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A Γ-Convergence Result for Thin Curved Films Bonded to a Fixed Substrate with a Noninterpenetration Constraint

Hamdi ZORGATI*

Abstract The behavior of a thin curved hyperelastic film bonded to a fixed substrate is described by an energy composed of a nonlinearly hyperelastic energy term and a debonding interfacial energy term. The author computes the Γ-limit of this energy under a noninterpenetration constraint that prohibits penetration of the film into the substrate without excluding contact between them.

Keywords Γ-Convergence, Thin curved films, Noninterpenetration constraint
2000 MR Subject Classification 49J45, 74K35

1 Introduction

The purpose of this article is to describe the debonding of a three-dimensional thin curved film from a large, rigid substrate when the thickness of the film goes to zero by means of rigorous convergence analysis.

In [5], A. Braides, I. Fonseca and G. Francfort studied the asymptotic behavior of heterogeneous thin films. They generalized the results obtained by H. Le Dret and A. Raoult in [14] for homogeneous membranes to the heterogeneous case via a compactness result using Γ-convergence arguments.

In [4], K. Bhattacharya, I. Fonseca and G. Francfort took up the work of A. Braides, I. Fonseca and G. Francfort and analyzed the asymptotic behavior of flat bonded thin films, one of them possibly rigid, i.e., a substrate, with a debonding interfacial energy. They studied the different limit behaviors resulting from different scaling in powers of the thickness of the films. They showed that when the interfacial energy is very strong, the limit deformations are continuous across the interface and independent of the thickness variable. In the case of weak interfacial energy, the limit deformations are not continuous across the interface while the independence of the thickness variable subsists in each film resulting in two decoupled Le Dret-Raoult membrane problem. The interfacial energy term explicitly contributes to the limit energy in only one case when it is of the same order of magnitude as the elastic energy. This debonding energy then couples two membrane energies.

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*Institut Für Mathematik, Universität Zürich, Winterthurerstr, 190 CH-8057 Zürich, Switzerland.
E-mail: hamdi.zorgati@math.unizh.ch

In the present work, we study the behavior of a thin curved film bonded to a rigid substrate with a curved upper surface. We suppose that in the reference configuration, contact between the film and the substrate takes place everywhere on the lower surface of the film. We impose a noninterpenetration condition on deformations. Noninterpenetration of matter is a basic physical requirement in solid mechanics. In the context of three-dimensional nonlinear elasticity, a first attempt by J. Ball in [3] was to impose the positivity of the determinant of the deformation gradient almost everywhere. In [8] and [9], P. G. Ciarlet and J. Nečas succeeded in imposing global injectivity by adding a condition on the deformed volume. The latter condition was generalized by Q. Tang in [16] to accommodate less regular deformations. For global injectivity in nonlinear elasticity, see also M. Giaquinta, G. Modica and J. Souček [13].

In our case, we will treat noninterpenetration between the film and the substrate by imposing that every point of the deformed body stays out of the interior of the substrate while allowing at the same time contact on the upper surface of the substrate. This condition seems reasonable from the physical point of view. We thus impose that the film deforms away from the substrate, without prohibiting contact between the two. Our approach is comparable to that of P. G. Ciarlet and J. Nečas in [6] and [7] for unilateral problems.

The equilibrium state of the film is described by the minimizers of an energy depending on the deformation of the film, over a space of admissible deformations which we choose in such a way that there is no interpenetration between the film and the substrate as explained above. We are interested in the asymptotic behavior of this energy and its minimizers, when they exist, when the thickness of the film tends to zero.

We thus consider a hyperelastic curved thin film occupying a domain $\widetilde{\Omega}^h$ of thickness h in contact on its lower surface $\widetilde{\omega}$ with a rigid substrate occupying a domain S. The behavior of this film undergoing a deformation $\tilde{\phi}$ is described by an energy $\tilde{e}^h$ composed of an elastic energy term $\widetilde{E}^h$ and an interfacial energy term $\widetilde{I}^h$. The latter term penalizes the debonding of the film from the substrate. The interfacial energy term admits a density depending on the jump of the deformation $\tilde{\phi}$ through the film-substrate interface. We are thus considering the energy

$$\tilde{e}^h = \widetilde{E}^h + \widetilde{I}^h$$

with

$$\widetilde{E}^h(\tilde{\phi}) = \int_{\widetilde{\Omega}^h} W(\nabla\tilde{\phi})\,dx,$$

$$\widetilde{I}^h(\tilde{\phi}) = h^\alpha \int_{\widetilde{\omega}} \Phi(|[\tilde{\phi}]|)\,d\sigma,$$

where W is the elastic energy density of the film, $h^\alpha\Phi$ is the interfacial energy density where α is a real number, $|[\tilde{\phi}]|$ is the norm of the jump of the deformation through the film-substrate interface and $d\sigma$ is the surface measure on the interface $\widetilde{\omega}$.

After setting the problem and rescaling the energy in order to work on a planar domain with constant thickness, we carry out a second change of variables that flattens the upper surface of the substrate in order to handle the noninterpenetration condition. Then, we compute the Γ-limit of the sequence of energies which describes the asymptotic behavior of almost minimizing sequences. Finally, we rewrite the limit model on the curved surface following [15].

2 Notations and Geometrical Preliminaries

Let (e_1, e_2, e_3) be the canonical orthonormal basis of the Euclidean space $\mathbb{R}^3$. We denote by $|v|$ the norm of a vector v in $\mathbb{R}^3$, by $u \cdot v$ the scalar product of two vectors in $\mathbb{R}^3$ and by $u \wedge v$ their vector product. Let $\mathbb{M}_{33}$ be the space of 3×3 real matrices endowed with the usual norm $|F| = \sqrt{\mathrm{tr}(F^T F)}$. We denote by $A = (a_1 \mid a_2 \mid a_3)$ the matrix in $\mathbb{M}_{33}$ whose i th column is a_i.

We consider a thin curved film of thickness $h > 0$ occupying at rest an open domain $\widetilde{\Omega}_h$. The reference configuration of the film is described as follows. We are thus given a surface $\widetilde{\omega}$, which is the lower surface of the film. This surface is a bounded two-dimensional C^2-submanifold of $\mathbb{R}^3$ and we assume for simplicity that it admits an atlas consisting of one chart. Let ψ be this chart, i.e., a C^2-diffeomorphism from a bounded open subset ω of $\mathbb{R}^2$ onto $\widetilde{\omega}$.

Let $a_\alpha(x) = \psi_{,\alpha}(x)$, $\alpha = 1, 2$, be the vectors of the covariant basis of the tangent plane $T_{\psi(x)}\widetilde{\omega}$ associated with the chart ψ, where $\psi_{,\alpha}$ denotes the partial derivative of ψ with respect to x_α. We assume that there exists $\delta > 0$ such that $|a_1(x) \wedge a_2(x)| \geq \delta$ on $\overline{\omega}$ and we define the unit normal vector $a_3(x) = \frac{a_1(x) \wedge a_2(x)}{|a_1(x) \wedge a_2(x)|}$, which belongs to $C^1(\overline{\omega}; \mathbb{R}^3)$. The vectors $a_1(x)$, $a_2(x)$ and $a_3(x)$ constitute the covariant basis at the point x. We define the contravariant basis by the relations $a^i(x) \cdot a_j(x) = \delta^i_j$, so that $a^\alpha(x) \in T_{\psi(x)}\widetilde{\omega}$ and $a^3(x) = a_3(x)$.

Next, we define a mapping $\Psi \colon \omega \times \mathbb{R} \to \mathbb{R}^3$ by

$$\Psi(x_1, x_2, x_3) = \psi(x_1, x_2) + x_3 a_3(x_1, x_2).$$

It is well known that there exists $h^* > 0$ such that for all $0 < h < h^*$, the restriction of Ψ to $\Omega_h = \omega \times]0, h[$ is a C^2-diffeomorphism on its image by the tubular neighborhood theorem. For such values of h, we set $\widetilde{\Omega}_h = \Psi(\Omega_h)$. Alternatively, we can write

$$\widetilde{\Omega}_h = \{\tilde{x} \in \mathbb{R}^3,\ \exists\, \tilde{\pi}(\tilde{x}) \in \widetilde{\omega},\ \tilde{x} = \tilde{\pi}(\tilde{x}) + \eta a_3(\psi^{-1}(\tilde{\pi}(\tilde{x}))) \text{ with } 0 < \eta < h\},$$

where $\tilde{\pi}$ denotes the orthogonal projection from $\widetilde{\Omega}_h$ onto $\widetilde{\omega}$, which is well defined and of class C^1 for $h < h^*$. Equivalently, every $\tilde{x} \in \widetilde{\Omega}_h$ can be written as

$$\tilde{x} = \tilde{\pi}(\tilde{x}) + [(\tilde{x} - \tilde{\pi}(\tilde{x})) \cdot a_3(\psi^{-1}(\tilde{\pi}(\tilde{x})))] \cdot a_3(\psi^{-1}(\tilde{\pi}(\tilde{x}))).$$

Thus, we have a curvilinear coordinate system in $\widetilde{\Omega}_h$ naturally associated with the chart ψ by

$$(x_1, x_2) = \psi^{-1}(\tilde{\pi}(\tilde{x})) \quad \text{and} \quad x_3 = (\tilde{x} - \tilde{\pi}(\tilde{x})) \cdot a_3(\psi^{-1}(\tilde{\pi}(\tilde{x}))).$$

For all $x \in \overline{\omega}$, we let $A(x) = (a_1(x) \mid a_2(x) \mid a_3(x))$. We note that $A(x)$ is an invertible matrix on $\overline{\omega}$, and that its inverse is given by $A(x)^{-1} = (a^1(x) \mid a^2(x) \mid a^3(x))^T$. We also note that $\det A(x) = |\operatorname{cof} A(x) \cdot e_3| = |a_1(x) \wedge a_2(x)| \geq \delta > 0$ on $\overline{\omega}$. We clearly have

$$\nabla\Psi(x_1, x_2, x_3) = A(x_1, x_2) + x_3 (a_{3,1}(x_1, x_2) \mid a_{3,2}(x_1, x_2) \mid 0).$$

The matrix $\nabla\Psi(x_1, x_2, x_3)$ is thus everywhere invertible in $\overline{\Omega}_h$ and its determinant is strictly positive, and therefore equal to the Jacobian of the change of variables, for h small enough. We assume that the substrate is infinite imposing that Ψ is the restriction to Ω_1 of a C^1-diffeomorphism $\overline{\Psi} : \mathbb{R}^3 \to \mathbb{R}^3$ such that

$$S^c = \overline{\Psi}\{x \in \mathbb{R}^3,\ x_3 > 0\}.$$

In the following, h denotes a generic sequence of real numbers in $]0, h^*[$ that tends to zero. The next convergences are easily established.

Lemma 2.1 *We have*

$$\begin{cases} \nabla\Psi^{-1} \circ \Psi(x_1, x_2, hx_3) \to A(x)^{-1}, \\ \det \nabla\Psi(x_1, x_2, hx_3) \to \det A(x), \end{cases}$$

uniformly on $\overline{\Omega}_1$ *when* $h \to 0$. *In particular,* $\inf_{\overline{\Omega}_1} \det \nabla\Psi(x_1, x_2, hx_3) \geq \frac{\delta}{2} > 0$ *for* h *small enough.*

3 The Three Dimensional and Rescaled Problems

We suppose that the film is made of a homogeneous hyperelastic material with an elastic internal energy density, $W : \mathbb{M}_{33} \to [0, +\infty[$, which is a continuous function verifying the following assumptions

$$\begin{cases} \exists\, c > 0,\ \exists\, p \in]1, +\infty[\,,\ \forall\, F \in \mathbb{M}_{33},\ |W(F)| \leq c(1 + |F|^p), \\ \exists\, \gamma > 0,\ \exists\, \beta \geq 0,\ \forall\, F \in \mathbb{M}_{33},\ W(F) \geq \gamma|F|^p - \beta, \\ \forall\, F, F' \in \mathbb{M}_{33},\ |W(F) - W(F')| \leq c(1 + |F|^{p-1} + |F'|^{p-1})|F - F'|. \end{cases} \tag{3.1}$$

The behavior of the film undergoing a deformation $\tilde{\phi}$ is described by the energy

$$\tilde{e}^h = \widetilde{E}^h + \widetilde{I}^h,$$

where

$$\widetilde{E}^h(\tilde{\phi}) = \int_{\widetilde{\Omega}^h} W(\nabla\tilde{\phi})\, d\tilde{x}, \quad \widetilde{I}^h(\tilde{\phi}) = h^\alpha \int_{\widetilde{\omega}} \Phi\big(|[\tilde{\phi}]|\big)\, d\tilde{\sigma}$$

with

$$[\tilde{\phi}] = \tilde{\phi}(\tilde{x}) - \tilde{x} \quad \text{for almost all } \tilde{x} \in \widetilde{\omega}.$$

The jump $[\tilde{\phi}]$ is well defined for $\tilde{\phi} \in W^{1,p}(\widetilde{\Omega}^h; \mathbb{R}^3)$ and belongs to $W^{1-\frac{1}{p},p}(\widetilde{\omega}; \mathbb{R}^3)$. It is zero if and only if the film remains bonded to the substrate. Note that if $p > 3$, then $[\tilde{\phi}] \in C^0(\overline{\widetilde{\omega}}; \mathbb{R}^3)$. In the sequel, we will assume that $p > 3$. In this case $\tilde{\phi} \in C^0(\overline{\widetilde{\Omega}^h}; \mathbb{R}^3)$ and its image $\tilde{\phi}(\widetilde{\Omega}^h)$ is unambiguously defined in the classical sense. The function $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ appearing in the interface energy term is supposed to be continuous, nondecreasing and verifying

$$\Phi(0) = 0 \quad \text{and} \quad \Phi(t) > 0 \quad \text{for } t \neq 0. \tag{3.2}$$

Since we make no quasiconvexity assumption on the density of the elastic energy, which would exclude some interesting examples from our study such as the Saint Venant-Kirchhoff material, we are not assured of the existence of solutions to the minimization problem: Find $\tilde{\varphi}(h) \in \widetilde{V}^h$ such that

$$\tilde{e}^h(\tilde{\varphi}(h)) = \inf_{\tilde{\phi} \in \widetilde{V}^h} \tilde{e}^h(\tilde{\phi})$$

with

$$\widetilde{V}^h = \{\tilde{\phi} \in W^{1,p}(\widetilde{\Omega}^h, \mathbb{R}^3),\ \tilde{\phi}(\widetilde{\Omega}^h) \subset \overline{S^c} \text{ and } \tilde{\phi}(x) = x \text{ on } \widetilde{\Gamma}\}, \quad p > 3,$$

where $\overline{S^c}$ represents the closure of the complement of the domain occupied by the substrate and $\widetilde{\Gamma}$ is the side surface of $\widetilde{\Omega}^h$. The noninterpenetration condition imposed on elements of $\widetilde{V}^h$ means that such deformations cannot map a point in $\widetilde{\Omega}^h$ into the interior of the substrate. On the other hand, such points may be mapped onto the boundary of the subset. Of course, points in $\widetilde{\omega}$ can be mapped onto $\widetilde{\omega}$, in which case there is contact between the film and the substrate. If $[\tilde{\phi}] = 0$, the film remains bonded, if $[\tilde{\phi}] \neq 0$ it is debonded, either by sliding on $\widetilde{\omega}$ or by moving into S^c. So, this condition prevents the penetration of the film into the substrate. We thus consider a diagonal minimizing sequence $\tilde{\varphi}_h$ for the sequence of energies $\tilde{e}^h$, which always exists, satisfying

$$\tilde{\varphi}_h \in \widetilde{V}^h, \quad \tilde{e}^h(\tilde{\varphi}_h) = \inf_{\tilde{\phi} \in \widetilde{V}^h} \tilde{e}^h(\tilde{\phi}) + h\varepsilon(h) \tag{3.3}$$

with $\varepsilon(h) \to 0$ when $h \to 0$. We start by flattening and rescaling the minimizing problem through a change of variables which enables us to work on a set that is independent of the thickness h. We proceed in two steps.

Let $\tilde{x} \in \widetilde{\Omega}^h$, there exists an $x \in \Omega_h$ such that $\tilde{x} = \Psi(x)$, where

$$\Omega_h = \big\{x \in \mathbb{R}^3,\ \exists\, x' \in \omega,\ x = x' + \eta e_3,\ 0 < \eta < h\big\}.$$

If $\tilde{\phi}_h$ is a deformation of the curved film in its reference configuration, we define for every $x \in \Omega_h$, $\phi_h : \Omega_h \to \mathbb{R}^3$ by

$$\phi_h(x) = \tilde{\phi}_h(\Psi(x)).$$

Knowing that for a deformation $\tilde{\phi} : \widetilde{\Omega}^h \to \mathbb{R}^3$ in membrane mode, the elastic energy is of the order of h when h tends to zero, we are interested in the limiting behavior of the energy per unit thickness, $\frac{1}{h}\tilde{e}^h(\tilde{\phi})$. For a deformation $\phi_h : \Omega_h \to \mathbb{R}^3$ we thus consider the rescaled energy

$$\begin{aligned} e^h(\phi_h) = \frac{1}{h}\tilde{e}^h(\tilde{\phi}) = \frac{1}{h}\int_{\Omega_h} & W(\nabla\phi_h(x)(\nabla\Psi)^{-1}(x)) \det \nabla\Psi(x)\, dx \\ & + h^{\alpha-1}\int_{\omega} \Phi(|[\phi_h]|)|\operatorname{cof}\nabla\Psi(x)e_3| d\sigma \end{aligned}$$

with

$$|[\phi_h]| = |\phi_h(x_1, x_2, 0) - \Psi(x_1, x_2, 0)|.$$

We define the map $z_h : \Omega_h \to \mathbb{R}^3$ by setting

$$z_h(x_1, x_2, x_3) = \Big(x_1, x_2, \frac{x_3}{h}\Big).$$

The map z_h sends Ω_h on

$$\Omega_1 = \{x \in \mathbb{R}^3,\ (x_1, x_2) \in \omega \text{ and } 0 < x_3 < 1\}.$$

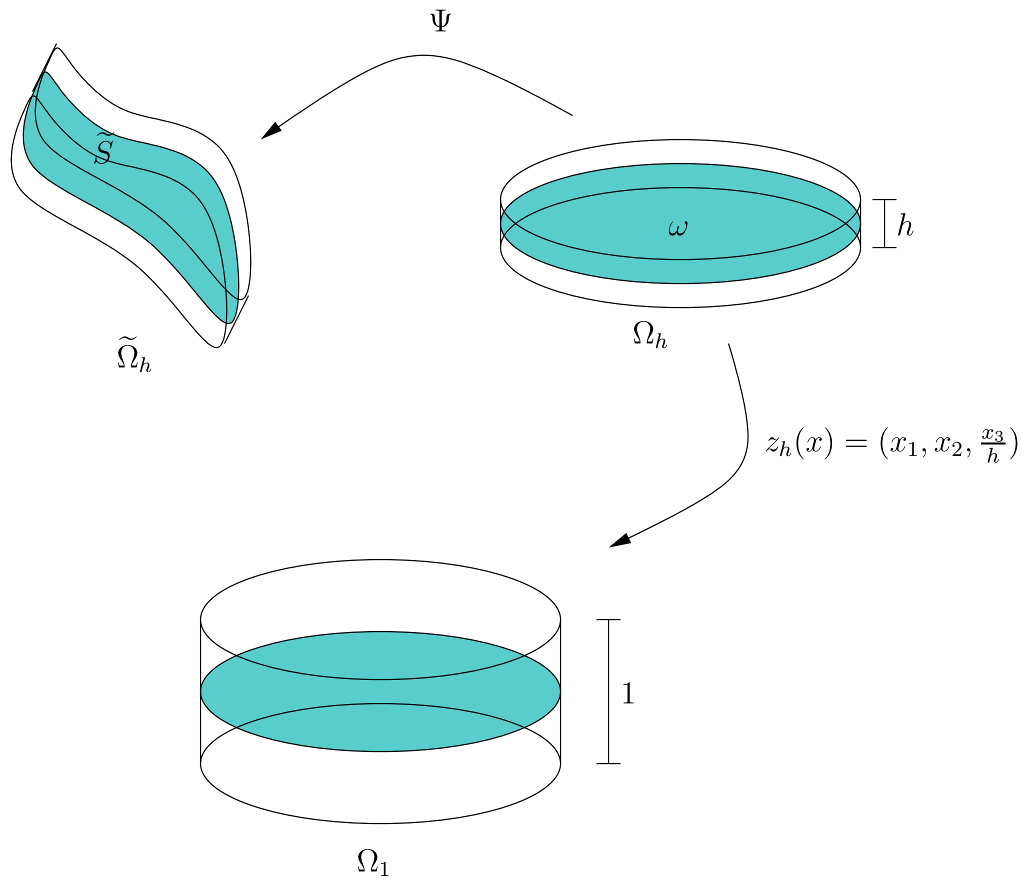

Figure 1

With every deformation ϕ of Ω_h, we associate a deformation $\phi_h : \Omega_1 \to \mathbb{R}^3$ defined by

$$\phi_h(x) = \phi(z_h^{-1}(x)).$$

We set $e(h)(\phi_h) = e^h(\phi)$. Thus, we have

$$e(h)(\phi) = E(h)(\phi) + I(h)(\phi)$$

with

$$E(h)(\phi) = \int_{\Omega_1} W\Big[\Big(\phi_{,1}(x)\,\Big|\,\phi_{,2}(x)\,\Big|\,\frac{1}{h}\phi_{,3}(x)\Big)A_h(x)\Big]\,d_h(x)\,dx,$$
$$I(h)(\phi) = h^{\alpha-1}\int_\omega \Phi(|[\phi]|)\,|\operatorname{cof}\nabla\Psi(x_1,x_2,hx_3)e_3|d\sigma,$$

where $[\phi]$ is defined as above, $d_h(x) = \det\nabla\Psi(x_1,x_2,hx_3)$ and $A_h(x) = \nabla\Psi^{-1}\circ\Psi(x_1,x_2,hx_3)$. We now let

$$\varphi(h) = \tilde{\varphi}_h(\Psi(z_h^{-1})).$$

Relation (3.3) becomes

$$\varphi(h) \in V(h), \quad e(h)(\varphi(h)) = \inf_{\phi \in V^h} e(h)(\phi) + \varepsilon(h) \tag{3.4}$$

with $\varepsilon(h) \to 0$ when $h \to 0$ and

$$V(h) = \{\phi \in W^{1,p}(\Omega_1;\mathbb{R}^3),\ \phi(\Omega_1) \subset \overline{S^c} \text{ and } \phi(x) = \Psi(x_1, x_2, hx_3) \text{ on } \partial\omega \times (0,1)\}.$$

The above change of variables enables us to work on a flat domain independent of the thickness of the curved film.

We now carry out another change of variables in the target space, which will enable us to flatten the upper surface of the substrate. This change of variables makes it possible to simplify the noninterpenetration condition and facilitates the computation of the upper bound of the Γ-limit. The noninterpenetration condition states that the deformed film stays outside of S. For all $x \in \Omega_1$ we set

$$\bar{\phi}(x) = \overline{\Psi}^{-1}(\phi(x)).$$

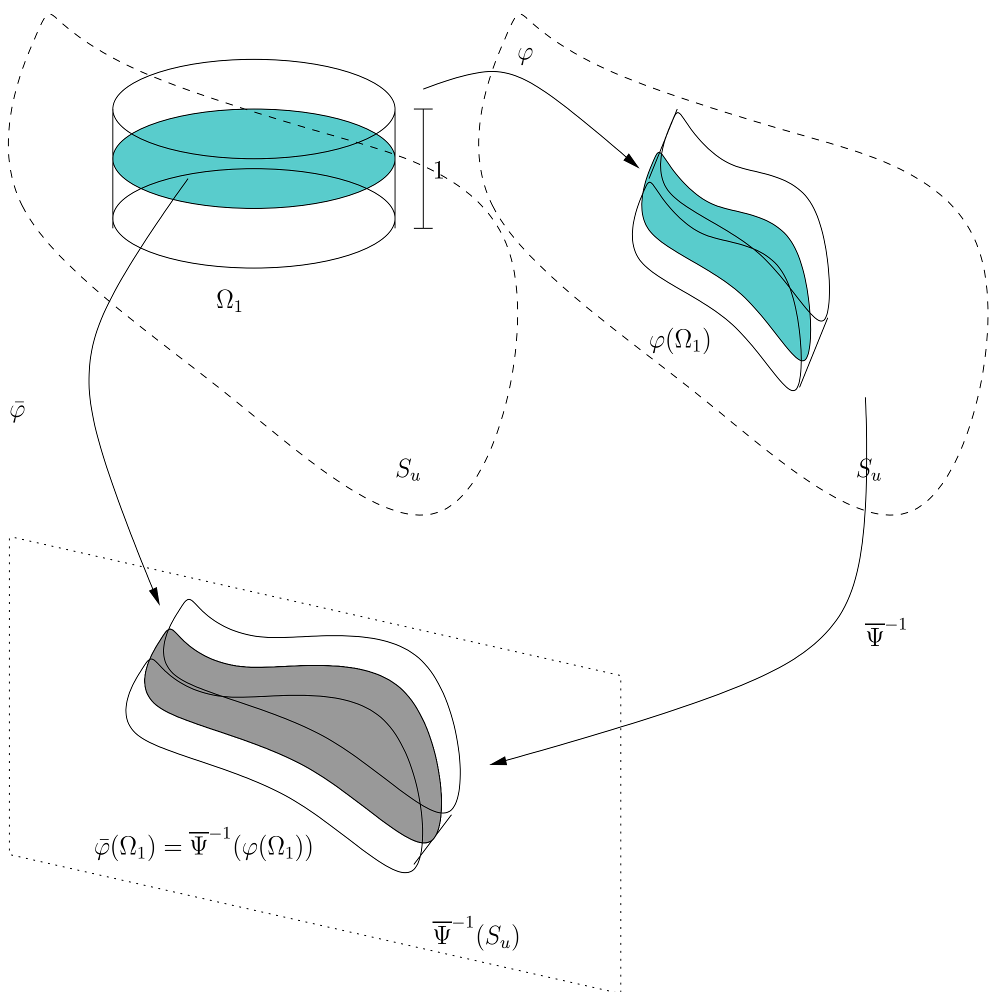

Figure 2

In terms of $\bar{\phi}$, the noninterpenetration condition simplifies as

$$\bar{\phi}_3 \geq 0.$$

The operator that associates the mapping $\overline{\Psi}^{-1}(\phi)$ with $\phi : \mathbb{R}^3 \to \mathbb{R}^3$ is a Nemytsky operator. In particular, since $\overline{\Psi}^{-1}$ is C^1, if $\phi \in W^{1,p}(\Omega_1;\mathbb{R}^3)$ then $\bar{\phi} \in W^{1,p}(\Omega_1;\mathbb{R}^3)$. Moreover we have

$$\nabla\phi(x) = \nabla\overline{\Psi}(\bar{\phi}(x))\nabla\bar{\phi}(x),$$

which is equivalent to

$$\partial_i\phi(x) = \nabla\overline{\Psi}(\bar{\phi}(x))\partial_i\bar{\phi}(x).$$

Finally, we obtain

$$\Big(\partial_\alpha\phi \,\Big|\, \frac{1}{h}\,\partial_3\phi\Big) = \nabla\overline{\Psi}(\bar{\phi})\Big(\partial_\alpha\bar{\phi}(x) \,\Big|\, \frac{1}{h}\,\partial_3\bar{\phi}\Big).$$

Let us define

$$\bar{e}(h)(\bar{\phi}) = \overline{E}(h)(\bar{\phi}) + \overline{I}(h)(\bar{\phi}) = e(h)(\phi)$$

with

$$\overline{E}(h)(\bar{\phi}) = E(h)(\phi) = \int_{\Omega_1} W\Big[\nabla\overline{\Psi}(\bar{\phi}(x))\Big(\bar{\phi}_{,1}(x)\,\Big|\,\bar{\phi}_{,2}(x)\,\Big|\,\frac{1}{h}\bar{\phi}_{,3}(x)\Big)A_h(x)\Big]\; d_h(x)\,dx,$$

$$\overline{I}(h)(\bar{\phi}) = I(h)(\phi) = h^{\alpha-1}\int_\omega \Phi(|[\overline{\Psi}(\bar{\phi})]|)\,|\operatorname{cof}\nabla\overline{\Psi}(x_1,x_2,hx_3)e_3|d\sigma,$$

where

$$[\overline{\Psi}(\bar{\phi})] = \overline{\Psi}(\bar{\phi}(x_1,x_2,0)) - \overline{\Psi}(x_1,x_2,0).$$

Setting

$$\bar{\varphi}(h)(x) = \overline{\Psi}^{-1}(\varphi(h)(x)),$$

relation (3.4) becomes

$$\bar{\varphi}(h) \in \overline{V}(h), \quad \bar{e}(h)(\bar{\varphi}(h)) = \inf_{\bar{\phi}\in\overline{V}(h)} \bar{e}(\bar{\phi}) + \varepsilon(h)$$

with $\varepsilon(h) \to 0$ when $h \to 0$ and

$$\overline{V}(h) = \{\bar{\phi} \in W^{1,p}(\Omega_1;\mathbb{R}^3),\ \bar{\phi}(\Omega_1) \subset \{x_3 \geq 0\} \text{ and } \bar{\phi}(x) = (x_1,x_2,hx_3) \text{ on } \partial\omega\times(0,1)\}.$$

4 Computation of the Γ-Limit

Before starting the computation of the Γ-limit of the sequence of energies $\bar{e}(h)$, we begin by extending this energy to $L^p(\Omega_1;\mathbb{R}^3)$ by setting

$$\text{for every } \bar{\phi} \in L^p(\Omega_1;\mathbb{R}^3), \quad \bar{e}^*(h)(\bar{\phi}) = \begin{cases} \bar{e}(h)(\bar{\phi}), & \text{if } \bar{\phi} \in \overline{V}(h), \\ +\infty, & \text{otherwise.} \end{cases}$$

The limit energy that we obtain by Γ-convergence is relaxed, i.e., the internal energy density is quasiconvexified. We cannot avoid this even if the three dimensional density is quasiconvex, since quasiconvexity is not retained by the density W_0 which will appear in the limit models (see [14]). We recall that the quasiconvex envelope of a function $W : \Omega \times \mathbb{R}^3 \times \mathbb{M}_{32} \to \mathbb{R}$ is given by

$$QW = \sup\{Z : \mathbb{M}_{32} \to \mathbb{R},\ Z \text{ quasiconvex and } Z \le W\},$$

and that a function $Z : \Omega \times \mathbb{R}^3 \times \mathbb{M}_{32} \to \mathbb{R}$ is quasiconvex if and only if

$$Z(x, y, A) \le \frac{1}{\operatorname{meas} D} \int_D Z(x, y, A + \nabla\theta(x))dx$$

for every bounded open set $D \subset \mathbb{R}^3$, every $A \in \mathbb{M}_{32}$ and every $\theta \in W_0^{1,\infty}(D; \mathbb{R}^3)$. The quasiconvex envelope of W may also be computed by the following representation formula (see [10])

$$QW(x, y, A) = \inf_{\theta \in W_0^{1,\infty}(D;\mathbb{R}^3)} \Big(\frac{1}{\operatorname{meas} D} \int_D W(x, y, A + \nabla\theta(x))dx \Big).$$

For every $F = (z_1 \mid z_2 \mid z_3) \in \mathbb{M}_{33}$ we denote by $\overline{F}$ the matrix in $\mathbb{M}_{32}$ defined by $\overline{F} = (z_1 \mid z_2)$. We introduce, in a similar fashion as in Acerbi, Buttazzo and Percivale [1] for nonlinearly elastic strings and Le Dret and Raoult [14] for membranes, the function $W_0 : \omega \times \mathbb{R}^3 \times \mathbb{M}_{32} \to \mathbb{R}$ defined by

$$W_0(x, y, \overline{F}) = \inf_{z \in \mathbb{R}^3} W(\nabla\overline{\Psi}(y)(\overline{F} \mid z)A_0(x))$$

with $A_0(x) = \nabla\overline{\Psi}^{-1}\big(\overline{\Psi}(x_1, x_2, 0)\big)$. This function is well defined thanks to the continuity of W and its growth and coercivity properties (3.1).

Proposition 4.1 *The function W_0 is continuous and verifies the following growth and coercivity properties*

$$\begin{cases} \exists\, c' > 0,\ \forall \overline{F} \in \mathbb{M}_{32},\ \forall y \in \mathbb{R}^3,\ \forall x \in \overline{\omega},\ |W_0(x, y, \overline{F}\,)| \le c'(1 + |\overline{F}\,|^p), \\ \exists\, \gamma' > 0,\ \exists\, \beta' \ge 0,\ \forall \overline{F} \in \mathbb{M}_{32},\ \forall y \in \mathbb{R}^3,\ \forall x \in \overline{\omega},\ W_0(x, y, \overline{F}\,) \ge \gamma'|\overline{F}\,|^p - \beta'. \end{cases}$$

Proof The function W_0 is upper semicontinuous as an infimum of continuous functions. To obtain the continuity of W_0, it is thus enough to show that it is lower semicontinuous. We consider a sequence $(x_n, y_n, \overline{F}_n) \in \overline{\omega} \times \mathbb{R}^3 \times \mathbb{M}_{32}$ converging to $(x, y, \overline{F})$ when $n \to +\infty$. Thanks to the coercivity of W, for all $z \in \mathbb{R}^3$ we have

$$\begin{aligned} \alpha|z|^p &= \alpha|\nabla\overline{\Psi}(y)(\nabla\overline{\Psi}(y))^{-1}(0 \mid z)A_0(x)(A_0(x))^{-1}|^p \\ &\le \alpha\|(\nabla\overline{\Psi})^{-1}\|_{L^\infty(\mathbb{R}^3)}\|(A_0)^{-1}\|_{L^\infty(\omega)}|\nabla\overline{\Psi}(y)(0 \mid z)A_0(x)|^p \\ &\le \|(\nabla\overline{\Psi})^{-1}\|_{L^\infty(\mathbb{R}^3)}\|(A_0)^{-1}\|_{L^\infty(\omega)}W(\nabla\overline{\Psi}(y)(0 \mid z)A_0(x)) + \beta. \end{aligned}$$

Consequently, there exists a compact set K such that for all N, the infimum W on $z \in \mathbb{R}^3$ is reached at a point $z^n \in K$. We proceed then as in [14]. We extract a subsequence still noted n such that $W_0(x_n, y_n, \overline{F}_n)$ converges when $n \to +\infty$, from which we extract another subsequence such that $z_n \to Z \in K$. Thanks to the continuity of W, we have

$$W_0(x_n, y_n, \overline{F}_n) = W(\nabla\overline{\Psi}(y_n)(\overline{F}_n \mid z_n)A_0(x_n)) \longrightarrow W(\nabla\overline{\Psi}(y)(\overline{F} \mid z)A_0(x)) \ge W_0(x, y, \overline{F}\,).$$

As this is true for any subsequence such that $W_0(x_n, y_n, \overline{F}_n)$ converges, we deduce that

$$\liminf\, W_0(x_n, y_n, \overline{F}_n) \geq W_0(x, y, \overline{F}).$$

Consequently W_0 is lower semicontinuous and thus continuous. Let us consider $(x, y, \overline{F}\,) \in \overline{\omega} \times \mathbb{R}^3 \times \mathbb{M}_{32}$ and $z_0 \in \mathbb{R}^3$ that achieves the minimum in the definition of W_0. We have

$$\begin{aligned} W_0(x, y, \overline{F}\,) &= W(\nabla\overline{\Psi}(y)(\overline{F} \mid z_0)A_0(x)) \geq \alpha|\nabla\overline{\Psi}(y)(\overline{F} \mid z_0)A_0(x)|^p - \beta \\ &\geq \alpha|\nabla\overline{\Psi}(y)(\overline{F} \mid 0)A_0(x)|^p - \beta. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} |\overline{F}|^p &= |\nabla\overline{\Psi}(y)(\nabla\overline{\Psi}(y))^{-1}(\overline{F} \mid 0)A_0(x)(A_0(x))^{-1}|^p \\ &\leq \|(\nabla\overline{\Psi})^{-1}\|_{L^\infty}\|(A_0)^{-1}\|_{L^\infty}|\nabla\overline{\Psi}(y)(\overline{F} \mid 0)A_0(x)|^p, \end{aligned}$$

which gives us the coercivity. Lastly, W_0 is nonnegative apart from a compact set on which it is bounded thanks to its continuity. For all $(x, y, \overline{F}\,)$ apart from this compact set, we have

$$\begin{aligned} |W_0(x, y, \overline{F}\,)| &= W_0(x, y, \overline{F}) = W(\nabla\overline{\Psi}(y)(\overline{F} \mid z_0)A_0(x)) \leq W(\nabla\overline{\Psi}(y)(\overline{F} \mid 0)A_0(x)) \\ &\leq c(1 + |\nabla\overline{\Psi}(y)(\overline{F} \mid 0)A_0(x)|^p) \leq c(1 + \|\nabla\overline{\Psi}\,\|_{L^\infty}\|A_0\|_{L^\infty}|\overline{F}|^p) \leq c\big(1 + |\overline{F}|^p\big). \end{aligned}$$

Thus the growth property holds true.

Next is a lemma that gives the behavior of deformations with bounded energy.

Lemma 4.1 *Let $\bar{\phi}(h) \in L^p(\Omega_1; \mathbb{R}^3)$ be a sequence verifying*

$$\bar{e}^*(h)(\bar{\phi}(h)) \leq c,$$

where c is a strictly positive constant independent of h. Then $\bar{\phi}(h)$ is uniformly bounded in $W^{1,p}(\omega; \mathbb{R}^3)$ and its limit points for the weak topology of $W^{1,p}(\omega; \mathbb{R}^3)$ belong to

$$\overline{V}_M = \{\bar{\phi} \in W^{1,p}(\Omega_1; \mathbb{R}^3),\ \bar{\phi}(\Omega_1) \subset \{x_3 \geq 0\},\ \bar{\phi}_{,3} = 0 \ \textit{and}\ \bar{\phi}(x) = (x_1, x_2, 0) \ \textit{on}\ \partial\omega \times (0,1)\}.$$

Moreover, in the case $\alpha < 1$, there is only one limit point,

$$\bar{\phi}(0)(x) = (x_1, x_2, 0) \quad \textit{in}\ \Omega_1.$$

Proof Let us consider $\bar{\phi}(h) \in L^p(\Omega_1; \mathbb{R}^3)$ verifying

$$\bar{e}^*(h)(\bar{\phi}(h)) \leq c < +\infty.$$

This implies that $\bar{\phi}(h) \in \overline{V}(h)$ and that

$$\bar{e}^*(h)(\bar{\phi}(h)) = \bar{e}(h)(\bar{\phi}(h)).$$

Thus, we have the following estimates

$$\overline{E}(h)(\bar{\phi}(h)) \leq c \quad \text{and} \quad \overline{I}(h)(\bar{\phi}(h)) \leq c. \tag{4.1}$$

The first estimate can also be written as

$$\int_{\Omega_1} W\Big(\nabla\overline{\Psi}(\bar{\phi}(x))\Big(\bar{\phi}_{,\alpha}(h)\,\Big|\,\frac{1}{h}\bar{\phi}_{,3}(h)\Big)A_h(x)\Big)\,d_h(x)dx \le c,$$

where $\bar{\phi}_{,\alpha}(h) = (\bar{\phi}_{,1}(h) \mid \bar{\phi}_{,2}(h)) \in \mathbb{M}_{32}$. Properties (3.1) of W imply that

$$\Big\|\nabla\overline{\Psi}(\bar{\phi})\Big(\bar{\phi}_{,\alpha}(h)\,\Big|\,\frac{1}{h}\bar{\phi}_{,3}(h)\Big)A_h\Big\|_{L^p(\Omega_1;\mathbb{R}^3)} \le c.$$

On the other hand, we have

$$\begin{aligned}&\Big\|\Big(\bar{\phi}_{,\alpha}(h)\,\Big|\,\frac{1}{h}\bar{\phi}_{,3}(h)\Big)\Big\|_{L^p(\Omega_1;\mathbb{R}^3)}\\ &= \Big\|(\nabla\overline{\Psi}(\bar{\phi}))^{-1}(\nabla\overline{\Psi}(\bar{\phi}))\Big(\bar{\phi}_{,\alpha}(h)\,\Big|\,\frac{1}{h}\bar{\phi}_{,3}(h)\Big)\,A_h^{-1}\,A_h\Big\|_{L^p(\Omega_1;\mathbb{R}^3)}\\ &\le \|(\nabla\overline{\Psi})^{-1}\|_{L^\infty(\Omega_1;\mathbb{M}_{33})}\|A_h^{-1}\|_{L^\infty(\Omega_1;\mathbb{M}_{33})}\Big\|\nabla\overline{\Psi}(\bar{\phi})\Big(\bar{\phi}_{,\alpha}(h)\,\Big|\,\frac{1}{h}\bar{\phi}_{,3}(h)\Big)A_h\Big\|_{L^p(\Omega_1;\mathbb{R}^3)} \le c.\end{aligned}$$

Thus, we have for h small enough

$$\|\nabla\bar{\phi}(h)\|_{L^p(\Omega_1;\mathbb{R}^3)} \le c$$

and the Poincaré inequality implies that $\bar{\phi}(h)$ is uniformly bounded in $W^{1,p}(\omega;\mathbb{R}^3)$. This implies that, for a subsequence h, there exists a $\bar{\phi}(0) \in W^{1,p}(\Omega_1;\mathbb{R}^3)$ such that

$$\bar{\phi}(h) \rightharpoonup \bar{\phi}(0) \quad \text{in } W^{1,p}(\Omega_1;\mathbb{R}^3).$$

Thus

$$\bar{\phi}(h) \to \bar{\phi}(0) \quad \text{in } L^p(\Omega_1;\mathbb{R}^3). \tag{4.2}$$

In addition, since

$$\frac{1}{h}\|\bar{\phi}_{,3}(h)\|_{L^p(\Omega_1;\mathbb{R}^3)} \le \Big\|\Big(\bar{\phi}_{,\alpha}(h)\,\Big|\,\frac{1}{h}\bar{\phi}_{,3}(h)\Big)\Big\|_{L^p(\Omega_1;\mathbb{R}^3)} \le c,$$

we have

$$\bar{\phi}_{,3}(0) = 0.$$

Since $p > 3$, $W^{1,p}(\Omega_1;\mathbb{R}^3)$ is compactly embedded in $C^0(\overline{\Omega}_1;\mathbb{R}^3)$. Hence,

$$\bar{\phi}(h) \to \bar{\phi}(0) \quad \text{uniformly on } \overline{\Omega}_1.$$

The noninterpenetration condition thus passes to the limit and

$$\bar{\phi}(0)(\overline{\Omega}_1) \subset \{x_3 \ge 0\}.$$

Thanks to the continuity of the trace operator, we have

$$\bar{\phi}(0) \in \overline{V}^0.$$

For $\alpha < 1$, since $\overline{I}(h)(\bar{\phi}(h)) \leq c$, we have

$$\lim_{h\to 0} \int_\omega \Phi(|[\overline{\Psi}(\bar{\phi}(h))]|)\,|\operatorname{cof}\nabla\overline{\Psi}(x_1,x_2,hx_3)\,e_3|\,dx = 0.$$

Fatou's lemma and the continuity of Φ imply that

$$\begin{aligned} 0 &= \liminf_{h\to 0} \int_\omega \Phi(|[\overline{\Psi}(\bar{\phi}(h))]|)\,|\operatorname{cof}\nabla\overline{\Psi}(x_1,x_2,hx_3)\,e_3|\,dx \\ &\geq \int_\omega \liminf_{h\to 0} \Phi(|[\overline{\Psi}(\bar{\phi}(h))]|)\,d_0(x)dx \geq \int_\omega \Phi\Big(\liminf_{h\to 0} |[\overline{\Psi}(\bar{\phi}(h))]|\Big)\,d_0(x)dx, \end{aligned}$$

from which it follows that, by (3.2)

$$\liminf_{h\to 0} |[\overline{\Psi}(\bar{\phi}(h))]| = 0,$$

almost everywhere in ω. But we have seen that $\bar{\phi}(h) \to \bar{\phi}(0)$ uniformly. Therefore

$$\overline{\Psi}(\bar{\phi}(0)(x_1,x_2,0)) = \overline{\Psi}(x_1,x_2,0) \quad \text{ on } \omega.$$

The injectivity of $\overline{\Psi}$ implies that

$$\bar{\phi}(0)(x_1,x_2,0) = (x_1,x_2,0) \quad \text{in } \omega.$$

Since $\bar{\phi}_{,3}(0) = 0$, we finally obtain

$$\bar{\phi}(0)(x) = (x_1,x_2,0) \quad \text{in } \Omega_1,$$

which completes the study in the case $\alpha < 1$.

We can now, compute the Γ-limit of our energy which is given by the following theorem.

Theorem 4.1 *The sequence of energies $\bar{e}^*(h)$ Γ-converges for the strong topology of $L^p(\Omega_1;\mathbb{R}^3)$ when $h \to 0$ to a functional $\bar{e}^*(0)$, defined by*

$$\bar{e}^*(0)(\bar{\phi}) = \begin{cases} \displaystyle\int_\omega QW_0(x,\bar{\phi}(x),(\bar{\phi}_{,1}(x) \mid \bar{\phi}_{,2}(x)))d_0(x)\,dx, & \text{if } \bar{\phi} \in \overline{V}_M, \\ +\infty, & \text{otherwise,} \end{cases}$$

for $\alpha > 1$,

$$\bar{e}^*(0)(\bar{\phi}) = \begin{cases} \displaystyle\int_\omega QW_0(x,\bar{\phi}(x),(\bar{\phi}_{,1}(x) \mid \bar{\phi}_{,2}(x)))d_0(x)\,dx & \\ \displaystyle+ \int_\omega \Phi(|[\overline{\Psi}(\bar{\phi})]|)d_0(x)dx, & \text{if } \bar{\phi} \in \overline{V}_M, \\ +\infty, & \text{otherwise,} \end{cases}$$

for $\alpha = 1$ and

$$\bar{e}^*(0)(\bar{\phi}) = \begin{cases} \displaystyle\int_\omega QW_0(x,\mathrm{id}(x),(e_1 \mid e_2))d_0(x)\,dx, & \text{if } \bar{\phi} = \mathrm{id}, \\ +\infty, & \text{otherwise,} \end{cases}$$

for $\alpha < 1$, with $d_0(x) = \det A(x)$, where $A(x) = \nabla\overline{\Psi}(x_1,x_2,0) = (a_1(x) \mid a_2(x) \mid a_3(x))$.

The proof of the theorem is a consequence of the following two propositions.

Proposition 4.2 *We have*

$$\bar{e}^*(0) \le \Gamma - \liminf \bar{e}^*(h).$$

Proof To obtain this, we have to show that for every $\bar{\phi}^0 \in L^p(\Omega_1; \mathbb{R}^3)$ and $\bar{\phi}(h) \in L^p(\Omega_1; \mathbb{R}^3)$ verifying

$$\bar{\phi}(h) \to \bar{\phi}^0 \quad \text{in } L^p(\Omega_1; \mathbb{R}^3),$$

we have

$$\liminf \bar{e}^*(h)(\bar{\phi}(h)) \ge \bar{e}^*(0)(\bar{\phi}^0).$$

The case when we have

$$\bar{e}^*(h)(\bar{\phi}(h)) = +\infty$$

is obvious. Let us thus consider $\bar{\phi}^0 \in \overline{V}_M$ for $\alpha \ge 1$ and $\bar{\phi}^0 = \mathrm{id}$ for $\alpha < 1$, with

$$\bar{e}^*(h)(\bar{\phi}(h)) < +\infty.$$

Thus, $\bar{\phi}(h) \in \overline{V}(h)$ and

$$\bar{\phi}(h) \rightharpoonup \bar{\phi}^0 \quad \text{in } W^{1,p}(\Omega_1; \mathbb{R}^3). \tag{4.3}$$

We propose to show that

$$\liminf \bar{e}(h)(\bar{\phi}(h)) \ge \bar{e}(0)(\bar{\phi}^0),$$

where

$$\bar{e}(0)(\bar{\phi}^0) = \int_\omega QW_0(x, \bar{\phi}^0(x), (\bar{\phi}^0_{,1}(x) \mid \bar{\phi}^0_{,2}(x)))d_0(x)\,dx + \delta(\alpha)\int_\omega \Phi(|[\overline{\Psi}(\,\overline{\phi^0}\,)]|)d_0(x)dx$$

with $\delta(\alpha) = 1$ if $\alpha = 1$ and $\delta(\alpha) = 0$ otherwise. We have

$$\begin{aligned}\overline{E}(h)(\bar{\phi}(h)) &= \int_{\Omega_1} W\Big[\nabla\overline{\Psi}(\bar{\phi}(h))\Big(\bar{\phi}_{,1}(h)\,\Big|\,\bar{\phi}_{,2}(h)\,\Big|\,\frac{1}{h}\bar{\phi}_{,3}(h)\Big)A_h\Big]\,d_h\,dx \\ &= \int_{\Omega_1} \Big\{W\Big[\nabla\overline{\Psi}(\bar{\phi}(h))\Big(\bar{\phi}_{,1}(h)\,\Big|\,\bar{\phi}_{,2}(h)\,\Big|\,\frac{1}{h}\bar{\phi}_{,3}(h)\Big)\;A_0\Big] + R(x,h,\bar{\phi}(h))\Big\}d_h\,dx,\end{aligned}$$

where

$$\begin{aligned}R(x,h,\bar{\phi}(h)) = {}& W\Big[\nabla\overline{\Psi}(\bar{\phi}(h))(x)\Big(\bar{\phi}_{,1}(h)(x)\,\Big|\,\bar{\phi}_{,2}(h)(x)\,\Big|\,\frac{1}{h}\bar{\phi}_{,3}(h)(x)\Big)\;A_h(x)\Big] \\ &- W\Big[\nabla\overline{\Psi}(\bar{\phi}(h))(x)\Big(\bar{\phi}_{,1}(h)(x)\,\Big|\,\bar{\phi}_{,2}(h)(x)\,\Big|\,\frac{1}{h}\bar{\phi}_{,3}(h)(x)\Big)\;A_0(x)\Big].\end{aligned}$$

Since

$$A_h \to A_0 \quad \text{in } C^0(\Omega_1)$$

and due to the third property of W in (3.1), we obtain that

$$\int_{\Omega_1} R(x,h,\bar{\phi}(h))d_h(x)\,dx \to 0 \quad \text{when } h \to 0. \tag{4.4}$$

Then, we have

$$\begin{aligned}\overline{E}(h)(\bar{\phi}(h)) &\geq \int_{\Omega_1}\{W_0(x,\bar{\phi}(h)(x),(\bar{\phi}_{,1}(h)(x)\mid\bar{\phi}_{,2}(h)(x)))+R(x,h,\bar{\phi}(h))\}d_h(x)dx\\ &\geq \int_{\Omega_1}\{QW_0(x,\bar{\phi}(h)(x),(\bar{\phi}_{,1}(h)(x)\mid\bar{\phi}_{,2}(h)(x)))+R(x,h,\bar{\phi}(h))\}d_h(x)dx,\end{aligned}$$

using the definition of W_0 and the quasiconvex envelop. Passing to the lim inf when h goes to zero, we obtain using (4.4)

$$\liminf \overline{E}(h)(\bar{\phi}(h)) \geq \liminf \int_{\Omega_1} QW_0(x,\bar{\phi}(h)(x),(\bar{\phi}_{,1}(h)(x)\mid\bar{\phi}_{,2}(h)(x)))d_h(x)dx.$$

The convergence of d_h to d_0 in $C^0(\Omega_1)$ implies that

$$\begin{aligned}&\liminf \int_{\Omega_1} QW_0(x,\bar{\phi}(h)(x),(\bar{\phi}_{,1}(h)(x)\mid\bar{\phi}_{,2}(h)(x)))d_h(x)dx\\ &= \liminf \int_{\Omega_1} QW_0(x,\bar{\phi}(h)(x),(\bar{\phi}_{,1}(h)(x)\mid\bar{\phi}_{,2}(h)(x)))d_0(x)dx.\end{aligned}$$

Let us consider the function $G : W^{1,p}(\Omega_1;\mathbb{R}^3) \to \mathbb{R}$ defined by

$$G(\bar{\phi}) = \int_{\Omega_1} QW_0(x,\bar{\phi}(x),(\bar{\phi}_{,1}(x)\mid\bar{\phi}_{,2}(x)))d_0(x)dx.$$

This function is lower semicontinuous for the weak topology of $W^{1,p}(\Omega_1;\mathbb{R}^3)$ thanks to the quasiconvexity of QW_0 and the fact that

$$0 \leq QW_0(x,y,\overline{F}) \leq c(1+|\overline{F}|^p)$$

(see [2, 10]). Since

$$\bar{\phi}(h) \rightharpoonup \bar{\phi}_0 \quad \text{in } W^{1,p}(\Omega_1;\mathbb{R}^3),$$

we have

$$\begin{aligned}\liminf \overline{E}(h)(\bar{\phi}(h)) &\geq \liminf G(\bar{\phi}(h)) \geq G(\bar{\phi}^0)\\ &= \int_\omega QW_0(x,\bar{\phi}^0(x),(\bar{\phi}^0_{,1}(x)\mid\bar{\phi}^0_{,2}(x)))d_0(x)dx. \qquad (4.5)\end{aligned}$$

Next, we treat the interfacial energy term. We have

$$\overline{I}(h)(\bar{\phi}(h)) = h^{\alpha-1}\int_\omega \Phi(|[\overline{\Psi}(\bar{\phi}(h))]|)\,d_0(x)dx.$$

There are three cases. Since the interfacial energy term is positive and $\delta(\alpha)=0$ for $\alpha\neq 1$, the case $\alpha \neq 1$ is obvious in the sense that

$$\liminf \overline{I}(h)(\bar{\phi}(h)) \geq 0 = \delta(\alpha)\int_\omega \Phi(|[\overline{\Psi}(\bar{\phi}^0)]|)d_0(x)dx.$$

If $\alpha = 1$, we have

$$\overline{I}(h)(\bar{\phi}(h)) = \int_\omega \Phi(|[\overline{\Psi}(\bar{\phi}(h))]|)\,d_0(x)dx.$$

By (4.3) and the compact embedding, we have that $\bar{\phi}(h)) \to \bar{\phi}^0$ uniformly in $\overline{\Omega}_1$. Thus, Fatou's lemma and the continuity of Φ imply that

$$\liminf \overline{I}(h)(\bar{\phi}(h)) \geq \int_\omega \liminf_{h\to 0} \Phi(|[\overline{\Psi}(\bar{\phi}(h))]|\, d_0(x)dx \geq \int_\omega \Phi(|[\overline{\Psi}(\bar{\phi}^0)]|)d_0(x)dx.$$

Finally, by (4.5) and the above estimates

$$\liminf \bar{e}(h)(\bar{\phi}(h)) \geq \liminf \overline{I}(h)(\bar{\phi}(h)) + \liminf \overline{E}(h)(\bar{\phi}(h)) \geq \bar{e}(0)(\bar{\phi}^0),$$

which implies that

$$\liminf \bar{e}^*(h)(\bar{\phi}(h)) \geq \bar{e}^*(0)(\bar{\phi}^0),$$

and thus

$$\Gamma - \liminf \bar{e}^*(h) \geq \bar{e}^*(0). \tag{4.6}$$

We pass to the computation of the upper bound of the Γ-limit. We will use the following lemma (see [14]).

Lemma 4.2 *Let $X \hookrightarrow Y$ be two Banach spaces such that X is reflexive and compactly embedded in Y. Consider a function $G : X \to \mathbb{R}$ such that $\forall\, v \in X$, $G(v) \geq g(\|v\|_X)$ where g verifies $g(t) \to +\infty$ when $t \to +\infty$. Let G^* be defined by $G^*(v) = G(v)$ if $v \in X$, $G^*(v) = +\infty$ otherwise. Let $\Gamma - G$ denote the sequential lower semicontinuous envelope of G for the weak topology of X and let $\Gamma - G^*$ denote the lower semicontinuous envelope of G^* for the strong topology of Y. Then*

$$\Gamma - G^* = (\Gamma - G)^*.$$

Proposition 4.3 *We have*

$$\Gamma - \limsup \bar{e}^*(h) \leq \bar{e}^*(0).$$

Proof To show this result, we have to find, for all $\bar{\phi}^0 \in L^p(\Omega_1; \mathbb{R}^3)$, a sequence of test-functions $\phi(h)$ converging to $\bar{\phi}^0$ in L^p strong, and verifying

$$\lim e^*(h)(\phi(h)) \leq e^*(0)(\bar{\phi}^0).$$

If $e^*(0)(\bar{\phi}^0) = +\infty$, there is nothing to prove. Hence, we need only to consider the cases $\bar{\phi}^0 \in \overline{V}_M$ for $\alpha \geq 1$ and $\bar{\phi}^0 = \mathrm{id}$ for $\alpha < 1$. Let $\bar{\phi}^0$ be such a deformation. We consider the function $h : \omega \times \mathbb{R}^3 \to \mathbb{R}$ defined by

$$h(x, z) = W[\nabla\overline{\Psi}(\bar{\phi}^0(x))(\bar{\phi}^0_{,1}(x) \mid \bar{\phi}^0_{,2}(x) \mid z + e_3)A_0(x)].$$

It is a Carathéodory function. Thus, the measurable selection lemma (see [12]), implies the existence of a measurable function ξ^0 such that

$$W_0(x, \bar{\phi}^0(x), (\bar{\phi}^0_{,1}(x) \mid \bar{\phi}^0_{,2}(x))) = W[\nabla\overline{\Psi}(\bar{\phi}^0(x))(\bar{\phi}^0_{,1}(x) \mid \bar{\phi}^0_{,2}(x) \mid \xi^0(x) + e_3)\; A_0(x)]$$

and using (3.1), we see that $\xi^0 \in L^p(\omega;\mathbb{R}^3)$. The density of $C_c^\infty(\omega;\mathbb{R}^3)$ in $L^p(\omega;\mathbb{R}^3)$ implies the existence of $\xi_\varepsilon^0 \in C_c^\infty(\omega;\mathbb{R}^3)$ verifying

$$\xi_\varepsilon^0 \to \xi^0 \quad \text{in } L^p(\omega;\mathbb{R}^3) \text{ when } \varepsilon \to 0. \tag{4.7}$$

In order to deal with the noninterpenetration constraint, we consider the sequence $\bar\phi_\varepsilon^0$ defined for every $x \in \omega$ by

$$\bar\phi_\varepsilon^0(x) = \bar\phi^0(x) + \varepsilon\,\mathrm{dist}(x,\partial\omega)e_3.$$

This sequence belongs to V_M since the distance to the boundary is Lipschitz, verifying $\bar\phi_\varepsilon^0(\Omega_1) \subset \{x_3 > 0\}$ and $\bar\phi_\varepsilon^0 \to \bar\phi^0$ in $W^{1,p}(\omega;\mathbb{R}^3)$ when $\varepsilon \to 0$. Thus, using the Lebesgue convergence theorem and (4.7), we have that for a subsequence still denoted by ε:

$$\int_\omega W[\nabla\overline{\Psi}(\bar\phi_\varepsilon^0(x))(\bar\phi_{\varepsilon,1}^0(x) \mid \bar\phi_{\varepsilon,2}^0(x) \mid \xi_\varepsilon^0(x) + e_3)A_0(x)]d_0(x)dx$$
$$\underset{\varepsilon\to 0}{\longrightarrow} \int_\omega W[\nabla\overline{\Psi}(\bar\phi^0(x))(\bar\phi_{,1}^0(x) \mid \bar\phi_{,2}^0(x) \mid \xi^0(x) + e_3)A_0(x)]d_0(x)dx.$$

In particular, for every $\eta > 0$, there exists an $\varepsilon(\eta) > 0$ such that for all $\varepsilon \le \varepsilon(\eta)$, we have

$$\int_\omega W[\nabla\overline{\Psi}(\bar\phi_\varepsilon^0(x))(\bar\phi_{\varepsilon,1}^0(x) \mid \bar\phi_{\varepsilon,2}^0(x) \mid \xi_\varepsilon^0(x) + e_3)A_0(x)]d_0(x)dx$$
$$\le \int_\omega W[\nabla\overline{\Psi}(\bar\phi^0(x))(\bar\phi_{,1}^0(x) \mid \bar\phi_{,2}^0(x) \mid \xi^0(x) + e_3)A_0(x)]d_0(x)dx + \eta$$

and thus

$$\int_\omega W[\nabla\overline{\Psi}(\bar\phi_\varepsilon^0(x))(\bar\phi_{\varepsilon,1}^0(x) \mid \bar\phi_{\varepsilon,2}^0(x) \mid \xi_\varepsilon^0(x) + e_3)A_0(x)]d_0(x)dx$$
$$\le \int_\omega W_0(x, \bar\phi^0(x), (\bar\phi_{,1}^0(x) \mid \bar\phi_{,2}^0(x)))d_0(x)dx + \eta.$$

Let us now set

$$\bar\phi_\varepsilon(h) = \bar\phi_\varepsilon^0 + hx_3\xi_\varepsilon^0 + hx_3e_3.$$

We fix $\varepsilon > 0$. Since $\bar\phi_\varepsilon^0 \in W^{1,p}(\omega;\mathbb{R}^3)$ with $p > 3$ and is thus continuous, for every compact subset $K \subset \omega$ there exists $c_K(\varepsilon) > 0$ such that $(\bar\phi_\varepsilon^0)_3 \ge c_K(\varepsilon)$ on K. For $h < \frac{c_K(\varepsilon)}{(1+\|\xi_\varepsilon^0\|_{L^\infty})}$ we have that $(\bar\phi_\varepsilon(h))_3 > 0$ on $K\times]0,1[$ and $(\bar\phi_\varepsilon(h))_3 = (\bar\phi_\varepsilon^0)_3 + hx_3 > 0$ also on $(\omega \setminus K)\times]0,1[$. Thus, $\bar\phi_\varepsilon(h) \in \overline{V}^h$ and

$$\bar\phi_\varepsilon(h) \underset{h\to 0}{\longrightarrow} \bar\phi_\varepsilon^0 \quad \text{strongly in } L^p(\Omega_1;\mathbb{R}^3).$$

Let us first study the interfacial energy term. By construction we have

$$|[\overline{\Psi}(\bar\phi_\varepsilon(h))]| = |[\overline{\Psi}(\bar\phi_\varepsilon^0)]|.$$

There are again three cases. The case $\alpha < 1$ is obvious.

Second case, $\alpha = 1$. We have

$$\begin{aligned}\overline{I}(h)(\bar\phi_\varepsilon(h)) &= \int_\omega \Phi(|[\overline{\Psi}(\bar\phi_\varepsilon(h))]|)\,|\operatorname{cof}\nabla\overline{\Psi}(x_1,x_2,hx_3)e_3|dx \\ &= \int_\omega \Phi(|[\overline{\Psi}(\bar\phi_\varepsilon^0)]|)\,|\operatorname{cof}\nabla\overline{\Psi}(x_1,x_2,hx_3)e_3|dx \\ &\underset{h\to 0}{\longrightarrow} \int_\omega \Phi(|[\overline{\Psi}(\bar\phi_\varepsilon^0)]|)\,d_0(x)dx = \delta(\alpha)\int_\omega \Phi(|[\overline{\Psi}(\bar\phi_\varepsilon^0)]|)\,d_0(x)dx.\end{aligned}$$

Third case, $\alpha > 1$. We have

$$\begin{aligned}\overline{I}(h)(\bar{\phi}_\varepsilon(h)) &= h^{\alpha-1}\int_\omega \Phi(|[\overline{\Psi}(\bar{\phi}_\varepsilon(h))]|)\,|\operatorname{cof}\nabla\overline{\Psi}(x_1,x_2,hx_3)e_3|dx \\ &= h^{\alpha-1}\int_\omega \Phi(|[\overline{\Psi}(\bar{\phi}^0_\varepsilon)]|)\,|\operatorname{cof}\nabla\overline{\Psi}(x_1,x_2,hx_3)e_3|dx \\ &\underset{h\to 0}{\longrightarrow} 0 = \delta(\alpha)\int_\omega \Phi(|[\overline{\Psi}(\bar{\phi}^0_\varepsilon)]|)d_0(x)dx.\end{aligned}$$

Thus, in all three cases

$$\overline{I}(h)(\bar{\phi}_\varepsilon(h)) \longrightarrow \overline{I}(0)(\bar{\phi}^0_\varepsilon), \tag{4.8}$$

where

$$\overline{I}(0)(\bar{\phi}) = \delta(\alpha)\int_\omega \Phi(|[\overline{\Psi}\circ\bar{\phi}]|)d_0(x)dx.$$

Next, we study the elastic energy term. We have

$$\Big((\bar{\phi}_\varepsilon(h))_{,1}\,\Big|\,(\bar{\phi}_\varepsilon(h))_{,2}\,\Big|\,\frac{1}{h}(\bar{\phi}_\varepsilon(h))_{,3}\Big) \to (\bar{\phi}^0_{\varepsilon,1}\mid\bar{\phi}^0_{\varepsilon,2}\mid\xi^0_\varepsilon + e_3)\quad\text{in } L^p(\Omega_1;\mathbb{R}^3) \tag{4.9}$$

and uniformly in Ω_1. The continuity of W and the convergence (4.9) imply that

$$\begin{aligned}&W\Big[\nabla\overline{\Psi}(\bar{\phi}_\varepsilon(h)(x))\Big((\bar{\phi}_\varepsilon(h))_{,1}\,\Big|\,(\bar{\phi}_\varepsilon(h))_{,2}\,\Big|\,\frac{1}{h}(\bar{\phi}_\varepsilon(h))_{,3}\Big)A_h(x)\Big] \\ &\to W[\nabla\overline{\Psi}(\bar{\phi}^0_\varepsilon(x))(\bar{\phi}^0_{\varepsilon,1}(x)\mid\bar{\phi}^0_{\varepsilon,2}(x)\mid\xi^0_\varepsilon(x)+e_3)\ A_0(x)]\quad\text{uniformly in }\Omega_1.\end{aligned} \tag{4.10}$$

Thus, we obtain that

$$\overline{E}(h)(\bar{\phi}_\varepsilon(h)) \to \int_\omega W[\nabla\overline{\Psi}(\bar{\phi}^0_\varepsilon(x))(\bar{\phi}^0_{\varepsilon,1}(x)\mid\bar{\phi}^0_{\varepsilon,2}(x)\mid\xi^0_\varepsilon(x)+e_3)A_0(x)]d_0(x)dx.$$

Consequently, using (4.8) we obtain

$$\begin{aligned}\bar{e}(h)(\bar{\phi}_\varepsilon(h)) &\to \int_\omega W[\nabla\overline{\Psi}(\bar{\phi}^0_\varepsilon(x))(\bar{\phi}^0_{\varepsilon,1}(x)\mid\bar{\phi}^0_{\varepsilon,2}(x)\mid\xi^0_\varepsilon(x)+e_3)A_0(x)]d_0(x)dx \\ &\quad + \delta(\alpha)\int_\omega \Phi(|[\overline{\Psi}(\bar{\phi}^0_\varepsilon)]|)d_0(x)dx \\ &\le \int_\omega W_0(x,\bar{\phi}^0(x),(\bar{\phi}^0_{,1}(x)\mid\bar{\phi}^0_{,2}(x)))d_0(x)dx + \delta(\alpha)\int_\omega \Phi(|[\overline{\Psi}(\bar{\phi}^0_\varepsilon)]|)d_0(x)dx + \eta.\end{aligned}$$

Let $G: W^{1,p}(\omega;\mathbb{R}^3)\to\mathbb{R}$ be defined by

$$G(\bar{\phi}) = \int_\omega W_0(x,\bar{\phi}(x),(\bar{\phi}_{,1}(x)\mid\bar{\phi}_{,2}(x)))d_0(x)dx.$$

We have just seen that

$$\Gamma - \limsup_{h\to 0}\bar{e}^*(h)(\bar{\phi}^0_\varepsilon) \le \lim_{h\to 0}\bar{e}^*(h)(\bar{\phi}_\varepsilon(h)) \le G(\bar{\phi}^0) + \delta(\alpha)\int_\omega \Phi(|[\overline{\Psi}(\bar{\phi}^0_\varepsilon)]|)d_0(x)dx + \eta.$$

Since $\bar{\phi}^0_\varepsilon \to \bar{\phi}^0$ in $W^{1,p}(\omega;\mathbb{R}^3)$ and the Γ-$\limsup$ is lower semicontinuous on $L^p(\omega;\mathbb{R}^3)$, it follows that

$$\Gamma - \limsup_{h\to 0}\bar{e}^*(h)(\bar{\phi}^0) \le G(\bar{\phi}^0) + \delta(\alpha)\liminf_{\varepsilon\to 0}\int_\omega \Phi(|[\overline{\Psi}(\bar{\phi}^0_\varepsilon)]|)d_0(x)dx + \eta.$$

By construction, $\bar{\phi}^0_\varepsilon \to \bar{\phi}^0$ uniformly on $\overline{\omega}$ and since Φ is continuous, it follows that

$$\int_\omega \Phi(|[\overline{\Psi}(\bar{\phi}^0_\varepsilon)]|)d_0(x)dx \to \int_\omega \Phi(|[\overline{\Psi}(\bar{\phi}^0)]|)d_0(x)dx.$$

Thus, we have proved that

$$\Gamma - \limsup_{h\to 0} \bar{e}^*(h)(\bar{\phi}^0) \leq H^*(\bar{\phi}^0) + \eta,$$

where $H^* : L^p(\omega;\mathbb{R}^3) \to \overline{\mathbb{R}}$ is defined by

$$H^*(\bar{\phi}) = \begin{cases} H(\bar{\phi}) := G(\bar{\phi}) + \delta(\alpha)\int_\omega \Phi(|[\overline{\Psi}(\bar{\phi})]|)d_0(x)dx, & \text{if } \bar{\phi} \in \overline{V}_M, \\ +\infty, & \text{otherwise} \end{cases}$$

(recall that $\overline{V}_M = \mathrm{id}$ and $\delta(\alpha) = 0$ for $\alpha < 1$). Since this is true for every $\eta > 0$, we obtain that

$$\Gamma - \limsup \bar{e}^*(h) \leq H^*. \tag{4.11}$$

In addition, the function I defined on $W^{1,p}(\omega;\mathbb{R}^3)$ by $I(\phi) = \int_\omega \Phi(|[\overline{\Psi}(\phi)]|)d_0(x)dx$ is continuous for the weak topology of $W^{1,p}(\omega;\mathbb{R}^3)$. Indeed, let $\phi_n \rightharpoonup \phi$ weakly in $W^{1,p}(\omega;\mathbb{R}^3)$. Since $p > 3$, we have that $\phi_n \to \phi$ uniformly in ω, and thus, $[\overline{\Psi}(\phi_n)] \to [\overline{\Psi}(\phi)]$ uniformly in ω. Thus, the continuity of Φ implies the continuity of I. Finally, the lower semicontinuous envelop of H is the function $\bar{e}(0)$ defined on $W^{1,p}(\omega;\mathbb{R}^3)$ by

$$\bar{e}(0)(\bar{\phi}) = \int_\omega QW_0(x, \bar{\phi}(x), (\bar{\phi}_{,1}(x) \mid \bar{\phi}_{,2}(x)))\det A(x)\,dx + \delta(\alpha)\int_\omega \Phi(|[\overline{\Psi}(\bar{\phi})]|)d_0(x)dx$$

(see [2, 11]). Applying the lower semicontinuous envelop in both sides of (4.11), using Lemma 4.2 and the lower semicontinuity of the Γ-$\limsup$ we obtain that

$$\Gamma - \limsup \bar{e}^*(h) \leq \bar{e}^*(0),$$

which completes the proof.

Proof of Theorem 4.1 The proof of the theorem is a direct consequence of the last two propositions.

Remark 4.1 It should be noted that, as opposed to the minimization problem for the energy $\bar{e}(h)$ where existence of solutions was not guaranteed, the minimization problem for the limit energy admits a solution thanks to the weak lower semicontinuity of the limit elastic energy term and to the coercivity.

As a consequence of the last theorem, we have the next corollary on the limit points of the diagonal minimizing sequence $\bar{\varphi}(h)$.

Corollary 4.1 *The diagonal minimizing sequence $\bar{\varphi}(h)$ of $\bar{e}(h)$ is bounded in $\overline{V}^h$ and its limit points for the weak topology of $W^{1,p}(\Omega_1;\mathbb{R}^3)$ minimizes the energy $\bar{e}(0)$ on $\bar{\phi} \in \overline{V}_M$ when $\alpha \geq 1$.*

Proof The proof of the corollary follows from Lemma 4.1 and the standard Γ-convergence argument.

5 The Curved Two-Dimensional Limit Model

Since the case $\alpha < 1$ is trivial, we will only consider the case $\alpha \geq 1$ in the sequel. Let us consider another chart $\psi' : \omega' \in \mathbb{R}^2 \to \widetilde{\omega}$. Working with this new chart, we obtain the same convergence results as previously, but this time it is written through the diffeomorphism Ψ'. Let us thus rewrite the limit model on the curved surface. As in [15], we consider for every unit vector e of S^2, a bounded open domain $O_e \subset e^\perp$ and we denote by π_e the orthogonal projection on this domain. We denote by $u \otimes v$ the tensor product of two vectors in $\mathbb{R}^3$. We extend any function $\chi \in W_0^{1,\infty}(O_e;\mathbb{R}^3)$ by setting

$$\chi_e(y) = \chi(\pi_e(y))$$

and we define for every $y \in O_e$,

$$D_{e^\perp}\chi(y) = \nabla\chi_e(y).$$

By associating to each deformation $\bar{\phi}$, first a deformation ϕ defined by

$$\phi(x) = \overline{\Psi}(\bar{\phi}(x)),$$

then a deformation $\tilde{\phi}$ defined on $\widetilde{\omega}$, setting

$$\tilde{x} = \overline{\Psi}(x) \quad \text{and} \quad \tilde{\phi}(\tilde{x}) = \phi(x),$$

we get the following theorem.

Theorem 5.1 *Any deformation $\tilde{\phi}$ associated to a minimizer $\bar{\phi}$ of the energy $\bar{e}(0)$, minimizes the energy $\tilde{e}(0)$ defined by*

$$\tilde{e}(0)(\tilde{\phi}) = \int_{\widetilde{\omega}} \widetilde{W}(a_3(\tilde{x}), \nabla\tilde{\phi}(\tilde{x}))\,d\tilde{x} + \delta(\alpha)\int_{\widetilde{\omega}} \Phi(|[\tilde{\phi}]|)d\tilde{x}$$

on

$$\widetilde{V} = \{\tilde{\phi} \in W^{1,p}(\widetilde{\omega};\mathbb{R}^3),\ \tilde{\phi}(\widetilde{\omega}) \subset \overline{S^c} \text{ and } \tilde{\phi}(\tilde{x}) = \tilde{x} \text{ on } \partial\widetilde{\omega}\},$$

where $a_3(\tilde{x})$ is the normal unit vector to $\widetilde{\omega}$ passing through $\tilde{x}$, $|[\tilde{\phi}]| = |\tilde{\phi}(\tilde{x}) - \tilde{x}|$, and $\widetilde{W} : S^2 \times \mathbb{M}_{33} \to \mathbb{R}$ denotes the membrane elastic energy density defined by

$$\widetilde{W}(e,F) = \inf_{\chi \in W_0^{1,\infty}(O_e;\mathbb{R}^3)} \Big[\frac{1}{\operatorname{meas} O_e}\int_{O_e}\Big[\inf_{z\in\mathbb{R}^3} W(F + z\otimes e + D_{e^\perp}\chi(y))\Big]dy\Big].$$

Proof Let us recall that in Corollary 4.1, we obtained that $\bar{\phi}$ minimizes the energy

$$\bar{e}(0)(\bar{\phi}) = \int_\omega QW_0(x, \bar{\phi}(x), (\bar{\phi}_{,1}(x) \mid \bar{\phi}_{,2}(x))) \det A(x)\,dx + \delta(\alpha)\int_\omega \Phi(|[\overline{\Psi}(\bar{\phi})]|)d_0(x)dx \quad \text{on } \overline{V}_M.$$

We use the change of variables to go back to the initial target space by setting for $x \in \omega$,

$$\phi(x) = \Psi\big(\bar{\phi}(x)\big) \quad \text{and} \quad e(0)(\phi) = \bar{e}(0)(\bar{\phi}).$$

We obtain that

$$e(0)(\phi) = \int_\omega QW_0(x, \overline{\Psi}^{-1}(\phi(x)), \nabla\overline{\Psi}^{-1}(\phi(x))(\phi_{,1}(x) \mid \phi_{,2}(x))) \det A(x)\, dx + \delta(\alpha)\int_\omega \Phi(|[\phi]|)d_0(x)dx$$

with

$$\phi \in V = \{\phi \in W^{1,p}(\omega;\mathbb{R}^3),\ \phi(\omega) \subset \overline{S^c} \text{ and } \phi(x) = \overline{\Psi}(x) \text{ on } \partial\omega\},$$
$$[\phi] = \phi(x_1, x_2, 0) - \overline{\Psi}(x_1, x_2, 0).$$

Then, we use a second change of variables in order to go back to the curved surface by setting for $x \in \omega$,

$$\tilde{x} = \overline{\Psi}(x) \quad \text{and} \quad \tilde{\phi}(\tilde{x}) = \phi(x).$$

Setting

$$\tilde{e}(0)(\tilde{\phi}) = e(0)(\phi),$$

we get

$$\tilde{e}(0)(\tilde{\phi}) = \int_{\widetilde{\omega}} QW_0(\overline{\Psi}^{-1}, \overline{\Psi}^{-1}(\tilde{\phi}), \nabla\overline{\Psi}^{-1}(\tilde{\phi})\nabla\tilde{\phi}(\overline{\Psi}_{,1}(\overline{\Psi}^{-1}) \mid \overline{\Psi}_{,2}(\overline{\Psi}^{-1})))\, d\tilde{x} + \delta(\alpha)\int_{\widetilde{\omega}} \Phi(|[\tilde{\phi}]|)d\tilde{x},$$

with

$$\phi \in \widetilde{V} = \{\tilde{\phi} \in W^{1,p}(\widetilde{\omega};\mathbb{R}^3),\ \tilde{\phi}(\widetilde{\omega}) \subset \overline{S^c} \text{ and } \tilde{\phi}(\tilde{x}) = \tilde{x} \text{ on } \partial\widetilde{\omega}\},$$
$$[\tilde{\phi}] = \tilde{\phi}(\tilde{x}) - \tilde{x}.$$

Then, we use the Dacorogna's integral representation for quasiconvex envelopes (see [10])

$$QW_0(x_0, x_1, F) = \inf_{\overline{\chi}\in W_0^{1,\infty}(O;\mathbb{R}^3)} \Big\{\frac{1}{\operatorname{meas} O}\int_O W_0(x_0, x_1, F + \nabla\overline{\chi}(\bar{y}))d\bar{y}\Big\},$$

so that

$$\begin{aligned} &QW_0(\overline{\Psi}^{-1}(\tilde{x}), \overline{\Psi}^{-1}(\tilde{\phi}(\tilde{x})), \nabla\overline{\Psi}^{-1}(\tilde{\phi}(\tilde{x}))\nabla\tilde{\phi}(\tilde{x})(\overline{\Psi}_{,1}(\overline{\Psi}^{-1}(\tilde{x})) \mid \overline{\Psi}_{,2}(\overline{\Psi}^{-1}(\tilde{x})))) \\ &= \inf_{\overline{\chi}\in W_0^{1,\infty}(O;\mathbb{R}^3)} \Big\{\frac{1}{\operatorname{meas} O}\int_O W_0(\overline{\Psi}^{-1}(\tilde{x}), \overline{\Psi}^{-1}(\tilde{\phi}(\tilde{x})), \nabla\overline{\Psi}^{-1}(\tilde{\phi}(\tilde{x}))\nabla\tilde{\phi}(\tilde{x}) \\ &\cdot (\overline{\Psi}_{,1}(\overline{\Psi}^{-1}(\tilde{x})) \mid \overline{\Psi}_{,2}(\overline{\Psi}^{-1}(\tilde{x}))) + \nabla\overline{\chi}(\overline{y}))d\bar{y}\Big\}. \end{aligned}$$

On the other hand, we have

$$W_0(x, y, \overline{F}) = \inf_{z\in\mathbb{R}^3} W(\nabla\overline{\Psi}(y)(\overline{F} \mid z)A_0(x)),$$

which gives

$$\begin{aligned} &QW_0(\overline{\Psi}^{-1}(\tilde{x}), \overline{\Psi}^{-1}(\tilde{\phi}(\tilde{x})), \nabla\overline{\Psi}^{-1}(\tilde{\phi}(\tilde{x}))\nabla\tilde{\phi}(\tilde{x})(\overline{\Psi}_{,1}(\overline{\Psi}^{-1}(\tilde{x})) \mid \overline{\Psi}_{,2}(\overline{\Psi}^{-1}(\tilde{x})))) \\ &= \inf_{\overline{\chi}\in W_0^{1,\infty}(O;\mathbb{R}^3)} \Big\{\frac{1}{\operatorname{meas} O}\int_O \inf_{z\in\mathbb{R}^3} W(\nabla\overline{\Psi}(\overline{\Psi}^{-1}(\tilde{\phi}(\tilde{x})))(\nabla\overline{\Psi}^{-1}(\tilde{\phi}(\tilde{x}))\nabla\tilde{\phi}(\tilde{x}) \\ &\cdot (\overline{\Psi}_{,1}(\overline{\Psi}^{-1}(\tilde{x})) \mid \overline{\Psi}_{,2}(\overline{\Psi}^{-1}(\tilde{x}))) + \nabla\overline{\chi}(\bar{y}) \mid z)A_0(\overline{\Psi}^{-1}(\tilde{x})))d\bar{y}\Big\}. \end{aligned}$$

We also have that

$$
\begin{aligned}
&(\nabla\overline{\Psi}^{-1}(\tilde{\phi}(\tilde{x}))\nabla\tilde{\phi}(\tilde{x})(\overline{\Psi}_{,1}(\overline{\Psi}^{-1}(\tilde{x})) \mid \overline{\Psi}_{,2}(\overline{\Psi}^{-1}(\tilde{x}))) + \nabla\overline{\chi}(\bar{y}) \mid z) \\
&= (\nabla\overline{\Psi}^{-1}(\tilde{\phi}(\tilde{x}))\nabla\tilde{\phi}(\tilde{x})(\overline{\Psi}_{,1}(\overline{\Psi}^{-1}(\tilde{x})) \mid \overline{\Psi}_{,2}(\overline{\Psi}^{-1}(\tilde{x}))) \mid 0) + (\nabla\overline{\chi}(\bar{y}) \mid 0) + (0 \mid z)
\end{aligned}
$$

and

$$
\nabla\overline{\Psi}(\overline{\Psi}^{-1}(\tilde{\phi}))(\nabla\overline{\Psi}^{-1}(\tilde{\phi})\nabla\tilde{\phi}(\overline{\Psi}_{,1}(\overline{\Psi}^{-1}) \mid \overline{\Psi}_{,2}(\overline{\Psi}^{-1})) \mid 0)A_0(\overline{\Psi}^{-1}) = \nabla\tilde{\phi}.
$$

As the matrix $\nabla\overline{\Psi}(\overline{\Psi}^{-1}(\tilde{\phi}(\tilde{x})))$ is invertible and independent of $\bar{y}$, the mapping

$$
\overline{\chi} \mapsto (\nabla\overline{\Psi}(\overline{\Psi}^{-1}(\tilde{\phi}(\tilde{x}))))^{-1}\overline{\chi}
$$

is a bijection between $W_0^{1,\infty}(O;\mathbb{R}^3)$ and $W_0^{1,\infty}(O;\mathbb{R}^3)$. This allows us to replace the term $\nabla\overline{\Psi}(\overline{\Psi}^{-1}(\tilde{\phi}(\tilde{x})))\nabla\overline{\chi}$ by $\nabla\overline{\chi}$ in the infimum. Similarly, we replace $\nabla\overline{\Psi}(\overline{\Psi}^{-1}(\tilde{\phi}(\tilde{x})))z$ by z. In addition we have

$$
\begin{aligned}
&\nabla\overline{\Psi}(\overline{\Psi}^{-1}(\tilde{\phi}(\tilde{x})))(\nabla\overline{\Psi}^{-1}(\tilde{\phi}(\tilde{x}))\nabla\tilde{\phi}(\tilde{x})(\overline{\Psi}_{,1}(\overline{\Psi}^{-1}(\tilde{x})) \mid \overline{\Psi}_{,2}(\overline{\Psi}^{-1}(\tilde{x}))) + \nabla\overline{\chi}(\bar{y}) \mid z)A_0(\overline{\Psi}^{-1}(\tilde{x})) \\
&= \nabla\tilde{\phi}(\tilde{x}) + (\nabla\overline{\Psi}(\overline{\Psi}^{-1}(\tilde{\phi}(\tilde{x})))\nabla\overline{\chi}(\bar{y}) \mid 0)A_0(\overline{\Psi}^{-1}(\tilde{x})) + (0 \mid \nabla\overline{\Psi}(\overline{\Psi}^{-1}(\tilde{\phi}(\tilde{x})))z)A_0(\overline{\Psi}^{-1}(\tilde{x})).
\end{aligned}
$$

Using the change of variables

$$
y = D\psi(\overline{\Psi}^{-1}(\tilde{x}))\bar{y},
$$

we obtain that

$$
(\nabla\overline{\chi}(\bar{y}) \mid 0)A_0(\overline{\Psi}^{-1}(\tilde{x})) = D_{a_3(\tilde{x})^\perp}\chi(y)
$$

and that $\chi \in W_0^{1,\infty}(O_{a_3(\tilde{x})};\mathbb{R}^3)$. Choosing $O = O_{a_3(\tilde{x})}$ and noting that

$$
(0 \mid z)A_0(\overline{\Psi}^{-1}(\tilde{x})) = z \otimes a_3(\tilde{x}),
$$

we get that

$$
\begin{aligned}
&QW_0(\overline{\Psi}^{-1}(\tilde{x}), \overline{\Psi}^{-1}(\tilde{\phi}(\tilde{x})), \nabla\overline{\Psi}^{-1}(\tilde{\phi}(\tilde{x}))\nabla\tilde{\phi}(\tilde{x})(\overline{\Psi}_{,1}(\overline{\Psi}^{-1}(\tilde{x})) \mid \overline{\Psi}_{,2}(\overline{\Psi}^{-1}(\tilde{x})))) \\
&= \inf_{\chi \in W_0^{1,\infty}(O_{a_3(\tilde{x})})} \Big\{ \frac{1}{\operatorname{meas}(O_{a_3(\tilde{x})})} \int_{O_{a_3(\tilde{x})}} \inf_{z \in \mathbb{R}^3} W(\nabla\tilde{\phi}(\tilde{x}) + D_{a_3(\tilde{x})^\perp}\chi(y) + z \otimes a_3(\tilde{x}))d\bar{y} \Big\},
\end{aligned}
$$

which gives us the result.

Remark 5.1 We note that the obtained limit energy does not depend on the coordinate system in which we write the energy. This underlines the intrinsic character of the limit minimization problem.

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