^{k(1-s)}T_8(\mathbf{k},s)ds,$$
$$\Phi_3(\mathbf{k}) = -\frac{i\kappa}{k_1}\int_1^\infty e^{\kappa(1-s)}T_3(\mathbf{k},s)ds + \frac{i\kappa}{k_2}\int_1^\infty e^{\kappa(1-s)}T_4(\mathbf{k},s)ds + \int_1^\infty e^{k(1-s)}T_8(\mathbf{k},s)ds.$$

Since

$$\begin{pmatrix} -1 & \frac{k_1-i}{k_2} & \frac{ik_1}{k} \\ \frac{ik_1-k_2^2}{k_1k_2} & 1 & \frac{ik_2}{k} \\ \frac{i\kappa}{k_1} & -\frac{i\kappa}{k_2} & 1 \end{pmatrix}^{-1} = \begin{pmatrix} -\frac{ik_1k_2^2}{\kappa k} & -\frac{ik_2(k_2^2+\kappa k)}{\kappa k} & -\frac{k_2^2(\kappa+k)}{\kappa k} \\ \frac{ik_2\left(k_1^2+k\kappa\right)}{\kappa k} & \frac{ik_1k_2^2}{\kappa k} & \frac{k_1k_2(\kappa+k)}{\kappa k} \\ -(\kappa+k) & -\frac{k_2(k+\kappa)}{k_1} & \frac{i\kappa(k+\kappa)}{k_1} \end{pmatrix},$$

we find that

$$\begin{pmatrix} \hat{\omega}^*_{-,1}(\mathbf{k}) \\ \hat{\omega}^*_{-,2}(\mathbf{k}) \\ \hat{u}^*_-(\mathbf{k}) \end{pmatrix} = \begin{pmatrix} -\frac{ik_1k_2^2}{\kappa k}\Phi_1(\mathbf{k}) - \frac{ik_2(k_2^2+\kappa k)}{\kappa k}\Phi_2(\mathbf{k}) - \frac{k_2^2(\kappa+k)}{\kappa k}\Phi_3(\mathbf{k}) \\ \frac{ik_2\left(k_1^2+k\kappa\right)}{\kappa k}\Phi_1(\mathbf{k}) + \frac{ik_1k_2^2}{\kappa k}\Phi_2(\mathbf{k}) + \frac{k_1k_2(\kappa+k)}{\kappa k}\Phi_3(\mathbf{k}) \\ -(\kappa+k)\Phi_1(\mathbf{k}) - \frac{k_2(k+\kappa)}{k_1}\Phi_2(\mathbf{k}) + \frac{i\kappa(k+\kappa)}{k_1}\Phi_3(\mathbf{k}) \end{pmatrix},$$

from which we get the following expressions for $\omega^*_{-,1}$, $\omega^*_{-,2}$, and u^*_-:

$$\omega^*_{-,1}(\mathbf{k}) = \left(1+\frac{2ik_2^2(\kappa+k)}{kk_1}\right)\int_1^\infty e^{\kappa(1-s)}T_3(\mathbf{k},s)ds - \frac{2ik_2\,(\kappa+k)}{k}\int_1^\infty e^{\kappa(1-s)}T_4(\mathbf{k},s)ds$$
$$- \frac{2k_2^2(k+\kappa)}{\kappa k}\int_1^\infty e^{k(1-s)}T_8(\mathbf{k},s)ds, \tag{182}$$
$$\omega^*_{-,2}(\mathbf{k}) = -\frac{2ik_2\,(\kappa+k)}{k}\int_1^\infty e^{\kappa(1-s)}T_3(\mathbf{k},s)ds + \left(1+\frac{2ik_1(\kappa+k)}{k}\right)\int_1^\infty e^{\kappa(1-s)}T_4(\mathbf{k},s)ds$$
$$+ \frac{2k_1k_2\,(k+\kappa)}{\kappa k}\int_1^\infty e^{k(1-s)}T_8(\mathbf{k},s)ds, \tag{183}$$
$$u^*_-(\mathbf{k}) = \frac{2\kappa^2\,(\kappa+k)}{k_1^2}\int_1^\infty e^{\kappa(1-s)}T_3(\mathbf{k},s)ds - \frac{2\kappa^2(\kappa+k)}{k_1k_2}\int_1^\infty e^{\kappa(1-s)}T_4(\mathbf{k},s)ds$$
$$+ \frac{i(\kappa+k)^2}{k_1}\int_1^\infty e^{k(1-s)}T_8(\mathbf{k},s)ds. \tag{184}$$

Substituting now (182)–(184) into (173)–(181) and then into (164)–(172) we get, after regrouping of the integrals the representation (44), (45). The detailed expressions of the integral kernels are given in the following subsections.

A.1 Integral kernels for $\hat{\omega}_{1,n,m}$

Substituting $\hat{\omega}^*_{-,1}$ given by (182) into (164) gives, after splitting the integral over $[1,\infty]$ into an integral over $[1,t]$ and over $[t,\infty]$, the representation in (44) for $\hat{\omega}_1$, with $\alpha_{1,3,1}(\mathbf{k},\sigma)=0$,

$$\alpha_{1,1,1}(\mathbf{k},\sigma)=-\frac{2ik_2\,(\kappa+k)}{k}\left(e^{-\kappa\sigma}-e^{-k\sigma}\right), \tag{185}$$

$$\alpha_{1,1,2}(\mathbf{k},\sigma)=e^{\kappa\sigma}-\left(1+\frac{2ik_2^2(\kappa+k)}{kk_1}\right)e^{-\kappa\sigma}+\frac{2ik_2^2(k+\kappa)}{kk_1}e^{-k\sigma}, \tag{186}$$

$$\alpha_{1,1,3}(\mathbf{k},\sigma)=-\frac{ik_2}{\kappa}e^{\kappa\sigma}+\frac{k_2(\kappa+k)^2}{\kappa k_1}e^{-\kappa\sigma}-\frac{2k_2(k+\kappa)}{k_1}e^{-k\sigma}, \tag{187}$$

$$\alpha_{1,2,1}(\mathbf{k},\sigma)=-\frac{2ik_2\,(\kappa+k)}{k}\left(e^{-\kappa\sigma}-e^{-k\sigma}\right), \tag{188}$$

$$\alpha_{1,2,2}(\mathbf{k},\sigma)=-\left(2+\frac{2ik_2^2(\kappa+k)}{kk_1}\right)e^{-\kappa\sigma}+\frac{2ik_2^2(k+\kappa)}{kk_1}e^{-k\sigma}, \tag{189}$$

$$\alpha_{1,2,3}(\mathbf{k},\sigma)=\frac{2k_2(\kappa+k)}{k_1}\left(e^{-\kappa\sigma}-e^{-k\sigma}\right), \tag{190}$$

$$\alpha_{1,3,2}(\mathbf{k},\sigma)=-\frac{\kappa}{1+k}e^{-\kappa\sigma}, \tag{191}$$

$$\alpha_{1,3,3}(\mathbf{k},\sigma)=-\frac{ik_2}{1+k}e^{-\kappa\sigma}. \tag{192}$$

A.2 Integral kernels for $\hat{\omega}_{2,n,m}$

Substituting $\hat{\omega}^*_{-,2}$ given by (183) into (165) gives the representation in (44) for $\hat{\omega}_2$, with $\alpha_{2,3,2}(\mathbf{k},\sigma)=0$,

$$\alpha_{2,1,1}(\mathbf{k},\sigma)=-e^{\kappa\sigma}+\left(1+\frac{2ik_1(\kappa+k)}{k}\right)e^{-\kappa\sigma}-\frac{2ik_1\,(k+\kappa)}{k}e^{-k\sigma}, \tag{193}$$

$$\alpha_{2,1,2}(\mathbf{k},\sigma)=\frac{2ik_2\,(\kappa+k)}{k}\left(e^{-\kappa\sigma}-e^{-k\sigma}\right), \tag{194}$$

$$\alpha_{2,1,3}(\mathbf{k},\sigma)=\frac{ik_1}{\kappa}e^{\kappa\sigma}-\frac{(\kappa+k)^2}{\kappa}e^{-\kappa\sigma}+2\,(k+\kappa)\,e^{-k\sigma}, \tag{195}$$

$$\alpha_{2,2,1}(\mathbf{k},\sigma)=\left(2+\frac{2ik_1(\kappa+k)}{k}\right)e^{-\kappa\sigma}-\frac{2ik_1\,(k+\kappa)}{k}e^{-k\sigma}, \tag{196}$$

$$\alpha_{2,2,2}(\mathbf{k},\sigma)=\frac{2ik_2\,(\kappa+k)}{k}\left(e^{-\kappa\sigma}-e^{-k\sigma}\right), \tag{197}$$

$$\alpha_{2,2,3}(\mathbf{k},\sigma)=-2(\kappa+k)\left(e^{-\kappa\sigma}-e^{-k\sigma}\right), \tag{198}$$

$$\alpha_{2,3,1}(\mathbf{k},\sigma)=\frac{\kappa}{1+k}e^{-\kappa\sigma}, \tag{199}$$

$$\alpha_{2,3,3}(\mathbf{k},\sigma)=\frac{ik_1}{1+k}e^{-\kappa\sigma}. \tag{200}$$

A.3 Integral kernels for $\hat{\omega}_{3,n,m}$

The representation of $\hat{\omega}_3$ in (44) is obtained using $\hat{\omega}_3=-ik_1\hat{u}_2+ik_2\hat{u}_1$, with $\alpha_{3,2,1}(\mathbf{k},\sigma)=0$,

$$\alpha_{3,1,1}(\mathbf{k},\sigma)=\frac{ik_2}{\kappa}\left(e^{\kappa\sigma}-e^{-\kappa\sigma}\right), \tag{201}$$

$$\alpha_{3,1,2}(\mathbf{k},\sigma) = \left(\frac{2i\kappa k_2^2(\kappa+k)}{kk_1}e^{-\kappa\sigma} - \frac{ik_2^2(\kappa+k)^2}{kk_1}e^{-k\sigma}\right) + \frac{ik_1}{\kappa}\left(e^{-\kappa\sigma} - e^{\kappa\sigma}\right) - \frac{k_2^2}{k}e^{k\sigma}, \tag{202}$$

$$\alpha_{3,1,3}(\mathbf{k},\sigma) = -\frac{2kk_2(\kappa+k)}{k_1}e^{-\kappa\sigma} + \frac{(k+\kappa)^2 k_2}{k_1}e^{-k\sigma} + ik_2e^{k\sigma}, \tag{203}$$

$$\alpha_{3,2,2}(\mathbf{k},\sigma) = \frac{2ik_2^2\kappa(\kappa+k)}{kk_1}\left(e^{-\kappa\sigma} - e^{-k\sigma}\right), \tag{204}$$

$$\alpha_{3,2,3}(\mathbf{k},\sigma) = -\frac{2k_2k(\kappa+k)}{k_1}e^{-\kappa\sigma} + \frac{2k_2\kappa(k+\kappa)}{k_1}e^{-k\sigma}, \tag{205}$$

$$\alpha_{3,3,1}(\mathbf{k},\sigma) = \frac{ik_2}{1+k}e^{-\kappa\sigma}, \tag{206}$$

$$\alpha_{3,3,2}(\mathbf{k},\sigma) = -\frac{k_2^2}{1+k}e^{-\kappa\sigma}, \tag{207}$$

$$\alpha_{3,3,3}(\mathbf{k},\sigma) = -\frac{i\kappa k_2}{1+k}e^{-\kappa\sigma}, \tag{208}$$

with $\beta_{3,1,1}(\mathbf{k},\sigma) = \beta_{3,2,1}(\mathbf{k},\sigma) = \beta_{3,3,1}(\mathbf{k},\sigma) = 0$,

$$\beta_{3,1,2}(\mathbf{k},\sigma) = ik_2\left[-\frac{ik_2}{k}e^{k\sigma} + \frac{k_2(\kappa+k)^2}{k_1k}e^{-k\sigma} - \frac{2\kappa k_2(\kappa+k)}{kk_1}e^{-\kappa\sigma}\right], \tag{209}$$

$$\beta_{3,1,3}(\mathbf{k},\sigma) = ik_2\left[-e^{k\sigma} + \frac{i(\kappa+k)^2}{k_1}e^{-k\sigma} - \frac{2ik(\kappa+k)}{k_1}e^{-\kappa\sigma}\right], \tag{210}$$

$$\beta_{3,2,2}(\mathbf{k},\sigma) = ik_2\left[-\frac{2\kappa k_2(\kappa+k)}{kk_1}\left(e^{-\kappa\sigma} - e^{-k\sigma}\right)\right], \tag{211}$$

$$\beta_{3,2,3}(\mathbf{k},\sigma) = ik_2\left[-\frac{2ik(\kappa+k)}{k_1}e^{-\kappa\sigma} + \frac{2i\kappa(\kappa+k)}{k_1}e^{-k\sigma}\right], \tag{212}$$

$$\beta_{3,3,2}(\mathbf{k},\sigma) = \frac{\left(k_2^2 - ik_1\right)}{1+k}e^{-\kappa\sigma}, \tag{213}$$

$$\beta_{3,3,3}(\mathbf{k},\sigma) = \frac{i\kappa k_2}{1+k}e^{-\kappa\sigma}, \tag{214}$$

with $\gamma_{3,1,1}(\mathbf{k},\sigma) = \gamma_{3,2,1}(\mathbf{k},\sigma) = \gamma_{3,3,1}(\mathbf{k},\sigma) = 0$,

$$\gamma_{3,1,2}(\mathbf{k},\sigma) = -ik_1\left[\frac{ik_2^2}{k(\kappa+k)}e^{k\sigma} + \frac{2\kappa k_2^2}{kk_1}e^{-\kappa\sigma} - \frac{k_2^2(\kappa+k)}{kk_1}e^{-k\sigma}\right], \tag{215}$$

$$\gamma_{3,1,3}(\mathbf{k},\sigma) = -ik_1\left[\frac{k_2}{\kappa+k}e^{k\sigma} + \frac{2ikk_2}{k_1}e^{-\kappa\sigma} - \frac{ik_2(\kappa+k)}{k_1}e^{-k\sigma}\right], \tag{216}$$

$$\gamma_{3,2,2}(\mathbf{k},\sigma) = -ik_1\left[\frac{2\kappa k_2^2}{kk_1}\left(e^{-\kappa\sigma} - e^{-k\sigma}\right)\right], \tag{217}$$

$$\gamma_{3,2,3}(\mathbf{k},\sigma) = -ik_1\left[\frac{2ikk_2}{k_1}e^{-\kappa\sigma} - \frac{2i\kappa k_2}{k_1}e^{-k\sigma}\right], \tag{218}$$

$$\gamma_{3,3,2}(\mathbf{k},\sigma) = \frac{\left(k_1k_2^2 - ik_1^2\right)}{(1+k)k}e^{-\kappa\sigma}, \tag{219}$$

$$\gamma_{3,3,3}(\mathbf{k},\sigma) = \frac{ik_1\kappa k_2}{(1+k)\,k}e^{-\kappa\sigma}. \tag{220}$$

A.4 Integral kernels for $\hat{u}_{1,n,m}$

Substituting $\hat{\omega}^*_{-,1}$, $\hat{\omega}^*_{-,2}$ and u^*_- given by (182), (183), and (184) into (170) gives the representation in (45) for $\hat{u}_1$, with

$$f_{1,1,1}(\mathbf{k},\sigma) = \frac{ik_1+1}{\kappa}\left(e^{\kappa\sigma}-e^{-\kappa\sigma}\right) + \frac{2\kappa(\kappa+k)}{k}\left(e^{-\kappa\sigma}-e^{-k\sigma}\right), \tag{221}$$

$$f_{1,1,2}(\mathbf{k},\sigma) = \frac{ik_2}{\kappa}\left(e^{\kappa\sigma}-e^{-\kappa\sigma}\right) + \frac{2\kappa k_2(\kappa+k)}{kk_1}\left(e^{-\kappa\sigma}-e^{-k\sigma}\right), \tag{222}$$

$$f_{1,1,3}(\mathbf{k},\sigma) = e^{\kappa\sigma} + \frac{i(\kappa+k)^2}{k_1}e^{-\kappa\sigma} - \frac{2i\kappa\,(k+\kappa)}{k_1}e^{-k\sigma}, \tag{223}$$

$$f_{1,2,1}(\mathbf{k},\sigma) = \frac{2\kappa\,(\kappa+k)}{k}\left(e^{-\kappa\sigma}-e^{-k\sigma}\right), \tag{224}$$

$$f_{1,2,2}(\mathbf{k},\sigma) = \frac{2\kappa k_2\,(\kappa+k)}{kk_1}\left(e^{-\kappa\sigma}-e^{-k\sigma}\right), \tag{225}$$

$$f_{1,2,3}(\mathbf{k},\sigma) = \frac{2ik(\kappa+k)}{k_1}e^{-\kappa\sigma} - \frac{2i\kappa(k+\kappa)}{k_1}e^{-k\sigma}, \tag{226}$$

$$f_{1,3,1}(\mathbf{k},\sigma) = \frac{(ik_1+1)}{1+k}e^{-\kappa\sigma}, \tag{227}$$

$$f_{1,3,2}(\mathbf{k},\sigma) = \frac{ik_2}{1+k}e^{-\kappa\sigma}, \tag{228}$$

$$f_{1,3,3}(\mathbf{k},\sigma) = -\frac{\kappa}{1+k}e^{-\kappa\sigma}, \tag{229}$$

and with

$$g_{1,1,1}(\mathbf{k},\sigma) = -\frac{ik_1}{k}e^{k\sigma} + \frac{(\kappa+k)^2}{k}e^{-k\sigma} - \frac{2\kappa(\kappa+k)}{k}e^{-\kappa\sigma}, \tag{230}$$

$$g_{1,1,2}(\mathbf{k},\sigma) = -\frac{ik_2}{k}e^{k\sigma} + \frac{k_2(\kappa+k)^2}{k_1k}e^{-k\sigma} - \frac{2\kappa k_2\,(\kappa+k)}{kk_1}e^{-\kappa\sigma}, \tag{231}$$

$$g_{1,1,3}(\mathbf{k},\sigma) = -e^{k\sigma} + \frac{i(\kappa+k)^2}{k_1}e^{-k\sigma} - \frac{2ik(\kappa+k)}{k_1}e^{-\kappa\sigma}, \tag{232}$$

$$g_{1,2,1}(\mathbf{k},\sigma) = -\frac{2\kappa(\kappa+k)}{k}\left(e^{-\kappa\sigma}-e^{-k\sigma}\right), \tag{233}$$

$$g_{1,2,2}(\mathbf{k},\sigma) = -\frac{2\kappa k_2\,(\kappa+k)}{kk_1}\left(e^{-\kappa\sigma}-e^{-k\sigma}\right), \tag{234}$$

$$g_{1,2,3}(\mathbf{k},\sigma) = -\frac{2ik(\kappa+k)}{k_1}e^{-\kappa\sigma} + \frac{2i\kappa(\kappa+k)}{k_1}e^{-k\sigma}, \tag{235}$$

$$g_{1,3,1}(\mathbf{k},\sigma) = -\frac{ik_1}{1+k}e^{-k\sigma}, \tag{236}$$

$$g_{1,3,2}(\mathbf{k},\sigma) = -\frac{ik_2}{1+k}e^{-k\sigma}, \tag{237}$$

$$g_{1,3,3}(\mathbf{k},\sigma) = \frac{k}{1+k}e^{-k\sigma}. \tag{238}$$

A.5 Integral kernels for $\hat{u}_{2,n,m}$

Substituting $\hat{\omega}^*_{-,1}$, $\hat{\omega}^*_{-,2}$ and u^*_{-} given by (182), (183), and (184) into (171) gives the representation in (45) for $\hat{u}_2$, with $f_{2,2,2}(\mathbf{k},\sigma) = f_{2,2,3}(\mathbf{k},\sigma) = f_{2,3,2}(\mathbf{k},\sigma) = f_{2,3,3}(\mathbf{k},\sigma) = 0$,

$$f_{2,1,1}(\mathbf{k},\sigma) = \frac{ik_2}{\kappa}\left(e^{\kappa\sigma} - e^{-\kappa\sigma}\right) + \frac{2\kappa k_2(\kappa+k)}{kk_1}\left(e^{-\kappa\sigma} - e^{-k\sigma}\right), \tag{239}$$

$$f_{2,1,2}(\mathbf{k},\sigma) = \frac{k_1 + ik_2^2}{\kappa k_1}\left(e^{\kappa\sigma} - e^{-\kappa\sigma}\right) + \frac{ik_2^2}{kk_1}\left(e^{-k\sigma} - e^{k\sigma}\right), \tag{240}$$

$$f_{2,1,3}(\mathbf{k},\sigma) = \frac{k_2}{k_1}\left(e^{\kappa\sigma} - e^{k\sigma}\right) + \frac{k_2}{k_1}\left(e^{-\kappa\sigma} - e^{-k\sigma}\right), \tag{241}$$

$$f_{2,2,1}(\mathbf{k},\sigma) = \frac{2\kappa k_2\left(\kappa+k\right)}{kk_1}\left(e^{-\kappa\sigma} - e^{-k\sigma}\right), \tag{242}$$

$$f_{2,3,1}(\mathbf{k},\sigma) = \frac{ik_2}{1+k}e^{-\kappa\sigma}, \tag{243}$$

with $g_{2,1,2}(\mathbf{k},\sigma) = g_{2,1,3}(\mathbf{k},\sigma) = g_{2,2,2}(\mathbf{k},\sigma) = g_{2,2,3}(\mathbf{k},\sigma) = 0$,

$$g_{2,1,1}(\mathbf{k},\sigma) = -\frac{ik_2}{k}e^{k\sigma} - \frac{2k_2\kappa(\kappa+k)}{kk_1}e^{-\kappa\sigma} + \frac{k_2(\kappa+k)^2}{kk_1}e^{-k\sigma}, \tag{244}$$

$$g_{2,2,1}(\mathbf{k},\sigma) = -\frac{2\kappa k_2(\kappa+k)}{kk_1}\left(e^{-\kappa\sigma} - e^{-k\sigma}\right), \tag{245}$$

$$g_{2,3,1}(\mathbf{k},\sigma) = -\frac{ik_2}{1+k}e^{-k\sigma}, \tag{246}$$

$$g_{2,3,2}(\mathbf{k},\sigma) = \frac{1}{1+k}\left[-\frac{ik_2^2}{k_1}e^{-k\sigma} + \frac{\left(k_1 + ik_2^2\right)}{k_1}e^{-\kappa\sigma}\right], \tag{247}$$

$$g_{2,3,3}(\mathbf{k},\sigma) = \frac{1}{1+k}\left[\frac{kk_2}{k_1}e^{-k\sigma} - \frac{\kappa k_2}{k_1}e^{-\kappa\sigma}\right], \tag{248}$$

and with $h_{2,1,1}(\mathbf{k},\sigma) = h_{2,2,1}(\mathbf{k},\sigma) = h_{2,3,1}(\mathbf{k},\sigma) = 0$,

$$h_{2,1,2}(\mathbf{k},\sigma) = \frac{ik_2^2}{k\left(\kappa+k\right)}e^{k\sigma} + \frac{2\kappa k_2^2}{kk_1}e^{-\kappa\sigma} - \frac{k_2^2(\kappa+k)}{kk_1}e^{-k\sigma}, \tag{249}$$

$$h_{2,1,3}(\mathbf{k},\sigma) = \frac{k_2}{\kappa+k}e^{k\sigma} + \frac{2ikk_2}{k_1}e^{-\kappa\sigma} - \frac{ik_2(\kappa+k)}{k_1}e^{-k\sigma}, \tag{250}$$

$$h_{2,2,2}(\mathbf{k},\sigma) = \frac{2\kappa k_2^2}{kk_1}\left(e^{-\kappa\sigma} - e^{-k\sigma}\right), \tag{251}$$

$$h_{2,2,3}(\mathbf{k},\sigma) = \frac{2ikk_2}{k_1}e^{-\kappa\sigma} - \frac{2i\kappa k_2}{k_1}e^{-k\sigma}, \tag{252}$$

$$h_{2,3,2}(\mathbf{k},\sigma) = \frac{\left(k_1 + ik_2^2\right)}{(1+k)\,k}e^{-\kappa\sigma}, \tag{253}$$

$$h_{2,3,3}(\mathbf{k},\sigma) = -\frac{\kappa k_2}{(1+k)\,k}e^{-\kappa\sigma}. \tag{254}$$

A.6 Integral kernels for $\hat{u}_{3,n,m}$

Substituting $\hat{\omega}^*_{-,1}$, $\hat{\omega}^*_{-,2}$ and u^*_- given by (182), (183), and (184) into (172) gives the representation in (45) for $\hat{u}_3$, with $f_{3,2,2}(\mathbf{k},\sigma) = f_{3,2,3}(\mathbf{k},\sigma) = f_{3,3,2}(\mathbf{k},\sigma) = f_{3,3,3}(\mathbf{k},\sigma) = 0$,

$$f_{3,1,1}(\mathbf{k},\sigma) = e^{\kappa\sigma} - \frac{i\left(\kappa+k\right)^2}{k_1}e^{-\kappa\sigma} + \frac{2ik\left(k+\kappa\right)}{k_1}e^{-k\sigma}, \tag{255}$$

$$f_{3,1,2}(\mathbf{k},\sigma) = \frac{k_2}{k_1}\left(e^{\kappa\sigma} - e^{k\sigma}\right) + \frac{k_2}{k_1}\left(e^{-\kappa\sigma} - e^{-k\sigma}\right), \tag{256}$$

$$f_{3,1,3}(\mathbf{k},\sigma) = \frac{ik}{k_1}\left(e^{k\sigma} - e^{-k\sigma}\right) + \frac{ik^2}{\kappa k_1}\left(e^{-\kappa\sigma} - e^{\kappa\sigma}\right), \tag{257}$$

$$f_{3,2,1}(\mathbf{k},\sigma) = -\frac{2i\kappa\left(k+\kappa\right)}{k_1}e^{-\kappa\sigma} + \frac{2ik\left(k+\kappa\right)}{k_1}e^{-k\sigma}, \tag{258}$$

$$f_{3,3,1}(\mathbf{k},\sigma) = -\frac{\kappa}{1+k}e^{-\kappa\sigma}, \tag{259}$$

with $g_{3,1,2}(\mathbf{k},\sigma) = g_{3,1,3}(\mathbf{k},\sigma) = g_{3,2,2}(\mathbf{k},\sigma) = g_{3,2,3}(\mathbf{k},\sigma) = 0$,

$$g_{3,1,1}(\mathbf{k},\sigma) = -e^{k\sigma} + \frac{2i\kappa(\kappa+k)}{k_1}e^{-\kappa\sigma} - \frac{i(\kappa+k)^2}{k_1}e^{-k\sigma}, \tag{260}$$

$$g_{3,2,1}(\mathbf{k},\sigma) = \frac{2i\kappa(\kappa+k)}{k_1}e^{-\kappa\sigma} - \frac{2ik(\kappa+k)}{k_1}e^{-k\sigma}, \tag{261}$$

$$g_{3,3,1}(\mathbf{k},\sigma) = \frac{k}{1+k}e^{-k\sigma}, \tag{262}$$

$$g_{3,3,2}(\mathbf{k},\sigma) = \frac{1}{1+k}\left[\frac{kk_2}{k_1}e^{-k\sigma} - \frac{\kappa k_2}{k_1}e^{-\kappa\sigma}\right], \tag{263}$$

$$g_{3,3,3}(\mathbf{k},\sigma) = \frac{ik^2}{(1+k)\,k_1}\left(e^{-k\sigma} - e^{-\kappa\sigma}\right), \tag{264}$$

and with $h_{3,1,1}(\mathbf{k},\sigma) = h_{3,2,1}(\mathbf{k},\sigma) = h_{3,3,1}(\mathbf{k},\sigma) = 0$,

$$h_{3,1,2}(\mathbf{k},\sigma) = \frac{k_2}{\kappa+k}e^{k\sigma} - \frac{2i\kappa k_2}{k_1}e^{-\kappa\sigma} + \frac{ik_2(\kappa+k)}{k_1}e^{-k\sigma}, \tag{265}$$

$$h_{3,1,3}(\mathbf{k},\sigma) = -\frac{ik}{\kappa+k}e^{k\sigma} + \frac{2k^2}{k_1}e^{-\kappa\sigma} - \frac{k(\kappa+k)}{k_1}e^{-k\sigma}, \tag{266}$$

$$h_{3,2,2}(\mathbf{k},\sigma) = -\frac{2i\kappa k_2}{k_1}e^{-\kappa\sigma} + \frac{2ikk_2}{k_1}e^{-k\sigma}, \tag{267}$$

$$h_{3,2,3}(\mathbf{k},\sigma) = \frac{2k^2}{k_1}\left(e^{-\kappa\sigma} - e^{-k\sigma}\right), \tag{268}$$

$$h_{3,3,2}(\mathbf{k},\sigma) = -\frac{\kappa k_2}{(1+k)k}e^{-\kappa\sigma}, \tag{269}$$

$$h_{3,3,3}(\mathbf{k},\sigma) = -\frac{ik^2}{(1+k)k}e^{-\kappa\sigma}. \tag{270}$$

Appendix B: Basic bounds

B.1 Continuity of semi-groups

We have:

Proposition 30. *Let α', β', $\gamma' \geq 0$ with $\alpha' - \beta' + \gamma' \geq 0$, and let $\mu > 0$. Then, we have the bound*

$$\frac{1}{1+|\mathbf{k}|^{\alpha'}} e^{\mu\Lambda_-(t-1)} |\Lambda_-|^{\beta'} \left(\frac{t-1}{t}\right)^{\gamma'} \leq \text{const. } \frac{1}{t^{\beta'}} \frac{1}{1+(|\mathbf{k}|\, t)^{\alpha'-\beta'+\gamma'}},$$

uniformly in $\mathbf{k} \in \mathbb{R}^2$ and $t \geq 1$. Similarly, for positive α', β', γ' with $\alpha' - \beta' + \gamma' \geq 0$ and $\mu > 0$, we have the bound

$$\frac{1}{1+|\mathbf{k}|^{\alpha'}} e^{-\mu k(t-1)} k^{\beta'} \left(\frac{t-1}{t}\right)^{\gamma'} \leq \text{const. } \frac{1}{t^{\beta'}} \frac{1}{1+(|\mathbf{k}|\, t)^{\alpha'-\beta'+\gamma'}},$$

uniformly in $\mathbf{k} \in \mathbb{R}^2$, $k = |\mathbf{k}| = \sqrt{k_1^2 + k_2^2}$ and $t \geq 1$.

Proof. For $1 \leq t \leq 2$ and $|\mathbf{k}| \leq 1$, we have that

$$\frac{1}{1+|\mathbf{k}|^{\alpha'}} e^{\mu\Lambda_-(t-1)} |\Lambda_-|^{\beta'} \left(\frac{t-1}{t}\right)^{\gamma'} \leq \text{const.} \leq \text{const. } \frac{1}{t^{\beta'}} \frac{1}{1+(|\mathbf{k}|\, t)^{\alpha'-\beta'+\gamma'}},$$

and that

$$\frac{1}{1+|\mathbf{k}|^{\alpha'}} e^{-\mu k(t-1)} k^{\beta'} \left(\frac{t-1}{t}\right)^{\gamma'} \leq \text{const.} \leq \text{const. } \frac{1}{t^{\beta'}} \frac{1}{1+(|\mathbf{k}|\, t)^{\alpha'-\beta'+\gamma'}}.$$

Next, for $1 \leq t \leq 2$ and $|\mathbf{k}| > 1$, we have that

$$\begin{aligned}
\frac{1}{1+|\mathbf{k}|^{\alpha'}} e^{\mu\Lambda_-(t-1)} |\Lambda_-|^{\beta'} \left(\frac{t-1}{t}\right)^{\gamma'} &\leq \text{const. } \frac{1}{1+|\mathbf{k}|^{\alpha'}} e^{\mu\Lambda_-(t-1)} \left(|\Lambda_-|\,(t-1)\right)^{\gamma'} \ |\Lambda_-|^{\beta'-\gamma'} \\
&\leq \text{const. } \frac{1}{1+|\mathbf{k}|^{\alpha'}} \ k^{\beta'-\gamma'} \leq \text{const. } \frac{1}{1+|\mathbf{k}|^{\alpha'-\beta'+\gamma'}} \\
&\leq \text{const. } \frac{1}{t^{\beta'}} \frac{1}{1+(|\mathbf{k}|\, t)^{\alpha'-\beta'+\gamma'}}.
\end{aligned}$$

and similarly that

$$\begin{aligned}
\frac{1}{1+|\mathbf{k}|^{\alpha'}} e^{-\mu k(t-1)} k^{\beta'} \left(\frac{t-1}{t}\right)^{\gamma'} &\leq \text{const. } \frac{1}{1+|\mathbf{k}|^{\alpha'}} e^{-\mu k(t-1)} \left(k\,(t-1)\right)^{\gamma'} \ k^{\beta'-\gamma'} \\
&\leq \text{const. } \frac{1}{1+|\mathbf{k}|^{\alpha'}} \ k^{\beta'-\gamma'} \leq \text{const. } \frac{1}{1+|\mathbf{k}|^{\alpha'-\beta'+\gamma'}} \\
&\leq \text{const. } \frac{1}{t^{\beta'}} \frac{1}{1+(|\mathbf{k}|\, t)^{\alpha'-\beta'+\gamma'}}.
\end{aligned}$$

Finally, for $t > 2$ and $\mathbf{k} \in \mathbb{R}^2$, we have

$$\begin{aligned}
&\left(1+(|\mathbf{k}|\, t)^{\alpha'-\beta'+\gamma'}\right) e^{\mu\Lambda_-(t-1)} |\Lambda_- t|^{\beta'} \left(\frac{t-1}{t}\right)^{\gamma'} \\
&\quad\leq \text{const. } \left(1+(|\mathbf{k}|\, t)^{\alpha'-\beta'+\gamma'}\right) e^{\frac{1}{2}\mu\Lambda_- t} |\Lambda_- t|^{\beta'} \\
&\quad\leq \text{const. } \left(1+(|\mathbf{k}|\, t)^{\alpha'-\beta'+\gamma'} \ e^{\frac{1}{2}\mu\Lambda_- t} |\Lambda_- t|^{\beta'}\right) \\
&\quad\leq \text{const. } \left(1+\frac{|\mathbf{k}|^{\alpha'-\beta'+\gamma'}}{|\Lambda_-|^{\alpha'-\beta'+\gamma'}} |\Lambda_- t|^{\alpha'-\beta'+\gamma'} |\Lambda_- t|^{\beta'} \ e^{\frac{1}{2}\mu\Lambda_- t}\right) \\
&\quad\leq \text{const. } \left(1+\frac{|\mathbf{k}|^{\alpha'-\beta'+\gamma'}}{|\Lambda_-|^{\alpha'-\beta'+\gamma'}}\right) \leq \text{const.},
\end{aligned}$$

and similarly that

$$\left(1+(|\mathbf{k}|\,t)^{\alpha'-\beta'+\gamma'}\right)e^{-\mu k(t-1)}\,(kt)^{\beta'}\left(\frac{t-1}{t}\right)^{\gamma'}$$
$$\leq \text{const.}\left(1+(|\mathbf{k}|\,t)^{\alpha'-\beta'+\gamma'}\right)e^{-\frac{1}{2}\mu kt}\,(kt)^{\beta'}\leq \text{const.}.$$

□

B.2 Convolution with the semi-group $e^{\Lambda_- t}$

In order to bound the integrals over the interval $[1,t]$, we systematically split them into integrals over $[1,\frac{1+t}{2}]$ and integrals over $[\frac{1+t}{2},t]$ and bound the resulting terms separately. We have:

Proposition 31. *Let $\alpha\geq 0$, $r\geq 0$ and $\delta\geq 0$ and $\gamma+1\geq\beta\geq 0$. Then,*

$$e^{\Lambda_-(t-1)}\int_1^{\frac{t+1}{2}}e^{|\Lambda_-|(s-1)}|\Lambda_-|^{\beta}\frac{(s-1)^{\gamma}}{s^{\delta}}\mu_{\alpha,r}(\mathbf{k},s)\;ds$$
$$\leq\begin{cases}\text{const.}\dfrac{1}{t^{\beta}}\bar{\mu}_{\alpha}(\mathbf{k},t), & \text{if } \delta>\gamma+1\\[2ex] \text{const.}\dfrac{\log(1+t)}{t^{\beta}}\bar{\mu}_{\alpha}(\mathbf{k},t), & \text{if } \delta=\gamma+1\\[2ex] \text{const.}\dfrac{t^{\gamma+1-\delta}}{t^{\beta}}\bar{\mu}_{\alpha}(\mathbf{k},t), & \text{if } \delta<\gamma+1\end{cases}\tag{271}$$

uniformly in $t\geq 1$ and $\mathbf{k}\in\mathbb{R}^2$.

Proof. We have that

$$e^{\Lambda_-(t-1)}\int_1^{\frac{t+1}{2}}e^{|\Lambda_-|(s-1)}|\Lambda_-|^{\beta}\frac{(s-1)^{\gamma}}{s^{\delta}}\mu_{\alpha,r}(\mathbf{k},s)\;ds$$
$$\leq e^{\Lambda_-(t-1)}e^{|\Lambda_-|\frac{t-1}{2}}|\Lambda_-|^{\beta}\mu_{\alpha,r}(\mathbf{k},1)\int_1^{\frac{t+1}{2}}\frac{(s-1)^{\gamma}}{s^{\delta}}\;ds$$
$$\leq \text{const.}\left(\frac{t-1}{t}\right)^{\gamma+1}e^{\Lambda_-\frac{t-1}{2}}|\Lambda_-|^{\beta}\mu_{\alpha,1}(\mathbf{k},1)\begin{cases}1, & \text{if } \delta>\gamma+1\\[1ex] \log(1+t), & \text{if } \delta=\gamma+1\\[1ex] t^{\gamma+1-\delta}, & \text{if } \delta<\gamma+1\end{cases}$$

The bounds in (271) now follow using Proposition 30. □

Proposition 32. *Let $\alpha\geq 0$, $r\geq 0$, $\delta\in\mathbb{R}$, and $\beta\in\{0,1\}$. Then,*

$$e^{\Lambda_-(t-1)}\int_{\frac{t+1}{2}}^{t}e^{|\Lambda_-|(s-1)}|\Lambda_-|^{\beta}\frac{1}{s^{\delta}}\mu_{\alpha,r}(\mathbf{k},s)\;ds\leq\frac{\text{const.}}{t^{\delta-1+\beta}}\mu_{\alpha,r}(\mathbf{k},t),\tag{272}$$

uniformly in $t\geq 1$ and $\mathbf{k}\in\mathbb{R}^2$.

Proof. If $\beta=0$, we have that

$$e^{\Lambda_-(t-1)}\int_{\frac{t+1}{2}}^{t}e^{|\Lambda_-|(s-1)}\frac{1}{s^{\delta}}\mu_{\alpha,r}(\mathbf{k},s)\;ds\leq\frac{\text{const.}}{t^{\delta}}\mu_{\alpha,r}(\mathbf{k},t)\int_{\frac{t+1}{2}}^{t}ds,$$

and (272) follows, and if $\beta = 1$, we have that

$$e^{\Lambda_-(t-1)} \int_{\frac{t+1}{2}}^{t} e^{|\Lambda_-|(s-1)} |\Lambda_-|^{\beta} \frac{1}{s^{\delta}} \mu_{\alpha,r}(\mathbf{k}, s)\ ds \le \frac{\text{const.}}{t^{\delta}} \mu_{\alpha,r}(\mathbf{k}, t) e^{\Lambda_-(t-1)} \int_{\frac{t+1}{2}}^{t} e^{|\Lambda_-|(s-1)} |\Lambda_-|\ ds$$
$$\le \frac{\text{const.}}{t^{\delta}} \mu_{\alpha,r}(\mathbf{k}, t),$$

and (272) follows. Using Hölder's inequality, the proposition can also be proved for intermediate values of β. □

Next, we have:

Proposition 33. *Let $\alpha \ge 0$, $r \ge 0$, $\delta > 1$, and $\beta \in \{0, 1\}$. Then,*

$$e^{|\Lambda_-|(t-1)} \int_t^{\infty} e^{\Lambda_-(s-1)} |\Lambda_-|^{\beta} \frac{1}{s^{\delta}} \mu_{\alpha,r}(\mathbf{k}, s)\ ds \le \frac{\text{const.}}{t^{\delta-1+\beta}} \mu_{\alpha,r}(\mathbf{k}, t), \tag{273}$$

$$\left| \frac{1+k}{2\kappa} \left(e^{\kappa(t-1)} - e^{-\kappa(t-1)} \right) \right| \int_t^{\infty} e^{\Lambda_-(s-1)} |\Lambda_-|^{\beta} \frac{1}{s^{\delta}} \mu_{\alpha,r}(\mathbf{k}, s)\ ds \le \frac{\text{const.}}{t^{\delta-2+\beta}} \mu_{\alpha,r}(\mathbf{k}, t), \tag{274}$$

uniformly in $t \ge 1$ and $\mathbf{k} \in \mathbb{R}^2$.

Proof. We first prove (273). If $\beta = 0$, we have that

$$e^{|\Lambda_-|(t-1)} \int_t^{\infty} e^{\Lambda_-(s-1)} \frac{1}{s^{\delta}} \mu_{\alpha,r}(\mathbf{k}, s)\ ds \le \mu_{\alpha,r}(\mathbf{k}, t) \int_t^{\infty} \frac{1}{s^{\delta}} ds,$$

and (273) follows, and if $\beta = 1$, we have that

$$e^{|\Lambda_-|(t-1)} \int_t^{\infty} e^{\Lambda_-(s-1)} |\Lambda_-| \frac{1}{s^{\delta}} \mu_{\alpha,r}(\mathbf{k}, s)\ ds \le \frac{1}{t^{\delta}} \mu_{\alpha,r}(\mathbf{k}, t) e^{|\Lambda_-|(t-1)} \int_t^{\infty} e^{\Lambda_-(s-1)} |\Lambda_-|\ ds$$
$$\le \frac{1}{t^{\delta}} \mu_{\alpha,r}(\mathbf{k}, t),$$

and (273) follows. We now prove (274). For $k \le 1$, we have that

$$\left| \frac{1+k}{2\kappa} \left(e^{\kappa(t-1)} - e^{-\kappa(t-1)} \right) \right| = \left| (1+k) e^{\kappa(t-1)} (t-1) \frac{\left(1 - e^{-2\kappa(t-1)}\right)}{(-2\kappa(t-1))} \right|$$
$$\le \text{const.} e^{|\Lambda_-|(t-1)} t.$$

The bound (274) now follows as in the proof of (273). For $k > 1$, we easily get that

$$\left| \frac{1+k}{2\kappa} \left(e^{\kappa(t-1)} - e^{-\kappa(t-1)} \right) \right| < \text{const.} e^{|\Lambda_-|(t-1)},$$

and the bound (274) now again follows as in the proof of (273). The proposition can also be proved for intermediate values of β. □

Proposition 34. *Let $\alpha \ge 0$, $r \ge 0$, $\delta > 1$, and $\beta \in \{0, 1\}$. Then,*

$$\left| \frac{1}{k+1} (K_3 - G_3) \right| \int_t^{\infty} e^{\Lambda_-(s-1)} |\Lambda_-|^{\beta} \frac{1}{s^{\delta}} \mu_{\alpha,r}(\mathbf{k}, s)\ ds \le \frac{\text{const.}}{t^{\delta-2+\beta}} \mu_{\alpha,r}(\mathbf{k}, t), \tag{275}$$

$$\left| \frac{k}{k_1} (K_3 - G_3) \right| \int_t^{\infty} e^{\Lambda_-(s-1)} |\Lambda_-|^{\beta} \frac{1}{s^{\delta}} \mu_{\alpha,r}(\mathbf{k}, s)\ ds \le \frac{\text{const.}}{t^{\delta-3+\beta}} \mu_{\alpha,r}(\mathbf{k}, t), \tag{276}$$

uniformly in $t \ge 1$ and $\mathbf{k} \in \mathbb{R}^2$, $k = |\mathbf{k}| = \sqrt{k_1^2 + k_2^2}$.

Proof. Firstly, we prove the bound (275). The representation of K_3 and G_3 gives

$$\left|\frac{1}{k+1}(K_3-G_3)\right| = \left|\frac{1}{2\kappa}(e^{\kappa(s-1)}-e^{-\kappa(s-1)})-\frac{1}{2k}(e^{k(s-1)}-e^{-k(s-1)})\right|$$
$$= \left|\frac{1}{2}\int_{-(t-1)}^{(t-1)}(e^{\kappa s}-e^{ks})ds\right| \leq \text{const.}e^{|\Lambda_-|(t-1)}t.$$

Thus, (275) follows from (273). Next, for $k \leq 1$, we have that

$$\left|\frac{k}{k_1}(K_3-G_3)\right| = \frac{1}{2}\left|\frac{(1+k)\,k}{k_1}\int_{-(t-1)}^{(t-1)}(e^{\kappa s}-e^{ks})ds\right|$$
$$= \frac{1}{2}\left|\frac{(1+k)\,k\,(k-\kappa)}{k_1}\int_{-(t-1)}^{(t-1)}se^{\kappa s}\frac{(1-e^{(k-\kappa)s})}{(k-\kappa)\,s}ds\right|$$
$$= \frac{1}{2}\left|\frac{i\,(1+k)\,k}{(k+\kappa)}\int_{-(t-1)}^{(t-1)}se^{\kappa s}\frac{(1-e^{(k-\kappa)s})}{(k-\kappa)\,s}ds\right|$$
$$\leq \text{const.}e^{|\Lambda_-|(t-1)}t^2.$$

For $k > 1$ we have

$$\left|\frac{k}{k_1}(K_3-G_3)\right| = \left|\frac{(1+k)\,k}{k_1}\frac{1}{2\kappa}(e^{\kappa(t-1)}-e^{k(t-1)}-e^{-\kappa(t-1)}+e^{-k(t-1)})\right.$$
$$\left.+\left(\frac{1}{2\kappa}-\frac{1}{2k}\right)(e^{k(t-1)}-e^{-k(t-1)})\right|$$
$$= \left|\frac{i\,(1+k)\,k(t-1)}{2\kappa(k+\kappa)}e^{\kappa(t-1)}\frac{(1-e^{(k-\kappa)(t-1)})}{(k-\kappa)\,(t-1)}\right.$$
$$-\frac{i\,(1+k)\,k(t-1)}{2\kappa(k+\kappa)}e^{-k(t-1)}\frac{\left(e^{(k-\kappa)(t-1)}-1\right)}{(k-\kappa)\,(t-1)}$$
$$\left.-\frac{i\,(1+k)\,k(t-1)}{\kappa(k+\kappa)}e^{k(t-1)}\frac{(1-e^{-2k(t-1)})}{-2k(t-1)}\right|$$
$$\leq \text{const.}e^{|\Lambda_-|(t-1)}t.$$

The bound (276) now follows as in the proof of (273). □

B.3 Convolution with the semi-group e^{-kt}

In order to bound the integrals over the interval $[1,t]$, we systematically split them into integrals over $[1,\frac{1+t}{2}]$ and integrals over $[\frac{1+t}{2},t]$ and bound the resulting terms separately. We have:

Proposition 35. *Let $\alpha \geq 0$, $r \geq 0$ and $\delta \geq 0$ and $\gamma + 1 \geq \beta \geq 0$. Then,*

$$e^{-k(t-1)} \int_1^{\frac{t+1}{2}} e^{k(s-1)} k^{\beta} \frac{(s-1)^{\gamma}}{s^{\delta}} \mu_{\alpha,r}(\mathbf{k},s)\ ds$$
$$\leq \begin{cases} \text{const.} \frac{1}{t^{\beta}} \bar{\mu}_{\alpha}(\mathbf{k},t), & \text{if } \delta > \gamma + 1 \\ \text{const.} \frac{\log(1+t)}{t^{\beta}} \bar{\mu}_{\alpha}(\mathbf{k},t), & \text{if } \delta = \gamma + 1 \\ \text{const.} \frac{t^{\gamma+1-\delta}}{t^{\beta}} \bar{\mu}_{\alpha}(\mathbf{k},t), & \text{if } \delta < \gamma + 1 \end{cases}$$

uniformly in $t \geq 1$ and $k = |\mathbf{k}| = \sqrt{k_1^2 + k_2^2}$, $\mathbf{k} \in \mathbb{R}^2$.

Proof. The proof is identical to the one of Proposition 31. □

Next, we have:

Proposition 36. *Let $\alpha \geq 0$, $r \geq 0$, $\delta \in \mathbb{R}$, and $\beta \in \{0, 1\}$. Then,*

$$e^{-k(t-1)} \int_{\frac{t+1}{2}}^{t} e^{k(s-1)} k^{\beta} \frac{1}{s^{\delta}} \mu_{\alpha,r}(\mathbf{k},s)\ ds \leq \frac{\text{const.}}{t^{\delta-1+\beta}} \mu_{\alpha,r}(\mathbf{k},t),$$

uniformly in $t \geq 1$ and $k = |\mathbf{k}| = \sqrt{k_1^2 + k_2^2}$, $\mathbf{k} \in \mathbb{R}^2$.

Proof. The proof is as for Proposition 32. Using Hölder's inequality the proposition can also be proved for intermediate values of β. □

Next, we have:

Proposition 37. *Let $\alpha \geq 0$, $r \geq 0$, $\delta > 1$, $\beta \in [0, 1]$ Then,*

$$e^{k(t-1)} \int_t^{\infty} e^{-k(s-1)} k^{\beta} \frac{1}{s^{\delta}} \mu_{\alpha,r}(\mathbf{k},s)\ ds \leq \frac{\text{const.}}{t^{\delta-1+\beta}} \mu_{\alpha,r}(\mathbf{k},t), \tag{277}$$

$$\left| \frac{k+\kappa}{k_1} (K_i - G_i) \right| \int_t^{\infty} e^{-k(s-1)} k^{\beta} \frac{1}{s^{\delta}} \mu_{\alpha,r}(\mathbf{k},s)\ ds \leq \frac{\text{const.}}{t^{\delta-2+\beta}} \mu_{\alpha,r}(\mathbf{k},t), \quad i = 1, 2, \tag{278}$$

$$\left| \frac{1+k}{2k} (e^{k(t-1)} - e^{-k(t-1)}) \right| \int_t^{\infty} e^{-k(s-1)} k^{\beta} \frac{1}{s^{\delta}} \mu_{\alpha,r}(\mathbf{k},s)\ ds \leq \frac{\text{const.}}{t^{\delta-2+\beta}} \mu_{\alpha,r}(\mathbf{k},t), \tag{279}$$

uniformly in $t \geq 1$ and $k = |\mathbf{k}| = \sqrt{k_1^2 + k_2^2}$, $\mathbf{k} \in \mathbb{R}^2$.

Proof. For $k < 1/t$ and $0 < \beta < 1$, we have that

$$e^{k(t-1)} \int_t^{\infty} e^{-k(s-1)} \frac{k^{\beta}}{s^{\delta}} \mu_{\alpha,r}(\mathbf{k},s)\ ds \leq \mu_{\alpha,r}(\mathbf{k},t) \int_t^{\infty} \frac{t^{-\beta}}{s^{\delta}}\ ds \leq \frac{\text{const.}}{t^{\delta-1+\beta}} \mu_{\alpha,r}(\mathbf{k},t),$$

and for $k \geq 1/t$ and $0 < \beta < 1$, we have that

$$e^{k(t-1)} \int_t^{\infty} e^{-k(s-1)} \frac{k^{\beta}}{s^{\delta}} \mu_{\alpha,r}(\mathbf{k},s)\ ds \leq \mu_{\alpha,r}(\mathbf{k},t) \frac{k^{\beta}}{t^{\delta}} e^{k(t-1)} \int_t^{\infty} e^{-k(s-1)}\ ds \leq \frac{k^{\beta}}{t^{\delta}} \frac{1}{k} \mu_{\alpha,r}(\mathbf{k},t)$$
$$= \frac{1}{t^{\delta}} \frac{1}{k^{1-\beta}} \mu_{\alpha,r}(\mathbf{k},t) \leq \frac{1}{t^{\delta-1+\beta}} \mu_{\alpha,r}(\mathbf{k},t),$$

and (277) follows. Next,

$$\begin{aligned}\left|\frac{k+\kappa}{k_1}(K_i-G_i)\right| &= \left|\frac{k+\kappa}{2k_1}(e^{-\kappa(t-1)}-e^{-k(t-1)})\right| \\ &= \left|\frac{i}{2}e^{-k(t-1)}(t-1)\frac{(e^{(k-\kappa)(t-1)}-1)}{(k-\kappa)\,(t-1)}\right| \\ &\leq \text{const.}e^{-k(t-1)}t,\end{aligned}$$

and the bound on (278) now immediately follows from (277). Finally, since for all $k \leq 1$, we have

$$\begin{aligned}\left|\frac{1+k}{2k}(e^{k(t-1)}-e^{-k(t-1)})\right| &= \left|(k+1)e^{k(t-1)}(t-1)\frac{(1-e^{-2k(t-1)})}{(-2k(t-1))}\right| \\ &\leq \text{const.}e^{k(t-1)}t,\end{aligned}$$

and, for $k > 1$,

$$\left|\frac{1+k}{2k}(e^{k(t-1)}-e^{-k(t-1)})\right| \leq \text{const.}e^{k(t-1)},$$

then the bound (279) now follows from the proof of (277). □

Appendix C: Diagonalization of the matrix L

In this section, we construct a matrix S, with the same block structure as L,

$$S = \begin{pmatrix} S_1 & 0 \\ S_3 & S_2 \end{pmatrix},$$

such that

$$S^{-1}LS = D = \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix},$$

with D_1 a diagonal 6×6 matrix with diagonal entries $0, 0, \kappa, \kappa, -\kappa, -\kappa$ and with D_2 a diagonal 3×3 matrix with diagonal entries $0, k, -k$. The matrix S_1 diagonalizes L_1. Namely, $D_1 = S_1^{-1}L_1S_1$, where S_1 is given in (161). The matrix S_2 diagonalizes L_2, namely, $D_2 = S_2^{-1}L_2S_2$, where S_2 is given in (162). We now compute S_3. Since S has to satisfy $LS = SD$, we find for S_3 the equation $L_3S_1 + L_2S_3 = S_3D_1$, which can be solved as follows.

Let $S_3 = S_2Z$, then we obtain the following equation for the matrix Z,

$$S_2^{-1}L_3S_1 = -D_2Z + ZD_1,$$

which can be solved for Z entry by entry, i.e.,

$$Z_{ij} = \frac{1}{-(D_2)_{ii} + (D_1)_{jj}}(S_2^{-1}L_3S_1)_{ij},$$

for $i = 1, 2, 3,\ j = 1, 2, \ldots, 6$. Explicitly, we have the 3×6 matrix

$$L_3S_1 = \begin{pmatrix} 0 & 0 & 0 & -\frac{i\kappa}{k_2} & 0 & \frac{i\kappa}{k_2} \\ 0 & 0 & \frac{i\kappa}{k_2} & 0 & -\frac{i\kappa}{k_2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

and therefore,

$$S_2^{-1}L_3S_1 = \begin{pmatrix} 0 & 0 & \frac{ik_1^2\kappa}{k^2k_2} & \frac{ik_1\kappa}{k^2} & -\frac{ik_1^2\kappa}{k^2k_2} & -\frac{ik_1\kappa}{k^2} \\ 0 & 0 & -\frac{\kappa}{2k} & \frac{k_1\kappa}{2kk_2} & \frac{\kappa}{2k} & -\frac{k_1\kappa}{2kk_2} \\ 0 & 0 & \frac{\kappa}{2k} & -\frac{k_1\kappa}{2kk_2} & -\frac{\kappa}{2k} & \frac{k_1\kappa}{2kk_2} \end{pmatrix},$$

which leads to

$$Z = \begin{pmatrix} 0 & 0 & \frac{ik_1^2}{k^2k_2} & \frac{ik_1}{k^2} & \frac{ik_1^2}{k^2k_2} & \frac{ik_1}{k^2} \\ 0 & 0 & -\frac{\kappa}{2k(\kappa-k)} & \frac{k_1\kappa}{2kk_2(\kappa-k)} & -\frac{\kappa}{2k(\kappa+k)} & \frac{k_1\kappa}{2kk_2(\kappa+k)} \\ 0 & 0 & \frac{\kappa}{2k(\kappa+k)} & -\frac{k_1\kappa}{2kk_2(\kappa+k)} & -\frac{\kappa}{2k(k-\kappa)} & \frac{k_1\kappa}{2kk_2(k-\kappa)} \end{pmatrix}.$$

We finally get for $S_3 = S_2Z$ the matrix in (162). We also need S^{-1}. We find that

$$S^{-1} = \begin{pmatrix} S_1^{-1} & 0 \\ (S^{-1})_3 & S_2^{-1} \end{pmatrix},$$

with $(S^{-1})_3 = -S_2^{-1}S_3S_1^{-1} = -ZS_1^{-1}$, from which we get (163).

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