Thomas Gauglhofer · Hugo Parlier

Minimal length of two intersecting simple closed geodesics

Abstract. On a hyperbolic Riemann surface, given two simple closed geodesics that intersect n times, we address the question of a sharp lower bound L_n on the length attained by the longest of the two geodesics. We show the existence of a surface S_n on which there exists two simple closed geodesics of length L_n intersecting n times and explicitly find L_n for $n \leq 3$.

1. Introduction

Extremal hyperbolic Riemann surfaces for a variety of geometric quantities are objects of active research. Well known cases include the study of surfaces with maximum size systoles [2, 4, 16], surfaces with largest embedded disk ([3, 10]) or more classically surfaces with maximum number of automorphisms (the study of Hurwitz surfaces and related topics). These subjects are related to the study of the simple length spectrum of a surface S, denoted $\Delta_0(S)$, which is the ordered set of lengths of (non-oriented, primitive) simple closed geodesics (with multiplicity). The question of interpreting the geometry of the surface through the values found in the simple length spectrum seems to be a very difficult subject. In particular, it is not clear whether the simple length spectrum determines a surface up to isometry.

One of the major tools used to approach these problems is the collar theorem, and in particular a corollary which states that two short simple closed geodesics (of length less than $2 \operatorname{arcsinh}(1)$) cannot intersect (see [7]). The bound is sharp because it can be realized on a particular torus with a cusp. The bound is never reached for a closed surface, but for any genus, is realized in the compactification of its Moduli space.

The goal of this article is to generalize this result by studying the relationship between the number of intersection points between two simple closed geodesics and the length of the geodesics. The surfaces we consider lie in the Moduli space of

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surfaces with boundary $\mathcal{M}_{g,k}$, where g is the genus and k is the number of simple closed boundary geodesics which we allow to be cusps (geodesics of length 0). The foundation of our study is found in the following theorem (Sect. 2).

Theorem 1.1. *On a hyperbolic Riemann surface S, let α and β be simple closed geodesics that intersect n times. Then there exists a universal constant L_n such that $\max\{\ell(\alpha), \ell(\beta)\} \geq L_n$ and $L_n \longrightarrow \infty$ when $n \longrightarrow \infty$. Furthermore, a surface S_n realizing the bound exists.*

By realizing the bound, we mean that on S_n there are two simple closed geodesics of length L_n that intersect n times. We further investigate the asymptotic behavior of L_n in the following proposition where we prove:

Proposition 1.2. *Let l_n be the positive solution of the equation*

$$l_n = 2n \operatorname{arcsinh} \left(\frac{1}{\sinh(l_n/2)} \right).$$

Then

$$l_n \leq L_n < 2l_n.$$

We are able to describe the surfaces explicitly for $n \in \{2, 3\}$, which gives us the following result.

Theorem 1.3. *The surfaces S_2 and S_3 are once-punctured tori and*

$$L_2 = 2 \operatorname{arccosh}(2), \quad L_3 = 2 \operatorname{arccosh}\left(\sqrt{\tfrac{1}{2}\left(7 + \tfrac{11}{3}\sqrt{\tfrac{11}{3}}\right)} \right).$$

As mentioned earlier, $L_1 = 2 \operatorname{arcsinh} 1$ and note that the value for L_2 was previously proved in [9] but the proof presented here is new. The surfaces S_1, S_2 and S_3 are all once-punctured tori ($S_1 = S_2 \neq S_3$). We show that they have non-trivial isometry groups. (A once-punctured torus is necessarily hyperelliptic, so by non-trivial isometry group we mean an isometry group not isomorphic to $\mathbb{Z}_2$). It seems reasonable to conjecture that for all n, S_n is also a once-punctured torus with a non-trivial isometry group.

This article is organized as follows. Section 2 is devoted to preliminaries and the proof of Theorem 1.1 and Proposition 1.2. The next two sections concern the exact values of L_2 and L_3 and are similar in nature. The final section discusses possible future directions for related questions.

2. Preliminaries and groundwork

We will be considering hyperbolic Riemann surfaces of finite area, with or without boundary. We allow boundary to be either cusps or simple closed geodesics. A *surface* will always designate a surface of this type. The signature of a surface

will be denoted (g, k) where g is the genus and k the number of boundary geo-desics (or cusps). A surface of signature $(0, 3)$ is called a pair of pants, a surface of signature $(1, 1)$ a one-holed torus, and a surface of signature $(0, 4)$ a four-holed sphere. We reserve the term *punctures* for cusps, and holes can be cusps as well as boundary geodesics. The Moduli space of surfaces with boundary will be denoted $\mathcal{M}_{g,k}$.

The set of interior simple closed geodesics of a surface S will be denoted $\mathbb{G}(S)$. The length of a simple closed geodesic α will be denoted $\ell(\alpha)$, although a geodesic and its length might not be distinguished. The term *geodesic* will sometimes be used instead of simple closed geodesic, but only if it is obvious in the context. Closed geodesics will be considered to be non-oriented (unless specified) and primitive, meaning that a closed geodesic cannot be written as the k-fold iterate of another closed geodesic. Seen this way, geodesics are point sets independent of parametrization. We denote $\text{int}(\alpha, \beta)$ the number of transver-sal intersection points between two simple closed geodesics α and β. We define the simple length spectrum $\Delta_0(S)$ as the ordered set of lengths of all interior simple closed geodesics. Notice that our definition takes into account multiplic-ity, namely that if there are n distinct simple closed geodesics of S with equal length ℓ, then the value ℓ will appear n times in $\Delta_0(S)$. Consider two surfaces S and $\tilde{S}$ with simple length spectra $\Delta_0(S) = \{\ell_1 \leq \ell_2 \leq \cdots\}$ and $\Delta_0(\tilde{S}) = \{\tilde{\ell}_1 \leq \tilde{\ell}_2 \leq \cdots\}$. The notation $\Delta_0(S) < \Delta_0(\tilde{S})$ is an abbreviation for $\ell_i < \tilde{\ell}_i$ for all $i \in \mathbb{N}^*$.

In order to describe the pasting of a simple closed geodesic, one generally uses *twist parameters*. The only use we will have of twist parameters is to describe what we call *without twist* or *zero-twist* and *half-twist*. Recall that a pair of pants has three disjoint unique simple geodesic paths between distinct boundary geodesics, called perpendiculars, which decompose the pair of pants into two isometric hyperbolic right-angled hexagons. If two pairs of pants are pasted along a geodesic α such that the endpoints of the perpendiculars coincide, then we refer to a pasting with *zero-twist* or *without twist*. The terminology is slightly different for one-holed tori. Consider a pair of pants with two boundary geodesics α_1 and α_2 of equal length, and paste α_1 and α_2 together in order to obtain a one-holed torus. If the common perpendicular a between α_1 and α_2 has its endpoints that coincide, then we refer to a pasting with *zero-twist* or *without twist*. If the endpoints of a are diametrically opposite on the geodesic formally known as α_1 or α_2, then we refer to a *half-twist*. Finally, if an interior simple closed geodesic α is said to be *pasted with a half-twist*, then we mean that it has been obtained from the construction described above.

A function $f_\alpha : \mathcal{M}_{g,k} \longrightarrow \mathbb{R}^+$ that associates to a closed geodesic α its length depending on the choice of metric is generally referred to as a length function. Length functions are well known to be analytic (one way of seeing this is via Fricke trace calculus, see for instance [1]). What is interesting to us is that the func-tion of an interior closed geodesic remains continuous, even if boundary length goes to 0.

The collar theorem (i.e. [11, 6, 15]) gives a very precise description of the geometry of surfaces around simple closed geodesics.

Theorem 2.1. *Let γ_1 and γ_2 be non-intersecting simple closed geodesics on S. Then the collars*

$$\mathcal{C}(\gamma_i) = \{p \in S \mid d_S(p, \gamma_i) \leq w(\gamma_i)\}$$

of widths

$$w(\gamma_i) = \operatorname{arcsinh}\left(1/\sinh\frac{\ell(\gamma_i)}{2}\right)$$

are pairwise disjoint for $i = 1, 2$. Furthermore, each $\mathcal{C}(\gamma_i)$ is isometric to the cylinder $[-w(\gamma_i), w(\gamma_i)] \times \mathbb{S}^1$ with the metric $ds^2 = d\rho^2 + \ell(\gamma_i)^2 \cosh^2\rho \, dt^2$.

This implies that a geodesic α that transversally intersects n times another geodesic β satisfies $\ell(\alpha) \geq 2n \, w(\beta)$.

Corollary 2.2. *If α and β are two simple closed geodesics such that*

$$\ell(\alpha) \leq \ell(\beta) \leq 2\operatorname{arcsinh}1,$$

then they do not intersect.

We shall also use the following result found in [14].

Theorem 2.3. *Let S be a surface of signature (g, k) with $k > 0$. Let $\gamma_1, \ldots, \gamma_k$ be the boundary geodesics of S. For $(\varepsilon_1, \ldots, \varepsilon_k) \in (\mathbb{R}^+)^k$ with at least one $\varepsilon_i \neq 0$, and $\varepsilon_j \leq \ell(\gamma_j)$ for all j, there exists a surface $\tilde{S}$ with boundary geodesics of length $\ell(\gamma_1) - \varepsilon_1, \ldots, \ell(\gamma_k) - \varepsilon_k$ such that all corresponding simple closed geodesics in $\tilde{S}$ are of length strictly less than those of $S(\Delta_0(\tilde{S}) < \Delta_0(S))$.*

There is an immediate corollary to this result which is very useful to our study.

Corollary 2.4. *If α and β are two simple closed geodesics that intersect n times on a surface with non-empty boundary and with at least one boundary geodesic not a cusp, then there exists a surface of same signature, with only cusps as boundary, containing two simple closed geodesics $\tilde{\alpha}$ and $\tilde{\beta}$ which intersect n times and such that $\ell(\tilde{\alpha}) < \ell(\alpha)$ and $\ell(\tilde{\beta}) < \ell(\beta)$.*

This corollary implies that we can limit ourselves to studying surfaces with cusp boundary.

We need to introduce the notion of a minimally embedded surface around two simple closed geodesics. Let α and β be two simple closed geodesics that intersect on a surface S. We define $S_{\alpha,\beta}$ to be the unique subsurface of S constructed as follows. For $\varepsilon < \min\{w(\alpha), w(\beta)\}$, consider the ε-tubular neighborhood of the set $\alpha \cup \beta$. Its boundary consists of a set of disjoint simple closed curves whose unique geodesic representatives we shall denote γ_i, $i = 1, \ldots, m$. Note that the γ_i may not all be distinct and that a γ_i may be the empty set if the curve it is representing is null-homotopic. Only one of the connected components obtained by cutting S along the set $\{\gamma_i\}_{i=1}^m$ contains α and β and it is this one that we denote $S_{\alpha,\beta}$. The collar theorem implies that any simple closed geodesic γ of S that does not cross

α or β is either a boundary geodesic of $S_{\alpha,\beta}$ or satisfies $\gamma \cap S_{\alpha,\beta} = \emptyset$, i.e., the complement of $\alpha \cup \beta$ on $S_{\alpha,\beta}$ is a set of topological disks and cylinders. Each of the cylinders has one of the boundary curves of $S_{\alpha,\beta}$ as base. We refer to $S_{\alpha,\beta}$ as the *geodesic subsurface filled by α and β*, or simply the *geodesic subsurface* $S_{\alpha,\beta}$. The surface $S_{\alpha,\beta}$ has a certain signature (g, k). Note, as an example, that if α and β intersect once, then $S_{\alpha,\beta}$ is necessarily of signature $(1, 1)$. In general, there is a relationship, which we give in the following lemma, between the number of intersection points between the two geodesics and the signature of their associated geodesic subsurface.

Lemma 2.5. *Let α and β be two simple closed geodesics that intersect n times on a surface, and suppose their associated geodesic subsurface $S_{\alpha,\beta}$ has signature (g, k).*
    Then the inequality $n \geq 2g + k - 2$ holds.

Proof. The complement of $\alpha \cup \beta$ on $S_{\alpha,\beta}$ is a set of k topological cylinders and p disks.

If we replace these k topological cylinders on $S_{\alpha,\beta}$ by topological disks, we get a closed topological surface S_g of genus g whose Euler characteristic is $\chi = 2 - 2g$. We can also calculate its Euler characteristic using the polyhedral formula $\chi = V - E + F$ with $E = 2n$ edges (the arcs of α and β between intersection points), $V = n$ vertices (the intersection points of α and β), and $F = p + k$ faces. Thus, we get the equality $2 - 2g = p + k - n$, which implies $n \geq 2g + k - 2$. $\square$

Theorem 2.6. *There exists a universal constant L_n such that $\max\{\ell(\alpha), \ell(\beta)\} \geq L_n$ for any two simple closed geodesics α and β that intersect n times on a hyperbolic compact Riemann surface. Furthermore, a surface S_n realizing the bound exists. Finally, $L_n \longrightarrow \infty$ when $n \longrightarrow \infty$.*

Proof. The idea of the proof is to show that, for every n, we are evaluating a continuous function on a finite set of compact sets. The function is the one that associates to a surface S the following value:

$$f(S) = \min_{\{\alpha,\beta \in \mathbb{G}(S)|\mathrm{int}(\alpha,\beta)=n\}} \max\{\ell(\alpha), \ell(\beta)\}.$$

For a given signature (g, k), $f : \mathcal{M}_{g,k} \longrightarrow \mathbb{R}^+$, is obviously continuous and bounded. (Mind that for certain signatures, f may not be defined, for instance on surfaces of signature $(0, 4)$, there are no pairs of simple closed geodesics that intersect an odd number of times.) Suppose α and β are two simple closed geodesics on a surface S that intersect n times. Consider the geodesic subsurface $S_{\alpha,\beta}$ filled by α and β. By lemma 2.5, the signature (g, k) of $S_{\alpha,\beta}$ is universally bounded by a function of n ($g \leq \frac{n+2}{2}$, $k \leq n + 2$). There are thus a finite number of possible signatures for $S_{\alpha,\beta}$, which we shall denote $(g_1, k_1), \ldots, (g_m, k_m)$. As any interior simple closed geodesic of $S_{\alpha,\beta}$ intersects either α or β, and as we are trying to minimize the lengths, the collar theorem ensures that the length of the systole of $S_{\alpha,\beta}$ is bounded from below (otherwise the maximum length of α and β would be unbounded). Denote by ϵ_n this lower bound. By corollary 2.4, as we are searching

for a minimal value among all surfaces, we can limit ourselves to searching among surfaces with all boundary geodesics being cusps. Denote by $\mathcal{M}^0_{g,k}$ the restricted set of surfaces of signature (g, k) with cusp boundary. Further denote by $\mathcal{M}^0_{(g,k),\epsilon_n}$ the subset of $\mathcal{M}^0_{g,k}$ with systole bounded below by ϵ_n. We are now searching among a finite set of such sets, namely for each (g_j, k_j), $j \in \{1, \ldots, m\}$, we need to study the set $\mathcal{M}^0_{(g_j,k_j),\epsilon_n}$. These sets are well-known to be compact (for surfaces without boundary see [13], and with boundary see [12]). As f is a continuous function that we allow to cover a finite number of compact sets, it follows that f admits a minimum, and the value of f in this point we denote L_n. A point in Moduli space which reaches the minimum is denoted S_n.

We now need to show that $L_n \longrightarrow \infty$ when $n \longrightarrow \infty$. Suppose this is not the case, meaning there exists some L such that $L_n < L$ for all n. This would mean that for any n, there exist two simple closed geodesics α_n and β_n on some surface S that intersect n times such that $\ell(\alpha_n) \leq \ell(\beta_n) \leq L$. By the collar theorem $\ell(\beta_n) \geq 2n \operatorname{arcsinh}(\frac{1}{\operatorname{arcsinh}(L/2)})$. But this is a contradiction, because for any L, n can be chosen so that this is not the case. The theorem is now proven. $\qquad \square$

To study the asymptotic behavior of L_n, we shall use the quantity l_n defined in the following proposition.

Proposition 2.7. *For $n \in \mathbb{N}$, let l_n be the positive solution of the equation*

$$l_n = 2n \operatorname{arcsinh}\left(\frac{1}{\sinh(l_n/2)}\right).$$

Then l_n is strictly increasing in n.

Proof. The equation $l_n = 2n \operatorname{arcsinh}(\frac{1}{\sinh(l_n/2)})$ is equivalent to $\sinh(\frac{l_n}{2n}) \sinh(\frac{l_n}{2}) = 1$. Suppose now that there is an $n \in \mathbb{N}$ such that $l_n \geq l_{n+1}$. Then $\sinh(\frac{l_n}{2}) \geq \sinh(\frac{l_{n+1}}{2})$ which implies therefore that $\sinh(\frac{l_n}{2n}) \leq \sinh(\frac{l_{n+1}}{2n+2})$.

But $\frac{l_n}{2n} \leq \frac{l_{n+1}}{2n+2}$ implies $l_n < l_{n+1}$ which leads to a contradiction. $\qquad \square$

Now the asymptotic behavior of L_n can be expressed as follows.

Proposition 2.8. $l_n \leq L_n < 2l_n$.

Proof. Let us begin by showing $l_n \leq L_n$.

If a simple closed geodesic α of length l_n intersects a simple closed geodesic β n times, then β is at least as long as $2n$ times the width of the collar of α. Thus $\ell(\beta) \geq l_n$. The width of the collar of α increases when α gets shorter, thus $l_n \leq L_n$.

It remains to show that $L_n < 2l_n$.

For $n \in \mathbb{N}$, let $\mathcal{Y}$ be a pair of pants whose boundary consists of a cusp and two boundary geodesics, α_1 and α_2, both of length l_n. Let us paste these two geodesics together (denote the resulting geodesic α) without twist. The common perpendicular between α_1 and α_2 is now a simple closed geodesic, which we shall denote δ. Notice that $\ell(\delta) = l_n/n$. For a given primitive parametrization of α and δ, consider the simple closed curve $\tilde{\beta} = \delta^n \alpha$ and its unique geodesic representative β.
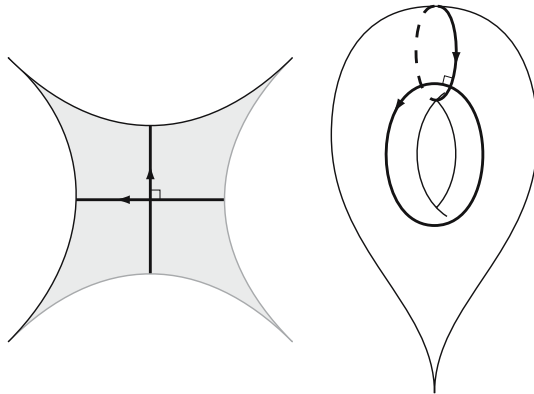
Fig. 1 Surface S_1 with geodesics in bold

By construction, $\ell(\beta) < \ell(\tilde{\beta}) = l_n + n(l_n/n) = 2l_n$. We have thus constructed a once-punctured torus with two interior geodesics α and β that satisfy $\mathrm{int}(\alpha, \beta) = n$ and $\max\{\ell(\alpha), \ell(\beta)\} < 2l_n$. It follows that $L_n < 2l_n$.   □

Finally, as an illustration of our investigation, let us give the value for L_1 and describe the surface S_1. Corollary 2.2 implies that $L_1 \geq 2\,\mathrm{arcsinh}1$. In fact, $L_1 = 2\,\mathrm{arcsinh}1$, and this can be shown by constructing the surface S_1 which realizes the bound L_1. Consider, in the hyperbolic plane, a quadrilateral with three right angles and one zero angle (a point at infinity). This quadrilateral can be chosen such that the two finite length adjacent sides are of length arcsinh1. By taking four copies of this quadrilateral, and pasting them together as in Fig. 1, one obtains a once-punctured torus with two simple closed geodesics of length $2\,\mathrm{arcsinh}1$ that intersect once.

This once-punctured torus is the only surface on which two intersecting geodesics can have length L_1. It is worth mentioning that this torus has other remarkable properties: it is the only once-punctured torus with an automorphism of order 4.

3. Finding S_2 and calculating L_2

Let us consider two simple closed geodesics α and β on a surface S that intersect twice in points A and B, and the geodesic subsurface $S_{\alpha,\beta}$. In order to distinguish possible signatures for the surface $S_{\alpha,\beta}$, let us give α and β orientations. Let C_β be a collar around β. The ordered pair of simple closed *oriented* geodesics (α, β) induces an orientation on C_β in both A and B. These orientations are either opposite (case 1) or the same (case 2). This is illustrated in Fig. 2.

In case 1, $S_{\alpha,\beta}$ is a surface of signature $(0, 4)$ obtained by cutting along the simple closed geodesics $\varepsilon_1, \varepsilon_2, \varepsilon_3$ and ε_4 homotopic to the simple closed curves $\tilde{\varepsilon}_1, \tilde{\varepsilon}_2, \tilde{\varepsilon}_3$ and $\tilde{\varepsilon}_4$ shown in Fig. 3.

In case 2, there are two possible signatures for the surface $S_{\alpha,\beta}$. Indeed, consider the simple closed curves $\tilde{\varepsilon}_1$ and $\tilde{\varepsilon}_2$ shown in Fig. 4. One of these curves may
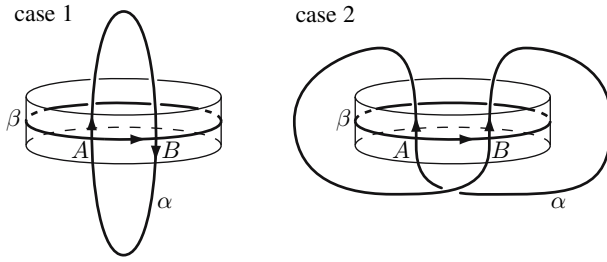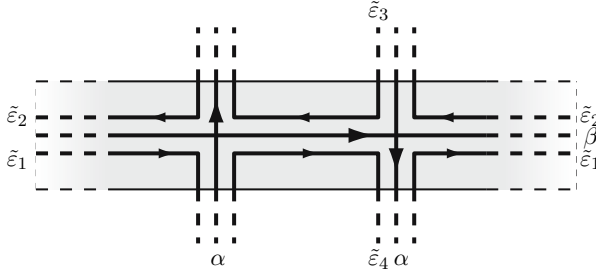
Fig. 2 The two cases for $n = 2$

Fig. 3 The simple closed curves $\tilde{\varepsilon}_1, \tilde{\varepsilon}_2, \tilde{\varepsilon}_3$ and $\tilde{\varepsilon}_4$ in case 1
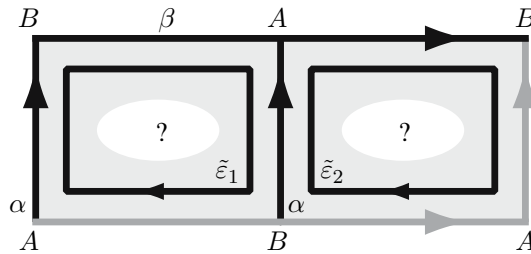

Fig. 4 The simple closed curves $\tilde{\varepsilon}_1$ and $\tilde{\varepsilon}_2$ in case 2

be null-homotopic, but not both because otherwise the surface would be a torus without holes, which of course cannot admit a hyperbolic metric.

If only one curve is not null-homotopic, say $\tilde{\varepsilon}_1$, we cut the surface S along the geodesic that is homotopic to $\tilde{\varepsilon}_1$ to obtain a surface of signature $(1, 1)$. If neither curve is null-homotopic, we cut the surface S along the two geodesics homotopic to $\tilde{\varepsilon}_1$ and $\tilde{\varepsilon}_2$ to obtain a surface of signature $(1, 2)$.

Therefore, in view of Corollary 2.4, S_2 is a sphere with four cusps, a torus with one cusp or a torus with two cusps.

First let us investigate geodesics intersecting twice on a four-holed sphere.

Proposition 3.1. *Let X be a four-holed sphere* (*where we allow the boundary geodesics to be cusps*)*. Let α and β be distinct interior simple closed geodesics of X. Then*

$$\max\{\ell(\alpha), \ell(\beta)\} \geq 4 \operatorname{arcsinh} 1 = 2 \operatorname{arccosh} 3.$$

Furthermore equality holds for a sphere with four cusps obtained by gluing two pairs of pants with two cusps and third boundary geodesic of length $2 \operatorname{arccosh} 3$ *without twist.*

Proof. By corollary 2.4 it suffices to show the result for X a sphere with four cusps. Suppose $\ell(\alpha) \geq \ell(\beta)$. Now suppose by contradiction that $\ell(\alpha) < 4 \operatorname{arcsinh} 1$. On a four-holed sphere, distinct interior simple closed geodesics cross at least twice. By the collar theorem, the length of any other interior simple closed geodesic must be strictly greater than four times the width $w(\alpha)$ of the half-collar around α, which by 2.1 is $w(\alpha) \geq \operatorname{arcsinh}(\frac{1}{\sinh(2 \operatorname{arcsinh} 1/2)}) = \operatorname{arcsinh} 1$. Thus $\ell(\beta) > 4 \operatorname{arcsinh} 1$, a contradiction. Thus equality can only be attained if both α and β are of length $4 \operatorname{arcsinh} 1$. It follows that a surface on which equality is reached has a simple closed geodesic α of length $4 \operatorname{arcsinh} 1$. If there is any twist around this geodesic, then all simple closed geodesics crossing α are of length strictly superior to $4 \operatorname{arcsinh} 1$ which concludes the argument. $\qquad\square$

Let us now consider the case of two geodesics that intersect twice on a one-holed torus. We recall that one-holed tori are hyperelliptic, and we shall refer to the three interior fixed points of the hyperelliptic involution as the Weierstrass points.

Definition 3.2. Let T be a one-holed torus and let α be an interior simple closed geodesic of T. We denote h_α the unique simple geodesic path which goes from boundary to boundary and intersects boundary at two right angles and does not cross α. We will refer to the geodesic path h_α as the height associated to α (see Fig. 5).

By using hyperbolic trigonometry, one can prove the following result (for a proof, see for instance [16]).

Lemma 3.3. *Let T be a one-holed torus. Let γ be an interior simple closed geodesic of T and denote its associated height h_γ. Then γ passes through exactly two of the three Weierstrass points and the remaining Weierstrass point is the midpoint of h_γ. Furthermore, the length of γ is directly proportional to the length of h_γ.*
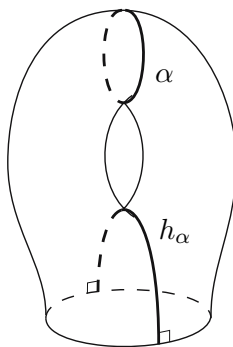

Fig. 5 The height h_α associated to α

The following proposition, slightly more general than what we require, has an interest in its own right.

Proposition 3.4. *Let T be a one-holed torus (where the boundary geodesic ε is allowed to be a cusp). Let α be an interior simple closed geodesic and let β be any other interior simple closed geodesic that intersects α twice. Then*

$$\ell(\beta) \geq 2 \operatorname{arccosh} \left(1 + \frac{\cosh \frac{\ell(\varepsilon)}{2} + 1}{2 \left(\cosh \frac{\ell(\alpha)}{2} - 1 \right)} \right).$$

Furthermore equality holds only when T is obtained by pasting α with a half-twist and β is the shortest simple closed geodesic that intersects α twice.

Proof. For a given α and ε, let β be the shortest simple closed geodesic β that crosses α twice. Now consider the family of tori obtained by twisting along α. The key to the proof is showing that β is shortest when α is pasted with a half-twist. In first instance, let us suppose that ε is not a cusp. Consider the height h_β associated to β. By Lemma 3.3, the length of h_β is proportional to the length of β, so minimizing the length of β is equivalent to minimizing the length of h_β. Denote by e_1, e_2 and a the three boundary to boundary geodesic perpendicular paths of the pair of pants $(\alpha, \alpha, \varepsilon)$ as indicated on Fig. 6.

Cutting T along α and path a one obtains a one-holed hyperbolic rectangle as in Fig. 6. (This particular way of viewing the one-holed torus is a central part of [5].) Notice that $\ell(e_1) = \ell(e_2)$, which can be seen either by using hyperbolic trigonometry or by using the hyperelliptic involution. By cutting along paths h_α, e_1, e_2 and a, one would obtain four isometric right-angled pentagons. The path h_β intersects α twice, and thus the two subpaths of h_β between α and ε are of length at least $\ell(e_1)(=\ell(e_2))$, and the subpath from α and back again is at least of length $\ell(a)$. Thus

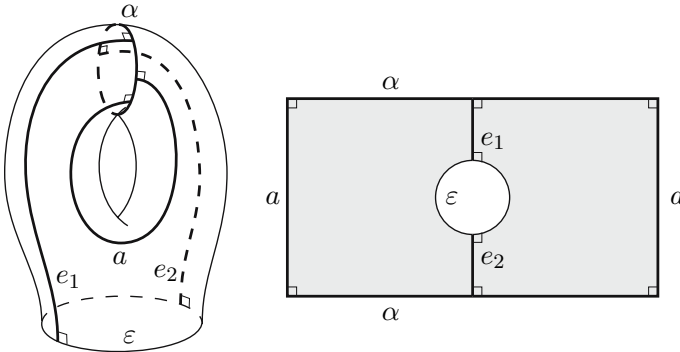$$\ell(h_\beta) \geq \ell(e_1) + \ell(e_2) + \ell(a).$$
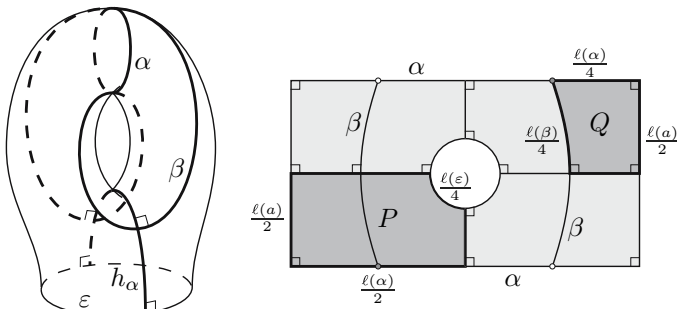

Fig. 6 The torus T cut along α and a

Fig. 7 The polygons P and Q

Equality only holds when the path h_β is exactly the path $e_1 \cup a \cup e_2$. This only occurs when the pasting is right, meaning when α is pasted with a half-twist. Now, when ε is a cusp, we cannot immediately assume that the optimal situation is when there is a half-twist, but this is true because of the continuity of lengths of interior closed curves when $\ell(\varepsilon)$ goes to 0.

We now need to calculate the length of β when α is pasted with a half-twist. For this we shall use the well known formulas for different types of hyperbolic polygons (see for instance [7, p. 454]). This can be done by considering two hyperbolic polygons inscribed in T.

The first one, denoted Q, is one of the hyperbolic quadrilaterals with three right angles delimited by arcs of paths a, α, β and h_α as in Fig. 7. The second polygon P is one the four isometric right-angled pentagons (or quadrilaterals with a point at infinity when $\ell(\varepsilon) = 0$) obtained by cutting T along α, a, e_1, e_2 and h_α (see Fig. 7). Using the formulas for a quadrilateral with three right angles, one obtains

$$\sinh \frac{\ell(\beta)}{4} = \sinh \frac{\ell(a)}{2} \, \cosh \frac{\ell(\alpha)}{4}.$$

Now using the formula for a right-angled pentagon with P we obtain

$$\sinh \frac{\ell(a)}{2} \, \sinh \frac{\ell(\alpha)}{2} = \cosh \frac{\ell(\varepsilon)}{4}.$$

Putting these two formulas together one obtains

$$\sinh \frac{\ell(\beta)}{4} = \frac{\cosh \frac{\ell(\varepsilon)}{4} \cosh \frac{\ell(\alpha)}{4}}{\sinh \frac{\ell(\alpha)}{2}}.$$

With a little manipulation one obtains

$$\cosh \frac{\ell(\beta)}{2} = 1 + \frac{\cosh \frac{\ell(\varepsilon)}{2} + 1}{2 \left(\cosh \frac{\ell(\alpha)}{2} - 1 \right)},$$

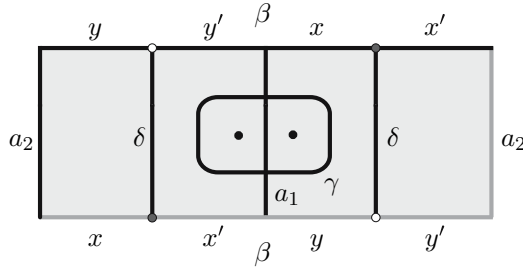which proves the result.   □

Fig. 8 Two intersections on a twice-punctured torus

There is an immediate corollary which gives a universal lower bound on the greatest of two lengths of two geodesics intersecting twice on a one-holed torus.

Corollary 3.5. *Let T be a one-holed torus (where the boundary geodesic ε is allowed to be a cusp). Let α and β be two interior simple closed geodesics that intersect twice. Then*

$$\max\{\ell(\alpha), \ell(\beta)\} \geq 2\arccosh 2.$$

Furthermore equality holds for a torus T with a cusp which contains a simple closed geodesic α of length $2\arccosh 2$ pasted with a half-twist, and taking β to be the shortest simple closed geodesic which intersects α twice.

Proof. Suppose that $\ell(\alpha) < 2\arccosh 2$. By Proposition 3.4, $\ell(\beta) > 2\arccosh 2$. Now if $\ell(\alpha) = 2\arccosh 2$, by proposition 3.4, $\ell(\beta) = 2\arccosh 2$ if and only if $\ell(\varepsilon) = 0$ and T is obtained by pasting α with a half-twist.   □

Note that the torus described in Corollary 3.5 is the same torus as S_1. To see this, we shall find two simple closed geodesics that intersect once, and both of length $2\arccosh\sqrt{2} = 2\arcsinh 1$. Consider the quadrilateral Q in Fig. 7 and in particular the diagonal of Q from top left to bottom right. Now consider the diagonals of each one of the four isometric copies of Q. Together these four geodesic paths form two simple closed geodesics, say γ_1 and γ_2, of equal length that intersect once. When $\ell(\epsilon) = 0$ and $\ell(\alpha) = 2\arccosh 2$, a quick calculation shows that $\ell(\gamma_1) = \ell(\gamma_2) = 2\arccosh\sqrt{2}$. As S_1 is unique up to isometry, the two tori are the same.

Theorem 3.6. *The surface S_2 is the one-holed torus described in Corollary 3.5 and $L_2 = 2\arccosh(2)$.*

Proof. In view of Proposition 3.1 and Corollary 3.5, we now know that S_2 is a torus with one or two punctures. Suppose S_2 is a torus with two punctures, i.e., the curves labeled $\tilde{\varepsilon}_1$ and $\tilde{\varepsilon}_2$ on Fig. 4 are homotopic to cusps.

Now suppose we have two simple closed geodesics α and β that intersect twice, with $\ell(\alpha) \geq \ell(\beta)$ and $\ell(\alpha) \leq 2\arccosh 2$. (If this is not possible, then necessarily S_2 is the once punctured torus of corollary 3.5.) The geodesic α is cut into two arcs by β, say a_1 and a_2. Suppose $\ell(a_1) \geq \ell(a_2)$. Consider the geodesic curves γ and δ as in Fig. 8. γ is the separating curve that intersects a_1 twice but doesn't

intersect a_2 or β, and δ is the curve that intersects β twice but doesn't intersect γ or α. Consider the lengths x, x', y, y' of the different arcs of β as labeled on Fig. 8. We have $\ell(\delta) < x + \ell(a_2) + y + x' + \ell(a_2) + y' \leq \ell(\alpha) + \ell(\beta) \leq 2\ell(\alpha)$. Notice that this implies the width of the collar around δ satisfies

$$w(\delta) > \operatorname{arcsinh}\left(\frac{1}{\sinh \ell(\alpha)}\right).$$

We can now apply the collar theorem to β, using the fact that β intersects both α and δ twice and α and δ do not intersect. The collar theorem 2.1 implies that the length of β satisfies the following inequality:

$$\ell(\beta) \geq 4w(\alpha) + 4w(\delta) > 2\operatorname{arccosh}2 \geq \ell(\alpha).$$

This proves the result. □

4. Finding S_3 and calculating L_3

Let α and β be two simple closed geodesics on a Riemann surface that intersect three times. Name the intersection points A, B and C and orient α and β such that A, B and C come in that order on α and on β. As in the case of two intersections, we consider a collar around β and the orientations induced on it in the different intersection points by the ordered pair of simple closed *oriented* geodesics (α, β). We distinguish two situations:

1. (α, β) induces opposite orientations in two of the three intersection points (without loss of generality we can assume that (α, β) induces opposite orientations in A and in B),
2. (α, β) induces the same orientation in A, in B and in C.

In the first situation, Lemma 4.1 will show that $\max\{\ell(\alpha), \ell(\beta)\} \geq 2\operatorname{arccosh}(3)$.

In the second situation, we will show that the optimal surface is a torus with a cusp containing two simple closed geodesics of lengths approximately $2\operatorname{arccosh}(2.648)$ intersecting one another three times.

Lemma 4.1. *Let α and β be two simple closed oriented geodesics on a Riemann surface that intersect three times in A, B and C, such that A, B, C are consecutive on both α and β.*

If the ordered pair (α, β) induces opposite orientations on the surface in A and in B, then $\max\{\ell(\alpha), \ell(\beta)\} \geq 2\operatorname{arccosh}(3)$.

Proof. Without loss of generality, we can assume, that the ordered pair (α, β) induces the same orientation on the surface in B and in C (cyclicly rename A, B and C, if needed). Comparing the lengths of the arcs between B and C, there are two possible situations:

1. The length $\overline{BC}_\alpha$ of the oriented geodesic arc from B to C on the geodesic α is smaller then $\overline{BC}_\beta$, the length of the oriented geodesic arc from B to C on the geodesic β.
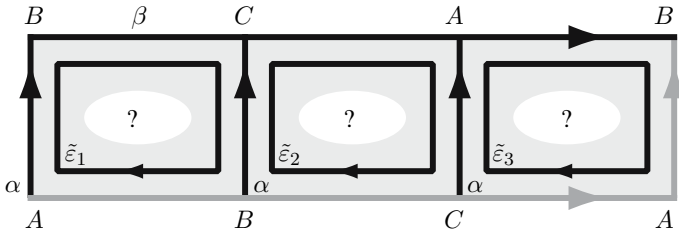
Fig. 9 The simple closed curves $\tilde{\varepsilon}_1$, $\tilde{\varepsilon}_2$ and $\tilde{\varepsilon}_3$

2. This is not the case, meaning $\overline{BC}_\alpha \geq \overline{BC}_\beta$.

   We now build the oriented closed curves $\tilde{\gamma}$ and $\tilde{\delta}$:

- In situation 1, we set $\tilde{\gamma} = \alpha$; in situation 2, $\tilde{\gamma}$ is obtained following α from A to B, then β from B to C and again α from C to A.
- In situation 1, $\tilde{\delta}$ is obtained following β from A to B, then α from B to C and again β from C to A; in situation 2, we set $\tilde{\delta} = \beta$.

   These two curves $\tilde{\gamma}$ and $\tilde{\delta}$ are thus homotopic to two simple closed oriented geodesics γ and δ intersecting one another twice such that

$$\max\{\ell(\alpha), \ell(\beta)\} \geq \max\{\ell(\gamma), \ell(\delta)\}.$$

   Furthermore, the ordered pair (γ, δ) induces opposite orientations in its two intersection points. Therefore $\max\{\ell(\gamma), \ell(\delta)\} \geq 2\operatorname{arccosh}(3)$ by Proposition 3.1.   □

Lemma 4.2. *Let S be a Riemann surface and let α and β be two oriented simple closed geodesics on S intersecting one another three times such that the ordered pair (α, β) induces the same orientation on S in every intersection. Name the intersections A, B, C such that they are consecutive on α. If A, B, C are also consecutive on β, then there is a torus with one cusp or a torus with two cusps containing two simple closed geodesics γ and δ which satisfy $\operatorname{int}(\gamma, \delta) = 3$ and $\max\{\ell(\alpha), \ell(\beta)\} \geq \max\{\ell(\gamma), \ell(\delta)\}$.*

Proof. The surface $S_{\alpha,\beta}$ is of signature $(1, 3)$, $(1, 2)$ or $(1, 1)$, as can easily be seen by considering whether the simple closed curves $\tilde{\varepsilon}_1$, $\tilde{\varepsilon}_2$ and $\tilde{\varepsilon}_3$ (shown in Fig. 9) are null-homotopic or not.

   If one of the curves $\tilde{\varepsilon}_1$, $\tilde{\varepsilon}_2$ or $\tilde{\varepsilon}_3$ is null-homotopic, corollary 2.4 proves the lemma. Otherwise, the optimal topological situation is a torus with three cusps (again due to corollary 2.4). On this surface, there is a simple closed geodesic η dividing the surface into X_η, a sphere with three cusps and boundary geodesic η, and T_η, a surface of signature $(1, 1)$. Notice that β is entirely contained in T_η as can be seen in Fig. 10.

   The intersection points between α and η will be denoted U, V, W and Y as in Fig. 10. First consider the geodesic arc of α from Y to W. There is a dividing geodesic ε on X_η, that does not intersect this arc. Cutting X_η along ε, we get a surface of signature $(0, 3)$. We can now diminish the length of ε in order to get another
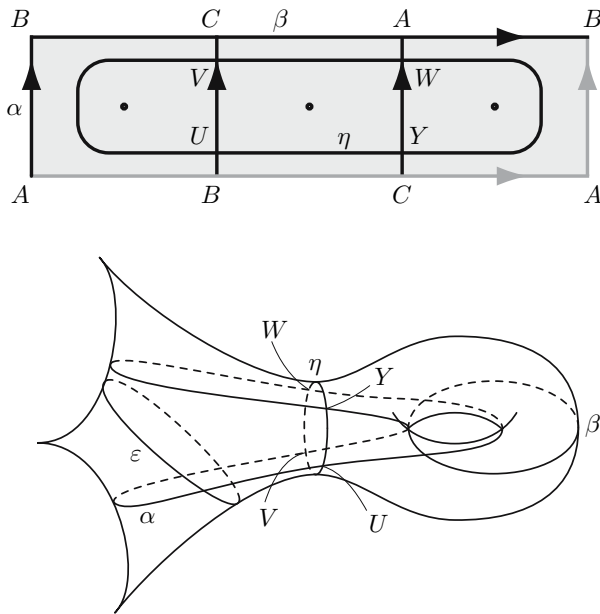
Fig. 10 Three intersections on a torus with three cusps

cusp. This surface of signature $(0, 3)$ with two cusps and the boundary geodesic η contains a geodesic arc from Y to W that is shorter than the original arc from Y to W on X_η (this is part of the statement of the technical lemma used in [14] in order to show Theorem 2.3).

Note the geodesic arc joining U and V intersects ε and its length goes thus to infinity during this process. Nevertheless, if we do the analogous process to reduce the length of the geodesic arc joining U and V, we get another surface of signature $(0, 3)$ with two cusps and a boundary geodesic of length $\ell(\eta)$ containing a geodesic arc from U to V that is shorter than the original arc from U to V on X_η. As a surface Y_η of signature $(0, 3)$ with two cusps and a boundary geodesic of length $\ell(\eta)$ is unique up to isometry, we can paste Y_η to T_η such that there are two geodesic arcs, one joining Y and W and the other U and V, that are each shorter than the original ones on X_η. The pasted surface is a torus with two cusps that contains a geodesic β and a curve $\tilde{\alpha}$ that intersect three times and such that $\ell(\alpha) \geq \ell(\tilde{\alpha})$. Therefore, the geodesic γ that is homotopic to $\tilde{\alpha}$ intersects the geodesic β (that we rename to δ) three times and $\max\{\ell(\alpha), \ell(\beta)\} \geq \max\{\ell(\gamma), \ell(\delta)\}$. □

Lemma 4.3. *If α and β are two simple closed geodesics on a one-holed torus satisfying* $\mathrm{int}(\alpha, \beta) = 3$, *then*

$$\max\{\ell(\alpha), \ell(\beta)\} \geq 2\,\mathrm{arccosh}\left(\sqrt{\frac{1}{2}\left(7 + \frac{11}{3}\sqrt{\frac{11}{3}}\right)}\right).$$

This bound is sharp and is reached by a unique once-punctured torus up to isometry.

Proof. We shall use the parameters for the set of isometry classes of one-holed tori found in [5]. Let (r, s, t) be a set of these parameters such that $1 < r \leq s \leq t \leq rs$ where r, s and t are the half-traces (hyperbolic cosines of half of the lengths) of the shortest three geodesics ϱ, σ and $\tau = (\varrho\sigma)^{-1}$. (In [5], half-traces are denoted traces, but we shall continue to use the term half-traces as it is more standard.) Then, the geodesics $\alpha = \varrho\sigma^{-1}$ and $\beta = \tau\varrho^{-1}$ intersect three times and α is the forth shortest simple closed geodesic (see [5] for details). The half-traces of α and β are $a = 2rs - t$ and $b = 2rt - s$.

For a fixed r, $\max\{a, b\} = b = 2rt - s$ is therefore minimal if $s = t$. In this case $0 = 2rst - r^2 - s^2 - t^2 = 2s^2(r - 1) - r^2$ and therefore $b^2 = s^2(2r - 1)^2 = \frac{r^2(2r-1)^2}{2(r-1)}$.

But for $r > 1$, this last quantity is minimal for

$$\frac{d}{dr} \frac{r^2(2r-1)^2}{2(r-1)} = 0 \iff \frac{r(2r-1)(6r^2 - 9r + 2)}{2(r-1)^2} = 0,$$

i.e., $r = \frac{1}{4}\left(3 + \sqrt{\frac{11}{3}}\right)$. Therefore $b^2 = s^2(2r - 1)^2 \geq \frac{1}{2}\left(7 + \frac{11}{3}\sqrt{\frac{11}{3}}\right)$.

There is a torus with one cusp on which there are two geodesics of lengths $2\operatorname{arccosh}\left(\sqrt{\frac{1}{2}\left(7 + \frac{11}{3}\sqrt{\frac{11}{3}}\right)}\right)$ intersecting one another three times. Its half-trace are necessarily $(r, s, t) = \left(\frac{1}{4}\left(3 + \sqrt{\frac{11}{3}}\right), \sqrt{\frac{13+7\sqrt{\frac{11}{3}}}{8}}, \sqrt{\frac{13+7\sqrt{\frac{11}{3}}}{8}}\right)$, up to a choice of curves ϱ, σ and τ. Therefore the bound is sharp and is attained by a unique once-punctured torus up to isometry. $\square$

It is worth noticing the torus described in this lemma is *not* S_1. As mentioned in the proof, its systole length is $2\operatorname{arccosh}\left(\frac{1}{4}\left(3 + \sqrt{\frac{11}{3}}\right)\right)$ and not $2\operatorname{arccosh}\sqrt{2} = 2\operatorname{arcsinh}1$.

Theorem 4.4. *The surface S_3 is the one-holed torus described in Lemma 4.3 and*

$$L_3 = 2\operatorname{arccosh}\left(\sqrt{\frac{1}{2}\left(7 + \frac{11}{3}\sqrt{\frac{11}{3}}\right)}\right).$$

Proof. By what precedes, S_3 is a torus with one or two punctures. As in the proof of theorem 3.6, let us suppose that there exists a twice-punctured torus T with two geodesics α and β that intersect three times, and both of length less or equal to $2\operatorname{arccosh}\left(\sqrt{\frac{1}{2}\left(7 + \frac{11}{3}\sqrt{\frac{11}{3}}\right)}\right)$. For the remainder of the proof, denote this constant k_3, as L_3 would be premature.

Both α and β are separated into three paths by each other, and let us denote these paths respectively a_1, a_2 and a_3 for α and b_1, b_2 and b_3 for β. The pasting condition implies that we are now in the situation illustrated in Fig. 11. On this figure, two additional simple closed curves have been added, and are denoted γ_α and γ_β. The
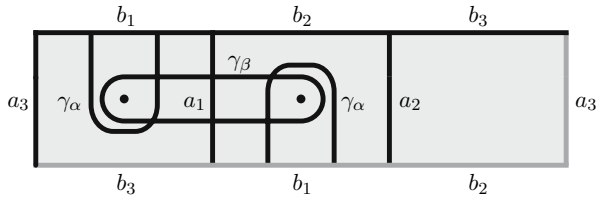
Fig. 11 Three intersections on a twice-punctured torus
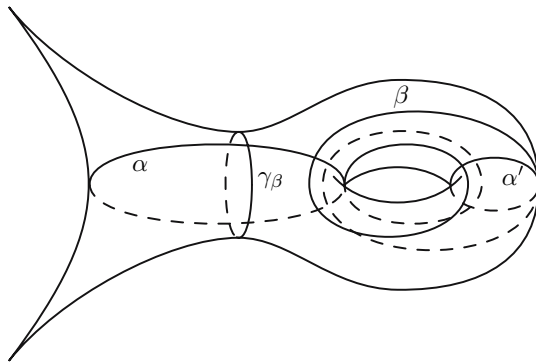
Fig. 12 The geodesics α and α'

curve γ_α is defined as the unique separating simple closed geodesic that does not intersect α and intersects β minimally (twice), and γ_β is defined symmetrically. We will use a rough upper-bound on the sum of their lengths. It is easy to see that

$$\ell(\gamma_\alpha) + \ell(\gamma_\beta) < (2\ell(a_1) + \ell(a_2) + \ell(a_3) + \ell(b_2) + \ell(b_3))$$
$$+ (\ell(a_2) + \ell(a_3) + 2\ell(b_1) + \ell(b_2) + \ell(b_3))$$
$$= 2\ell(\alpha) + 2\ell(\beta).$$

This implies that $\min\{\ell(\gamma_\alpha), \ell(\gamma_\beta)\} \leq 2\max\{\alpha, \beta\} \leq 2k_3$. So far, we have made no particular assumptions on α and β, so without loss of generality we can suppose that α and β are such that $\ell(\gamma_\beta) \leq 2k_3$.

Denote by α' the unique simple closed geodesic of T that intersects neither α nor γ_β. Notice that α' intersects β three times. We shall now find an upper-bound on the length of α'. By cutting along α and α', one obtains two (isometric) pairs of pants. Consider one of them as in Fig. 13. We denote by h_α the shortest non-trivial path from α and back again. Notice that

$$\ell(h_\alpha) \leq \frac{\ell(\gamma_\beta)}{2} \leq k_3. \tag{1}$$

Consider the lengths l and l' in Fig. 13. Once again, we shall make use of the formulas for hyperbolic polygons. Using the hyperbolic trigonometry formulas for a pentagon with right angles, we obtain

$$\sinh \frac{\ell(h_\alpha)}{2} \sinh l' = \cosh \frac{\ell(\alpha')}{2}.$$
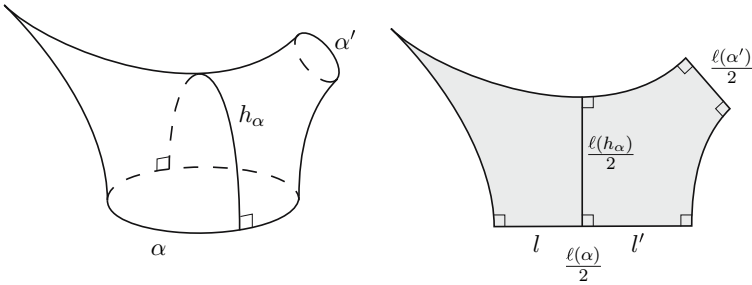
Fig. 13 Bounding the length of α'

Using the formulas for a quadrilateral with three right angles and one zero angle, one obtains $\sinh l \ \sinh \frac{\ell(h_\alpha)}{2} = 1$ and equation (1) leads to

$$l > \frac{1}{\sinh \frac{k_3}{2}} \ .$$

From these equations, and our initial hypothesis on the lengths of α and β, we obtain the following bound on the length of α':

$$\ell(\alpha') < 2 \operatorname{arccosh}\left(\sinh \frac{k_3}{2} \ \sinh \left(\frac{k_3}{2} - \operatorname{arcsinh} \frac{1}{\sinh \frac{k_3}{2}} \right) \right).$$

This implies that the collar width of α' satisfies $w(\alpha') > 0.25$. As β intersects both α and α' three times and α and α' are disjoint, we have that

$$\ell(\beta) \geq 6w(\alpha) + 6w(\alpha').$$

As $\ell(\alpha) \leq k_3$ implies $w(\alpha) > 0.3$, we now have $\ell(\beta) > 1.5 + 1.8 > k_3$ which contradicts the hypotheses. Thus S_3 is a once-punctured torus and we can apply Lemma 4.3. □

5. Concluding remarks

The surfaces $S_1 = S_2$ and S_3 are specific once-punctured tori. Both admit automorphisms distinct from the hyperelliptic involution. S_1 admits a number of automorphisms both conformal and anticonformal. Using the main result of [8], S_3 admits an orientation reversing involution because it can be obtained by pasting a simple closed geodesic with a half-twist, but does not admit a non-trivial conformal automorphism. This is not so surprising seeing as there are only two isometry classes of once-punctured tori that admit a non-trivial conformal automorphism, namely S_1 and the torus with largest automorphism group, often called the Modular torus.

Finding S_k for $k \geq 4$ seems like a difficult problem, but can we say something about the set of S_k? For higher intersection number, it is not clear whether or not S_k even has boundary (recall that two simple closed geodesics can fill closed surfaces if they are allowed sufficiently many intersection points). In spite of this

remark, it seems reasonable to conjecture that S_k is always a once-punctured torus. Furthermore, due to the existence on S_k of geodesics of equal length, it also seems reasonable to conjecture that the S_k all have non-trivial automorphism groups. Supposing that the S_k are all once-punctured tori, are they all found in a finite set of isometry classes of once-punctured tori?

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