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Phase Turbulence in the Complex Ginzburg-Landau Equation via Kuramoto–Sivashinsky Phase Dynamics

Guillaume van Baalen[⋆]

Département de Physique Théorique, Université de Genève, Switzerland.
E-mail: guillaume.vanbaalen@physics.unige.ch

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Abstract: We study the Complex Ginzburg-Landau initial value problem

$$\partial_t u = (1+i\alpha)\,\partial_x^2 u + u - (1+i\beta)\,u\,|u|^2\,, \quad u(x,0)=u_0(x)\,, \qquad \text{(CGL)}$$

for a complex field $u \in \mathbf{C}$, with $\alpha, \beta \in \mathbf{R}$. We consider the Benjamin–Feir linear instability region $1+\alpha\beta = -\varepsilon^2$ with $\varepsilon \ll 1$ and $\alpha^2 < 1/2$. We show that for all $\varepsilon \le \mathcal{O}(\sqrt{1-2\alpha^2}\;L_0^{-32/37})$, and for all initial data u_0 sufficiently close to 1 (up to a global phase factor $e^{i\,\phi_0}$, $\phi_0 \in \mathbf{R}$) in the appropriate space, there exists a unique (spatially) periodic solution of space period L_0. These solutions are small *even* perturbations of the traveling wave solution, $u = (1+\alpha^2\,s)\,e^{i\,\phi_0 - i\beta\,t}\,e^{i\alpha\,\eta}$, and s, η have bounded norms in various L^p and Sobolev spaces. We prove that $s \approx -\frac{1}{2}\,\eta''$ apart from $\mathcal{O}(\varepsilon^2)$ corrections whenever the initial data satisfy this condition, and that in the linear instability range $L_0^{-1} \le \varepsilon \le \mathcal{O}(L_0^{-32/37})$, the dynamics is essentially determined by the motion of the phase alone, and so exhibits 'phase turbulence'. Indeed, we prove that the phase η satisfies the Kuramoto–Sivashinsky equation

$$\partial_t \eta = -\left(\tfrac{1+\alpha^2}{2}\right)\triangle^2\eta - \varepsilon^2\triangle\eta - (1+\alpha^2)\,(\eta')^2 \qquad \text{(KS)}$$

for times $t_0 \le \mathcal{O}(\varepsilon^{-52/5}\;L_0^{-32/5})$, while the amplitude $1+\alpha^2\,s$ is essentially constant.

Contents

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1. Introduction

1.1. Generalities about the Ginzburg-Landau equation. The Complex Ginzburg-Landau equation (CGL) admits explicit traveling wave solutions of the form

$$u(x,t) = c(p)\ \exp\left(i\,(\phi_0 + p\,x - \omega(p)\,t)\right)\ , \tag{1.1}$$

with $\phi_0 \in \mathbf{R}$, $p \in [-1, 1]$, $c(p) = \sqrt{1-p^2}$ and $\omega(p) = \alpha\ p^2 + \beta\ (1-p^2)$. For all α, β with $1 + \alpha\ \beta > 0$, there exists a parameter $p_E = p_E(\alpha, \beta)$, with $p_E \to 0$ as $1 + \alpha\ \beta \to 0^+$ such that traveling wave solutions (1.1) with $|p| \geq p_E(\alpha, \beta)$ are linearly unstable, a phenomenon called 'sideband' or 'Eckhaus' instability, while those with $|p| \leq p_E$ are linearly stable (see e.g. [CH93] and the references therein). When $1 + \alpha\beta < 0$, all traveling wave solutions are linearly unstable, a phenomenon called 'Benjamin–Feir' or 'Benjamin–Feir–Newell' instability (see e.g. [BF67] and [New74]).

In this paper, we consider the case $1+\alpha\ \beta = -\varepsilon^2$. When ε is small enough, numerical simulations on finite domains (see e.g. [MHAM97] and the references therein) indicate that the dynamics of the phase is turbulent, the phase evolving irregularly, (with fluctuations of order ε^2 around the global phase ϕ_0), while the amplitude of u is constant up to $\mathcal{O}(\varepsilon^4)$ corrections. This type of behavior is called 'phase turbulence'. The persistence of phase turbulence on infinite domains is not known, while its existence on finite domains is, to our knowledge, not proven rigorously.

As ε increases (or the domain is larger), 'amplitude' or 'defect' turbulence occurs, the amplitude of u vanishing at some instants and places, called 'defects' or 'phase slips' (see also [EGW95]). Note that 'phase' and 'amplitude' turbulence may coexist at the same time in the α, β parameter space, depending on initial conditions, in which case one speaks of 'bichaos'.

The 'amplitude' turbulence regime is technically difficult because the phase is not well defined when the amplitude vanishes. In this paper, we concentrate on the easier phase turbulence regime and prove that for the particular case[1] $p = 0$, phase turbulence occurs for small initial perturbations of the traveling wave $e^{i\ \phi_0 - i\beta\ t}$ on domains of size L_0 for all $\alpha^2 < 1/2$ and for all $\varepsilon \leq \varepsilon_0(L_0, \alpha)$ with $\varepsilon_0(L_0, \alpha) \to 0$ as $L_0 \to \infty$ or $\alpha^2 \to 1/2$, see Fig. 1.1. We restrict ourselves to *even* perturbations for concision, though general perturbations could be treated as well (see Remark 2.4 below). We believe that the restriction $\alpha^2 < 1/2$ could be weakened to some extent (see the discussion at the end of Sect. 1.4), at the price of unwanted additional technical difficulties.

[1] The case $p \neq 0$ should give a similar result but is more challenging.

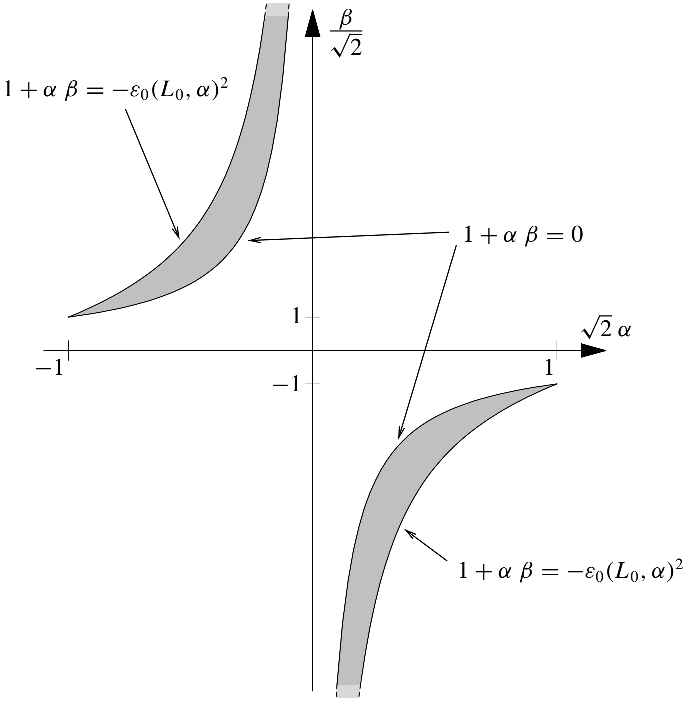

Fig. 1.1. Parameter space for (CGL). Linear instability occurs for $1+\alpha\,\beta<0$, and phase turbulence is shown in this paper to occur in shaded region

1.2. Setting. We consider perturbations of the solution $e^{i\,\phi_0-i\beta\,t}$ of (CGL) which are of the form[2]

$$u(x,t)=(1+\alpha^2\,s(x,t))\,e^{i\,\phi_0-i\beta t}\,e^{i\alpha\,\eta(x,t)}\,, \tag{1.2}$$

for (small) $s,\eta\in\mathbf{R}$. To state our results, we introduce the following scalings[3]

$$\eta(x,t)=\tfrac{1}{4}\,\hat{\varepsilon}^2\,\hat{\eta}(\hat{x},\hat{t})\,, \tag{1.3}$$

$$s(x,t)=\,\hat{\varepsilon}^4\,\hat{s}(\hat{x},\hat{t})\,, \tag{1.4}$$

with $\chi=\frac{4}{1+\alpha^2}$, $\hat{\varepsilon}=\sqrt{\frac{\chi}{2}}\,\varepsilon$, $\hat{x}=\hat{\varepsilon}\,x$ and $\hat{t}=\frac{2}{\chi}\,\hat{\varepsilon}^4\,t$.

We consider the initial value problem (CGL) with $\eta(x,0)=\eta_0(x)$ and $s(x,0)=s_0(x)$, where η_0 and s_0 are *even* periodic functions of period L_0, or equivalently, in terms of the 'hat' variables, $\hat{\eta}_0$ and $\hat{s}_0$ are *even* periodic functions of period $L=\hat{\varepsilon}\,L_0$. To state our conditions on the initial data $\hat{s}_0$ and $\hat{\eta}_0$, we introduce the Banach space $\mathcal{W}_{0,\sigma}$ obtained by completing $\mathcal{C}^\infty_{\rm per}([-L/2,L/2],\mathbf{R})$ under the norm $\|\cdot\|_\sigma=\|\cdot\|_{{\rm L}^2([-L/2,L/2])}+\|\cdot\|_{\mathcal{W},\sigma}$, where $\|\cdot\|_{\mathcal{W},\sigma}$ is a sup norm with algebraic weight (going like $|k|^\sigma$ at infinity) on the Fourier transform, see Sect. 2.3 for details. Essentially $\mathcal{W}_{0,\sigma}$ consists of functions in ${\rm L}^2([-L/2,L/2],\mathbf{R})$, whose Fourier transform decays (at least) like $|k|^{-\sigma}$ as $|k|\to\infty$

[2] The α factors in front of s and η are only a convenient normalization.

[3] They will be justified in the next subsection.

(this is a regularity assumption). Since we consider only real valued functions, we will from now on write $\mathrm{L}^2([-L/2, L/2])$ instead of $\mathrm{L}^2([-L/2, L/2], \mathbf{R})$. We will also often use the shorthand notation L^2 for $\mathrm{L}^2([-L/2, L/2])$, while we will always write $\mathrm{L}^2([-L_0/2, L_0/2])$ to avoid confusion.

We postpone the precise definition of the class $\mathcal{C}$ of admissible initial conditions to the end of Sect. 2.3 (see Definition 2.8). At this point, we will only say that if $\hat{\eta}_0$ and $\hat{s}_0$ are admissible initial conditions, then $\hat{\eta}_0' \in \mathcal{W}_{0,\sigma}$ and $\hat{s}_0 \in \mathcal{W}_{0,\sigma-1}$, and

$$\hat{\eta}_0(0) = 0 \, , \quad \|\hat{\eta}_0'\|_\sigma \leq c_{\eta_0} \, \rho \, , \quad \left\| \hat{s}_0 - \frac{\hat{\varepsilon}^2 \, \hat{s}_0''}{2} \right\|_{\sigma-1} \leq c_{s_0} \, \rho^3 \, , \tag{1.5}$$

for $\rho = K \, L^{8/5}$, and

$$\left\| \hat{s}_0 - \frac{\hat{\varepsilon}^2 \, \hat{s}_0''}{2} + \frac{\hat{\eta}_0''}{8} + \frac{\hat{\varepsilon}^2 \, (\hat{\eta}_0')^2}{32} \right\|_{\mathrm{L}^2} \leq \lambda_{2,0} \left(\frac{\hat{\varepsilon}}{\hat{\varepsilon}_0} \right)^2 \varepsilon^2 \, c_{s_0} \, \rho^3 \, . \tag{1.6}$$

The class $\mathcal{C}$ of admissible initial conditions is characterized by the different parameters in (1.5) and (1.6), which we now describe. The parameter L is the (space) period (in the scaled variables) of the solution. The constant K is essentially the same as that of [CEES93] in their discussion of the Kuramoto–Sivashinsky equation,

$$\partial_{\hat{t}} \hat{\eta}_c = -\Delta^2 \hat{\eta}_c - \Delta \hat{\eta}_c - \frac{1}{2} (\hat{\eta}_c')^2 \, , \tag{1.7}$$

where it appears in the bound $\lim_{t\to\infty} \|\hat{\eta}_c'(\cdot, t)\|_{\mathrm{L}^2} \leq K \, L^{8/5}$ for symmetric periodic solutions. Therefore, K is independent of α, ε and L. The parameters α and $\hat{\varepsilon}$ are those of (CGL), with $\hat{\varepsilon}^2 = -2 \, \frac{1+\alpha \, \beta}{1+\alpha^2}$, while $\hat{\varepsilon}_0$ is the maximal value of $\hat{\varepsilon}$ for which our results hold. The parameters c_{η_0} and c_{s_0} measure the size of the initial perturbation. Note that only $\hat{\eta}_0'$ and $\hat{\eta}_0(0)$ appear in the conditions. We can motivate this by noting that (CGL) has a $U(1)$ symmetry (the global phase factor $e^{i \, \phi_0}$). Expressing all constraints in terms of $\hat{\eta}_0'$ and $\hat{\eta}_0(0)$ is a convenient way to take this invariance into account. The condition $\eta_0(0) = 0$ can always be satisfied, up to a redefinition of the global phase ϕ_0. Furthermore, this condition is preserved by the evolution (see e.g. (1.17)). We will prove that if $\hat{\eta}_0$ and $\hat{s}_0$ are in the class $\mathcal{C}$, the (CGL) dynamics (which has a complex function as initial condition) is increasingly well approximated as $\hat{\varepsilon} \to 0$ by the Kuramoto–Sivashinsky dynamics (1.7), which has a real function as initial condition. For this to hold, $\hat{s}_0$ and $\hat{\eta}_0$ have to be tightly related as $\hat{\varepsilon} \to 0$. This relation is quantified by (1.6), which says that, up to $\mathcal{O}(\varepsilon^4)$ corrections, $\hat{s}_0$ and $\hat{\eta}_0$ are related by

$$\hat{s}_0 = -\frac{1}{8} \, \hat{G} \, \hat{\eta}_0'' - \frac{\hat{\varepsilon}^2}{32} \, \hat{G} \, (\hat{\eta}_0')^2 \, ,$$

where $\hat{G}$ is the operator with symbol $\hat{G}(k) = (1 + \frac{\hat{\varepsilon}^2}{2} \, k^2)^{-1}$, i.e. the inverse of the (positive) operator $1 - \frac{\hat{\varepsilon}^2}{2} \, \partial_{\hat{x}}^2$.

1.3. Main results and their physical discussion. Our main results are twofold. We first have an existence and uniqueness result for the solutions of (CGL), see Theorem 1.1 below, and then an approximation result in Theorem 1.2.

From now on, we will denote generic constants by the letters C and c. We will use the letter c with different labels to recall the quantity on which the bound is. By constants, we mean quantities which do not depend on α, $\hat{\varepsilon}$, L and σ in the ranges

$$0 \le \hat{\varepsilon} \le 1\,, \quad \alpha^2 < 1/2\,, \quad L > 2\pi \quad \text{and}\ \sigma \le \sigma_0$$

for some finite $\sigma_0 > \frac{11}{2}$.

Theorem 1.1. *Let $\alpha^2 < 1/2$, $\sigma > \frac{11}{2}$, $c_{s_0} > 0$, $c_{\eta_0} > 0$, $\lambda_{1,0} < \min\left(\frac{2}{3}, \frac{1-2\,\alpha^2}{1-\alpha^2}\right)$, $\lambda_{2,0} > 0$ and $L > 2\,\pi$. There exist constants K and c_ε such that for all $m_\varepsilon \ge 4$, for any $\hat{\varepsilon} \le \hat{\varepsilon}_0 = c_\varepsilon\,\sqrt{1-2\alpha^2}\;\rho^{-m_\varepsilon}$ and for all $\hat{\eta}_0$ and $\hat{s}_0$ in the class $\mathcal{C}$, the solution of (CGL) with parameters α and $\beta = -\frac{2+(1+\alpha^2)\,\hat{\varepsilon}^2}{2\,\alpha}$ exists for all times, is of the form (1.2) and satisfies*

$$\sup_{\hat{t}\ge 0} \|\hat{\eta}(\cdot,\hat{t})'\|_\sigma \le c_\eta\,\rho\,, \quad \sup_{\hat{t}\ge 0} \|\hat{s}(\cdot,\hat{t})\|_{\sigma-1} \le c_s\,\rho^3, \tag{1.8}$$

$$\sup_{\hat{t}\ge 0} \left\| \hat{s}(\cdot,\hat{t}) + \frac{1}{8}\,\hat{G}\,\hat{\eta}''(\cdot,\hat{t}) + \frac{\hat{\varepsilon}^2}{32}\,\hat{G}\,(\hat{\eta}')^2(\cdot,\hat{t}) \right\|_{\mathrm{L}^2} \le \left(\frac{\hat{\varepsilon}}{\hat{\varepsilon}_0}\right)^4 c_\eta\,\rho\,, \tag{1.9}$$

with $\rho = K\;L^{8/5}$, $c_\eta > 1 + c_{\eta_0}$ and $c_s > c_{s_0}$. This solution is unique among functions satisfying (1.8).

Our results are valid for any $\hat{\varepsilon} \le \hat{\varepsilon}_0 = c_\varepsilon\,\sqrt{1-2\alpha^2}\;\rho^{-4}$ and for any $L > 2\,\pi$. Since $L = \hat{\varepsilon}\;L_0$ and $\rho = K\;L^{8/5}$, we see that the applicability range is

$$C\;L_0^{-1} \le \varepsilon \le C'\sqrt{1-2\alpha^2}\;L_0^{-32/37}\,.$$

The lower bound is the linear instability condition.

In terms of the original variables, Theorem 1.1 shows that solutions of (CGL) of the form (1.2) exist, and that (see Appendix G for details)

$$\sup_{t\ge 0} \|\eta(\cdot,t)'\|_{\mathrm{L}^2([-L_0/2,L_0/2])} \le C\;\varepsilon^{5/2-1/m_\varepsilon}\,, \tag{1.10}$$

$$\sup_{t\ge 0} \|s(\cdot,t)\|_{\mathrm{L}^2([-L_0/2,L_0/2])} \le C\;\varepsilon^{7/2-3/m_\varepsilon}\,, \tag{1.11}$$

$$\sup_{t\ge 0}\ \sup_{x\in[-L_0/2,L_0/2]} |\eta(x,t)| \le C\;\varepsilon^{2-13/(8\,m_\varepsilon)}\,, \tag{1.12}$$

$$\sup_{t\ge 0}\ \sup_{x\in[-L_0/2,L_0/2]} |s(x,t)| \le C\;\varepsilon^{4-4/m_\varepsilon}\,. \tag{1.13}$$

The inequalities (1.12) and (1.13) quantify the 'physical intuition' $\eta = \mathcal{O}(\varepsilon^2)$ and $s = \mathcal{O}(\varepsilon^4)$, see Sect. 1.1.

Inequalities (1.8) or (1.10)–(1.13) also show that the solutions belong to a (local) attractor, while (1.9) shows that on that attractor, the 'amplitude' s satisfies $s = -\frac{1}{2}\,\eta'' + \mathcal{O}(\varepsilon^2)$. The attractor is thus well approximated by the graph $s = -\frac{1}{2}\,\eta''$ in the s, η space. This result was discovered at a heuristic level by Kuramoto and Tsuzuki in [KT76].

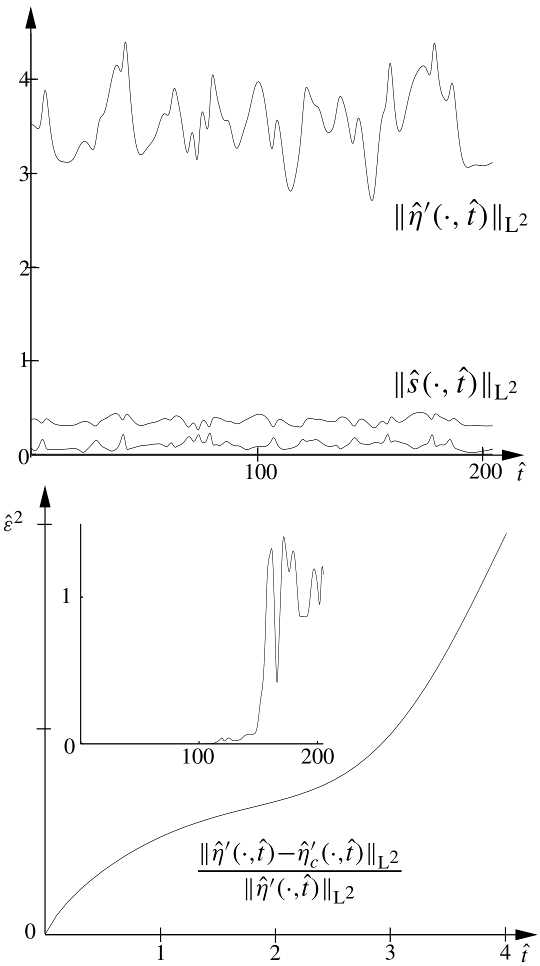

Fig. 1.2. Numerical results for $\hat\varepsilon = 10^{-3}$, $\alpha = 10^{-2}$ and $L_0 = 10^4 \cdot 2\pi$

We do not expect the bounds (1.8) and (1.9) to be optimal. Numerical simulations show that $\hat\eta'$ and $\hat s$ are uniformly bounded in space and time, at least for a large range of $L = \varepsilon\, L_0$. This suggests that $\|\hat\eta'\|_{L^2}$ and $\|\hat s\|_{L^2}$ should both scale with L like $\sqrt{L}$ and not like $L^{8/5}$ and $L^{24/5}$, hence we should have $\rho \sim \sqrt{L}$. In the upper panel of Fig. 1.2, we display as a function of $\hat t \in [0, 200]$ (by decreasing size) the typical behavior of $\|\hat\eta'(\cdot,\hat t)\|_{L^2}$, $\|\hat s(\cdot,\hat t)\|_{L^2}$ and $\hat\varepsilon^{-4}\|\hat s(\cdot,\hat t) + \frac{1}{8}\,\hat G\,\hat\eta''(\cdot,\hat t) + \frac{\hat\varepsilon^2}{32}\,\hat G\,(\hat\eta')^2(\cdot,\hat t)\|_{L^2}$ in units proportional to $\sqrt{\varepsilon L_0}$.

We now show that the dynamics of the phase on the attractor is well approximated by the Kuramoto–Sivashinsky equation.

Theorem 1.2. *Under the assumptions of Theorem 1.1, there exists a constant c_t such that if $\hat t_1 \le c_t\,\rho^{-4}$, then for all $\hat t_0 \ge 0$,*

$$\sup_{0\le\hat t\le\hat t_1} \|\hat\eta(\cdot,\hat t_0+\hat t)' - \hat\eta_c(\cdot,\hat t)'\|_{\mathrm{L}^2} \le \left(\frac{\hat\varepsilon}{\hat\varepsilon_0}\right)^2 c_\eta\,\rho\,, \tag{1.14}$$

where $\hat\eta_c$ satisfies the Kuramoto–Sivashinsky equation (1.7) with $\hat\eta_c(\hat x,0)=\hat\eta(\hat x,\hat t_0)$.

In physical terms, Theorem 1.2 says that on each time interval $[t_0, t_0+t_1]$, the distance between η and the solution of the Kuramoto–Sivashinsky equation with initial condition $\eta(t_0)$ is small compared to the size of the attractor (see (1.14)), at least for time intervals of length t_1 of order $\varepsilon^{-4}\,\rho^{-4}=\varepsilon^{-52/5}\,L_0^{-32/5}$. This result gives a rigorous foundation to the heuristic derivation in [KT76]) of the Kuramoto–Sivashinsky equation as a phase equation for the Complex Ginzburg-Landau equation near the Benjamin–Feir line (see also [Man90]). Furthermore, if ε is sufficiently small, the amplitude $1+\alpha^2\,s$ does not vanish by (1.13). This proves that the solution exhibits phase turbulence for all times, the solutions of the Kuramoto–Sivashinsky equation being believed to be chaotic.

The bound (1.14) for $\hat t_1 \le c_t\,\rho^{-4}$ is again certainly not optimal. Numerical simulations show that $\hat t_1$ scales like L^{-2} (this is in agreement with $\rho\sim\sqrt{L}$). In the lower panel of Fig. 1.2, we show in the large plot $\frac{\|\hat\eta'(\cdot,\hat t)-\hat\eta_c'(\cdot,\hat t)\|_{\mathrm{L}^2}}{\|\hat\eta'(\cdot,\hat t)\|_{\mathrm{L}^2}}$ in units of $\hat\varepsilon^2$ for short times (large times are displayed in small inserted plot in absolute units).

In the remainder of this section, we derive the dynamical equations for $\hat s$ and $\hat\eta$, then we discuss informally these equations to motivate the analytical treatment that we will present in the next sections. In particular, we will explain the particular choice of the scalings (1.3) and (1.4). We will treat the phase dynamical equation in Sect. 2, while the treatment of the dynamical equation for s is postponed to Sect. 3, s being 'slaved' to η by that equation.

1.4. Derivation of the amplitude and phase equations. The ansatz (1.2) leads, after separation of the real and imaginary parts of equation (CGL), to

$$\partial_t s = s'' - 2s - \eta'' - (\eta')^2 - \alpha^2\left(3s^2+\alpha^2 s^3 + 2s'\eta' + s\eta'' + s(\eta')^2\right), \tag{1.15}$$

$$\partial_t\eta = \eta'' + \alpha^2 s'' - 2\alpha\beta s - \alpha^2\left((\eta')^2 + \alpha\beta s^2 - \frac{2s'\eta'}{1+\alpha^2 s} + \frac{\alpha^2 s s''}{1+\alpha^2 s}\right). \tag{1.16}$$

Since these equations preserve the subspace of functions that are *even* in the space variable, we restrict ourselves to that particular case. We also use $\alpha, -\frac{1+\varepsilon^2}{\alpha}$ as parameters instead of α,β as it allows to emphasize the dependence on the small parameter ε. Finally, as the right-hand sides of (1.15) and (1.16) contain only (space) derivatives of the function η, we introduce the odd function μ (the phase derivative) by

$$\eta(x,t) = \int_0^x \mathrm{d}y\;\mu(y,t)\,, \tag{1.17}$$

and obtain

$$\partial_t s = s'' - 2\,s - \mu' - \mu^2 - \alpha^2\,\left(3\,s^2 + 2\,s'\,\mu + s\,\mu' + s\,\mu^2 + \alpha^2\,s^3\right)\,, \tag{1.18}$$

$$\begin{aligned}\partial_t\mu = {} & \mu'' + \alpha^2\,s''' + 2\,(1+\varepsilon^2)\,s' - \alpha^2\,(\mu^2)' \\ & +\alpha^2\,\left((1+\varepsilon^2)\,s^2 + \frac{2s'\,\mu}{1+\alpha^2\,s} - \frac{\alpha^2\,s\,s''}{1+\alpha^2\,s}\right)'\,.\end{aligned} \tag{1.19}$$

We expect $\partial_t s, s' \ll s, \mu' \ll \mu \ll 1$ when $\varepsilon \ll 1$. We then have

$$\partial_t s = s'' - 2\, s - \mu' - \mu^2 + f_s(s,\mu)\,, \tag{1.20}$$

$$\begin{aligned}\partial_t \mu = &-\Big(s'' - 2\, s - \mu' - \mu^2\Big)' + (1+\alpha^2)\, s''' + 2\,\varepsilon^2\, s' \\ &-2\,(1+\alpha^2)\,\mu\,\mu' + f_\mu(s,\mu)'\,,\end{aligned} \tag{1.21}$$

where $f_s(s,\mu)$, respectively $f_\mu(s,\mu)$, is defined as the function appearing in the second line of (1.18) resp. (1.19). The $-2s$ term in (1.20) strongly damps s, which therefore is 'slaved' to μ. Indeed, as we will show in Sect. 3, for given μ satisfying appropriate bounds, the map $\mu \mapsto s(\mu)$ defined by the (global and strong) solution of (1.20) is well defined and Lipschitz in μ. Furthermore, to third order in ε, the map is given by the solution s_1 of $s_1'' - 2\, s_1 - \mu' - \mu^2 = 0$, which can be represented as

$$s_1(\mu) = -\frac{1}{2}\, G\, \left(\mu' + \mu^2\right)\,, \tag{1.22}$$

where G is the operator of convolution with the fundamental solution $\mathcal{G}$ of $\mathcal{G}(x) - \frac{1}{2}\mathcal{G}''(x) = \delta(x)$. Note that G acts multiplicatively in Fourier space, with symbol $(1 + \frac{k^2}{2})^{-1}$, in particular, $G\, f$ has two more derivatives than f. As we will also show in Sect. 3, $s(\mu)$ will have the same structure as $s_1(\mu)$, that is, the $\mathcal{G}$–convolution of another map with the same regularity as μ'. As such, $s(\mu)$ is once more differentiable than μ, due to the regularizing properties of G, and $s(\mu) = s(\eta')$ is as regular as η. This is reasonable, since from $u = (1+\alpha^2\, s)\, e^{-i\beta\, t + i\alpha\eta}$ we see that s and η should have both the same degree of regularity as u.

Inserting (1.22) into (1.21) and neglecting f_μ leads to the (modified) Kuramoto–Sivashinsky equation for the phase

$$\partial_t \mu = -\tfrac{1+\alpha^2}{2} G\mu'''' - \varepsilon^2 G\mu'' - 2(1+\alpha^2)\mu\mu' - \varepsilon^2 G(\mu^2)' - \tfrac{1+\alpha^2}{2} G(\mu^2)'''\,, \tag{1.23}$$

from which we recover the Benjamin–Feir linear instability criterion $1 + \alpha\beta < 0$. Namely, linear stability analysis in Fourier space (set $\mu = \varepsilon_0\, e^{ikx + \lambda(k)t}$ with $\varepsilon_0 \ll 1$) gives the dispersion relation

$$\lambda(k) = \frac{\varepsilon^2\, k^2 - k^4\, \left(\frac{1+\alpha^2}{2}\right)}{1 + \frac{k^2}{2}} = \frac{-(1+\alpha\beta)\, k^2 - k^4\, \left(\frac{1+\alpha^2}{2}\right)}{1 + \frac{k^2}{2}}\,.$$

This shows that there are linearly unstable modes for $|k| \le \varepsilon \ll 1$, growing at most like $e^{\varepsilon^4 t}$. This suggests that the dynamics of (1.23) should be dominated by the dynamics of the Fourier modes in the small $|k|$ region, the high $|k|$ modes being slaved to them. For $|k| \ll 1$, we have $G \approx 1$, and neglecting the last two terms of (1.23), we get the Kuramoto–Sivashinsky equation in derivative form

$$\partial_t \mu \approx -\tfrac{1+\alpha^2}{2}\, \mu'''' - \varepsilon^2 \mu'' - 2(1+\alpha^2)\, \mu\mu'\,. \tag{1.24}$$

Defining

$$\mu(x,t) = \tfrac{1}{4}\, \hat{\varepsilon}^3\, \hat{\mu}(\hat{x},\hat{t})\,, \tag{1.25}$$

with $\chi = \frac{4}{1+\alpha^2}$, $\hat{\varepsilon} = \sqrt{\frac{\chi}{2}}\, \varepsilon$, $\hat{x} = \hat{\varepsilon}\, x$, and $\hat{t} = \frac{2}{\chi}\, \hat{\varepsilon}^4\, t$, we get from (1.24)

$$\partial_t \hat{\mu} = -\hat{\mu}'''' - \hat{\mu}'' - \hat{\mu}\, \hat{\mu}'\,, \tag{1.26}$$

which is the original Kuramoto–Sivashinsky equation in derivative form. This justifies the scalings (1.3). Equation (1.26) possesses an universal attractor of finite radius in $L^2([-L/2, L/2])$ with periodic boundary conditions (see e.g. [CEES93]), hence we can expect μ to be of size ε^3 times a typical solution in that attractor.

From (1.22), we get (the μ–dependence of s_1 is implicit here for concision)

$$s_1(x,t) = -\frac{\hat{\varepsilon}^4}{32}\,\hat{G}\,\left(4\hat{\mu}'(\hat{x},\hat{t}) + \hat{\varepsilon}^2\,\hat{\mu}(\hat{x},\hat{t})^2\right) \equiv \hat{\varepsilon}^4\,\hat{s}_1(\hat{x},\hat{t})\,, \tag{1.27}$$

where $\hat{G}$ is the convolution operator with the fundamental solution $\hat{\mathcal{G}}$ of $\hat{\mathcal{G}}(x) - \frac{\varepsilon^2}{2}\hat{\mathcal{G}}''(x) = \delta(x)$. As above, $\hat{G}$ acts multiplicatively in Fourier space, with symbol $\hat{G}(k) = (1 + \frac{\varepsilon^2 k^2}{2})^{-1}$. Equation (1.27) motivates the scalings $s(x,t) = \hat{\varepsilon}^4\,\hat{s}(\hat{x},\hat{t})$ we introduced for s in (1.4).

We now apply (1.25) and (1.4) to (1.20) and (1.21). From now on we drop the hats. Then s and μ satisfy the following equations:

$$\partial_t s = -\frac{\chi}{\varepsilon^4}\,\mathcal{L}_s\,s + \frac{\chi}{\varepsilon^4}\,r_1(\mu) - \frac{\alpha^2\,\chi}{8}(2s'\mu + s\mu') + F_3(s,\mu)\,, \tag{1.28}$$

$$\partial_t \mu = -\mathcal{L}_\mu\,\mu - \mu\mu' + \varepsilon^2\,F_0(s,\mu)' + \varepsilon^2\,\chi\,\mathcal{L}_{\mu,r}\,r_2'\,, \tag{1.29}$$

where $\mathcal{L}_s$, $\mathcal{L}_\mu$ and $\mathcal{L}_{\mu,r}$ are multiplicative operators in Fourier space, with symbols given by

$$\mathcal{L}_s(k) = 1 + \frac{\varepsilon^2\,k^2}{2}\,,$$

$$\mathcal{L}_\mu(k) = \frac{k^4 - k^2}{1 + \frac{\varepsilon^2\,k^2}{2}}\,, \tag{1.30}$$

$$\mathcal{L}_{\mu,r}(k) = 2\,\frac{2 + \varepsilon^2\,(1+\alpha^2) - \alpha^2\,\varepsilon^2\,k^2}{1 + \frac{\varepsilon^2\,k^2}{2}}\,, \tag{1.31}$$

while r_1, r_2, F_3 and F_0 are defined by

$$r_1(\mu) = -\frac{1}{32}(4\mu' + \varepsilon^2\mu^2)\,, \tag{1.32}$$

$$r_2 = \frac{r}{\varepsilon^4} - \frac{r_1(\mu)}{\varepsilon^4}, \tag{1.33}$$

$$F_3(s,\mu) = -\chi\alpha^2\left(\tfrac{3}{2}s^2 + \varepsilon^2\tfrac{1}{32}s\mu^2 + \tfrac{\alpha^2}{2}\varepsilon^4 s^3\right)\,, \tag{1.34}$$

$$F_0(s,\mu) = \chi\alpha^2\Big(\big(2 + \varepsilon^2(1+\alpha^2)\big)s^2 + \frac{s'\mu - 2\,\varepsilon^2\alpha^2 ss''}{1 + \varepsilon^4\alpha^2 s}\Big) - \frac{1}{4}G\mu^2 - \frac{1}{4}G(\mu^2)''\,, \tag{1.35}$$

where the auxiliary variable r and the operator G are defined by

$$G(k) = \frac{1}{1+\frac{\varepsilon^2\,k^2}{2}}\,, \tag{1.36}$$

$$r = s - \tfrac{\varepsilon^2}{2}s''\,, \tag{1.37}$$

and satisfy $s = G\,r$.

We will prove that (1.28) defines a map $\mu \mapsto s(\mu)$ for all μ in an open ball of $\mathcal{W}_\sigma$, and that this map has indeed 'the same properties' as $G\, r_1(\mu)$, e.g. in terms of regularity. This is so essentially because for $\varepsilon \ll 1$, we have $\frac{\chi}{\varepsilon^4}\mathcal{L}_s \gg 1$, so that by Duhamel's formula, $s \sim \mathcal{L}_s^{-1}\, r_1(\mu) + \mathcal{O}(\varepsilon^4) = G\, r_1(\mu) + \mathcal{O}(\varepsilon^4)$ (see Sect. 3). At the same time, as a dynamical variable, r_2 satisfies

$$\partial_t r_2 = -\frac{\chi}{\varepsilon^4}\, G\, \mathcal{L}_r\, r_2 + \frac{\chi}{16}\mu\, \mathcal{L}_{\mu,r}\, r_2' + \frac{1}{\varepsilon^4}\, F_6(s,\mu)\ , \tag{1.38}$$

where $\mathcal{L}_r$ is the multiplicative operator in Fourier space with symbol

$$\mathcal{L}_r(k) = 1 + \left(\tfrac{3}{2} + \varepsilon^2\left(\tfrac{1+\alpha^2}{4}\right)\right)\varepsilon^2\, k^2 + \left(\tfrac{1-\alpha^2}{4}\right)\varepsilon^4\, k^4\ ,$$

and

$$F_6(s,\mu) = \mathcal{L}_s\left(F_3(s,\mu) + F_4(s,\mu)\right) + F_7(s,\mu) + F_8(\mu)\ , \tag{1.39}$$

$$F_4(s,\mu) = -\tfrac{\alpha^2\,\chi}{8}(2s'\mu + s\mu')\ , \tag{1.40}$$

$$F_7(s,\mu) = \frac{\varepsilon^2}{8}\left(\partial_x + \frac{\varepsilon^2\,\mu}{2}\right)\left(F_0(s,\mu)'\right)\ , \tag{1.41}$$

$$F_8(s,\mu) = -\frac{1}{8}\left(\partial_x + \frac{\varepsilon^2\,\mu}{2}\right)\left(\mathcal{L}_\mu\,\mu + \mu\,\mu'\right)\ .$$

Once s is considered as a given map $\mu \mapsto s(\mu)$, (1.38) defines the map $\mu \mapsto r_2(\mu)$ through a *linear* equation for r_2. By the same mechanism as for s, we have $r_2 \sim (G\ \mathcal{L}_r)^{-1}\ F_6(s,\mu) \sim G\ F_6(s,\mu)$ if $\alpha^2 < 1$ (see Sect. 4). The restriction $\alpha^2 < 1$ is necessary here to make $\mathcal{L}_r$ positive definite. For technical reasons, we have in fact to restrict $\alpha^2 < 1/2$ to prove theorems 1.1 and 1.2. We believe that the results of these theorems could be extended to part of the $\alpha^2 > 1/2$ region by exploiting the following argument. If $\alpha^2 > 1$, Eq. (1.38) for r_2 is linearly unstable at high frequencies. However, the linear coupling of r_2 to μ through (1.29) stabilizes r_2. To see this, we introduce the vector $\mathbf{v} = (\mu, r_2)$, and consider (1.29) and (1.38) simultaneously, as a vector dynamical system of the form

$$\partial_t\mathbf{v} = \mathcal{L}_M\mathbf{v} + f(\mathbf{v})\ , \tag{1.42}$$

for a (nonlinear) vector map f, where $\mathcal{L}_M$ is the operator with (matrix) symbol

$$\mathcal{L}_M(k) = \begin{pmatrix} -\mathcal{L}_\mu(k) & \varepsilon^2\,\chi\,\mathcal{L}_{\mu,r}(k)\,ik \\ -\frac{1}{8\,\varepsilon^4}\,\mathcal{L}_\mu(k)\,ik & -\frac{\chi}{\varepsilon^4}G\mathcal{L}_r(k) \end{pmatrix}\ .$$

The stability of (1.42) at high frequency is then determined by the eigenvalues $\lambda_\pm(k)$ of $\mathcal{L}_M(k)$ for large k. Since[4]

$$\lambda_\pm(k) \to -(1 \pm i|\alpha|)\,\frac{k^2}{\hat{\varepsilon}^2}$$

as $k \to \infty$, (1.42) is stable at high frequency, the real part of the eigenvalues $\lambda_\pm(k)$ of $\mathcal{L}(k)$ being negative for large k. However to exploit this would force us to solve (1.29) and (1.38) simultaneously, which is technically (and notationally) more difficult, see

[4] This is the analogon of $(1 + i\alpha)\,u''$ in (CGL).

[GvB02] for a similar problem. Instead, in our approach the system (1.28), (1.29) and (1.38) is considered as a 'main' equation, (1.29), of the form

$$\partial_t \mu = -\mathcal{L}_\mu \, \mu - \mu\mu' + \varepsilon^2 \, F(\mu)' \,, \tag{1.43}$$

supplied with two 'auxiliary' equations, (1.28) and (1.38), which can be solved independently.

We will first study (1.43) for a general class of map $F(\mu)$ in Sect. 2 below, because it explains the choice of the functional space, and which properties of the solutions of the amplitude equations (1.28) and (1.38) are needed. Then, in Sect. 3 and 4, we will show that the solutions of the amplitude equations (1.28) and (1.38) exist and satisfy the 'right' properties.

2. The Phase Equation

2.1. Strategy. Having argued that $r_2 = r_2(\mu)$, we rewrite (1.29) as

$$\partial_t \mu = -\mathcal{L}_\mu \, \mu - \mu\mu' + \varepsilon^2 \, F(\mu)' \,, \quad \mu(x, 0) = \mu_0(x) \,, \tag{2.1}$$

where μ_0 is a given (odd) space periodic function of period L for some given L. Since (2.1) preserves the mean of μ over $[-L/2, L/2]$, and since μ_0 is the space derivative of a space periodic function, we restrict ourselves to μ_0 which have zero mean over $[-L/2, L/2]$.

We will show that the term $\varepsilon^2 \, F(\mu)'$ is in some sense negligible. If $\varepsilon = 0$, then $\mathcal{L}_\mu = \partial_x^4 + \partial_x^2 \equiv \mathcal{L}_{\mu,c}$, and (2.1) is the Kuramoto–Sivashinsky equation. If $F = 0$ and $\varepsilon > 0$, (in this case, $\mathcal{L}_\mu$ is of smaller order than $\partial_x^4 + \partial_x^2$), this situation can still be easily handled by the techniques of [CEES93] or [NST85], which show that equation (2.1) possesses a universal attractor of finite radius in $\mathrm{L}^2([-L/2, L/2])$ if $F = 0$. A key ingredient of that proof is the observation that the trilinear form $\int \mathrm{d}x \; \mu^2\mu'$ vanishes for periodic functions. However, in general, $\varepsilon^2 \int \mathrm{d}x \; \mu \; F(\mu)'$ will not vanish, and might even not exist at all for $\mu \in \mathrm{L}^2$.

We will explain precisely below how we circumvent this, but the mechanism is indeed quite simple. If the n^{th} Fourier coefficients of μ were vanishing for all $n \geq \frac{\delta}{q}$ with $1 \ll \delta \ll 1/\varepsilon$, we would have e.g. $\|\mu''\|_{\mathrm{L}^2} \leq \delta^2 \, \|\mu\|_{\mathrm{L}^2}$, which would (presumably) give $\varepsilon^2 \int \mathrm{d}x \; \mu \; F(\mu)' \sim \varepsilon^2\delta^2 \; \|\mu\|_{\mathrm{L}^2}^2$. For ε sufficiently small, this would only give a small blur to the attractor of the true Kuramoto–Sivashinsky equation.

Evidently, we cannot expect the high–n Fourier modes to vanish, so we will have to treat them separately. On that matter, we want to point out that contrary to the 'true' Kuramoto–Sivashinsky equation (1.26), where the linear operator $\mathcal{L}_{\mu,c}$ acting on μ on the r.h.s. is of fourth order, $\mathcal{L}_\mu$ is only of second order due to the regularizing properties of G. From the point of view of derivatives of μ, it is easy to see that $\varepsilon^2 \; \hat{s}_1'$ and $\varepsilon^2 \; \hat{s}_1''$ contain at most first derivatives of μ, hence we expect $\varepsilon^2 \; F(\mu)'$ to contain at most second order derivatives of μ, and we see that at high frequencies, (1.43) is more similar to the well studied equation $\dot{u} = u'' + f(u, u', u'')$ (see e.g. [BKL94]) than to the Kuramoto–Sivashinsky equation.

Note that the term $F(\mu)'$ is 'irrelevant' due to its prefactor ε^2, while $\mu \; \mu'$ is certainly not. Indeed, it would be catastrophic to solve (2.1) by successive approximations, beginning with the solution of the equation with $-\mu\mu' + \varepsilon^2 \; F(\mu) = 0$, inserting that solution into the nonlinear terms and solving again the linear inhomogeneous problem. This would lead to (apparently) exponentially growing modes, because the linear operator $\mathcal{L}_\mu$ is not positive definite at small frequencies. Solving (2.1) iteratively as

$$\partial_t \mu_{n+1} = -\mathcal{L}_\mu \, \mu_{n+1} - \mu_{n+1}\mu_{n+1}' + \varepsilon^2 \, F(\mu_n)' ,$$

for $n \geq 0$ is a much better choice. We therefore consider the following class of equations

$$\partial_t \mu = -\mathcal{L}_\mu \, \mu - \mu\mu' + \varepsilon^2 \, g' , \quad \mu(x,0) = \mu_0(x) , \tag{2.2}$$

for some given time dependent and spatially periodic perturbation g and periodic initial data μ_0.

From this (informal) discussion, we see that we should treat the small n Fourier coefficients with an L^2–like norm as in [CEES93] or [NST85], and the high n modes as in e.g. [BKL94]. In the next three subsections, we implement this idea. We first show L^2 estimates for (2.2) in Subsect. 2.2. Then in Subsect. 2.3 we define functional spaces similar to those of [BKL94], and prove inequalities in these spaces, which will allow us to prove the 'high frequency estimates' in Subsect. 2.4. In Subsect. 2.5, we will prove that the full phase equation has a solution if $\mu \mapsto F(\mu)$ is a well behaved Lipschitz map, and finally, in Subsect. 2.6, we will show how the phase equation relates to the Kuramoto–Sivashinsky equation.

2.2. Coercive functional method, L^2 *estimates.* The initial value problem (2.2) is globally well posed in $L^2([-L/2, L/2])$ if the perturbation g is periodic and in L^2 for all $t \geq 0$. The local uniqueness/existence theory follows from standard techniques (see e.g. [Tem97]), whereas the global existence follows from the a priori estimate

$$\|\mu(\cdot,t)\|_{L^2}^2 \leq e^t \, \|\mu(\cdot,0)\|_{L^2}^2 + 2\,\varepsilon^4 \, \left(e^t - 1\right) \sup_{0\leq s\leq t} \|g(\cdot,s)\|_{L^2}^2 . \tag{2.3}$$

Namely, denoting by $\int$ the integral over $[-L/2, L/2]$, using Young's inequality, integration by parts and the fact that $\int \mu^2\mu' = 0$ by periodicity of μ, we have

$$\partial_t \int \mu^2 \leq -2 \int \mu \, \mathcal{L}_\mu \, \mu + \frac{1}{2} \int (\mu')^2 + 2\,\varepsilon^4 \int g^2 \leq \int \mu^2 + 2\,\varepsilon^4 \int g^2 ,$$

from which (2.3) follows immediately. As a much stronger result, we can in fact prove that the L^2–norm of the solution stays bounded for all $t \geq 0$. To do this, we adapt the strategy of [NST85] and [CEES93] to our setting. We first need a technical result.

Proposition 2.1. *Let* $(v, w) = \int vw$, $(v, w)_{\gamma\phi} = \int v(\mathcal{L}_\mu + \gamma\phi')w$ *and*

$$\mathcal{L}_v(k) = \sqrt{\frac{1}{3}\frac{1+k^4}{1+\frac{\varepsilon^2 k^2}{2}}} . \tag{2.4}$$

For all $L \geq 2\pi$*, there exist a constant* K *and an antisymmetric periodic function* ϕ *such that for all* $\gamma \in [\frac{1}{4}, 1]$ *and* $\varepsilon \leq L^{-2/5}$*, and for every antisymmetric periodic function* v*, one has*

$$\frac{3}{4} \, (\mathcal{L}_v \, v, \mathcal{L}_v \, v) \leq (v, v)_{\gamma\phi} \leq \|\phi'\|_\infty \, (v, v) + (v'', v'') ,$$

$$(\phi, \phi)_{\gamma\phi} \leq K \; L^{16/5} \quad \textit{and} \quad (\phi, \phi) \leq \tfrac{4}{3} \, L^3 .$$

The proof, which follows closely [CEES93] is relegated to Appendix A. We then have the

Theorem 2.2. *There exists a constant K such that the solution μ of (2.2) is periodic, antisymmetric, and satisfies*

$$\sup_{t\geq 0}\|\mu(\cdot,t)\|_{L^2} \leq \rho + \|\mu_0\|_{L^2} + 4\,\varepsilon^2 \sup_{t\geq 0}\|\mathcal{L}_v^{-1}\, g(\cdot,t)'\|_{L^2}\ ,$$

where $\rho = K\ L^{8/5}$, if μ_0 and g' are antisymmetric (spatially) periodic functions of period L,

Proof. Note first that $\mathcal{L}_v^{-1}\,\partial_x$ is a bounded operator on L^2 with norm ≤ 2 (see Lemma F.1 in Appendix F), then local existence in L^2 follows from the above argument. Next, following [NST85] with the modifications of [CEES93], we write $\mu(x,t) = v(x,t) + \phi(x)$ for some constant periodic function ϕ to be chosen later on. Denoting by $\int$ the integral over $[-L/2, L/2]$, using integration by parts, that $\int v^2 v'$ vanishes because v is periodic and the inner products defined in Proposition 2.1, we get from (2.2)

$$\frac{1}{2}\partial_t(v,v) = -(v,v)_{\phi/2} - (v,\phi)_\phi + \varepsilon^2\,(v,g')\ . \tag{2.5}$$

Next, we use that $(\mathcal{L}_v\ v, \mathcal{L}_v\ v) \geq \frac{4}{3}\Big(\frac{\sqrt{\varepsilon^4+4}-2}{\varepsilon^4}\Big)\ (v,v) \equiv c_v^2\ (v,v)$, Young's inequality and Proposition 2.1 to get from (2.5),

$$\begin{aligned}\partial_t(v,v) &\leq -2\,(v,v)_{\phi/2} + \tfrac{2}{3}(v,v)_\phi + \tfrac{3}{2}(\phi,\phi)_\phi + 2\,\varepsilon^2\,(v,g')\\ &\leq -\tfrac{4}{3}\,(v,v)_{\phi/4} + \tfrac{3}{2}(\phi,\phi)_\phi + 2\,\varepsilon^2\,(v,g')\\ &\leq -(\mathcal{L}_v\,v,\mathcal{L}_v\,v) + \tfrac{3}{2}(\phi,\phi)_\phi + 2\,\varepsilon^2\,(\mathcal{L}_v\,v,\mathcal{L}_v^{-1}\,g')\\ &\leq -\frac{c_v^2}{2}\,(v,v) + \tfrac{3}{2}(\phi,\phi)_\phi + 2\,\varepsilon^4\,\|\mathcal{L}_v^{-1}\,g'\|_{L^2}^2\ .\end{aligned} \tag{2.6}$$

Since $v(x,t) = \mu(x,t) - \phi(x)$ we conclude that

$$\|\mu(\cdot,t)-\phi(\cdot)\|_{L^2}^2 \leq \|\mu_0-\phi\|_{L^2}^2 + \frac{3}{c_v^2}(\phi,\phi)_\phi + \frac{4\,\varepsilon^4}{c_v^2}\sup_{t\geq 0}\|\mathcal{L}_v^{-1}\,g(\cdot,t)'\|_{L^2}^2\ .$$

Finally, since $\frac{2}{c_v} \leq 4$, we have

$$\sup_{t\geq 0}\|\mu(\cdot,t)\|_{L^2} \leq \|\mu_0\|_{L^2} + \rho + 4\,\varepsilon^2\sup_{t\geq 0}\|\mathcal{L}_v^{-1}\,g(\cdot,t)'\|_{L^2}\ ,$$

where $\rho = 2\,\|\phi\|_{L^2} + 4\,\sqrt{(\phi,\phi)_\phi}$. Furthermore, by Proposition 2.1, we have $\rho < \infty$, since $\|\phi\|_{L^2} = \sqrt{(\phi,\phi)} < \infty$ and $(\phi,\phi)_\phi < \infty$. This completes the proof of the theorem. □

Corollary 2.3. *The antisymmetric solution of the Kuramoto–Sivashinsky equation with periodic boundary conditions on* $[-L/2, L/2]$

$$\partial_t\mu = -\mu'''' - \mu'' - \mu\,\mu'\ ,\quad \mu(x,0) = \mu_0(x)\ , \tag{2.7}$$

stays in a ball of radius $\mathcal{O}(L^{8/5})$ in L^2 as $L \to \infty$.

Proof. This result was already established in [NST85] and [CEES93]. To prove it, we only have to note that (2.7) corresponds to (2.2) with $\varepsilon = 0$, and that Theorem 2.2 is uniformly valid in $\varepsilon \le 1$. $\square$

Remark 2.4. The proof of Theorem 2.2 is the only point in this paper where we need s, respectively μ, to be spatially even, resp. odd, functions. The theorem holds also in the general (non symmetric) case. The proof can be obtained as a straightforward extension of the result of [CEES93] for the Kuramoto–Sivashinsky equation in the non symmetric case.

If $\varepsilon = 0$, Theorem 2.2 shows that the solution of (2.2) stays in a ball in L^2, centered on 0 and of radius $\|\mu_0\|_{\mathrm{L}^2} + \rho$ for all $t \ge 0$, with $\rho = \mathcal{O}(L^{8/5})$ as $L \to \infty$. When $\varepsilon \neq 0$, the radius of the ball widens to lowest order like $\varepsilon^2 \sup_{t\ge 0}\|g(\cdot,t)\|_{\mathrm{L}^2}$.

2.3. Functional spaces, definitions and properties. In this section, we explain how to treat the high frequency part of the solution of (2.2). This development is inspired by [BKL94] (see also [GvB02] for similar definitions).

Let $L \ge 2\pi$ and $q \equiv \frac{2\pi}{L} \le 1$. We define the Fourier coefficients f_n of a function $f : [-L/2, L/2] \to \mathbf{R}$ by

$$f_n = \frac{1}{L}\int_{-L/2}^{L/2} \mathrm{d}x \; e^{-iqnx} f(x) \;, \qquad \text{so that} \quad f(x) = \sum_{n\in\mathbf{Z}} e^{iqnx} \; f_n \;,$$

and $P_<$, $P_>$, the projectors on the small/high frequency part by

$$P_< f(x) = \sum_{|n|\le\frac{\delta}{q}} e^{iqnx} f_n \quad , \quad P_> f(x) = \sum_{|n|>\frac{\delta}{q}} e^{iqnx} f_n \;,$$

where the parameter $\delta \ge 2$ will be chosen later. We also define the L^p and l^p norms as

$$\|f\|_{\mathrm{L}^p}^p = \int_{-L/2}^{L/2} \mathrm{d}x \; |f(x)|^p \;, \quad \|f\|_{l^p}^p = \sum_{n\in\mathbf{Z}} |f_n|^p \;, \quad \|f\|_{l^\infty} = \sup_{n\in\mathbf{Z}} |f_n| \;,$$

and $\|f\|_{\mathrm{L}^\infty} = \operatorname{ess\,sup}_{x\in[-L/2,L/2]} |f(x)|$. We will use Plancherel's equality $\|f\|_{\mathrm{L}^2} = \sqrt{L}\;\|f\|_{l^2}$ without notice. Finally, for $\sigma \ge 0$, $\delta \ge 2$, we define the norm $\|\cdot\|_{\mathcal{N},\sigma}$ by

$$\|f\|_{\mathcal{N},\sigma} = \frac{\sqrt{\delta}}{q} \sup_{n\in\mathbf{Z}} \big(1 + (\tfrac{qn}{\delta})^2\big)^{\frac{\sigma}{2}} \; |f_n| \;.$$

With a different normalization, the norm $\|\cdot\|_{\mathcal{N},\sigma}$ was introduced in [BKL94] to study the long time asymptotics of solutions of $\dot{u} = u'' + f(u, u', u'')$, where f is some (polynomial) nonlinearity. From the point of view of the nonlinearity, our situation is similar to the case treated there, but our linear operator $\mathcal{L}_\mu$ is not positive definite as $-\Delta$ was in their case. The potentially exponentially growing modes correspond to $|n| \le \frac{1}{q}$, and we saw in Sect. 2.2 that their l^2 norm was bounded. Since there are only a finite (but large) number of linearly unstable modes, changing the definition of the $\|\cdot\|_{\mathcal{N},\sigma}$–norm on these modes to an l^2–like norm will give an equivalent norm which is better suited to our case. Thus we define the norms $\|\cdot\|_{\mathcal{W},\sigma}$ and $\|\cdot\|_\sigma$ by

$$\|f\|_{\mathcal{W},\sigma} = \frac{\sqrt{\delta}}{q} \sup_{|n|>\frac{\delta}{q}} \left(1+(\tfrac{qn}{\delta})^2\right)^{\frac{\sigma}{2}} |f_n| , \tag{2.8}$$

$$\|f\|_\sigma = \|f\|_{\mathrm{L}^2} + \|f\|_{\mathcal{W},\sigma} . \tag{2.9}$$

While $\|\cdot\|_{\mathcal{W},\sigma}$ is clearly *not* a norm, $\|\cdot\|_\sigma$ is a norm which is equivalent to $\|\cdot\|_{\mathcal{N},\sigma}$ for $\sigma \geq 1$. Indeed, easy calculations lead to $\|f\|_\sigma \leq \left(1+\pi\sqrt{2}\right)\|f\|_{\mathcal{N},\sigma}$ and

$$\|f\|_{\mathcal{N},\sigma} \leq \sqrt{2^\sigma L\,\delta}\,\|f\|_{\mathrm{L}^2} + \|f\|_{\mathcal{W},\sigma} \leq \left(1+\sqrt{2^\sigma L\,\delta}\right)\|f\|_\sigma .$$

We point out also that if $\sigma > \frac{1}{2}$, the $\|\cdot\|_{\mathcal{W},\sigma}$–semi–norm is a *decreasing* function of δ. Indeed, we have (here the norms carry an additional index to specify the value of δ)

$$\|f\|_{\mathcal{W},\sigma,\delta_1} \leq 2^{\frac{\sigma}{2}} \left(\frac{\delta_0}{\delta_1}\right)^{\sigma-\frac{1}{2}} \|f\|_{\mathcal{W},\sigma,\delta_0} , \tag{2.10}$$

for all $\delta_1 \geq \delta_0 \geq 2$. As δ will be fixed later on, the additional index is suppressed to simplify the notation. On the other hand, $\|\cdot\|_\sigma$ is an *non–decreasing* function of σ, since, for all $\sigma_1 \geq \sigma_0$,

$$\|f\|_{\sigma_0} \leq \|f\|_{\sigma_1} . \tag{2.11}$$

We now define the functional spaces

Definition 2.5. *Denoting by* $\mathcal{C}^\infty_{0,\mathrm{per}}([-L/2,L/2],\mathbf{R})$ *the set of infinitely differentiable periodic real valued functions on* $[-L/2,L/2]$*, we define the (Banach) space* $\mathcal{W}_{0,\sigma}$ *as the completion of* $\mathcal{C}^\infty_{0,\mathrm{per}}([-L/2,L/2],\mathbf{R})$ *under the norm* $\|\cdot\|_\sigma$*, and* $\mathcal{B}_{0,\sigma}(r) \subset \mathcal{W}_{0,\sigma}$ *the open ball of radius r centered on* $0 \in \mathcal{W}_{0,\sigma}$.

Up to now, we considered functions depending on the space variable only. We extend the definition 2.9 to functions $f : [-L/2,L/2]\times[0,\infty) \to \mathbf{R}$ by

$$|\!|\!| f |\!|\!|_\sigma = \sup_{t\geq 0} \|f(\cdot,t)\|_\sigma .$$

The same convention applies for L^p and l^p norms. Finally, we make the following definition.

Definition 2.6. *Let* $\Omega = [-L/2,L/2]\times\mathbf{R}^+$ *and* $\mathcal{C}^\infty_{\mathrm{per}}(\Omega,\mathbf{R})$ *denote the set of infinitely differentiable functions on* Ω *compactly supported on* $\mathbf{R}^+$ *and satisfying* $f(-L/2,t) = f(L/2,t)$ *for all* $t \in \mathbf{R}^+$*. We define the (Banach) space* $\mathcal{W}_\sigma$ *as the completion of* $\mathcal{C}^\infty_{\mathrm{per}}(\Omega,\mathbf{R})$ *under the norm* $|\!|\!|\cdot|\!|\!|_\sigma$*, and* $\mathcal{B}_\sigma(r) \subset \mathcal{W}_\sigma$ *the open ball of radius r centered on* $0 \in \mathcal{W}_\sigma$.

The spaces $\mathcal{W}_\sigma$ satisfy nice properties under derivation and multiplication. Space deri-vation maps $\mathcal{W}_\sigma$ to $\mathcal{W}_{\sigma-1}$ essentially with a factor δ on the norms, while multipli-cation maps $\mathcal{W}_\sigma \times \mathcal{W}_\sigma$ to $\mathcal{W}_\sigma$ with essentially a factor $\sqrt{\delta}$. Furthermore, in the spaces $\mathcal{W}_\sigma$, it is very easy to quantify the regularising effects of the evolution equation (2.2) on the inhomogeneous term g' (or the nonlinearity $F(\mu)'$). For precise statement on these results, see Lemma B.1, and Propositions B.2 and B.3 in Appendix B. The following proposition, which follows directly from Lemma B.1 relate $\mathcal{W}_\sigma$ to more well known spaces:

Proposition 2.7. *For all $\sigma > \frac{5}{2}$, $\mathcal{W}_\sigma \subset \mathrm{L}^\infty(\mathbf{R}^+, W_{2,2}([-L/2, L/2]))$, the Banach space of functions on Ω which are (together with their space derivatives up to order 2) uniformly (in time) bounded in $\mathrm{L}^2([-L/2, L/2])$.*

We can now define the class $\mathcal{C}$ of initial conditions for which Theorems 1.1 and 1.2 hold.

Definition 2.8. *We say that η_0 and s_0 are in the class $\mathcal{C}$ if $\eta_0' \in \mathcal{W}_{0,\sigma}$ and $s_0 \in \mathcal{W}_{0,\sigma-1}$, if*

$$\eta_0(0) = 0\ , \quad \|\eta_0'\|_\sigma \leq c_{\eta_0}\,\rho\ , \quad \Big\| s_0 - \frac{\hat{\varepsilon}^2\, s_0''}{2} \Big\|_{\sigma-1} \leq c_{s_0}\,\rho^3\ , \tag{2.12}$$

for $\rho = K\ L^{8/5}$, $c_{\eta_0} > 0$ and $c_{s_0} > 0$ and if

$$4\varepsilon^2 \|\chi \mathcal{L}_{\mu,r}\mathcal{L}_v^{-1} r_{2,0}'\|_{\mathrm{L}^2} + \varepsilon^2 \|\chi \mathcal{L}_{\mu,r}\mathcal{L}_\mu^{-1} r_{2,0}'\|_{\mathcal{W},\sigma} \leq \lambda_{1,0} c_{\eta_0}\rho\ , \tag{2.13}$$

$$\varepsilon^2 \|r_{2,0}\|_{\mathrm{L}^2} + \varepsilon^2 \|\chi \mathcal{L}_{\mu,r}\mathcal{L}_v^{-1} r_{2,0}'\|_{\mathrm{L}^2} \leq \lambda_{2,0}\Big(\frac{\varepsilon}{\varepsilon_0}\Big)^2 c_{s_0}\rho^3\ , \tag{2.14}$$

with $\lambda_{2,0} > 0$, $\lambda_{1,0} < \min\big(\frac{2}{3}, \frac{1-2\,\alpha^2}{1-\alpha^2}\big)$ and $r_{2,0} = \frac{1}{\varepsilon^4}\big(s_0 - \frac{\varepsilon^2}{2} s_0'' + \frac{1}{8}\eta_0'' + \frac{\varepsilon^2}{32}(\eta_0')^2\big)$.

Note that (2.14) is stronger than (2.13) as $\varepsilon \to 0$, while it is the contrary as $\varepsilon \to \varepsilon_0$.

2.4. High frequency estimates. By Theorem 2.2, the solution μ of (2.2) exists and is bounded in L^2 for all $t \geq 0$ if $\|\mu_0\|_{\mathrm{L}^2} + \|\!|\mathcal{L}_v^{-1}\, g'\|\!|_{\mathrm{L}^2} < \infty$. We will now show that upon further restrictions on μ_0 and g, the solution has bounded $\|\cdot\|_\sigma$–norm for all $t \geq 0$. Namely, setting

$$c_0 = 1 + \tfrac{\|\mu_0\|_\sigma}{\rho} + \tfrac{4\varepsilon^2}{\rho}\|\!|\mathcal{L}_v^{-1} g'\|\!|_{\mathrm{L}^2} + \tfrac{\varepsilon^2}{\rho}\Big\|\!\Big| \tfrac{g'}{\mathcal{L}_\mu}\Big\|\!\Big|_{\mathcal{W},\sigma} \quad \text{and} \quad \xi = \tfrac{C_m c_0 \rho}{\sqrt{2\delta}}\ , \tag{2.15}$$

we have the following theorem.

Theorem 2.9. *Let c_0 and ξ be defined by (2.15), and assume that the initial condition μ_0 and the function g satisfy $\xi < \frac{1}{4}$. Then the solution μ of (2.2) satisfies*

$$\|\!|\mu\|\!|_\sigma \leq \left(\frac{1 - \sqrt{1 - 4\xi}}{2\xi}\right) c_0\ \rho\ . \tag{2.16}$$

Remark 2.10. Note that c_0 is implicitly dependent of δ (because the norm $\|\cdot\|_\sigma$ is). If μ_0 and g are given, c_0 is a *non-increasing* function of δ (see (2.10)). Hence we can surely satisfy $\xi < \frac{1}{4}$ by taking δ sufficiently large.

Proof of Theorem 2.9. Let $d_0 = \|\!|\mu\|\!|_{\mathrm{L}^2}$, $\sigma_0 = 0$, $\sigma_1 = \frac{1}{2}$ and, for all $n \geq 1$, define

$$\sigma_{n+1} = \begin{cases} \sigma_n + 1 & \text{if } \sigma_n + 1 < \sigma \\ \sigma & \text{if } \sigma_n + 1 \geq \sigma \end{cases}\ .$$

We will now show inductively that $d_n \equiv \|\!|\mu\|\!|_{\sigma_n}$ are bounded for all $n \geq 1$, and that $\|\!|\mu\|\!|_\sigma = \lim_{n\to\infty} d_n$ satisfies (2.16). The first step is to note that by Theorem 2.2, we have

$$d_0 = \|\!|\mu\|\!|_{\mathrm{L}^2} \leq \rho + \|\mu_0\|_{L^2} + 4\,\varepsilon^2\ \|\!|\mathcal{L}_v^{-1}\ g'\|\!|_{\mathrm{L}^2} \leq c_0\ \rho\ . \tag{2.17}$$

To bound $|||\mu|||_{\mathcal{W},\sigma}$, we use Duhamel's representation formula for the solution of (2.2),

$$\mu(x,t) = e^{-\mathcal{L}_\mu t}\mu_0 + \varepsilon^2 \int_0^t \mathrm{d}s\; e^{-\mathcal{L}_\mu (t-s)} g'(x,s) + T(\mu)(x,t) \;, \tag{2.18}$$
$$T(\mu)(x,t) = -\int_0^t \mathrm{d}s\; e^{-\mathcal{L}_\mu (t-s)} (\mu\mu')(\cdot, s) \;.$$

Since $\mu\,\mu' = \frac{1}{2}(\mu^2)'$, using Propositions B.2 and B.3, we get for all $n \geq 1$ the bound

$$|||T(\mu)|||_{\mathcal{W},\sigma_n+1} \leq \frac{1}{\sqrt{2}\,\delta} |||\mu^2|||_{\mathcal{W},\sigma_n} \leq \frac{C_m}{\sqrt{2\delta}}\, d_n^2 \;.$$

Using again Proposition B.2 and the definitions (2.15), we get for all $n \geq 0$ the bound

$$\frac{d_{n+1}}{c_0\,\rho} \leq 1 + \xi \left(\frac{d_n}{c_0\,\rho}\right)^2 ,$$

because $\rho + \|\mu_0\|_{\sigma_n} + 4\varepsilon^2 |||\mathcal{L}_v^{-1} g'|||_{\mathrm{L}^2} + \varepsilon^2 |||\frac{g'}{\mathcal{L}_\mu}|||_{\mathcal{W},\sigma_n} \leq c_0\,\rho$ for all $n \geq 0$. Note that since $\xi < \frac{1}{4}$, the (infinite) sequence $\tilde{d}_{n+1} = 1 + \xi\, \tilde{d}_n^2$, $\tilde{d}_0 = 1$, is increasing and satisfies $\tilde{d}_n \leq \lim_{n\to\infty} \tilde{d}_n = \tilde{d}_\infty \equiv \frac{1-\sqrt{1-4\xi}}{2\xi}$, hence $|||\mu|||_\sigma \leq \tilde{d}_\infty\, c_0\,\rho$, from which (2.16) follows immediately. □

2.5. Existence and uniqueness of the solution of the phase equation. Let $\tilde{\mu} \in \mathcal{W}_\sigma$ and $\mu_0 \in \mathcal{B}_{0,\sigma}(c_{\eta 0}\,\rho) \subset \mathcal{W}_{0,\sigma}$. We consider the equation

$$\partial_t f = -\mathcal{L}_\mu f - f\, f' + \varepsilon^2\, F(\tilde{\mu})' \;, \quad f(x,0) = \mu_0(x) \;. \tag{2.19}$$

By Theorem 2.9, f exists if $\|\mu_0\|_\sigma + |||\mathcal{L}_v^{-1}\, F(\tilde{\mu})'|||_{\mathrm{L}^2} + |||\mathcal{L}_\mu^{-1}\, F(\tilde{\mu})'|||_{\mathcal{W},\sigma} < \infty$, in which case, we define the map $(\tilde{\mu}, \mu_0) \mapsto \mathcal{F}(\tilde{\mu}, \mu_0)$, by $\mathcal{F}(\tilde{\mu}, \mu_0) \equiv f$. We will show that for fixed μ_0, $\tilde{\mu} \mapsto \mathcal{F}(\tilde{\mu}, \mu_0)$ is a contraction in the ball $\mathcal{B}_\sigma(c_\eta\,\rho)$ if the following condition holds.

Condition 2.11. *There exist constants $\lambda_1 < 1$ and $\lambda_2 > 0$ such that for all $c_\eta > \frac{c_{\eta 0}+1}{1-\lambda_1}$, there exists a constant ε_0 such that for all $\varepsilon \leq \varepsilon_0$ and for all $\mu_i \in \mathcal{B}_\sigma(c_\eta\,\rho)$ the following bounds hold:*

$$4\,\varepsilon^2\, |||\mathcal{L}_v^{-1}\, F(\mu_i)'|||_{\mathrm{L}^2} + \varepsilon^2 \left|\left|\left| \frac{F(\mu_i)'}{\mathcal{L}_\mu} \right|\right|\right|_{\mathcal{W},\sigma} \leq \lambda_1\, c_\eta\, \rho \;, \tag{2.20}$$

$$\varepsilon^2\, |||\mathcal{L}_v^{-1}\, \Delta F'|||_{\mathrm{L}^2} + \varepsilon^2 \left|\left|\left| \frac{\Delta F'}{\mathcal{L}_\mu} \right|\right|\right|_{\mathcal{W},\sigma} \leq \lambda_1\, |||\mu_1 - \mu_2|||_\sigma \;, \tag{2.21}$$

$$\varepsilon^2\, |||r_2(\mu_i)|||_{\mathrm{L}^2} + \varepsilon^2\, |||\mathcal{L}_v^{-1}\, F(\mu_i)'|||_{\mathrm{L}^2} \leq \lambda_2 \left(\frac{\varepsilon}{\varepsilon_0}\right)^2 c_s\, \rho^3 \;, \tag{2.22}$$

where $\Delta F = F(\mu_1) - F(\mu_2)$.

We prove that this condition holds in Sect. 4. The proof requires bounds on s and r_2. We will now motivate briefly why this condition is a natural one.

We recall that $F(\mu) = F_0(s(\mu), \mu) + \chi\mathcal{L}_{\mu,r}\, r_2$. If we consider (2.20)–(2.22) only at $t = 0$, and set $F_0 = 0$, we see that by Definition 2.8, the bounds (2.20)–(2.22) are satisfied with $\lambda_1 = \lambda_{1,0} < 1$ and $\lambda_2 = \lambda_{2,0} > 0$. We will see in Sect. 4 that r_2 satisfies the same kind of bounds as those of Definition 2.8 for any time $t > 0$. On the other hand, if $s = s_1(\mu)$, or equivalently $r_2 = 0$, we have $F(\mu) = F_0(s_1(\mu), \mu)$, and (see Appendix C or the beginning of Sect. 4) we can satisfy Condition 2.11 for any $\lambda_1 < 1$ and $\varepsilon_0 = c_\varepsilon\, \delta^{-5/4}\, \rho^{-1/2}$ if c_ε is sufficiently small (depending on λ_1). To apply Theorem 2.9, we need $\xi = \frac{C_m\, c_0\, \rho}{\sqrt{2}\, \delta} < \frac{1}{4}$, and from (2.20), we have $c_0 < c_\eta$, hence we can satisfy $\xi < \frac{1}{4}$ by choosing $\delta = c_\delta\, \rho^2$ for some constant c_δ. This implies also that we should take (at least) $\varepsilon_0 = c_\varepsilon\, \rho^{-m_\varepsilon}$ with $m_\varepsilon \geq 3$.

We then have the following proposition

Proposition 2.12. *Let $c_\eta > \frac{1+c_{\eta 0}}{1-\lambda_1}$, and assume that Condition 2.11 holds with ε_0 sufficiently small. Then there exists a constant c_δ sufficiently large such that if $\delta = c_\delta\, \rho^2$ and $\varepsilon \leq \varepsilon_0$, then for all $\mu_0 \in \mathcal{B}_{0,\sigma}(c_{\eta 0}\, \rho)$, it holds*

$$|||\mathcal{F}(\tilde{\mu}_i, \mu_0)|||_\sigma < c_\eta\, \rho\ . \tag{2.23}$$

Proof. The proof follows from Theorem 2.9. We first note that $\mathcal{F}(\tilde{\mu}_i, \mu_0)$ satisfies (2.2) with $g = F(\tilde{\mu})$, and define $c_0(\tilde{\mu})$ and $\xi(\tilde{\mu})$ as in (2.15). Then by Condition 2.11, for all $\mu \in \mathcal{B}_\sigma(c_\eta\, \rho)$ and $\mu_0 \in \mathcal{B}_{0,\sigma}(c_{\eta 0}\, \rho)$, we have $c_0(\tilde{\mu}) < \lambda\, c_\eta$ with $\lambda = \lambda_1 + \frac{1+c_{\eta 0}}{c_\eta} < 1$. Choosing c_δ sufficiently large, we have $\xi(\tilde{\mu}) < \frac{1}{4}$. The proof is then completed noting that by Theorem 2.9, we have $|||\mathcal{F}(\tilde{\mu}, \mu_0)|||_\sigma \leq \left(\frac{1-\sqrt{1-4\xi(\tilde{\mu})}}{2\xi(\tilde{\mu})}\right) c_0(\tilde{\mu})\, \rho < c_\eta\, \rho$. $\square$

Proposition 2.13. *Let c_η, c_δ and ε_0 be given by Proposition 2.12, and assume that for all $\tilde{\mu}_1, \tilde{\mu}_2 \in \mathcal{B}_\sigma(c_\eta\, \rho)$ we have $|||\mathcal{F}(\tilde{\mu}_i, \mu_0)|||_\sigma < c_\eta\, \rho$ for all $\mu_0 \in \mathcal{B}_{0,\sigma}(c_\eta\, \rho)$. Then there exists a time t_0 such that*

$$\sup_{0 \leq t \leq t_0} \|\mathcal{F}(\tilde{\mu}_1, \mu_0)(\cdot, t) - \mathcal{F}(\tilde{\mu}_2, \mu_0)(\cdot, t)\|_\sigma < \sup_{0 \leq t \leq t_0} \|\tilde{\mu}_1(\cdot, t) - \tilde{\mu}_2(\cdot, t)\|_\sigma\ .$$

Proof. The proof, being very similar to the estimates leading to (2.23), can be found in Appendix D. Note that here we only asked for $\mu_0 \in \mathcal{B}_{0,\sigma}(c_\eta \rho)$ and not for $\mu_0 \in \mathcal{B}_{0,\sigma}(c_{\eta 0} \rho)$. $\square$

We now deduce from Propositions 2.12 and 2.13 existence, uniqueness, and estimates for the solution of the phase equation.

Theorem 2.14. *Let c_η, c_δ and ε_0 be given by Proposition 2.12. Then for all $T \geq 0$, the solution μ of (2.1) exists for all $0 \leq t \leq T$ and satisfies*

$$\sup_{0 \leq t \leq T} \|\mu(\cdot, t)\|_\sigma \leq c_\eta\, \rho\ , \tag{2.24}$$

for all $\mu_0 \in \mathcal{B}_{0,\sigma}(c_{\eta 0}\, \rho)$.

Proof. Let $\mathcal{F}(\tilde{\mu}, \mu_0)$ be defined by the solution of (2.19). By Proposition 2.12, we know that $\|\|\mathcal{F}(\tilde{\mu})\|\|_\sigma < c_\eta\ \rho$ if $\|\|\tilde{\mu}\|\|_\sigma \le c_\eta\ \rho$. Hence, we can apply Proposition 2.13 and get that $\tilde{\mu} \mapsto \mathcal{F}(\tilde{\mu}, \mu_0)$ is a contraction for $0 \le t \le t_0$ in the ball of radius $c_\eta\ \rho$. Thus $\tilde{\mu} \mapsto \mathcal{F}(\tilde{\mu}, \mu_0)$ has a unique fixed point $\mu_\star$ in that ball. By easy arguments (see e.g. [GvB02]), this fixed point is the unique *strong* solution of (2.1) for $0 \le t \le t_0$. Furthermore, since the image of $\tilde{\mu} \mapsto \mathcal{F}(\tilde{\mu}, \mu_0)$ is in a ball of radius $c_\eta\ \rho$, $\mu_\star$ satisfies (2.24) with $T = t_0$.

We can now show inductively that $\mu_\star$ exists for all $t \ge 0$ and satisfies (2.24) for all $T \ge 0$. Define $t_n = (n+1)t_0$ for $n \ge 1$, and suppose that $\mu_\star$ exists on $0 \le t \le t_{n-1}$ and satisfies (2.24) with $T = t_{n-1}$. By Proposition 2.12, we know that for $t_{n-1} \le t \le t_n$, the solution $\mathcal{F}(\tilde{\mu}, \mu_\star(\cdot, t_{n-1}))$ of

$$\partial_t \mu = -\mathcal{L}_\mu\ \mu - \mu\ \mu' + \varepsilon^2\ F(\tilde{\mu})' , \quad \mu(x, t_0) = \mu_\star(x, t_{n-1}) \tag{2.25}$$

is in a ball of size $c_\eta\ \rho$ if $\tilde{\mu}$ is in a ball of size $c_\eta\ \rho$ for $t_{n-1} \le t \le t_n$, because it is the *continuation* of a solution of (2.19), beginning with μ_0 in $t = 0$, with $\tilde{\mu}(x, t) = \mu_\star(x, t)$ for $0 \le t \le t_{n-1}$. Shifting the origin of time to t_{n-1} and replacing μ_0 by $\mu_\star(\cdot, t_{n-1})$, we see that the conditions of Proposition 2.13 are satisfied, hence $\tilde{\mu} \mapsto \mathcal{F}(\tilde{\mu}, \mu_\star(\cdot, t_{n-1}))$ is a contraction for $t_{n-1} \le t \le t_n$. As above, this implies that there exists an unique fixed point $\mu_\star$ which is the unique *strong* solution of 2.1 on $0 \le t \le t_n$ and satisfies (2.24) with $T = t_n$. □

2.6. Consequences. Up to now, we did not use (2.22) of Condition 2.11. This inequality has two important consequences which are proved in Theorems 2.15 and 2.16 below. The first one is that s (if it exists) and η are related by $s = -\frac{1}{8}\ \eta'' = -\frac{1}{8}\ \mu'$ up to corrections of order ε^2 and the second one concerns the relation with the Kuramoto–Sivashinsky equation. Once these theorems are proved, we will only have to prove the bound on $\hat{s}$ to complete the proof of Theorems 1.1 and 1.2.

Theorem 2.15. *There exists a constant $c_\varepsilon > 0$ sufficiently small such that if Condition 2.11 is satisfied with $\varepsilon_0 \le c_\varepsilon\ \rho^{-4}$, δ is given by Proposition 2.12 and $\varepsilon \le \varepsilon_0$, then it holds*

$$\frac{\|\|s + \frac{1}{8}\ G\ \mu' + \frac{\varepsilon^2}{32}\ G\ (\mu)^2\|\|_{\mathrm{L}^2}}{c_\eta\ \rho} \le \left(\frac{\varepsilon}{\varepsilon_0}\right)^4 .$$

Proof. The proof is very simple. We use that $s + \frac{1}{32} G\ (4\mu' + \varepsilon^2\ \mu^2) = \varepsilon^4\ G\ r_2$, and that by assumption (see (2.22)), we have

$$\varepsilon^4\ \|r_2\|_{\mathrm{L}^2} \le \lambda_2 \left(\frac{\varepsilon}{\varepsilon_0}\right)^2 \varepsilon^2 c_s\ \rho^3 \le \left(\frac{\varepsilon}{\varepsilon_0}\right)^4 c_\eta\ \rho \left(\frac{c_\varepsilon\ \lambda_2\ c_s}{c_\eta}\right),$$

choosing c_ε sufficiently small achieves the proof. □

We next show that the solution μ_c of the Kuramoto–Sivashinsky equation (in derivative form) captures the dynamics of the (derivative of the) phase for short times (then $-\frac{1}{8}\ \mu_c'$ captures the dynamics of the amplitude by Theorem 2.15). To state the result, we introduce the operator $\mathcal{L}_{\mu,c} = \partial_x^4 + \partial_x^2$. We have the following theorem.

Theorem 2.16. *Let μ and μ_c be the solutions of*

$$\begin{aligned} \partial_t \mu &= -\mathcal{L}_\mu \, \mu - \mu\mu' + \varepsilon^2 \, F(\mu)' \,, \quad && \mu(x,0) = \mu_0(x) \,, \\ \partial_t \mu_c &= -\mathcal{L}_{\mu,c} \, \mu_c - \mu_c \mu_c' \,, && \mu_c(x,0) = \mu_0(x). \end{aligned}$$

There exist constants c_ε and c_t such that if Condition 2.11 holds with $\varepsilon_0 \le c_\varepsilon \, \rho^{-4}$, then

$$\sup_{0 \le t \le t_0} \frac{\|\mu(\cdot,t) - \mu_c(\cdot,t)\|_{\mathrm{L}^2}}{c_\eta \, \rho} \le \left(\frac{\varepsilon}{\varepsilon_0}\right)^2 , \tag{2.26}$$

for all $t_0 \le c_t \, \rho^{-4}$ and for all $\varepsilon \le \varepsilon_0$ if c_ε and c_t are sufficiently small.

Although this theorem compares μ with μ_c on the time interval $[0, c_t \, \rho^{-4}]$, it is also valid on any interval of the form $[t_0, t_0 + c_t \, \rho^{-4}]$ if μ and μ_c are equal at time t_0, and thus implies directly Theorem 1.2 (see the remark after the proof).

Proof of Theorem 2.16. Let $\mu_\pm = \mu \pm \mu_c$ and $\mathcal{L}_- \equiv \mathcal{L}_\mu - \mathcal{L}_{\mu,c} = \frac{\varepsilon^2}{2} \mathcal{L}_\mu \partial_x^2$. Note that μ_c exists and satisfies $|||\mu_c|||_\sigma \le c_\eta \, \rho$. Furthermore, μ_- satisfies

$$\frac{1}{2} \partial_t (\mu_-, \mu_-) = -(\mu_-, \mathcal{L}_{\mu,c} \mu_-) - \frac{1}{4} (\mu_-, \mu_+' \mu_-) + \varepsilon^2 \Big(\mu_-, F(\mu)' - \frac{\mathcal{L}_\mu}{2} \mu'' \Big) \,.$$

Next, we define the operator $\mathcal{L}_{v,c}$ by $\mathcal{L}_{v,c}(k) = \sqrt{\frac{1}{3}(1+k^4)}$. Using $\frac{1}{2}\mathcal{L}_{v,c}^2 + \mathcal{L}_v^2 - 2\mathcal{L}_{\mu,c} \le \frac{7}{6}$, the Cauchy–Schwartz inequality and defining $\zeta = \frac{7}{6} + \frac{\|\mu_+'\|_{\mathrm{L}^\infty}}{2}$, we get

$$\begin{aligned} \partial_t (\mu_-, \mu_-) &\le \zeta (\mu_-, \mu_-) + \varepsilon^4 \Big(\|\mathcal{L}_{v,c}^{-1} \mathcal{L}_\mu \mu''\|_{\mathrm{L}^2}^2 + \|\mathcal{L}_v^{-1} F(\mu)'\|_{\mathrm{L}^2}^2 \Big) \\ &\le \zeta \; (\mu_-, \mu_-) + \left(\frac{\varepsilon}{\varepsilon_0}\right)^4 \left(C \, c_\varepsilon^4 \, \rho^2 + C' \, \rho^6 \right) , \end{aligned}$$

for some constants C, C'. The second inequality follows from Condition 2.8, $\varepsilon_0 \le c_\varepsilon \, \rho^{-4}$ and $\|\mathcal{L}_{v,c}^{-1} \, \mathcal{L}_\mu \, \mu''\|_{\mathrm{L}^2} \le \sqrt{3} \|\mu''''\|_{\mathrm{L}^2} \le C \, \delta^4 \, \|\mu\|_\sigma$ (see Lemma B.1).

Let $t_0 \le c_t \, \rho^{-4}$, since $\zeta \le \frac{7}{6} + C \, c_\eta \, \rho \, \delta^{3/2} = c_\zeta \, \rho^4$, we have

$$\sup_{0 \le t \le t_0} \frac{\|\mu(\cdot,t) - \mu_c(\cdot,t)\|_{\mathrm{L}^2}}{c_\eta \, \rho} \le \left(\frac{\varepsilon}{\varepsilon_0}\right)^2 \left(C \, c_\varepsilon^2 + C' \right) \sqrt{e^{c_\zeta \, c_t} - 1} \le \left(\frac{\varepsilon}{\varepsilon_0}\right)^2 ,$$

if c_ε and c_t are sufficiently small. □

Note that in the proof we only used global bounds on the solutions and that the initial condition is absent from the estimations, thus the theorem generalizes immediately to intervals of the form $[t_0, t_0 + c_t \, \rho^{-4}]$.

3. The Amplitude Equation

This section is devoted to the study of the 'amplitude' equation (1.28). Using the definitions and properties of the norms $\|\!|\cdot|\!\|_\sigma$ of Sects. 2.3, we will show that for given μ with $\|\!|\mu|\!\|_\sigma$ not too large, the solution of (1.28) is determined by a well defined Lipschitz map of μ. As in Sect. 2, Eq. (1.28) suggests that we study

$$\partial_t s = -\frac{\chi}{\varepsilon^4}\big(s - \frac{\varepsilon^2}{2}s''\big) - \tfrac{\alpha^2\chi}{8}(2s'\nu + s\nu') + \frac{\chi}{\varepsilon^4}f\ , \quad s(x,0) = s_0(x)\ , \tag{3.1}$$

for given s_0, ν and f. Since $\|\!|\nu|\!\|_\sigma < \infty$, (3.1) is a linear (in s) inhomogeneous heat equation with bounded coefficients, hence the local existence and uniqueness of the solution in L^2 is known by classical arguments (see e.g. [Tem97]). For later reference, we state the

Condition 3.1. *There exist constants $c_\delta, c_\varepsilon, c_{s_0}, c_\eta$ and c_f such that $\delta = c_\delta\,\rho^2$, $\varepsilon \le \varepsilon_0 = c_\varepsilon \rho^{-3}$, $s_0 \in \mathcal{W}_{0,\sigma-1}$, $\nu \in \mathcal{B}_\sigma(c_\eta\,\rho)$ and $G^{1/2}f \in \mathcal{W}_{\sigma-1}$.*

Proposition 3.2. *If Condition 3.1 holds with c_ε sufficiently small, then there exist a constant $\lambda > 1$ such that the solution s of (3.1) satisfies*

$$\|\!|s|\!\|_{\sigma-1} \le \lambda\,\big(\|s_0\|_{\sigma-1} + \|G^{1/2}f\|_{\sigma-1}\big)\ . \tag{3.2}$$

Proof. As in the proof of Theorem 2.9, let $d_0 \equiv \|\!|s|\!\|_{\mathrm{L}^2}$, $\sigma_0 = 0$, $\sigma_1 = \frac{1}{2}$ and, for all $n \ge 1$, define $d_n \equiv \|\!|s|\!\|_{\sigma_n}$, where

$$\sigma_{n+1} = \begin{cases} \sigma_n + 1 & \text{if } \sigma_n + 1 < \sigma - 1 \\ \sigma & \text{if } \sigma_n + 1 \ge \sigma - 1 \end{cases}\ .$$

Multiplying (3.1) with s, integrating over one period, using Young's inequality, noting that $\int s(2s'\nu + s\nu') = \int (s^2\nu)' = 0$ because s and ν are periodic, and finally integrating the differential inequality, we get immediately that

$$\|\!|s|\!\|_{\mathrm{L}^2} \le \|s_0\|_{\mathrm{L}^2} + \frac{\varepsilon^4}{\chi}\,\|\!|G^{1/2}\,f|\!\|_{\mathrm{L}^2}\ . \tag{3.3}$$

From Duhamel's representation formula, we get $s(x,t) = e^{-\mathcal{L}t}s_0(x) + \mathcal{T}(s,f)(s,t)$, where $\mathcal{L} = \frac{\chi}{\varepsilon^4}\mathcal{L}_s$ and $\mathcal{T}$ is given by

$$\mathcal{T}(s,f)(s,t) = \int_0^t \mathrm{d}\tau\ e^{-\mathcal{L}(t-\tau)}\Big(\tfrac{\alpha^2\chi}{8}\big(2\partial_x\big(s\nu\big) - s\partial_x\nu\big) + f\Big)(x,\tau)\ .$$

Using (3.3) and the inequalities

$$\Big\|\!\Big|\int_0^t \mathrm{d}\tau\ e^{-\mathcal{L}(t-\tau)}\ f(x,\tau)\Big|\!\Big\|_{\mathcal{W},\sigma} \le \frac{\varepsilon^4}{\chi}\|\!|G\,f|\!\|_{\mathcal{W},\sigma} \le \frac{\varepsilon^4}{\chi}\|\!|G^{1/2}\,f|\!\|_{\mathcal{W},\sigma}\ , \tag{3.4}$$

we get that for any $n \ge 1$, we have (recall that $d_n = \|\!|s|\!\|_{\sigma_n}$)

$$d_n \le \|s_0\|_{\sigma-1} + \|G^{1/2}f\|_{\sigma-1} + C\varepsilon^4\Big(\|\!|G(s\nu)'|\!\|_{\mathcal{W},\sigma_n} + \|\!|G(s\nu')|\!\|_{\mathcal{W},\sigma_n}\Big)\ . \tag{3.5}$$

Using $\varepsilon^2\|Gf'\|_{\sigma_n} \le 2\|f\|_{\sigma_n-1}$ and $\varepsilon\|Gf\|_{\sigma_n} \le 2\|f\|_{\sigma_n-1}$, we see that the r.h.s. of (3.5) involves only d_{n-1}, which shows that the d_n are bounded for all $n \ge 1$, which gives

$|||s|||_{\sigma-1} < \infty$. Using $\varepsilon \| Gf' \|_{\sigma-1} \leq 2 \| f \|_{\sigma-1}$, $\| Gf \|_{\sigma-1} \leq \| f \|_{\sigma-1}$ and Proposition B.3, we get from (3.5) and $\nu \in \mathcal{B}_\sigma(c_\eta\, \rho)$ the inequality

$$|||s|||_{\sigma-1} \leq \|s_0\|_{\sigma-1} + \|G^{1/2} f\|_{\sigma-1} + \big(C\, \varepsilon^3\, \alpha^2\, \sqrt{\delta}\, (1+\varepsilon\, \delta)\, \rho\big)\, |||s|||_{\sigma-1}\,.$$

Choosing c_ε sufficiently small in Condition 3.1 completes the proof. □

We are now in position to prove that the solution of (1.28) exists if ε_0 is sufficiently small.

Theorem 3.3. *Let c_{r_1} and c_η be given by Proposition C.1 and Theorem 2.14, and $c_{s_0} > 0$. There exist constants $c_s > c_{r_1} + c_{s_0}$ and c_ε such that for all $\varepsilon \leq c_\varepsilon\, \rho^{-3}$, for all $\mu \in \mathcal{B}_\sigma(c_\eta\, \rho)$ and for all $s_0 \in \mathcal{B}_{0,\sigma-1}(c_{s_0}\, \delta\, \rho)$, the solution s of (1.28) with $s(x,0) = s_0(x)$ exists and is unique in $\mathcal{B}_{\sigma-1}(c_s\, \delta\, \rho)$. As such, it defines the map $\mu \mapsto s(\mu)$, which, for all $\mu_i \in \mathcal{B}_\sigma(c_\eta\, \rho)$, satisfies*

$$|||s(\mu_i)|||_{\sigma-1} \leq c_s\, \delta\, \rho\,, \tag{3.6}$$

$$|||s(\mu_1) - s(\mu_2)|||_{\sigma-1} \leq c_s\, \delta\, |||\mu_1 - \mu_2|||_\sigma\,. \tag{3.7}$$

Proof. For all $\tilde{s} \in \mathcal{W}_{\sigma-1}$, define $T(\tilde{s}, \mu)$ as the solution of (3.1) with $\nu = \mu$ and $f = r_1(\mu) + \frac{\varepsilon^4}{\chi} F_3(\tilde{s}, \mu)$. By Proposition 3.2, $T(\tilde{s}, \mu)$ is well defined if $\|s_0\|_{\sigma-1} + |||r_1(\mu)|||_{\sigma-1} + |||F_3(\tilde{s}, \mu)\|_{\sigma-1} < \infty$. To show that $s(\mu)$ exists, is unique and satisfies (3.6), we only have to show that if ε is sufficiently small, $\tilde{s} \mapsto T(\tilde{s}, \mu)$ is a contraction in $\mathcal{B}_{\sigma-1}(c_s\, \delta\, \rho) \subset \mathcal{W}_{\sigma-1}$. Using Propositions 3.2 and C.1 and the assumption on s_0, we have

$$|||T(s,\mu)|||_{\sigma-1} \leq \lambda\, (c_{r_1} + c_{s_0})\, \delta\, \rho + \frac{\lambda\, \varepsilon^4}{\chi}\, |||F_3(s,\mu)|||_{\sigma-1}\,, \tag{3.8}$$

$$|||T(s_1,\mu) - T(s_2,\mu)|||_{\sigma-1} \leq \frac{\lambda\, \varepsilon^4}{\chi}\, |||F_3(s_1,\mu) - F_3(s_2,\mu)|||_{\sigma-1}\,. \tag{3.9}$$

The contraction property follows immediately from Proposition C.4 if c_ε is sufficiently small and $c_s > \lambda(c_{r_1} + c_{s_0} + \zeta)$. Hence, the map $s \mapsto T(s,\mu)$ has a unique fixed point $s^\star(\mu)$. This fixed point satisfies (3.6) and is a strong solution of (1.28) (see also [GvB02]).

For (3.7), we define $\mu_\pm = \mu_1 \pm \mu_2$, $s_1 = s^\star(\mu_1)$, $s_2 = s^\star(\mu_2)$ and $s_\pm = s_1 \pm s_2$. First, we note that s_- satisfies (3.1) with $\nu = \mu_+$ and $f = F_3(s_1,\mu_1) - F_3(s_2,\mu_2) - \frac{\alpha^2\, \chi\, \varepsilon^4}{16}(2s_+'\mu_- + s_+\mu_-')$. Next, for all $0 < \zeta_1 < 1$, we have the estimations

$$\varepsilon^4 |||G^{1/2}(s_+\mu_-)')|||_{\sigma-1} \leq C\varepsilon^3\sqrt{\delta}|||s_+|||_{\sigma-1}|||\mu_-|||_\sigma \leq \zeta_1\delta|||\mu_-|||_\sigma\,,$$
$$\varepsilon^4 |||G^{1/2} s_+\mu_-'|||_{\sigma-1} \leq C\varepsilon^4\delta^{3/2}|||s_+|||_{\sigma-1}|||\mu_-|||_\sigma \leq \zeta_1\delta|||\mu_-|||_\sigma\,,$$

if $\varepsilon \leq c_\varepsilon\, \rho^{-3}$ with c_ε sufficiently small. Finally, writing $F_3(s_1,\mu_1) - F_3(s_2,\mu_2) = F_3(s_1,\mu_1) - F_3(s_1,\mu_2) + F_3(s_1,\mu_2) - F_3(s_2,\mu_2)$, using Propositions 3.2, C.1 and C.4, we conclude that

$$|||s_-|||_{\sigma-1} \leq \lambda\, (c_{r_1} + 2\zeta_1 + \zeta)\, \delta\, |||\mu_-|||_\sigma + \zeta\, |||s_-|||_{\sigma-1}\,.$$

Since $\zeta < 1$, the proof is completed choosing c_s sufficiently large. □

We end this section by proving that $\mu \mapsto r(\mu) = s(\mu) - \frac{\varepsilon^2}{2} s(\mu)''$ satisfies essentially the same bounds as $\mu \mapsto s(\mu)$.

Corollary 3.4. *Assume that* $r_0 \in \mathcal{B}_{0,\sigma-1}(c_{r_1}\ \delta\ \rho)$. *Then there exists a constant* $c_r > c_{r_1} + c_{s_0}$ *such that* $\mu \mapsto r(\mu)$ *satisfies*

$$\||r(\mu)\||_{\sigma-1} \leq c_r\ \delta\ \rho\ , \tag{3.10}$$

$$\||r(\mu_1) - r(\mu_2)\||_{\sigma-1} \leq c_r\ \delta\ \||\mu_1 - \mu_2\||_\sigma\ , \tag{3.11}$$

if the conditions of Theorem 3.3 are satisfied.

Proof. The proof, being very similar to the ones of Proposition 3.2 and Theorem 3.3 is outlined in Appendix E. □

4. The Condition 2.11, Properties of $\mu \mapsto F(\mu)$ and $\mu \mapsto r_2(\mu)$

We recall that $F(\mu) = F_0(s(\mu), \mu) + \chi\ \mathcal{L}_{\mu,r}\ r_2(\mu)$, where r_2 is defined in (1.33). If $r_2 = 0$, Condition 2.11 can be satisfied if $\varepsilon_0 \leq c_\varepsilon\ \rho^{-3}$ with c_ε sufficiently small. Namely, from Theorem C.2, Appendix C, using also $\|\mathcal{L}_v^{-1}\ f'\|_{\mathrm{L}^2} \leq 2\ \|f\|_{\mathrm{L}^2} \leq 2\ \|f\|_{\sigma-2}$, we have

$$4\varepsilon^2\||\mathcal{L}_v^{-1}F_0(\mu_i)'\||_{\mathrm{L}^2} + \varepsilon^2\left\|\left|\frac{F_0(\mu_i)'}{\mathcal{L}_\mu}\right|\right\|_{\mathcal{W},\sigma} \leq C\varepsilon^2\delta^{5/2}\rho^2\ , \tag{4.1}$$

$$\varepsilon^2\||\mathcal{L}_v^{-1}\Delta F_0'\||_{\mathrm{L}^2} + \varepsilon^2\left\|\left|\frac{\Delta F_0'}{\mathcal{L}_\mu}\right|\right\|_{\mathcal{W},\sigma} \leq C\varepsilon^2\delta^{5/2}\rho\||\mu_1 - \mu_2\||_\sigma\ , \tag{4.2}$$

where $\Delta F_0 = F_0(\mu_1) - F_0(\mu_2)$. Since $\delta = c_\delta\ \rho^2$, we see that for $\varepsilon_0 = c_\varepsilon\ \rho^{-3}$, the contribution of F_0 to the bounds (2.20)–(2.22) can be made arbitrarily small, choosing c_ε sufficiently small, independently of ρ, or of the size of the system L. So what we need is more detailed information on r_2. Note that r_2 inherits the bounds of r and r_1, but with a factor ε^{-4}, so that we have to work a little more to show that the bounds on r_2 are finite as $\varepsilon \to 0$, and that (2.20)–(2.22) are also satisfied when the contribution of r_2 is taken into account. The essential input will be that as a dynamical variable, r_2 satisfies (1.38) with $r_2(x, 0) = r_{2,0}(x) \equiv \frac{r_0}{\varepsilon^4} - \frac{r_1(\mu_0)}{\varepsilon^4}$. Since we know that $s(\mu)$ exists, we can view (1.38) as a linear inhomogeneous equation for r_2 and derive bounds from it. These bounds are proved in the four following lemmas, where, for convenience, we write $F_6(s(\mu), \mu) = F_6(\mu)$.

Lemma 4.1. *If* r_2 *solves (1.38) with* $r_2(x, 0) = r_{2,0}(x)$, *then for all* $\mu \in \mathcal{B}_\sigma(c_\eta\rho)$, *one has*

$$\||r_2(\mu)\||_{\mathrm{L}^2} \leq \|r_{2,0}\|_{\mathrm{L}^2} + \||F_6(\mu)\||_{\mathrm{L}^2}\ , \tag{4.3}$$

$$\||\chi\mathcal{L}_{\mu,r}\mathcal{L}_v^{-1}r_2(\mu)'\||_{\mathrm{L}^2} \leq \||\chi\mathcal{L}_{\mu,r}\mathcal{L}_v^{-1}r_{2,0}'\||_{\mathrm{L}^2} + \sqrt{2}\||\mathcal{L}_{\mu,r}\mathcal{L}_v^{-1}F_6(\mu)'\||_{\mathrm{L}^2}\ . \tag{4.4}$$

Proof. Let $r_4 = \chi\mathcal{L}_{\mu,r}\mathcal{L}_v^{-1}r_2'$, then using Young's inequality and Proposition F.2 (see Appendix F) we get that r_2 and r_4 satisfy

$$\partial_t(r_2, r_2) \leq -\frac{\chi}{\varepsilon^4}(r_2, r_2) + \frac{2}{\chi\varepsilon^4}\|F_6(\mu)\|_{\mathrm{L}^2}^2\ ,$$

$$\partial_t(r_4, r_4) \leq -\frac{\chi}{\varepsilon^4}(r_4, r_4) + \frac{2\chi}{\varepsilon^4}\|\mathcal{L}_{\mu,r}\mathcal{L}_v^{-1}F_6(\mu)'\|_{\mathrm{L}^2}^2\ .$$

Integrating these differential inequalities completes the proof. □

Lemma 4.2. *Assume that the solution r_2 of (1.38) satisfies $\varepsilon^2 \,|||\chi\, \mathcal{L}_{\mu,r}\, \mathcal{L}_v^{-1}\, r_2(\mu)'|||_{\mathrm{L}^2} \leq c_\eta\, \rho$. Then for all $\gamma > 1$, there exist constants C and c_ε such that for all $\mu \in \mathcal{B}_\sigma(c_\eta\, \rho)$ and for all $\varepsilon \leq c_\varepsilon$ with c_ε sufficiently small, one has*

$$\Big|\Big|\Big| \frac{\chi\mathcal{L}_{\mu,r}}{\mathcal{L}_\mu} r_2(\mu)' \Big|\Big|\Big|_{\mathcal{W},\sigma} \leq \gamma\Big(\| \frac{\chi\mathcal{L}_{\mu,r}}{\mathcal{L}_\mu} r'_{2,0}\|_{\mathcal{W},\sigma} + \Big|\Big|\Big| \frac{\mathcal{L}_{\mu,r}}{\mathcal{L}_\mu} \frac{F_6(\mu)'}{G\mathcal{L}_r} \Big|\Big|\Big|_{\mathcal{W},\sigma} + C\varepsilon^2 c_\eta \rho \Big) .$$

Proof. We define

$$r_3 = P_>\Big(\frac{\chi\mathcal{L}_{\mu,r}}{\mathcal{L}_\mu} r'_2 \Big) , \tag{4.5}$$

and we note that $|||r_3|||_{\mathcal{W},\sigma} < \infty$, because

$$|||r_3|||_{\mathcal{W},\sigma} = \Big|\Big|\Big| \frac{\chi\mathcal{L}_{\mu,r}}{\mathcal{L}_\mu} r'_2 \Big|\Big|\Big|_\sigma \leq \frac{C}{\delta} |||r_2|||_{\sigma-1} \leq \frac{C}{\delta\varepsilon^4}\Big(|||r(\mu)|||_{\sigma-1} + |||r_1(\mu)|||_{\sigma-1} \Big) .$$

On the other hand, r_3 satisfies

$$\partial_t r_3 = -\tfrac{\chi}{\varepsilon^4} G\mathcal{L}_r r_3 + \tfrac{\chi}{16} P_>\Big(\tfrac{\mathcal{L}_{\mu,r}}{\mathcal{L}_\mu} \big(\mu\mathcal{L}_\mu r_3\big)' \Big) + \tfrac{\chi}{\varepsilon^4} F_9(\mu, P_< r_2) , \tag{4.6}$$

with $r_3(x, 0) = r_{3,0}(x)$ and

$$F_9(\mu, P_< r_2) = \frac{\chi\varepsilon^4}{16} P_> \frac{\mathcal{L}_{\mu,r}}{\mathcal{L}_\mu} \big(\mu\mathcal{L}_{\mu,r} P_< r'_2\big)' + P_> \frac{\mathcal{L}_{\mu,r}}{\mathcal{L}_\mu} F_6(\mu)' .$$

Using that $|||r_3|||_{\mathrm{L}^2} \leq c_\eta\rho$ by hypothesis on r_2, and $\varepsilon^2\|\mathcal{L}_\mu f\|_{\sigma-2} \leq 2\delta^2\|f\|_\sigma$, we get

$$\varepsilon^2\delta^{-3}|||\mu\mathcal{L}_\mu r_3|||_{\mathcal{W},\sigma-3} \leq C\delta^{-5/2}|||\mu|||_\sigma |||r_3|||_\sigma \leq Cc_\eta\rho + C|||r_3|||_{\mathcal{W},\sigma} .$$

Using this estimate, Duhamel's formula for the solution of (4.6) and Lemma F.1, we get

$$|||r_3|||_{\mathcal{W},\sigma} \leq \|r_{3,0}\|_{\mathcal{W},\sigma-1} + \Big|\Big|\Big| \frac{F_9(\mu, P_< r_2)}{G\mathcal{L}_r} \Big|\Big|\Big|_{\mathcal{W},\sigma} + C\varepsilon^2 c_\eta\rho + C\varepsilon^2|||r_3|||_{\mathcal{W},\sigma} .$$

This gives an estimation on $|||r_3|||_{\mathcal{W},\sigma}$ if $\varepsilon \leq c_\varepsilon$ with c_ε sufficiently small. Using Lemma F.1, $\|P_<\mathcal{L}_v f\|_\sigma = \|P_<\mathcal{L}_v f\|_{\mathrm{L}^2} \leq 4\delta^2\|f\|_{\mathrm{L}^2}$ and $\|P_< f\|_\sigma = \|P_< f\|_{\mathrm{L}^2}$, we get

$$\begin{aligned}\frac{\chi\varepsilon^4}{16}\Big|\Big|\Big| \frac{\mathcal{L}_{\mu,r}}{\mathcal{L}_\mu G\mathcal{L}_r}\big(\mu\mathcal{L}_{\mu,r}P_< r'_2\big)' \Big|\Big|\Big|_{\mathcal{W},\sigma} &\leq C\varepsilon^4\delta^{-1/2}\rho|||\chi P_<\mathcal{L}_{\mu,r}\mathcal{L}_v^{-1} r'_2|||_\sigma \\ &= C\varepsilon^4|||\chi\mathcal{L}_{\mu,r}\mathcal{L}_v^{-1} r'_2|||_{\mathrm{L}^2} \leq C\varepsilon^2 c_\eta\rho .\end{aligned}$$

Choosing c_ε sufficiently small completes the proof. □

Lemma 4.3. *Let $r_2(\mu_i)$ be the solution of (1.38) with $\mu = \mu_i$, and define $\Delta r_2 = r_2(\mu_1) - r_2(\mu_2)$ and $\Delta F_6 = F_6(\mu_1) - F_6(\mu_2)$. Assume that*

$$4\varepsilon^2|||\chi\mathcal{L}_{\mu,r}\mathcal{L}_v^{-1} r_2(\mu_i)'|||_{\mathrm{L}^2} + \varepsilon^2\Big|\Big|\Big| \frac{\chi\mathcal{L}_{\mu,r}}{\mathcal{L}_\mu} r_2(\mu_i)' \Big|\Big|\Big|_{\mathcal{W},\sigma} \leq c_\eta\rho . \tag{4.7}$$

Then for all $\mu_i \in \mathcal{B}_\sigma(c_\eta\, \rho)$, there exists a constant C such that

$$|||\chi\mathcal{L}_{\mu,r}\mathcal{L}_v^{-1}\Delta r'_2|||_{\mathrm{L}^2} \leq |||\chi\mathcal{L}_{\mu,r}\mathcal{L}_v^{-1}\Delta F'_6|||_{\mathrm{L}^2} + C\delta^{5/2}\rho|||\mu_1 - \mu_2|||_\sigma . \tag{4.8}$$

Proof. The proof follows from Lemma 4.1, with the replacements $r_{2,0} = 0$, $r_2 \leftrightarrow \Delta r_2$, $F_6 \leftrightarrow \Delta F_6 + \Delta\mu\mathcal{L}_{\mu,r}\Delta r_+'$. We estimate the additional term by

$$\frac{\varepsilon^4}{8}\|\chi\mathcal{L}_{\mu,r}\mathcal{L}_v^{-1/2}(\Delta\mu\mathcal{L}_{\mu,r}r_+{}')'\|_{L^2} \leq 8\varepsilon^4\|\Delta\mu\mathcal{L}_{\mu,r}r_+{}'\|_{L^2} .$$

Using (4.7), defining r_3 in terms of r_+ as in (4.5) in terms of r_2, we have

$$\begin{aligned}\varepsilon^4\|\Delta\mu\mathcal{L}_{\mu,r}r_+'\|_{L^2} &\leq \varepsilon^4\|\Delta\mu P_<\mathcal{L}_{\mu,r}\mathcal{L}_v\mathcal{L}_v^{-1}r_+'\|_{L^2} + \varepsilon^4\|\Delta\mu\mathcal{L}_\mu r_3\|_{L^2}\\ &\leq C\varepsilon^4\delta^{5/2}\|\Delta\mu\|_\sigma\|\mathcal{L}_{\mu,r}\mathcal{L}_v^{-1}r_+'\|_{L^2} + \varepsilon^4\|\Delta\mu\mathcal{L}_\mu r_3\|_{\sigma-2}\\ &\leq C\varepsilon^2\delta^{5/2}\|\Delta\mu\|_\sigma c_\eta\rho + C\varepsilon^2\delta^{5/2}\|\Delta\mu\|_\sigma\|r_3\|_\sigma ,\end{aligned}$$

since $\|P_<\mathcal{L}_v f\|_{L^2} \leq 3\delta^2\|f\|_{L^2}$, $\varepsilon^2\|\mathcal{L}_\mu f\|_{\sigma-2} \leq 2\delta^2\|f\|_\sigma$ and $\varepsilon^2\|r_3\|_\sigma \leq c_\eta\rho$ and the proof is completed. □

Lemma 4.4. *Let $r_2(\mu_i)$ be the solution of (1.38) with $\mu = \mu_i$, and define $\Delta r_2 = r_2(\mu_1) - r_2(\mu_2)$ and $\Delta F_6 = F_6(\mu_1) - F_6(\mu_2)$. Assume that*

$$4\varepsilon^2|||\chi\mathcal{L}_{\mu,r}\mathcal{L}_v^{-1}r_2(\mu_i)'|||_{L^2} + \varepsilon^2\left|\left|\left|\frac{\chi\mathcal{L}_{\mu,r}}{\mathcal{L}_\mu}r_2(\mu_i)'\right|\right|\right|_{\mathcal{W},\sigma} \leq c_\eta\rho , \tag{4.9}$$

$$\varepsilon^2|||\chi\mathcal{L}_{\mu,r}\mathcal{L}_v^{-1}\Delta r_2(\mu_i)'|||_{L^2} \leq |||\mu_1 - \mu_2|||_\sigma . \tag{4.10}$$

Then for all $\gamma > 1$ and for all $\mu_i \in \mathcal{B}_\sigma(c_\eta\ \rho)$ and for all $\varepsilon \leq c_\varepsilon$ sufficiently small, there exists a constant C such that

$$\left|\left|\left|\frac{\chi\mathcal{L}_{\mu,r}}{\mathcal{L}_\mu}\Delta r_2'\right|\right|\right|_{\mathcal{W},\sigma} \leq \gamma\left(\left|\left|\left|\frac{\mathcal{L}_{\mu,r}}{\mathcal{L}_\mu G\mathcal{L}_r}\Delta F_6'\right|\right|\right|_{\mathcal{W},\sigma} + C|||\mu_1 - \mu_2|||_\sigma\right) . \tag{4.11}$$

Proof. The proof follows from Lemma 4.2, with the replacements of the proof of Lemma 4.3 for ΔF_6 and $\mu\mathcal{L}_{\mu,r}r_2'$, we omit the details. □

We can now show that Condition 2.11 is satisfied if $\varepsilon \leq \varepsilon_0 \leq c_\varepsilon\sqrt{1 - 2\,\alpha^2}\ \rho^{-4}$ with c_ε sufficiently small, $\alpha^2 < \frac{1}{2}$ and if the initial data μ_0 and s_0 are in the class $\mathcal{C}$.

Proposition 4.5. *Let $\alpha^2 < \frac{1}{2}$, $\delta = c_\delta\ \rho^2$, and assume that $r_{2,0}$ is an admissible initial condition. For all $\gamma > 1$, there exist a constant c_ε sufficiently small such that for all $\varepsilon \leq \varepsilon_0 \leq c_\varepsilon\sqrt{1 - 2\,\alpha^2}\ \rho^{-4}$ there exist constants $\lambda_1 < 1$ and $\lambda_2 < 1$ such that for all $\mu \in \mathcal{B}_\sigma(c_\eta\ \rho)$, one has*

$$4\,\varepsilon^2\ |||\mathcal{L}_v^{-1}\ F(\mu_i)'|||_{L^2} + \varepsilon^2\ \left|\left|\left|\frac{F(\mu_i)'}{\mathcal{L}_\mu}\right|\right|\right|_{\mathcal{W},\sigma} \leq \lambda_1\ c_\eta\ \rho , \tag{4.12}$$

$$4\,\varepsilon^2\ |||\mathcal{L}_v^{-1}\ \Delta F'|||_{L^2} + \varepsilon^2\ \left|\left|\left|\frac{\Delta F'}{\mathcal{L}_\mu}\right|\right|\right|_{\mathcal{W},\sigma} \leq \lambda_1\ |||\mu_1 - \mu_2|||_\sigma , \tag{4.13}$$

$$\varepsilon^2\ |||r_2(\mu)|||_{L^2} + \varepsilon^2\ |||\mathcal{L}_v^{-1}\ F(\mu)'|||_{L^2} \leq \lambda_2\left(\frac{\varepsilon}{\varepsilon_0}\right)^2 c_s\ \rho^3 . \tag{4.14}$$

Proof. We recall that $F = F_0 + \chi \mathcal{L}_{\mu,r} r_2$. The logical order of the proof is to start with (4.14), which follows from Definition 2.8, Lemma 4.1, Proposition C.6 and (4.1) if c_ε is sufficiently small. Using the same results, we get $\varepsilon^2 \,|||\chi\, \mathcal{L}_{\mu,r}\, \mathcal{L}_v^{-1}\, r_2(\mu)'|||_{L^2} \leq c_\eta\, \rho$ (this is needed in Lemma 4.2), and (4.12) then follows from Definition 2.8, Lemma 4.2, Proposition C.6 and (4.1), again if c_ε is sufficiently small. Now, since $\lambda_1 < 1$, we get (4.7) from (4.12), so the hypothesis of Lemma 4.3 is fulfilled, which in turn shows that both hypotheses (4.9) and (4.10) of Lemma 4.4 are fulfilled, and then (4.13) follows from Definition 2.8, Lemmas 4.3 and 4.4, Proposition C.6 and (4.2), again if c_ε is sufficiently small. □

We end this paper by noting that the class $\mathcal{C}$ is almost preserved by the time evolution, in the sense that the solution of the Complex Ginzburg Landau equation with corresponding initial data exists for all times and is for all times in a (larger) class $\mathcal{C}'$ characterized by the same constants as those of $\mathcal{C}$, except for $c_s, c_\eta, \lambda_1, \lambda_2$ which are larger than $c_{s_0}, c_{\eta_0}, \lambda_{1,0}, \lambda_{2,0}$.

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A. Coercive Functional for the Phase

Proposition A.1. *Let* $(v, w) = \int vw$, $(v, w)_{\gamma\phi} = \int v(\mathcal{L}_\mu + \gamma\phi')w$ *and*

$$\mathcal{L}_v(k) = \sqrt{\frac{1}{3}\frac{1+k^4}{1+\frac{\varepsilon^2 k^2}{2}}}\,. \tag{A.1}$$

For all $L \geq 2\pi$, there exist a constant K and an antisymmetric periodic function ϕ such that for all $\gamma \in [\frac{1}{4}, 1]$ and $\varepsilon \leq L^{-2/5}$, and for every antisymmetric periodic function v, one has

$$\frac{3}{4}\,(\mathcal{L}_v\, v, \mathcal{L}_v\, v) \leq (v, v)_{\gamma\phi} \leq \|\phi'\|_\infty\,(v, v) + (v'', v'')\,,$$
$$(\phi, \phi)_{\gamma\phi} \leq K\, L^{16/5} \quad \text{and} \quad (\phi, \phi) \leq \tfrac{4}{3}\, L^3\,.$$

Proof. The proof is based on a similar result of [CEES93] for $\mathcal{L}_{\mu,c} = \partial_x^4 + \partial_x^2$. We will need some technical alterations of their proof to take into account that $\mathcal{L}_\mu$ is of lower order than $\mathcal{L}_{\mu,c}$. However, by (1.31) and (1.36), the two operators are equal in the limit $\varepsilon \to 0$, so we will recover their result as a particular case. As we will see, in the statement of Proposition 2.1, the restriction $\varepsilon \leq (\pi L^{2/5})^{-1}$ is a convenient one because then we can use the same function ϕ as that defined in [CEES93]. We will see later that we need a much stronger restriction on ε anyway.

The proof really amounts to construct the function ϕ. Let $q \equiv \frac{2\pi}{L} \leq 1$ and M be the smallest integer (strictly) larger than $\frac{1}{2}\, L^{7/5}$. We define ϕ by

$$\phi(x) = \sum_{n\in\mathbf{Z}} e^{iqnx}\, \phi_n\,,$$

where the Fourier coefficients ϕ_n are given by

$$\phi_n = \begin{cases} 0\,, & n = 0 \\ \frac{4\,i}{qn}\,, & 1 \le |n| \le \;2M \\ \frac{4\,i\;f(|n|/2M-1)}{qn}\,, & \text{otherwise} \end{cases}\;,$$

where f is a non-increasing $\mathcal{C}^1$ function satisfying $f(0) = 1$, $f'(0) = 0$ and

$$f \ge 0,\;\; \sup|f'| < 1, \quad \int_0^\infty \mathrm{d}k\;(1+k)^2\;|f(k)|^2 < \infty\,.$$

The proof then follows from the three technical lemmas below. □

Lemma A.2. *There exists a constant K such that the function ϕ defined above satisfies*

$$(\phi,\phi) \le \tfrac{4}{3}\,L^3\,,\;\;(\phi,\phi)_{\gamma\phi} \le K\;L^{16/5} \quad \textit{and} \quad (v,v)_{\gamma\phi} \le K\;L^{7/5}\;\|v\|^2_{\mathrm{L}^2} + \|v''\|^2_{\mathrm{L}^2}$$

for all periodic antisymmetric functions v.

Proof. For the first inequality, we have

$$(\phi,\phi) = \frac{4\pi}{q}\sum_{n=1}^{\infty}|\phi_n|^2 \le \frac{4^3\,\pi}{q^3}\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{1}{6}\left(\frac{4\pi}{q}\right)^3 = \frac{4}{3}\,L^3\,.$$

For the second inequality, we use that ϕ is periodic, so that $\int\phi^2\phi' = 0$, giving

$$(\phi,\phi)_{\gamma\phi} = (\phi,\mathcal{L}_\mu\phi) = \frac{4\pi}{q}\sum_{n=1}^{\infty}\mathcal{L}_\mu(qn)\;|\phi_n|^2\,,$$

where $\mathcal{L}_\mu$ is defined in (1.30). Since $\mathcal{L}_\mu(qn) \le (qn)^4$ and $M < L^{7/5}$, we get

$$\begin{aligned}(\phi,\phi)_{\gamma\phi} = (\phi,\mathcal{L}_\mu\phi) &= \frac{4\pi}{q}\sum_{n=1}^{\infty}\mathcal{L}_\mu(qn)\;|\phi_n|^2 \\ &\le 4^3\pi\;q\left(\sum_{n=1}^{2M}n^2 + (2M)^2\sum_{n=1}^{\infty}\Big(1+\frac{n}{2M}\Big)^2 f\Big(\frac{n}{2M}\Big)^2\right) \\ &\le C\;L^{16/5}\left(1+\int_0^\infty \mathrm{d}k\;(1+k)^2\;f(k)^2\right)\,.\end{aligned}$$

Finally, using again $\mathcal{L}_\mu(qn) \le (qn)^4$, we have

$$(v,v)_{\gamma\phi} \le \|\phi'\|_{\mathrm{L}^\infty}\;\|v\|^2_{\mathrm{L}^2} + \|v''\|^2_{\mathrm{L}^2}\,.$$

Using the Cauchy–Schwartz inequality, we have

$$\begin{aligned}\|\phi'\|_{\mathrm{L}^\infty} &\leq 2\sum_{n=1}^{\infty}|qn|\,|\phi_n| \leq 16M + 2\sum_{n=1}^{\infty}\left|f\left(\frac{n}{2M}\right)\right| \\ &\leq 16M + 4M\int_0^\infty \mathrm{d}k\,\left(\frac{1+k}{1+k}\right)|f(k)| \\ &\leq C\,L^{7/5}\left(1+\sqrt{\int_0^\infty \mathrm{d}k\;(1+k)^2\,|f(k)|^2}\right)\,.\end{aligned}$$

This completes the proof of the lemma. □

Lemma A.3. *For all $L \geq 2\pi$, for all $\gamma \in [\frac{1}{4}, 1]$ and for all $\varepsilon \leq \frac{1}{Mq}$, one has*

$$(v, v)_{\gamma\phi} \geq \frac{3}{4}(\mathcal{L}_v\; v, \mathcal{L}_v\; v)\;. \tag{A.2}$$

Proof. Following [CEES93] one shows first that

$$(v, v)_{\gamma\phi} = 2L\left[\sum_{n>0}(\mathcal{L}_\mu(qn) + \gamma\psi_{2n})v_n^2 + 2\gamma\sum_{k>m>0} v_k v_m(\psi_{|k+m|} - \psi_{|k-m|})\right]\,,$$

where $\psi_n = -iqn\;\phi_n$. Then one notices that for $0 \leq \varepsilon \leq 1$, one has

$$\mathcal{L}_\mu(qn) + \gamma\psi_{2n} \geq \tau(qn)^2 \equiv \frac{1}{2}\frac{1+(qn)^4}{1+\frac{\varepsilon^2(qn)^2}{2}} \geq \tau_1(qn)^2 \equiv \frac{1}{2}\frac{(qn)^4}{1+\frac{\varepsilon^2(qn)^2}{2}}\;.$$

The definition of τ here is different from that of [CEES93], except in the $\varepsilon = 0$ limit. We now define $w_n = v_n\tau_n$ (in particular $w = \sqrt{\frac{3}{2}}\;\mathcal{L}_v\; v$), and the operator Γ by

$$(w, \Gamma w) = L\sum_{0<m<k} w_k\frac{\psi_{|k+m|} - \psi_{|k-m|}}{\tau_k\tau_m}w_m\;, \tag{A.3}$$

and get

$$(v, v)_{\gamma\phi} \geq (w\;,\;(\mathrm{Id} + 2\gamma\Gamma)\;w) \geq \frac{1}{2}\;(w, w) = \frac{3}{4}\;(\mathcal{L}_v\; v, \mathcal{L}_v\; v)\;,$$

since (see Lemma A.4 below), the Hilbert–Schmidt norm of $2\gamma\Gamma$ is less than $\frac{1}{2}$. □

Lemma A.4. *Let the operator Γ be defined by (A.3). For all $L \geq 2\pi$, for all $\gamma \in [\frac{1}{4}, 1]$ and for all $\varepsilon \leq \frac{1}{Mq}$, the Hilbert–Schmidt norm of $2\gamma\Gamma$.*

Proof. Note again that $\varepsilon \leq \frac{1}{Mq} = (\pi L^{2/5})^{-1}$ is only a convenient restriction (see remark above). To prove this lemma, it is sufficient (see [CEES93]) to show that

$$\|\Gamma\|_{\mathrm{HS}}^2 \equiv \sum_{0<m<k} \left| \frac{\psi_{|k+m|} - \psi_{|k-m|}}{\tau_k \; \tau_m} \right|^2 < \frac{1}{16} \, . \tag{A.4}$$

By definition of ϕ, for all $k > m > 0$, we have

$$\begin{aligned} |\psi_{k-m} - \psi_{k+m}| &= 0 \, , \quad \text{if } k+m \leq 2M \, , \\ |\psi_{k-m} - \psi_{k+m}| &\leq 4 \min \left\{ 1, \frac{m}{M} \right\} \, , \quad \text{for all } k > m \, . \end{aligned}$$

We distinguish two sets of summation indices $S = S_{\mathrm{I}} \cup S_{\mathrm{II}}$ in the sum (A.4),

$$\begin{aligned} S_{\mathrm{I}} &= \left\{ (m, k) \in \mathbf{N}^2 \text{ s.t. } M+1 \leq m \text{ and } m+1 \leq k \right\} \, , \\ S_{\mathrm{II}} &= \left\{ (m, k) \in \mathbf{N}^2 \text{ s.t. } 1 \leq m \leq M \text{ and } 2M - m + 1 \leq k \right\} \, , \end{aligned}$$

and write $\|\Gamma\|_{\mathrm{HS}}^2 = T_{\mathrm{I}} + T_{\mathrm{II}}$ accordingly.

In the region I, we have $m > M$, and using $\varepsilon \leq \frac{1}{Mq}$ and $\frac{1}{\tau(k)} \leq \frac{1}{\tau_1(k)}$, we get

$$T_{\mathrm{I}} \leq 16 \sum_{m=M+1}^{\infty} \frac{1}{\tau(qm)^2} \sum_{k=m+1}^{\infty} \frac{1}{\tau(qk)^2} \leq 16 \int_M^{\infty} \mathrm{d}m \; \frac{1}{\tau_1(qm)^2} \int_m^{\infty} \mathrm{d}k \; \frac{1}{\tau_1(qk)^2} \, ,$$

whereas in the region II, we have $m \leq M$ and $k \geq M+1$, and using again $\varepsilon \leq \frac{1}{Mq}$ and $\frac{1}{\tau(k)} \leq \frac{1}{\tau_1(k)}$, we get

$$\begin{aligned} T_{\mathrm{II}} &\leq \frac{16}{M^2} \sum_{m=1}^{M} \frac{m^2}{\tau(qm)^2} \sum_{k=2M-m+1}^{\infty} \frac{1}{\tau(qk)^2} \leq \frac{16}{M^2} \sum_{m=1}^{M} \frac{m^2}{\tau(qm)^2} \int_M^{\infty} \mathrm{d}k \; \frac{1}{\tau_1(qk)^2} \\ &\leq \frac{160}{3} \frac{1}{M^5 \, q^4} \sum_{m=1}^{M} \left(\frac{1}{q^2} \frac{q^2 \, m^2}{1 + m^4 \, q^4} + \frac{1}{2 \, M^2 \, q^4} \frac{q^4 \, m^4}{1 + m^4 \, q^4} \right) \\ &\leq \frac{160}{3} \frac{1}{M^5 \, q^4} \left(\frac{1}{q^2} \int_0^{\infty} \frac{\mathrm{d}m}{1 + q^2 \, m^2} + \frac{1}{2 \, M \, q^4} \right) \, . \end{aligned}$$

Collecting these results, we get $\|\Gamma\|_{\mathrm{HS}}^2 \leq \frac{80\,\pi}{3} \, \frac{1}{q^7 \, M^5} + \frac{440}{9} \, \frac{1}{q^8 \, M^6}$. This bound is worse than that of [CEES93] by numerical factors only (in their bound $\frac{80\,\pi}{3}$ is replaced by $\frac{128}{3}$ and $\frac{440}{9}$ by $\frac{16}{3}$), but is uniform in $\varepsilon \leq \frac{1}{Mq}$. This motivates the restriction $\varepsilon \leq \frac{1}{Mq}$. The proof is then completed using $M > \frac{1}{2} \, L^{7/5}$. □

B. Properties of the Spaces $\mathcal{W}_\sigma$

Lemma B.1. *Let $\sigma \geq \frac{3}{2}$. There exists a constant C such that for all $n \leq \sigma - \frac{3}{2}$ and for all $m \leq \sigma - 1$, we have*

$$\|f^{(m)}\|_{\sigma-m} + \|G\, f^{(m)}\|_{\sigma-m} \;\leq\; C\,\delta^{m} \quad \|f\|_{\sigma}\,, \tag{B.1}$$

$$\|f^{(n)}\|_{\mathrm{L}^\infty} + \|G\, f^{(n)}\|_{\mathrm{L}^\infty} \quad\leq\; C\,\delta^{n+\frac{1}{2}}\, \|f\|_{\sigma}\,, \tag{B.2}$$

where $f^{(m)}$ is the m^{th} order spatial derivative of f.

Proof. Throughout the proof, we use that G acts multiplicatively in Fourier space, $(G\ f)_n = G(qn)\ f_n$ with $G(k) \leq 1$ (see 1.36), so that G is a bounded operator in the l^p and $\|\cdot\|_\sigma$ norms. Using that $\|f\|_{\mathrm{L}^\infty} \leq \|f\|_{l^1}$, and that the space derivative commutes with G, we see that we need only prove (B.1) and (B.2) for the terms without G, and with L^∞ replaced by l^1 in (B.2). In the sequel, we denote by K the operator with symbol $K(k) = |k|$.

For (B.1), we use that $|x| \leq \sqrt{1+x^2}$ and that $\|\cdot\|_{\mathrm{L}^2} = \sqrt{L}\|\cdot\|_{l^2}$ to show that

$$\begin{aligned}
\|f^{(m)}\|_{\sigma-m} &= \|f^{(m)}\|_{\mathcal{W},\sigma-m} + \|f^{(m)}\|_{\mathrm{L}^2} \leq \delta^m\ \|f\|_{\mathcal{W},\sigma} + \|f^{(m)}\|_{\mathrm{L}^2} \\
&\leq \delta^m \big(\|f\|_{\mathcal{W},\sigma} + \|P_< f\|_{\mathrm{L}^2}\big) + \sqrt{L}\ \delta^m \|(1+(K/\delta)^2)^{m/2}\ P_> f\|_{l^2} \\
&\leq \delta^m \|f\|_\sigma + \delta^m \Big(\int_{-\infty}^{\infty} \frac{2\pi\ \mathrm{d}x}{(1+x^2)^{\sigma-m}}\Big)^{1/2} \|f\|_{\mathcal{W},\sigma} \leq C\delta^m \|f\|_\sigma\ .
\end{aligned}$$

For (B.2), using the Cauchy–Schwartz inequality, we have

$$\|P_< f\|_{l^1} \leq \Big(\frac{2\,\delta}{q}\Big)^{1/2} \|P_< f\|_{l^2} \leq \sqrt{\delta\ L}\ \|P_< f\|_{l^2} \leq \sqrt{\delta}\ \|P_< f\|_{\mathrm{L}^2}\ , \tag{B.3}$$

so that

$$\begin{aligned}
\|f^{(n)}\|_{l^1} &\leq \|K^n\ P_< f\|_{l^1} + \|K^n\ P_> f\|_{l^1} \leq \delta^{n+\frac{1}{2}}\ \|f\|_{\mathrm{L}^2} + \|K^n\ P_> f\|_{l^1} \\
&\leq \delta^{n+\frac{1}{2}}\ \|f\|_{\mathrm{L}^2} + \delta^{n+\frac{1}{2}} \int_{-\infty}^{\infty} \frac{\mathrm{d}x}{(1+x^2)^{\frac{\sigma-n}{2}}}\ \|f\|_{\mathcal{W},\sigma} \leq C\ \delta^{n+\frac{1}{2}}\ \|f\|_\sigma\ .
\end{aligned}$$

The proof is completed. □

Proposition B.2. *Let $\delta \geq 2$, then there exist constants C_1 and C_2 such that*

$$\begin{aligned}
\Big\|e^{-\mathcal{L}_\mu t}\ f(\cdot)\Big\|_{\mathcal{W},\sigma} &\leq \|f(\cdot)\|_{\mathcal{W},\sigma}\ , \\
\Big\|\int_0^t \mathrm{d}s\ e^{-\mathcal{L}_\mu(t-s)}\ g'(\cdot,s)\Big\|_{\mathcal{W},\sigma} &\leq \sup_{0\leq s\leq t} \Big\|\frac{g'(\cdot,s)}{\mathcal{L}_\mu}\Big\|_{\mathcal{W},\sigma}\ , \\
\Big\|\frac{f' + G\ g'}{\mathcal{L}_\mu}\Big\|_{\mathcal{W},\sigma} &\leq \frac{C_1}{\delta}\ \|f\|_{\mathcal{W},\sigma-1} + \frac{C_2}{\delta^3}\ \|g\|_{\mathcal{W},\sigma-3}\ ,
\end{aligned} \tag{B.4}$$

where $\mathcal{L}_\mu$ is defined in (1.30) and $e^{-\mathcal{L}_\mu t}$ is the propagation Kernel associated with $\partial_t f = -\mathcal{L}_\mu f$.

Proof. The propagation Kernel $e^{-\mathcal{L}_\mu t}$ acts as $\left(e^{-\mathcal{L}_\mu t}\ f\right)_n = e^{-\mathcal{L}_\mu(qn)t}\ f_n$ in Fourier space. For $\varepsilon \le 1$ and $|k| \ge \delta \ge 2$, by (1.30), one has $\mathcal{L}_\mu(k) \ge \mathcal{L}_\mu(\delta) \ge 4$, which gives

$$\sup_{t\ge 0} \left\| e^{-\mathcal{L}_\mu t}\ f(\cdot) \right\|_{\mathcal{W},\sigma} \le e^{-\mathcal{L}_\mu(\delta)\ t}\ \|f(\cdot)\|_{\mathcal{W},\sigma} \le \|f(\cdot)\|_{\mathcal{W},\sigma}\ .$$

Next, we use that

$$\begin{aligned}\left\| \int_0^t \mathrm{d}s\ e^{-\mathcal{L}_\mu(t-s)} g'(\cdot,s) \right\|_{\mathcal{W},\sigma} &\le \sup_{|n|\ge\frac{\delta}{q}} \mathcal{L}_\mu(qn) \int_0^t \mathrm{d}s\ e^{-\mathcal{L}_\mu(qn)\ (t-s)} \left\| \frac{g'(\cdot,s)}{\mathcal{L}_\mu} \right\|_{\mathcal{W},\sigma} \\ &\le \sup_{|n|\ge\frac{\delta}{q}} \left(1 - e^{-\mathcal{L}_\mu(qn)\ t}\right) \sup_{0\le s\le t} \left\| \frac{g'(\cdot,s)}{\mathcal{L}_\mu} \right\|_{\mathcal{W},\sigma}\ .\end{aligned}$$

Since $1 - e^{-\mathcal{L}_\mu(qn)\ t} \le 1$ for $qn \ge \delta \ge 2$, the proof of (B.4) is completed.

Finally, we have

$$\begin{aligned}\left\| \frac{f'}{\mathcal{L}_\mu} \right\|_{\mathcal{W},\sigma} &\le \frac{\sqrt{\delta}}{q} \sup_{|n|\ge\frac{\delta}{q}} \left(1 + (\tfrac{qn}{\delta})^2\right)^{\sigma/2} \left(1 + \frac{\varepsilon^2 (qn)^2}{2}\right) \frac{|qn|\ |f_n|}{(qn)^4 - (qn)^2} \\ &\le \frac{1}{\delta}\ \|f\|_{\mathcal{W},\sigma-1} \left(\sup_{x\ge 1} \frac{\sqrt{1+x^2}}{x}\right)\left(\sup_{x\ge 2} \frac{1+x^2}{x^2-1}\right), \\ \left\| \frac{G\ g'}{\mathcal{L}_\mu} \right\|_{\mathcal{W},\sigma} &\le \frac{\sqrt{\delta}}{q} \sup_{|n|\ge\frac{\delta}{q}} \left(1 + (\tfrac{qn}{\delta})^2\right)^{\sigma/2} \frac{|qn|\ |g_n|}{(qn)^4 - (qn)^2} \\ &\le \frac{1}{\delta^3}\ \|g\|_{\mathcal{W},\sigma-3} \left(\sup_{x\ge 1} \frac{\sqrt{1+x^2}}{x}\right)^3 \left(\sup_{x\ge 2} \frac{x^4}{x^4-x^2}\right).\end{aligned}$$

This completes the proof. □

Proposition B.3. *Let* $\|u\|_{\sigma_1} < \infty$, $\|v\|_{\sigma_2} < \infty$ *and* $\sigma = \min(\sigma_1,\sigma_2) \ge \frac{3}{2}$. *Then there exists a constant* C_m *depending only on* σ *such that*

$$\|uv\|_\sigma \le C_m\ \sqrt{\delta}\ \|u\|_{\sigma_1}\ \|v\|_{\sigma_2}\ , \tag{B.5}$$

$$\left\| \frac{u}{1+v} \right\|_\sigma \le \frac{\|u\|_{\sigma_1}}{1 - C_m\ \sqrt{\delta}\ \|v\|_{\sigma_2}}\ , \tag{B.6}$$

provided $C_m\ \sqrt{\delta}\ \|v\|_{\sigma_2} < 1$. *If* $\sigma \le 1$, *we have the two particular cases*

$$\|uv\|_{\mathcal{W},\frac{1}{2}} \le C_m\ \sqrt{\delta}\ \|u\|_1\ \|v\|_1 \quad \text{and} \quad \|uv\|_{\mathcal{W},0} \le C_m\ \sqrt{\delta}\ \|u\|_{\mathrm{L}^2}\ \|v\|_{\mathrm{L}^2}\ . \tag{B.7}$$

Proof. We first note that if $\sigma = \min(\sigma_1,\sigma_2) \ge \frac{3}{2}$, by Lemma B.1, we have

$$\|uv\|_{\mathrm{L}^2} \le \|u\|_{\mathrm{L}^\infty}\ \|v\|_{\mathrm{L}^2} \le C\ \sqrt{\delta}\ \|u\|_{\sigma_1}\ \|v\|_{\sigma_2}\ .$$

So the L^2 part of (B.5) is proved. For the $\|\cdot\|_{\mathcal{W},\sigma}$ part of (B.5) and for (B.7), we write $u = u_< + u_>$, where $u_< = P_< u$ and $u_> = P_> u$ and the same for v. Then we have

$$\|uv\|_{\mathcal{W},\sigma} \le \|uv\|_{\mathcal{N},\sigma} \le \|u_< v_<\|_{\mathcal{N},\sigma} + \|u_< v_>\|_{\mathcal{N},\sigma} + \|u_> v_<\|_{\mathcal{N},\sigma} + \|u_> v_>\|_{\mathcal{N},\sigma}\ .$$

Clearly, $\|P_> f\|_{\mathcal{N},\sigma} \leq \|f\|_{\mathcal{W},\sigma} \leq \|f\|_\sigma$, so that we can apply Lemma B.4 below to the last term (see [BKL94] for the original version of the lemma). The first three terms in turn are bounded using Lemma B.5.

To prove (B.6), we write a geometric series for $\frac{1}{1+v}$ and use (B.5) inductively, getting

$$\left\|\frac{u}{1+w}\right\|_\sigma \leq \sum_{m\geq 0} \|uv^m\|_\sigma \leq \|u\|_{\sigma_1} \sum_{m\geq 0} \left(C_m \sqrt{\delta}\, \|v\|_{\sigma_2}\right)^m .$$

Summing the series since $C_m \sqrt{\delta}\, \|v\|_{\sigma_2} < 1$ completes the proof. □

Lemma B.4. *Let $\sigma_1, \sigma_2 \geq \frac{3}{2}$ and $\sigma = \min(\sigma_1, \sigma_2) \geq \frac{3}{2}$, then there exists a constant c_b depending only on σ such that*

$$\|uv\|_{\mathcal{N},\sigma} \leq c_b \sqrt{\delta}\, \|u\|_{\mathcal{N},\sigma_1} \|v\|_{\mathcal{N},\sigma_2} , \tag{B.8}$$

and if $\sigma < 1$, we have the two particular cases

$$\|uv\|_{\mathcal{N},\frac{1}{2}} \leq c_b\sqrt{\delta}\|u\|_{\mathcal{N},1}\|v\|_{\mathcal{N},1} \quad \textit{and} \quad \|uv\|_{\mathcal{N},0} \leq c_b\sqrt{\delta}\|u\|_{\mathrm{L}^2}\|v\|_{\mathrm{L}^2} . \tag{B.9}$$

Proof. We begin with the second inequality of (B.9). We have

$$\|uv\|_{\mathcal{N},0} = \frac{\sqrt{\delta}}{q} \sup_{n\in\mathbf{Z}} \sum_{m\in\mathbf{Z}} |u_n|\, |v_{m-n}| \leq \sqrt{\delta}\, L\, \|u\|_{l^2} \|v\|_{l^2} \leq \sqrt{\delta}\, \|u\|_{\mathrm{L}^2} \|v\|_{L^2} .$$

Next, we define $p = \frac{q}{\delta}$, $x = pn$ and $y = pm$. We have

$$\begin{aligned} \|uv\|_{\mathcal{N},\sigma} &\leq \frac{1}{\sqrt{\delta}} \sup_{n\in\mathbf{Z}} \frac{(1+(pn)^2)^{\frac{\sigma}{2}}}{p} \sum_{m\in\mathbf{Z}} |u_m||v_{n-m}| \\ &\leq \sqrt{\delta}\|u\|_{\mathcal{N},\sigma_1}\|v\|_{\mathcal{N},\sigma_2} \sup_{x\in\mathbf{R}} \int_{-\infty}^{\infty} \mathrm{d}y \left(\frac{1}{1+y^2}\frac{1+x^2}{1+(x-y)^2}\right)^{\frac{\sigma}{2}} , \end{aligned} \tag{B.10}$$

$$\|uv\|_{\mathcal{N},\frac{1}{2}} \leq \sqrt{\delta}\|u\|_{\mathcal{N},1}\|v\|_{\mathcal{N},1} \sup_{x\in\mathbf{R}} \int_{-\infty}^{\infty} \mathrm{d}y \frac{1}{(1+y^2)^{\frac{1}{2}}} \frac{(1+x^2)^{\frac{1}{4}}}{(1+(x-y)^2)^{\frac{1}{2}}} . \tag{B.11}$$

To see that both integrals in (B.10) and (B.11) are uniformly bounded in $x \in \mathbf{R}$, we can assume without loss of generality that $x \geq 0$, and use the uniform bounds

$$\begin{aligned} y \in (-\infty, x/2] &\Rightarrow \frac{1+x^2}{1+(x-y)^2} \leq 4 \text{ and } \frac{1}{1+(x-y)^2} \leq \frac{1}{1+y^2} , \\ y \in [x/2, \infty) &\Rightarrow \frac{1+x^2}{1+y^2} \leq 4 \text{ and } \frac{1}{1+y^2} \leq \frac{1}{1+(x-y)^2} , \end{aligned}$$

from which we get the desired result. □

Lemma B.5. *Let $\sigma \geq 0$, then there exists a constant C depending only on σ such that*

$$\|(P_> u)\, (P_< v)\|_{\mathcal{N},\sigma} \leq C \sqrt{\delta}\, \|u\|_\sigma \|v\|_\sigma , \tag{B.12}$$

$$\|(P_< u)\, (P_< v)\|_{\mathcal{N},\sigma} \leq C \sqrt{\delta}\, \|u\|_\sigma \|v\|_\sigma . \tag{B.13}$$

Proof. Let $p = \frac{q}{\delta}$. Using that $|p(n-m)| \le 1$ implies $1 + (pn)^2 \le 3(1 + (pm)^2)$ we get easily

$$\|(P_> u)(P_< v)\|_{\mathcal{N},\sigma} \le 3 \sup_{n \in \mathbf{Z}} \sum_{|pm| \ge 1,\ |p(n-m)| \le 1} |v_{n-m}| \Big(\frac{\sqrt{\delta}}{q} \ (1 + (pm)^2)^{\frac{\sigma}{2}} \ |u_m| \Big) .$$

The proof of (B.12) then follows since $\|v \star u\|_{l^\infty} \le \|v\|_{l^1} \|u\|_{l^\infty}$, using $\|u\|_{\mathcal{W},\sigma} \le \|u\|_\sigma$ and $\|P_< v\|_{l^1} \le \sqrt{\delta}\, \|v\|_{\mathrm{L}^2}$ (see (B.3) above). Similarly, since $|pm| \le 1$ and $|p(m-n)| \le 1$ implies $|pn| \le 2$, we have

$$\|(P_< u)(P_< v)\|_{\mathcal{N},\sigma} \le 5^{\frac{\sigma}{2}} \ \sqrt{\delta}\ L \ \sup_{n \in \mathbf{Z}} \sum_{|pm| \le 1,\ |p(m-n)| \le 1} |u_m| \ |v_{n-m}| \ .$$

The proof is completed since $L\|u \star v\|_{l^\infty} \le L\|u\|_{l^2}\|v\|_{l^2} = \|u\|_{\mathrm{L}^2}\|v\|_{\mathrm{L}^2} \le \|u\|_\sigma \|v\|_\sigma$. □

C. Bounds on Nonlinear Terms

C.1. Bounds on r_1.

Proposition C.1. *Assume that $\|\mu_1\|_\sigma \le c_\eta\, \rho$, $\|\mu_2\|_\sigma \le c_\eta\, \rho$, $\delta = c_\delta\, \rho^2$ and $\varepsilon \le 1$, and let r_1 be defined by (1.32). Then there exists a constant c_{r_1} such that*

$$\|r_1(\mu)\|_{\sigma-1} \le c_{r_1}\ \delta\ \rho\ , \tag{C.1}$$

$$\|r_1(\mu_1) - r_1(\mu_2)\|_{\sigma-1} \le c_{r_1}\ \delta\ \|\mu_1 - \mu_2\|_{\sigma-1}\ . \tag{C.2}$$

Proof. Using Lemma B.1, Proposition B.3 and $\mu_1^2 - \mu_2^2 = (\mu_1 - \mu_2)(\mu_1 + \mu_2)$, we have

$$\|r_1(\mu_i)\|_{\sigma-1} \le \delta\ \|\mu_i\|_\sigma + \|\mu_i^2\|_\sigma \le \delta\ \|\mu\|_\sigma \big(1 + \frac{C_m}{\sqrt{\delta}}\ \|\mu\|_\sigma\big)\ ,$$

$$\|r_1(\mu_1) - r_1(\mu_2)\|_\sigma \le \delta\ \|\mu_1 - \mu_2\|_\sigma \big(1 + \frac{C_m}{\sqrt{\delta}}\ \|\mu_1 + \mu_2\|_\sigma\big)\ .$$

The proof is completed noting that $\|\mu_i\|_\sigma \le c_\eta\ \rho$ and $\delta = c_\delta\ \rho^2$. □

C.2. Bounds on F_0.

Theorem C.2. *Let $\delta = c_\delta\, \rho^2$, and suppose that for all μ, $\mu_i \in \mathcal{B}_\sigma(c_\eta\, \rho)$, we have $r(\mu) \in \mathcal{B}_{\sigma-1}(c_r\, \delta\, \rho)$ and $\|r(\mu_1) - r(\mu_2)\|_{\sigma-1} \le c_r\, \delta\, \|\mu_1 - \mu_2\|_\sigma$. Let $\Delta F_0 = F_0(\mu_1) - F_0(\mu_2)$. Then there exist constants c_ε and c_{F_0} such that for all $\varepsilon \le c_\varepsilon\, \rho^{-1}$ we have*

$$\|F_0(\mu)\|_{\sigma-2} + \left\| \frac{F_0(\mu)'}{\mathcal{L}_\mu} \right\|_{\mathcal{W},\sigma} \le c_{F_0} \delta^{5/2} \rho^2\ , \tag{C.3}$$

$$\|\Delta F_0\|_{\mathrm{L}^2} + \left\| \frac{\Delta F_0'}{\mathcal{L}_\mu} \right\|_{\mathcal{W},\sigma} \le c_{F_0} \delta^{5/2} \rho \|\mu_1 - \mu_2\|_\sigma\ . \tag{C.4}$$

Proof. We first note that since $s = G\, r$, from the definition of G (see (1.36)) we have

$$\|s\|_{\sigma-1} + \left\|\varepsilon s'\right\|_{\sigma-1} + \left\|\varepsilon^2 s''\right\|_{\sigma-1} \le 5\, \|r\|_{\sigma-1} \le C\, \delta\, \rho\, . \tag{C.5}$$

Since $\|r\|_{\sigma-1} \le c_r\, \delta\, \rho$ and $\mu \in \mathcal{B}_\sigma(c_\eta\, \rho)$, for all $\varepsilon \le c_\varepsilon\, \rho^{-1}$ with c_ε sufficiently small, we have

$$1 - C_m\, \varepsilon^4\, \alpha^2\, \sqrt{\delta}\, \|r\|_{\sigma-1} \ge 1/2 \quad \text{and} \quad \varepsilon^3\, \sqrt{\delta}\, \|\mu\|_\sigma \le C\, . \tag{C.6}$$

Let $\sigma_i = \sigma - i$, $i = 1, 2, 3$. Using (C.5), (C.6) and Proposition B.3, we easily show that

$$\left\|s^2\right\|_{\sigma_1} + \left\|\frac{s(\varepsilon^2 s)''}{1+\varepsilon^4\alpha^2 s}\right\|_{\sigma_1} + \left\|\frac{\mu s(\varepsilon^4 s)'}{1+\varepsilon^4\alpha^2 s}\right\|_{\sigma_1} \le C\delta^{5/2}\rho^2\, , \tag{C.7}$$

$$\left\|\frac{\mu s'}{1+\varepsilon^4\alpha^2 s}\right\|_{\sigma_2} \le C\delta^{5/2}\rho^2\, , \tag{C.8}$$

$$\|\mu^2\|_{\sigma_3} + \|(\mu^2)''\|_{\sigma_3} + \|\mu r'\|_{\sigma_3} + \|\mu' r\|_{\sigma_3} + \|\mu' s\|_{\sigma_3} \le C\delta^{5/2}\rho^2\, , \tag{C.9}$$

$$\left\|\mu''\, (\varepsilon^2 s')\right\|_{\sigma_3} \le C\delta^{7/2}\rho^2\, . \tag{C.10}$$

Applying Lemma B.1 for the two last terms of $F_0(\mu)$, and using (C.7) and (C.8), we see that $\|F_0(\mu)\|_{\sigma-2} \le C\delta^{5/2}\rho^2$, which proves the first part of (C.3).

To prove the remainder of (C.3), we use that

$$s'\mu = G\left(\mu\, r' + 2\mu'\, r - 2\mu'\, s - \frac{1}{2}\mu''\, (\varepsilon^2\, s')\right),$$

which follows from easy algebra[5], to get $F_0(s, \mu) = F_1(s, \mu) + G\, F_2(s, r, \mu)$, where

$$F_1(s, \mu) = \chi\alpha^2\left(\left(2 + \varepsilon^2(1+\alpha^2)\right)s^2 - \frac{\alpha^2(\varepsilon^4 s')s\mu}{1+\varepsilon^4\alpha^2 s} - \frac{2\alpha^2 s(\varepsilon^2 s'')}{1+\varepsilon^4\alpha^2 s}\right),$$

$$F_2(s, r, \mu) = -\frac{1}{4}\mu^2 - \frac{1}{4}(\mu^2)'' + \frac{\chi\alpha^2}{2}\left(2\mu r' + 4\mu' r - 4\mu' s - \mu''(\varepsilon^2 s')\right).$$

We then use Proposition B.2 which gives

$$\left\|\frac{F_0(\mu)'}{\mathcal{L}_\mu}\right\|_{\mathcal{W},\sigma} \le \frac{C_1}{\delta}\, \|F_1(\mu)\|_{\mathcal{W},\sigma-1} + \frac{C_2}{\delta^3}\, \|F_2(\mu)\|_{\mathcal{W},\sigma-3}\, .$$

Using (C.9) and (C.10) for the F_2–term and (C.7) for the F_1–term completes the proof of (C.3), while equalities like $a_1 b_1 - a_2 b_2 = (a_1 - a_2)b_1 + (b_1 - b_2)a_2$ and $\frac{a_1}{b_1} - \frac{a_2}{b_2} = \frac{a_1 - a_2}{b_1} + \frac{a_2}{b_2}\frac{b_2 - b_1}{b_1}$ show that the estimates needed to prove (C.4) are similar to those for (C.3), we omit the details. □

[5] act on both sides of this equation with $\left(1 - \frac{\varepsilon^2}{2}\partial_x^2\right)$ and use that $\varepsilon^2\, s'' = 2s - 2r$ and $\varepsilon^2\, s''' = 2s' - 2r'$.

Corollary C.3. *Let* $\alpha < 1$ *and* $F_7(\mu) = \frac{\varepsilon^2}{8}\left(\partial_x + \frac{\varepsilon^2\,\mu}{2}\right)F_0(\mu)'$ *as in (1.41). Then there exist constants* c_ε *and* c_{F_7} *such that for all* $\varepsilon \le c_\varepsilon\,\rho^{-2}$ *the following bounds hold:*

$$\left\|\mathcal{L}_{\mu,r}\mathcal{L}_v^{-1}F_7(\mu_i)'\right\|_{\mathrm{L}^2} + \left\|\frac{\mathcal{L}_{\mu,r}}{\mathcal{L}_\mu G\mathcal{L}_r}F_7(\mu_i)'\right\|_{\mathcal{W},\sigma} \le c_{F_7}\delta^{5/2}\rho^2\,, \tag{C.11}$$

$$\left\|\mathcal{L}_{\mu,r}\mathcal{L}_v^{-1}\Delta F_7'\right\|_{\mathrm{L}^2} + \left\|\frac{\mathcal{L}_{\mu,r}}{\mathcal{L}_\mu G\mathcal{L}_r}\Delta F_7'\right\|_{\mathcal{W},\sigma} \le c_{F_7}\delta^{5/2}\rho\|\Delta\mu\|_\sigma\,, \tag{C.12}$$

where $\Delta F_7 = F_7(\mu_1) - F_7(\mu_2)$ *and* $\Delta\mu = \mu_1 - \mu_2$.

Proof. We first use that $\|\mathcal{L}_{\mu,r}\ \mathcal{L}_v^{-1}\ f'\|_{\mathrm{L}^2} \le 16\|f\|_{\mathrm{L}^2}$ (see Lemma F.1 in Appendix F), so that for the L^2 bounds in (C.11) and (C.12), we need only bound $\|F_7(\mu_i)\|_{\mathrm{L}^2}$ and $\|F_7(\mu_1) - F_7(\mu_2)\|_{\mathrm{L}^2}$. Then we have also $\|f\|_{\mathrm{L}^2} \le \|f\|_{\sigma'}$ for any $\sigma' > 0$, from which we get

$$\|F_7(\mu_i)\|_{\mathrm{L}^2} \le \varepsilon^2\|F_0(\mu_i)''\|_{\sigma-4} + \varepsilon^4 C\sqrt{\delta}\rho\|F_0(\mu_i)'\|_{\sigma-3}\,.$$

Using $\|f'\|_{\sigma-3} \le \delta\|f\|_{\sigma-2}$, $\|F_0(\mu_i)\|_{\sigma-2} \le c_{F_0}\ \delta^{5/2}\ \rho^2$ and $\varepsilon \le c_\varepsilon\ \rho^{-2}$ gives the desired result. Similarly, since

$$\mu_1\ F_0(\mu_1)' - \mu_2\ F_0(\mu_2)' = \frac{1}{2}\Delta\mu\ F_+' + \frac{1}{2}(\mu_1+\mu_2)\ F_-'\,, \tag{C.13}$$

where $\Delta\mu = \mu_1 - \mu_2$ and $F_\pm = F_0(\mu_1) \pm F_0(\mu_2)$, we also have

$$\|\Delta F_7\|_{\mathrm{L}^2} \le C_1\ \|F_-\|_{\sigma-2} + C_2\ \varepsilon^2\ \|F_+\|_{\sigma-2}\ \|\mu_1 - \mu_2\|_\sigma\,.$$

The proof of (C.12) is completed noting that $\|F_-\|_{\sigma-2} \le c_{F_0}\ \delta^{5/2}\ \rho\ \|\mu_1 - \mu_2\|_\sigma$, and using again $\varepsilon \le c_\varepsilon\ \rho^{-2}$.

For the remainder of the proof of (C.11), we use Lemma F.1 to get

$$\varepsilon^2\left\|\frac{\mathcal{L}_{\mu,r}}{G\mathcal{L}_r}f''\right\|_{\mathcal{W},\sigma} \le C\ \|f\|_{\mathcal{W},\sigma} \quad\text{and}\quad \left\|\frac{\mathcal{L}_{\mu,r}}{\mathcal{L}_\mu G\mathcal{L}_r}f'\right\|_{\mathcal{W},\sigma} \le C\delta^{-3}\|f\|_{\mathcal{W},\sigma-3}$$

for some constant C, and we conclude that

$$\begin{aligned}\left\|\frac{\mathcal{L}_{\mu,r}}{\mathcal{L}_\mu G\mathcal{L}_r}F_7(\mu_i)'\right\|_{\mathcal{W},\sigma} &\le C\left(\left\|\frac{F_0(\mu_i)'}{\mathcal{L}_\mu}\right\|_{\mathcal{W},\sigma} + \varepsilon^4\delta^{-3}\|\mu_i F_0(\mu_i)'\|_{\sigma-3}\right)\\ &\le C\left(\left\|\frac{F_0(\mu_i)'}{\mathcal{L}_\mu}\right\|_{\mathcal{W},\sigma} + \|F_0(\mu_i)\|_{\sigma-2}\right)\,,\end{aligned}$$

which gives the desired result. The proof of the remainder of (C.12) is very similar (use e.g. (C.13) and proceed as above), we omit the details. □

C.3. Bounds on F_3 and F_4.

Proposition C.4. *Let $c_\eta, c_s > 0$ and $\delta = c_\delta \rho^2 > 2$. For all $0 < \zeta < 1$, there exists a constant c_ε such that for all $\varepsilon \le c_\varepsilon\, \rho^{-3}$, for all $\mu_i \in \mathcal{B}_\sigma(c_\eta\, \rho)$ and for all $s_i \in \mathcal{B}_{\sigma-1}(c_s\, \delta\, \rho)$ the following bounds hold:*

$$\frac{\varepsilon^4}{\chi}\|F_3(s_i, \mu_i)\|_{\sigma-1} \le \zeta \delta \rho\ ,$$
$$\frac{\varepsilon^4}{\chi}\|F_3(s_1, \mu_i) - F_3(s_2, \mu_i)\|_{\sigma-1} \le \zeta \|s_1 - s_2\|_{\sigma-1}\ ,$$
$$\frac{\varepsilon^4}{\chi}\|F_3(s_i, \mu_1) - F_3(s_i, \mu_2)\|_{\sigma-1} \le \zeta \delta \|\mu_1 - \mu_2\|_{\sigma-1}\ .$$

Proof. The proof is very easy. For instance, we have

$$\varepsilon^4\|s_i^2\|_{\sigma-1} + \varepsilon^6\|s_i\mu_i^2\|_{\sigma-1} + \varepsilon^8\|s_i^3\|_{\sigma-1} \le C\big(\varepsilon^4\delta^{3/2}\rho + \varepsilon^6\delta\rho^2 + \varepsilon^8\delta^3\rho^2\big)\delta\rho\ ,$$

which can be made arbitrarily small choosing c_ε sufficiently small. □

Proposition C.5. *Let $F_5(s,\mu) = F_3(s,\mu) + F_4(s,\mu)$. There exist constants c_ε and c_{F_5} such that for all $\varepsilon \le c_\varepsilon\, \rho^{-2}$, for all $\mu_i \in \mathcal{B}_\sigma(c_\eta\, \rho)$ and for all maps s satisfying $\|\mathcal{L}_s s(\mu_i)\|_{\sigma-1} \le c_s\, \delta\, \rho$ and $\|\mathcal{L}_s(s(\mu_1) - s(\mu_2))\|_{\sigma-1} \le c_s\, \delta\, \|\mu_1 - \mu_2\|$ the following bounds hold:*

$$\|\mathcal{L}_s F_5(s(\mu_i), \mu_i)\|_{\sigma-3} \le c_{F_5}\delta^{5/2}\rho^2\ , \tag{C.14}$$
$$\|\mathcal{L}_s F_5(s(\mu_1), \mu_1) - \mathcal{L}_s F_5(s(\mu_2), \mu_2)\|_{\sigma-3} \le c_{F_3}\delta^{5/2}\rho\|\mu_1 - \mu_2\|_\sigma\ . \tag{C.15}$$

Proof. We first note that since $\mathcal{L}_s = 1 - \frac{\varepsilon^2}{2}\partial_x^2$, we have

$$\|\mathcal{L}_s\ f\|_{\sigma-3} \le \Big(1 + C\ \varepsilon^2\ \delta^2\Big)\ \|f\|_{\sigma-1} \le C\ \|f\|_{\sigma-1}\ .$$

Then, as in the proof of Proposition C.4, for the contribution of F_3, we have

$$\|s_i^2\|_{\sigma-1} + \varepsilon^2\|s_i\mu_i^2\|_{\sigma-1} + \varepsilon^4\|s_i^3\|_{\sigma-1} \le C\delta^{5/2}\rho^2\Big(1 + \varepsilon^2\frac{\rho}{\sqrt{\delta}} + \varepsilon^4\delta^{3/2}\rho\Big)\ ,$$

where $s(\mu_i) = s_i$. For the contribution of F_4, we have

$$\|\mathcal{L}_s(s(\mu_i)\mu_i')\|_{\sigma-3} \le C\sqrt{\delta}\|s(\mu_i)\|_{\sigma-1}\|\mu_i'\|_{\sigma-1} \le C\delta^{5/2}\rho^2\ ,$$

and for the other term in F_4, we use

$$\mathcal{L}_s(s'\mu) = \mu(\mathcal{L}_s s') + 2\mu'(\mathcal{L}_s s) + s'(\mathcal{L}_s\mu) - 2s\mu' - s'\mu\ ,$$

and get $\|\mathcal{L}_s(s(\mu_i)'\mu_i)\|_{\sigma-3} \le C\delta^{5/2}\rho^2$. The proof of (C.15) is similar, we omit the details. □

C.4. Bounds on F_6.

Proposition C.6. *Let $\alpha^2 < 1$, $\delta = c_\delta\,\rho^2$ and $\Delta F_6 = F_6(\mu_1) - F_6(\mu_2)$. For all $\zeta < 1$, there exist constants c_ε and c_{F_6} such that for all $\varepsilon \le c_\varepsilon\,\rho^{-2}$ and for all $\mu_i \in \mathcal{B}_\sigma(c_\eta\,\rho)$, the following bounds hold:*

$$\varepsilon^2 \left\| \frac{\mathcal{L}_{\mu,r}}{\mathcal{L}_\mu G \mathcal{L}_r} F_6(\mu_i)' \right\|_{\mathcal{W},\sigma} \le \Big(\max\big(\tfrac{1}{3}, \tfrac{\alpha^2}{1-\alpha^2}\big) + \zeta\Big) c_\eta \rho\,, \tag{C.16}$$

$$\varepsilon^2 \left\| \frac{\mathcal{L}_{\mu,r}}{\mathcal{L}_\mu G \mathcal{L}_r} \Delta F_6' \right\|_{\mathcal{W},\sigma} \le \Big(\max\big(\tfrac{1}{3}, \tfrac{\alpha^2}{1-\alpha^2}\big) + \zeta\Big) \|\mu_1 - \mu_2\|_\sigma\,, \tag{C.17}$$

$$\|F_6(\mu_i)\|_{\mathrm{L}^2} \le c_{F_6}\Big(\delta^5 \rho + \delta^{5/2}\rho^2\Big)\,, \tag{C.18}$$

$$\Big\|\mathcal{L}_{\mu,r}\mathcal{L}_v^{-1} F_6(\mu_i)'\Big\|_{\mathrm{L}^2} \le c_{F_6}\Big(\delta^4 \rho + \delta^{5/2}\rho^2\Big)\,, \tag{C.19}$$

$$\Big\|\mathcal{L}_{\mu,r}\mathcal{L}_v^{-1} \Delta F_6'\Big\|_{\mathrm{L}^2} \le c_{F_6}\Big(\delta^4 + \delta^{5/2}\rho\Big)\|\mu_1 - \mu_2\|_\sigma\,. \tag{C.20}$$

Proof. It is crucial for the phase equation that the prefactor of $c_\eta\rho$ in (C.16) and of $\|\mu_1 - \mu_2\|_\sigma$ in (C.17) is smaller than 1. Using Lemma F.1 of Appendix F, we see that the term $\frac{1}{8}\mathcal{L}_\mu\mu'$ in F_6 gives the leftmost contribution in (C.16)–(C.20), since

$$\begin{aligned}
\frac{\varepsilon^2}{8}\left\|\frac{\mathcal{L}_{\mu,r}}{G\,\mathcal{L}_r}\,\mu''\right\|_{\mathcal{W},\sigma} &\le \max\Big(\frac{1}{3}, \frac{\alpha^2}{1-\alpha^2}\Big)\,\|\mu\|_{\mathcal{W},\sigma}\,,\\
\|\mathcal{L}_{\mu,r}\,\mathcal{L}_v^{-1}\,\mathcal{L}_\mu\,\mu''\|_{\mathrm{L}^2} &\le C\|(1+\partial_x^4)\,\mu\|_{\mathrm{L}^2} \le C\,\delta^4\,\|\mu\|_\sigma \le C\,\delta^4\,\rho\,,\\
\|\mathcal{L}_\mu\,\mu'\|_{\mathrm{L}^2} &\le C\,\|(1-\partial_x^2)^{5/2}\,\mu\|_{\mathrm{L}^2} \le C\,\|\mu\|_\sigma \le C\,\delta^5\,\rho\,,
\end{aligned}$$

while using $\|\mathcal{L}_{\mu,r}\,\mathcal{L}_v^{-1}\,f'\|_{\mathrm{L}^2} \le 16\|f\|_{\mathrm{L}^2}$, Corollary C.3 and Proposition C.5 above, it is easy to see that F_3, F_4 and F_7 give a contribution to ζ in (C.16) and (C.17) which can be made arbitrarily small choosing c_ε sufficiently small, and part of the rightmost contribution in (C.18)–(C.20).

It remains to bound the contribution of $F_{10}(\mu) = F_8(\mu) - \frac{1}{8}\,\mathcal{L}_\mu\,\mu'$. For the proof of (C.16), we first note that by inequality (F.5) of Lemma F.1, it is sufficient to bound $\|F_{10}(\mu_i)\|_{\mathcal{W},\sigma-3}$, on which we have

$$\begin{aligned}
\left\|\frac{\varepsilon^2}{2}\mu_i\mathcal{L}_\mu\,\mu_i\right\|_{\mathcal{W},\sigma-3} &\le C_m\,\sqrt{\delta}\,\|\mu_i\|_\sigma\,\|\varepsilon^2\mathcal{L}_\mu\,\mu_i\|_{\sigma-2} \le C\,\delta^{5/2}\,\rho^2\,,\\
\left\|\Big(\partial_x + \frac{\varepsilon^2\,\mu_i}{2}\Big)\mu_i\mu_i'\right\|_{\mathcal{W},\sigma-3} &\le C_1\,\delta^{5/2}\,\rho^2 + \varepsilon^2\,C_2\,\delta^2\,\rho^3 \le C\,\delta^{5/2}\,\rho^2\,,
\end{aligned}$$

which gives an arbitrarily small contribution to ζ in (C.16) if c_ε is sufficiently small. To get the contribution of F_{10} to (C.18) and (C.19), we use

$$\begin{aligned}
\|F_{10}(\mu_i)\|_{\mathrm{L}^2} &\le \varepsilon^2\|\mu_i\mathcal{L}_\mu\mu_i\|_{\mathrm{L}^2} + \|(\mu_i\mu_i')'\|_{\mathrm{L}^2} + \varepsilon^2\|\mu_i^2\mu_i'\|_{\mathrm{L}^2}\\
&\le C_1\sqrt{\delta}\|\mu_i\|_\sigma\|\varepsilon^2\mathcal{L}_\mu\mu_i\|_{\sigma-2} + C_2\delta^{5/2}\|\mu_i\|_\sigma^2 + \varepsilon^2 C_3\delta^2\|\mu_i\|_\sigma^3\\
&\le C\delta^{5/2}\|\mu_i\|_\sigma^2\Big(1 + \frac{\|\mu_i\|_\sigma}{\sqrt{\delta}}\Big) \le C\delta^{5/2}\rho^2\,.
\end{aligned}$$

The proof of (C.20) and (C.17) are similar to the above, we omit the details. □

D. Proof of Proposition 2.12

Before proving Proposition 2.12, we prove a simpler lemma.

Lemma D.1. *Let δ and ε_0 be given by Proposition 2.12, and let $\mathcal{F}(\tilde{\mu}, \mu_0)$ be given by the solution of (2.19). Assume that $|||\mathcal{F}(\tilde{\mu}, \mu_0)|||_\sigma \leq c_\eta\, \rho$, and that (2.21) holds with $\lambda_1 < 1$ for all $\varepsilon \leq \varepsilon_0$, for all $\tilde{\mu} \in \mathcal{B}_\sigma(c_\eta\, \rho)$, and for all $\mu_0 \in \mathcal{B}_{0,\sigma}(c_\eta\, \rho)$. Then for all $0 < c_\lambda < 1$, there exists a $t_1 > 0$ such that for all $\tilde{\mu}_i \in \mathcal{B}_\sigma(c_\eta\, \rho)$, it holds*

$$\sup_{0 \leq t \leq t_1} \|\mathcal{F}(\tilde{\mu}_1, \mu_0)(\cdot, t) - \mathcal{F}(\tilde{\mu}_2, \mu_0)(\cdot, t)\|_{\mathrm{L}^2} \leq c_\lambda \lambda_1 \sup_{0 \leq t \leq t_1} \|\tilde{\mu}_1(\cdot, t) - \tilde{\mu}_2(\cdot, t)\|_\sigma\ .$$

Proof. Let $\mu_i = \mathcal{F}(\tilde{\mu}_i, \mu_0)$, $i = 1, 2$ and $\mu_\pm = \mathcal{F}(\tilde{\mu}_1, \mu_0) \pm \mathcal{F}(\tilde{\mu}_2, \mu_0)$. We have

$$\partial_t \mu_- = -\mathcal{L}_\mu \mu_- - \frac{1}{2}(\mu_+ \mu_-)' + \varepsilon^2 \Delta F'\ , \quad \mu_-(x, 0) = 0\ , \tag{D.1}$$

where $\Delta F = F(\tilde{\mu}_1) - F(\tilde{\mu}_2)$. Multiplying (D.1) by μ_-, integrating over $[-L/2, L/2]$, using Young's inequality and $\mathcal{L}_v^2 - 2\mathcal{L}_\mu \leq 1$, we get

$$\begin{aligned} \partial_t(\mu_-, \mu_-) &= -2(\mu_-, \mathcal{L}_\mu\, \mu_-) - \frac{1}{2}(\mu_-, \mu_+'\, \mu_-) + 2\, \varepsilon^2\, (\mu_-, \Delta F') \\ &\leq (1 + \frac{1}{2}\|\mu_+'\|_{\mathrm{L}^\infty})\, (\mu_-, \mu_-) + \varepsilon^4\, \|\mathcal{L}_v^{-1}\, \Delta F'\|_{\mathrm{L}^2}^2\ . \end{aligned}$$

By (2.21), we have $\varepsilon^2 \|\mathcal{L}_v^{-1} \Delta F'\|_{\mathrm{L}^2} \leq \lambda_1\, \|\tilde{\mu}_1 - \tilde{\mu}_2\|_\sigma$ with $\lambda_1 < 1$, so that

$$\sup_{0 \leq t \leq t_1} \|\mu_-(\cdot, t)\|_{\mathrm{L}^2} \leq \lambda_1 \sqrt{\tfrac{e^{\zeta t_1} - 1}{\zeta}} \sup_{0 \leq s \leq t_1} \|\tilde{\mu}_1(\cdot, s) - \tilde{\mu}_2(\cdot, s)\|_\sigma\ ,$$

where $\zeta = 1 + \frac{1}{2}\|\mu_+'\|_{\mathrm{L}^\infty} \leq 1 + C\, c_\eta\, \delta^{3/2}\, \rho$. Setting $t_1 = \frac{1}{\zeta} \ln\bigl(1 + c_\lambda^2\, \zeta\bigr)$ completes the proof. $\square$

Proposition 2.12 is then an easy consequence of the following proposition.

Proposition D.2. *There exist constants c_δ sufficiently large and c_λ sufficiently small such that if t_1 is given by Lemma D.1, and $\mathcal{F}(\tilde{\mu}, \mu_0)$ (the solution of (2.19)) satisfies $|||\mathcal{F}(\tilde{\mu}, \mu_0)|||_\sigma \leq c_\eta\, \rho$, and (2.21) holds with $\lambda_1 < 1$ for all $\varepsilon \leq \varepsilon_0$, for all $\tilde{\mu} \in \mathcal{B}_\sigma(c_\eta\, \rho)$, and for all $\mu_0 \in \mathcal{B}_{0,\sigma}(c_\eta\, \rho)$, then there exists a constant $0 < \lambda < 1$ such that for all $\tilde{\mu}_i \in \mathcal{B}_\sigma(c_\eta\, \rho)$, it holds*

$$\sup_{0 \leq t \leq t_1} \|\mathcal{F}(\tilde{\mu}_1, \mu_0)(\cdot, t) - \mathcal{F}(\tilde{\mu}_2, \mu_0)(\cdot, t)\|_\sigma \leq \lambda \sup_{0 \leq t \leq t_1} \|\tilde{\mu}_1(\cdot, t) - \tilde{\mu}_2(\cdot, t)\|_\sigma\ .$$

Proof. We will use the same definitions as in Lemma D.1 above, and $\Delta F = F(\tilde{\mu}_1) - F(\tilde{\mu}_2)$. We first note that we have

$$\sup_{0 \leq t \leq t_1} \|\mu_\pm(\cdot, t)\|_\sigma = \sup_{0 \leq t \leq t_1} \|\mathcal{F}(\tilde{\mu}_1, \mu_0)(\cdot, t) \pm \mathcal{F}(\tilde{\mu}_2, \mu_0)(\cdot, t)\|_\sigma \leq 2 c_\eta \rho\ .$$

Then we use that Duhamel's representation formula for the solution of (D.1) gives

$$\mu_-(x,t) = -\frac{1}{2}\int_0^t \mathrm{d}s\; e^{-\mathcal{L}_\mu(t-s)}(\mu_-\mu_+)'(x,s) + \varepsilon^2 \int_0^t \mathrm{d}s\; e^{-\mathcal{L}_\mu(t-s)} \Delta F'(x,s)\;,$$

from which we get, using Condition 2.8, Propositions B.2 and B.3, and Lemma D.1 that

$$\sup_{0\le t\le t_1} \|\mu_-(\cdot,t)\|_\sigma \le \lambda_1(1+c_\lambda) \sup_{0\le t\le t_1} \|\tilde{\mu}_-(\cdot,t)\|_{\mathcal{W},\sigma} + \frac{C\rho}{\sqrt{\delta}} \sup_{0\le t\le t_1} \|\mu_-(\cdot,t)\|_\sigma\;,$$

for some $\lambda_1 < 1$. Since $\delta = c_\delta\, \rho^2$, choosing c_δ sufficiently large and c_λ sufficiently small completes the proof. □

E. Further Properties of the Amplitude Equation

Corollary E.1. *Assume that $\|r_0\|_{\sigma-1} \le c_{s_0}\, \delta\, \rho$. Then there exist constants $c_r > c_s$ and c_ε such that for all $\varepsilon \le c_\varepsilon \rho^{-2}$ and for all $\mu \in \mathcal{B}_\sigma(c_\eta \rho)$, we have*

$$|||r(\mu)|||_{\sigma-1} \le c_r\, \delta\, \rho\;, \tag{E.1}$$

$$|||r(\mu_1) - r(\mu_2)|||_{\sigma-1} \le c_r\, \delta\, |||\mu_1 - \mu_2|||_\sigma\;. \tag{E.2}$$

Proof. As a first step, we note that $|||r(\mu)|||_{\sigma-3}$ is finite, because

$$|||r(\mu)|||_{\sigma-3} \le |||s(\mu)|||_{\sigma-3} + \varepsilon^2 |||s(\mu)''|||_{\sigma-3} \le \bigl(1+\varepsilon^2\,\delta^2\bigr)c_s\delta\rho\;,$$

since $|||s(\mu)|||_{\sigma-1}$. Using $\varepsilon \le c_\varepsilon \rho^{-2}$, we also have

$$|||r(\mu)|||_{\sigma-1} \le |||r(\mu)|||_{\sigma-3} + |||r(\mu)|||_{\mathcal{W},\sigma-1} \le \zeta_1 c_s\delta\rho + |||r(\mu)|||_{\mathcal{W},\sigma-1}\;, \tag{E.3}$$

for some $\zeta_1 > 1$, while using Propositions C.1 and C.4, we have

$$\|r_0\|_{\sigma-1} + \|r_1\|_{\sigma-1} + \frac{\varepsilon^4}{\chi}|||F_3(s,\mu)|||_{\sigma-1} \le (c_{s_0} + c_{r_1} + \zeta)\delta\rho \le \zeta_2 c_s\delta\rho\;,$$

for some $\zeta_2 > 1$. Then, as in the proof of Proposition 3.2, we have that for all $\sigma' \le \sigma - 1$,

$$\begin{aligned}|||r(\mu)|||_{\sigma'} &\le \zeta_3 c_s\delta\rho + \varepsilon^4\bigl(|||s\mu'|||_{\sigma-1} + |||(s\mu)'|||_{\sigma'}\bigr)\\ &\le \zeta_4 c_s\delta\rho + \varepsilon^4 |||(s\mu)'|||_{\sigma'}\;,\end{aligned}$$

for some $\zeta_4 > 1$, since $\varepsilon^4 c_s \delta^{3/2}\|\mu\|_\sigma$ is arbitrarily small if c_ε is sufficiently small. And now, we use that $s\mu = G\bigl(r\mu - \varepsilon^2 s'\mu' - \frac{\varepsilon^2}{2}s\mu''\bigr)$, from which we get

$$\begin{aligned}|||r(\mu)|||_{\sigma'} &\le \zeta_4 c_s\delta\rho + \varepsilon^6 |||G(2s'\mu' + s\mu'')'|||_{\sigma-1} + \varepsilon^4 |||G(r(\mu)\mu)'|||_{\sigma'}\\ &\le \zeta_4 c_s\delta\rho + \varepsilon^4\,\bigl(2|||s'\mu'|||_{\sigma-2} + |||s\mu''|||_{\sigma-2}\bigr) + \varepsilon^4 |||G(r(\mu)\mu)'|||_{\sigma'}\\ &\le \zeta_5 c_s\delta\rho + \varepsilon^4 |||G(r(\mu)\mu)'|||_{\sigma'}\;,\end{aligned} \tag{E.4}$$

for some $\zeta_5 > 1$, since $\varepsilon^4\delta^{5/2}|||\mu|||_\sigma$ is arbitrarily small if c_ε is sufficiently small.

Since $|||G(r(\mu)\mu)'|||_{\sigma'} \le 2|||r(\mu)\mu|||_{\sigma'-1}$, we use (E.4) with $\sigma' = \sigma - 2$, and then with $\sigma' = \sigma - 1$ to conclude that $|||r(\mu)|||_{\sigma-1}$ is finite, and then we have

$$|||r(\mu)|||_{\sigma-1} \le \zeta_5 c_s\delta\rho + \varepsilon^3 |||r(\mu)\mu|||_{\sigma-1} \le \zeta_5 c_s\delta\rho + \bigl(C\varepsilon^3\sqrt{\delta}|||\mu|||_\sigma\bigr)|||r(\mu)|||_{\sigma-1}\;.$$

Since $\varepsilon^3\sqrt{\delta}|||\mu|||_\sigma$ is arbitrarily small if c_ε is sufficiently small, the proof of (E.1) is completed. The proof of (E.2) is similar, we omit the details. □

F. Coercive Functionals and Other Properties for the Amplitude Equation

We begin with a preliminary lemma.

Lemma F.1. *For all $\varepsilon^2 \le 1$ and $\alpha^2 < 1/2$, there exist a constant C such that*

$$\frac{\varepsilon^2}{8} \left\| \frac{\mathcal{L}_{\mu,r}}{G\,\mathcal{L}_r} f'' \right\|_{\mathcal{W},\sigma} \le \max\Big(\frac{1}{3}, \frac{\alpha^2}{1-\alpha^2}\Big) \, \|f\|_{\mathcal{W},\sigma} \, , \tag{F.1}$$

$$\|\mathcal{L}_{\mu,r} \, f\|_\sigma \le C \, \|f\|_\sigma \, , \tag{F.2}$$

$$\|\mathcal{L}_v^{-1} \, f'\|_{\mathrm{L}^2} \le C \, \|f\|_{\mathrm{L}^2} \, , \tag{F.3}$$

$$\|(1-\partial_x^2)^{-1} \, \mathcal{L}_v \, f\|_{\mathrm{L}^2} \le \|f\|_{\mathrm{L}^2} \, , \tag{F.4}$$

$$\left\| \frac{\mathcal{L}_{\mu,r}}{\mathcal{L}_\mu \, G \, \mathcal{L}_r} f' \right\|_{\mathcal{W},\sigma} \le C \, \delta^{-3} \, \|f\|_{\mathcal{W},\sigma-3} \, . \tag{F.5}$$

Proof. In terms of the Fourier coefficients, we have

$$\frac{\varepsilon^2}{8} \left(\frac{\mathcal{L}_{\mu,r}}{G\,\mathcal{L}_r} f'' \right)_n = -\left(\frac{\varepsilon^2 \, (qn)^2}{8} \, \frac{\mathcal{L}_{\mu,r}(qn)}{G(qn)\, \mathcal{L}_r(qn)} \right) f_n \, ,$$

and

$$\frac{\varepsilon^2 \, k^2}{8} \, \frac{\mathcal{L}_{\mu,r}(k)}{G(k) \, \mathcal{L}_r(k)} = \frac{\xi^2 \, (\lambda^2 - \alpha^2 \, \xi^2)}{1 + (2+\lambda^2) \, \xi^2 + (1-\alpha^2) \, \xi^4} \, ,$$

with $\xi = \frac{\varepsilon^2 \, k^2}{2}$ and $\lambda^2 = 1 + \varepsilon^2 \left(\frac{1+\alpha^2}{2}\right)$. Then as a function of ξ, we have

$$-\frac{\alpha^2}{1-\alpha^2} \le \frac{\xi^2 \, (\lambda^2 - \alpha^2 \, \xi^2)}{1 + (2+\lambda^2) \, \xi^2 + (1-\alpha^2) \, \xi^4} \le \frac{\lambda^4}{\lambda^4 + 4\,\lambda^2 + 4\alpha^2} \le \frac{1}{3} \, ,$$

where the last inequality comes from the fact that $\varepsilon^2 \le 1$ and $\alpha^2 < 1$ imply that $1 \le \lambda^2 \le 2$. This proves (F.1). For (F.2), we have $\left(\mathcal{L}_{\mu,r} \; f\right)_n = \mathcal{L}_{\mu,r}(qn) \; f_n$, and

$$|\mathcal{L}_{\mu,r}(k)| = 4 \, \frac{|\lambda^2 - \alpha^2 \, \xi^2|}{1+\xi^2} \le 4 \, \max(\alpha^2, \lambda^2) \le 8 \, ,$$

using the same notations. For (F.3) and (F.4), we use that

$$|ik \, \mathcal{L}_v(k)^{-1}| \le \sqrt{\frac{3\,k^2\big(1+\frac{k^2}{2}\big)}{1+k^4}} \le 2 \;\text{ and }\; \left| \frac{\mathcal{L}_v(k)}{1+k^2} \right| \le 1 \, .$$

For (F.5), we have

$$\left(\frac{\mathcal{L}_{\mu,r}}{\mathcal{L}_\mu G \mathcal{L}_r} f' \right)_n = i \; qn \; \left(\frac{\mathcal{L}_{\mu,r}(qn)}{\mathcal{L}_\mu(qn) G(qn) \mathcal{L}_r(qn)} \right) f_n \, ,$$

and for $|qn| = |k| \ge \delta \ge 2$, we have

$$\left| \frac{\mathcal{L}_{\mu,r}(k)}{\mathcal{L}_\mu(k) G(k) \mathcal{L}_r(k)} \right| \le \frac{8}{k^4} \sup_{|k|\ge\delta} \left| \frac{k^4}{k^4-k^2} \right| \sup_{|\xi|\ge 0} \left| \frac{(1+\xi^2)(\lambda^2-\alpha^2\xi^2)}{1+(2+\lambda^2)\xi^2+(1-\alpha^2)\xi^4} \right| .$$

The second supremum is finite if $\alpha^2 < 1$. Now, let $(K^{-4} \; f)_n \equiv (qn)^{-4} \; f_n$. We have $\|K^{-4} \; f'\|_{\mathcal{W},\sigma} = \delta^{-3} \, \|f\|_{\mathcal{W},\sigma-3}$, which completes the proof of (F.5). □

Proposition F.2. *Let* $\delta = c_\delta \rho^2$, $c_\eta > 0$ *and* $\alpha^2 < 1$. *There exists a constant* c_ε *such that for all* $\varepsilon \le c_\varepsilon\ \rho^{-2}$ *and for all* $\mu \in \mathcal{B}_\sigma(c_\eta\ \rho)$, *we have*

$$\int r_2 G\mathcal{L}_r r_2 - \frac{\varepsilon^4}{16}\int r_2\mu\mathcal{L}_{\mu,r} r_2' \ge \frac{3}{4}\int r_2^2\ , \tag{F.6}$$

$$\int r_4 G\mathcal{L}_r r_4 - \frac{\varepsilon^4}{16}\int r_4\mathcal{L}_{\mu,r}\mathcal{L}_v^{-1}\Big(\mu\mathcal{L}_v r_4\Big)' \ge \frac{3}{4}\int r_4^2\ . \tag{F.7}$$

Proof. We notice first that $\mathcal{L}_{\mu,r} r_2' = a_1 G r_2' - a_2 \frac{\varepsilon^2}{2} G r_2'''$ with $a_1 = 4 + 2\varepsilon^2(1+\alpha^2)$ and $a_2 = 4\alpha^2$. Since (by Fourier transform) $\|Gf\|_{\mathrm{L}^2} \le \|f\|_{\mathrm{L}^2}$ and $\|\varepsilon^2 G f''\|_{\mathrm{L}^2} \le 2\|f\|_{\mathrm{L}^2}$, there exists a constant C such that

$$\left|\int \mu r_2 \mathcal{L}_{\mu,r} r_2'\right| \le \|\mu\|_{\mathrm{L}^\infty}\|r_2\|_{\mathrm{L}^2}\|r_2'\|_{\mathrm{L}^2} \le C\rho\sqrt{\delta}\Big(\|r_2\|_{\mathrm{L}^2}^2 + \|r_2'\|_{\mathrm{L}^2}^2\Big)\ ,$$

and we get

$$\left|\frac{\varepsilon^4}{16}\int r_2\mu\mathcal{L}_{\mu,r} r_2'\right| \le C\varepsilon^2\rho\sqrt{\delta}\left(\int r_2^2 + \frac{\varepsilon^2}{2}\int (r_2')^2\right)\ .$$

Let now $a_3 = 3 + \varepsilon^2\Big(\frac{1+\alpha^2}{2}\Big)$ and $a_4 = 1 - \alpha^2$. We have

$$\int r_2 G\mathcal{L}_r r_2 - \frac{\varepsilon^4}{16}\int r_2\mu\mathcal{L}_{\mu,r} r_2' \ge \gamma\int r_2^2\ ,$$

where

$$\gamma = \min_{\xi\in\mathbf{R}}\left(\frac{1 + a_3\ \xi^2 + a_4\ \xi^4}{1+\xi^2} - C\varepsilon^2\rho\sqrt{\delta}(1+\xi^2)\right)\ .$$

Since $a_3 \ge 3$ and $a_4 > 0$, choosing c_ε sufficiently small completes the proof of (F.6). The proof of (F.7) is similar. We first use

$$\begin{aligned}\int r_4\mathcal{L}_{\mu,r}\mathcal{L}_v^{-1}\Big(\mu\mathcal{L}_v^{1/2} r_4\Big)' &= -\int\Big(\mathcal{L}_{\mu,r}\mathcal{L}_v^{-1} r_4\Big)'\mu\mathcal{L}_v r_4 = \int f\mu(1-\partial_x^2)g\\ &= \int f\mu g + f'\mu g' + f\mu' g\ ,\end{aligned}$$

where $f = \mathcal{L}_{\mu,r}\mathcal{L}_v^{-1} r_4'$ and $g = (1-\partial_x^2)^{-1}\mathcal{L}_v r_4$. Let $f^{(m)}$ be the m^{th} order spatial derivative of f. Then we have $\|f^{(m)}\|_{\mathrm{L}^2} \le 16\ \|r_4^{(m)}\|_{\mathrm{L}^2}$ and $\|g^{(m)}\|_{\mathrm{L}^2} \le \|r_4^{(m)}\|_{\mathrm{L}^2}$. Furthermore, we have $\|\mu'\|_{\mathrm{L}^\infty} \le C\delta^{3/2}\rho$ and $\|\mu\|_{\mathrm{L}^\infty} \le C\delta^{1/2}\rho \le C\delta^{3/2}\rho$. Using these inequalities, we get

$$\begin{aligned}\frac{\varepsilon^4}{16}\left|\int r_4\mathcal{L}_{\mu,r}\mathcal{L}_v^{-1}\Big(\mu\mathcal{L}_v^{1/2} r_4\Big)'\right| &\le C\varepsilon^4\delta^{3/2}\rho\big(\|r_4\|_{\mathrm{L}^2}^2 + \|r_4\|_{\mathrm{L}^2}\ \|r_4'\|_{\mathrm{L}^2} + \|r_4'\|_{\mathrm{L}^2}^2\big)\\ &\le C\varepsilon^2\delta^{3/2}\rho\left(\int r_4^2 + \frac{\varepsilon^2}{2}\int (r_4')^2\right)\ .\end{aligned}$$

As above, choosing c_ε sufficiently small completes the proof of (F.7). □

G. Discussion

The proofs of this section follow from definitions and proofs of Sect. 2.3 which should be read first. By (1.17), we have

$$\eta(x,t) = \frac{\hat{\varepsilon}^2}{4} \int_0^{\hat{\varepsilon}\, x} \mathrm{d}z\; \hat{\mu}(z,\hat{t})\ ,$$

and if $\delta = c_\delta \rho^2$ and $\hat{\varepsilon} \leq c_\varepsilon\ \rho^{-m_\varepsilon}$ with $m_\varepsilon \geq 4$, we get

$$\|\eta\|_{\mathrm{L}^\infty([-L_0/2,L_0/2])} \leq \hat{\varepsilon}^2\ \frac{\hat{\varepsilon}\ L_0}{2}\ \|\hat{\mu}\|_{\mathrm{L}^\infty} \leq c_\eta\ \hat{\varepsilon}^2\ L\ \rho \leq C\ \varepsilon^{2-13/(8\ m_\varepsilon)}\ ,$$
$$\|s\|_{\mathrm{L}^\infty([-L_0/2,L_0/2])} \leq \hat{\varepsilon}^4\ \|\hat{s}\|_{\mathrm{L}^\infty} \leq C\ \varepsilon^4\ \delta^{3/2}\ \rho \leq C\ \varepsilon^{4-4/m_\varepsilon}\ ,$$

since $\rho \leq c_\varepsilon\ \hat{\varepsilon}^{-1/m_\varepsilon}$. We also have

$$\|\eta'\|_{\mathrm{L}^2([-L_0/2,L_0/2])} \leq \hat{\varepsilon}^{5/2}\ \|\hat{\mu}\|_{\mathrm{L}^2} \leq C\ \varepsilon^{5/2}\ \rho \leq C\ \varepsilon^{5/2-1/m_\varepsilon}\ , \tag{G.1}$$

$$\|\eta'\|_{\mathrm{L}^\infty([-L_0/2,L_0/2])} \leq \hat{\varepsilon}^3\ \|\hat{\mu}\|_{\mathrm{L}^\infty} \leq C\ \varepsilon^3\ \sqrt{\delta}\ \rho \leq C\ \varepsilon^{3-2/m_\varepsilon}\ , \tag{G.2}$$

$$\|s\|_{\mathrm{L}^2([-L_0/2,L_0/2])} \leq \hat{\varepsilon}^{7/2}\ \|\hat{s}\|_{\mathrm{L}^2} \leq C\ \varepsilon^{7/2}\ \delta\ \rho \leq C\ \varepsilon^{7/2-3/m_\varepsilon}\ . \tag{G.3}$$

Various other estimates, e.g. on higher order derivatives can be obtained in a similar way.

References

[BKL94] Bricmont, J., Kupiainen, A., Lin, G.: Renormalization Group and Asymptotics of Solutions of Nonlinear Parabolic Equations. Commun. Pure Appl. Math. **47**(6), 893–922 (1994)

[GvB02] van Baalen, G.: Stationary solutions of the Navier-Stokes equations in a half-plane downstream of an obstacle: ‘universality’ of the wake. Nonlinearity **15**(2), 315–366 (2002)

[CH93] Cross, M., Hohenberg, P.: Rev. Mod. Phys. **65**, 851 (1993)

[CEES93] Collet, P., Eckmann, J.-P., Epstein, H., Stubbe, J.: A global attracting set for the Kuramoto-Sivashinsky equation. Commun. Math. Phys. **152**(1), 203–214 (1993)

[KT76] Kuramoto, Y., Tsuzuki, T.: Persistent Propagation of Concentration Waves in Dissipative Media Far from Thermal Equilibrium. Prog. Theor. Phys. **55**, 356–369 (1976)

[EGW95] Eckmann, J.-P., Gallay, Th., Wayne, C.E.: Phase slips and the Eckhaus instability. Nonlinearity **8**(6), 943–961 (1995)

[Man90] Manneville, P.: *Dissipative structures and weak turbulence*. Boston, MA: Academic Press, 1990

[MHAM97] Montagne, R., Hernández-García, E., Amengual, A., San Miguel, M.: Wound-up phase turbulence in the complex Ginzburg-Landau equation. Phys. Rev. E (3) **56**(1), part A, 151–167 (1997)

[New74] Newell, A.: Lect. Appl. Math. **15**, 157 (1974)

[NST85] Nicolaenko, B., Scheurer, B., Temam, R.: Phys. D **16**(2), 155–183 (1985)

[Tem97] Temam, R.: *Infinite-dimensional dynamical systems in mechanics and physics*. Second edition, New York: Springer, 1997

[BF67] Benjamin, T.B., Feir, J.: J. Fluid Mech. **27**, 417 (1967)

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