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Correctors for Some Asymptotic Problems

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Abstract—In the theory of anisotropic singular perturbation boundary value problems, the solution u_ε does not converge, in the H^1-norm on the whole domain, towards some u_0. In this paper we construct correctors to have good approximations of u_ε in the H^1-norm on the whole domain. Since the anisotropic singular perturbation problems can be connected to the study of the asymptotic behaviour of problems defined in cylindrical domains becoming unbounded in some directions, we transpose our results for such problems.

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1. INTRODUCTION

Let $\mathcal{O} = (-1,1) \times \omega$ be a bounded open subset of $\mathbb{R}^{p+1}$, $p \geq 1$, ω being a bounded open subset of $\mathbb{R}^p$. We denote by $x = (X_1, X_2)$ the points of $\mathcal{O}$ with

$$X_1 = x_1, \qquad X_2 = (x_1', \dots, x_p').$$

With this notation we set

$$\nabla u = \big(\partial_{x_1} u, \partial_{x_1'} u, \dots, \partial_{x_p'} u\big)^{\mathrm{T}} = \begin{pmatrix} \partial_{X_1} u \\ \nabla_{X_2} u \end{pmatrix},$$

where

$$\nabla_{X_2} u = \big(\partial_{x_1'} u, \dots, \partial_{x_p'} u\big)^{\mathrm{T}}.$$

For $f \in L^2(\mathcal{O})$ and $\varepsilon > 0$, there exists a unique solution u_ε (in a weak sense) of

$$\begin{cases} u_\varepsilon \in H_0^1(\mathcal{O}), \\ -\varepsilon^2 \partial_{X_1}^2 u_\varepsilon - \Delta_{X_2} u_\varepsilon = f \quad \text{in} \ \ \mathcal{O}. \end{cases} \tag{1.1}$$

We denote by Δ_{X_2} the Laplace operator defined by

$$\Delta_{X_2} = \partial_{x_1'}^2 + \ldots + \partial_{x_p'}^2.$$

For a.e. $X_1 \in (-1,1)$ one can define a solution u_0 to

$$\begin{cases} u_0(X_1, \cdot\,) \in H_0^1(\omega), \\ -\Delta_{X_2} u_0(X_1, \cdot\,) = f(X_1, \cdot\,) \quad \text{in} \ \ \omega. \end{cases} \tag{1.2}$$

It is shown in [3, 4] that

$$u_\varepsilon \to u_0 \qquad \text{in} \ \ L^2(\mathcal{O}) \quad \text{as} \ \ \varepsilon \to 0. \tag{1.3}$$

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Even if $\nabla_{X_2} u_\varepsilon \to \nabla_{X_2} u_0$ in $(L^2(\mathcal{O}))^p$ (see [3, 4]), one cannot expect in general that

$$u_\varepsilon \to u_0 \qquad \text{in } H^1(\mathcal{O}). \tag{1.4}$$

Indeed, if, for instance, f is independent of X_1, then so is u_0 and clearly, for $f \neq 0$, $u_0 \notin H_0^1(\mathcal{O})$ when u_ε does belong to $H_0^1(\mathcal{O})$, which makes (1.4) impossible. The goal of this paper is to "correct" $u_\varepsilon - u_0$ by a simple function w_ε which gives the behaviour of $u_\varepsilon - u_0$ near the end sections $\{-1, 1\} \times \omega$ and is such that

$$u_\varepsilon - u_0 - w_\varepsilon \to 0 \qquad \text{in } H_0^1(\mathcal{O}). \tag{1.5}$$

Due to the uniqueness of a solution of (1.1), one has (see Lemma 3.4)

$$u_\varepsilon(-X_1, X_2) = u_\varepsilon(X_1, X_2),$$

and this clearly implies that

$$\frac{\partial u_\varepsilon}{\partial_{X_1}}(0, X_2) = 0. \tag{1.6}$$

Thus, to study and correct the behaviour of $u_\varepsilon - u_0$, one can consider u_ε as the solution to

$$\begin{cases} -\varepsilon^2 \partial_{X_1}^2 u_\varepsilon - \Delta_{X_2} u_\varepsilon = f & \text{in } \Omega = (0,1) \times \omega, \\ u_\varepsilon = 0 & \text{on } \partial\Omega \setminus \{0\} \times \omega, \\ \dfrac{\partial u_\varepsilon}{\partial_{X_1}} = 0 & \text{on } \{0\} \times \omega. \end{cases} \tag{1.7}$$

This is what we will do in the next section. Note that this is inspired by [5], where a similar analysis was carried out for the Stokes problem. In Section 3 we will transpose our results—via a scaling argument—to the Dirichlet problem set in cylinders becoming infinite in various directions.

For more details about the anisotropic singular perturbation problems, as well as for details on the problems considered in Section 3, we refer the reader to [1–6, 8, 9]. The classic singular perturbation problems are dealt with in [10].

2. THE CASE OF ANISOTROPIC PROBLEMS IN ONE DIRECTION

Let Ω be defined as

$$\Omega = (0,1) \times \omega,$$

where ω is a bounded domain of $\mathbb{R}^p$, and V be the space

$$V = \big\{ v \in H^1(\Omega) \mid v = 0 \text{ on } \partial\Omega \setminus \{0\} \times \omega \big\}. \tag{2.1}$$

There exists a unique solution u_ε to

$$\begin{cases} u_\varepsilon \in V, \\ \displaystyle\int_\Omega \big(\varepsilon^2 \partial_{X_1} u_\varepsilon \partial_{X_1} v + \nabla_{X_2} u_\varepsilon \cdot \nabla_{X_2} v\big)\, dx = \int_\Omega f v\, dx \qquad \forall\, v \in V. \end{cases} \tag{2.2}$$

Clearly (2.2) is the weak formulation of (1.7). We assume that $f \in L^2(\Omega)$, and the existence of a unique solution to (2.2) follows from the Lax–Milgram theorem. The weak formulation of (1.2) reads for a.e. $X_1 \in (0,1)$ as

$$\begin{cases} u_0(X_1, \cdot) \in H_0^1(\omega), \\ \displaystyle\int_\omega \nabla_{X_2} u_0(X_1, \cdot) \cdot \nabla_{X_2} v\, dX_2 = \int_\omega f(X_1, \cdot)\, v\, dX_2 \qquad \forall\, v \in H_0^1(\omega). \end{cases} \tag{2.3}$$

In the case where

$$f \in L^2(\Omega), \qquad \partial_{X_1} f \in L^2(\Omega), \tag{2.4}$$

one can show (see [4]) that

$$u_0 \in H^1(\Omega).$$

Now (see, for instance, [2]) if $v \in V$, then for a.e. $X_1 \in (0,1)$ one has

$$v(X_1, \cdot) \in H_0^1(\omega). \tag{2.5}$$

Using this test function in (2.3), one derives, after an integration in X_1, that

$$\int_\Omega \nabla_{X_2} u_0 \cdot \nabla_{X_2} v \, dx = \int_\Omega f v \, dx \qquad \forall\, v \in V. \tag{2.6}$$

In order to construct a corrector for u_ε, we denote by S_ℓ the half-cylinder

$$S_\ell = (\ell, +\infty) \times \omega,$$

where $\ell \in \mathbb{R}$. Then if $\rho\colon [0,+\infty) \to [0,1]$ is the function defined by

$$\rho(x) = \begin{cases} 1 - x & \text{on } [0,1], \\ 0 & \text{on } (1,+\infty), \end{cases}$$

we introduce u as the solution to

$$\begin{cases} u \in H_0^1(S_0), \\ \displaystyle\int_{S_0} \nabla u \cdot \nabla v \, dx = \int_{S_0} \nabla(\rho(X_1) u_0) \cdot \nabla v \, dx \qquad \forall\, v \in H_0^1(S_0). \end{cases} \tag{2.7}$$

Since $u_0 \in H^1(\Omega)$, the existence and uniqueness of u follows from the Lax–Milgram theorem. Then we set

$$w(X_1, X_2) = u(X_1, X_2) - \rho(X_1) u_0(X_1, X_2) = u - \rho u_0 \tag{2.8}$$

and denote by w_ε the function defined as

$$w_\varepsilon(X_1, X_2) = w\left(\frac{1 - X_1}{\varepsilon}, X_2\right). \tag{2.9}$$

Note that $w \in H^1(S_0)$ and satisfies in a weak sense

$$\begin{cases} \Delta w = 0 & \text{in } S_0, \\ w = -u_0 \quad \text{on } \{0\} \times \omega, & w = 0 \quad \text{on } (0,+\infty) \times \partial\omega. \end{cases} \tag{2.10}$$

2.1. Some preliminary results. We denote by Ω_- the domain defined by

$$\Omega_- = (-1, 0) \times \omega.$$

For $v \in V$ we define by $\widehat{v}$ the function given by

$$\widehat{v}(X_1, X_2) = \begin{cases} v(X_1, X_2), & X_1 \ge 0, \\ v(-X_1, X_2), & X_1 < 0. \end{cases} \tag{2.11}$$

Then we have

Lemma 2.1. *For every $v \in V$ the following equality holds*:

$$\int_{\Omega} \left(\varepsilon^2 \partial_{X_1} w_\varepsilon \partial_{X_1} v + \nabla_{X_2} w_\varepsilon \cdot \nabla_{X_2} v\right) dx = -\int_{\Omega_-} \left(\varepsilon^2 \partial_{X_1} w_\varepsilon \partial_{X_1} \widehat{v} + \nabla_{X_2} w_\varepsilon \cdot \nabla_{X_2} \widehat{v}\right) dx.$$

Proof. For $\ell > 0$ we set $\Omega_\ell = (0,\ell)\times\omega$. Then first note that for $v \in V$ we have $\widehat{v}(1-\varepsilon X_1, X_2) \in H_0^1(\Omega_{2/\varepsilon})$. Thus we derive, from (2.10),

$$\int_{\Omega_{2/\varepsilon}} \nabla w \cdot \nabla \widehat{v}(1-\varepsilon X_1, X_2)\, dx = 0,$$

whence

$$\int_{\Omega_{1/\varepsilon}} \nabla w \cdot \nabla \widehat{v}(1-\varepsilon X_1, X_2)\, dx = -\int_{\Omega_{2/\varepsilon}\setminus\Omega_{1/\varepsilon}} \nabla w \cdot \nabla \widehat{v}(1-\varepsilon X_1, X_2)\, dx. \tag{2.12}$$

Making the change of variable $X_1' = 1 - \varepsilon X_1$ in the integrals above, we obtain respectively

$$\int_{\Omega_{1/\varepsilon}} \nabla w \cdot \nabla \widehat{v}(1-\varepsilon X_1, X_2)\, dx = \frac{1}{\varepsilon}\int_{\Omega} \left(\varepsilon^2 \partial_{X_1} w_\varepsilon\, \partial_{X_1} v + \nabla_{X_2} w_\varepsilon \cdot \nabla_{X_2} v\right) dx$$

and

$$\begin{aligned}\int_{\Omega_{2/\varepsilon}\setminus\Omega_{1/\varepsilon}} \nabla w \cdot \nabla \widehat{v}(1-\varepsilon X_1, X_2)\, dx &= \int_{\Omega_{2/\varepsilon}\setminus\Omega_{1/\varepsilon}} \left(-\varepsilon \partial_{X_1} w\, \partial_{X_1} \widehat{v}(X_1', X_2) + \nabla_{X_2} w \cdot \nabla_{X_2} \widehat{v}(X_1', X_2)\right) dx \\ &= \frac{1}{\varepsilon}\int_{\Omega_-} \left(\varepsilon^2 \partial_{X_1} w_\varepsilon\, \partial_{X_1} \widehat{v} + \nabla_{X_2} w_\varepsilon \cdot \nabla_{X_2} \widehat{v}\right) dx.\end{aligned}$$

The lemma follows from (2.12).

We will also need the following lemma.

Lemma 2.2. *There exist positive constants $C > 0$ and $\alpha > 0$ independent of ε such that*

$$\int_{S_{1/\varepsilon}} |\nabla w|^2\, dx \le C e^{-\frac{\alpha}{\varepsilon}} \int_{S_0} |\nabla w|^2\, dx. \tag{2.13}$$

Proof. Without loss of generality, we assume that $\varepsilon < 1$. Let $\gamma_\varepsilon\colon \mathbb{R} \to \mathbb{R}$ be a continuous function such that $\gamma_\varepsilon = 0$ in $\left(-\infty, \frac{1}{\varepsilon} - 1\right)$, $\gamma_\varepsilon = 1$ in $\left(\frac{1}{\varepsilon}, +\infty\right)$ and γ_ε is linear in $\left[\frac{1}{\varepsilon} - 1, \frac{1}{\varepsilon}\right]$. Since $\gamma_\varepsilon(X_1) w \in H_0^1(S_0)$, we have, by (2.10),

$$\int_{S_0} \nabla w \cdot \nabla(\gamma_\varepsilon(X_1) w)\, dx = 0. \tag{2.14}$$

Thus

$$\begin{aligned}\int_{S_0} |\nabla w|^2 \gamma_\varepsilon(X_1)\, dx &= -\int_{S_{1/\varepsilon-1}\setminus S_{1/\varepsilon}} \partial_{X_1} w\, \partial_{X_1}\gamma_\varepsilon(X_1) w\, dx \le \int_{S_{1/\varepsilon-1}\setminus S_{1/\varepsilon}} |\partial_{X_1} w|\, |w|\, dx \\ &\le \frac{1}{2}\int_{S_{1/\varepsilon-1}\setminus S_{1/\varepsilon}} |\partial_{X_1} w|^2\, dx + \frac{1}{2}\int_{S_{1/\varepsilon-1}\setminus S_{1/\varepsilon}} |w|^2\, dx.\end{aligned}$$

Applying the Poincaré inequality in X_2 to the last integral, we get for some constant C_ω

$$\int\limits_{S_{1/\varepsilon-1}\setminus S_{1/\varepsilon}} |w|^2\,dx = \int\limits_{1/\varepsilon-1}^{1/\varepsilon}\int\limits_{\omega} |w|^2\,dx \le C_\omega \int\limits_{1/\varepsilon-1}^{1/\varepsilon}\int\limits_{\omega} |\nabla_{X_2} w|^2\,dx.$$

This leads to

$$\int\limits_{S_{1/\varepsilon}} |\nabla w|^2\,dx \le \frac{\max(1,C_\omega)}{2}\int\limits_{S_{1/\varepsilon-1}\setminus S_{1/\varepsilon}} |\nabla w|^2\,dx$$
$$= \frac{\max(1,C_\omega)}{2}\int\limits_{S_{1/\varepsilon-1}} |\nabla w|^2\,dx - \frac{\max(1,C_\omega)}{2}\int\limits_{S_{1/\varepsilon}} |\nabla w|^2\,dx,$$

and thus

$$\int\limits_{S_{1/\varepsilon}} |\nabla w|^2\,dx \le r\int\limits_{S_{1/\varepsilon-1}} |\nabla w|^2\,dx,$$

where $r = \frac{\max(1,C_\omega)}{2+\max(1,C_\omega)}$. Iterating $\left[\frac{1}{\varepsilon}\right]$ times this formula ($\left[\frac{1}{\varepsilon}\right]$ is the integer part of $\frac{1}{\varepsilon}$), we obtain

$$\int\limits_{S_{1/\varepsilon}} |\nabla w|^2\,dx \le r^{[\frac{1}{\varepsilon}]}\int\limits_{S_{1/\varepsilon-[1/\varepsilon]}} |\nabla w|^2\,dx.$$

Since $\frac{1}{\varepsilon}-1 < \left[\frac{1}{\varepsilon}\right] \le \frac{1}{\varepsilon}$, we deduce

$$\int\limits_{S_{1/\varepsilon}} |\nabla w|^2\,dx \le \frac{1}{r}e^{\ln r\,\frac{1}{\varepsilon}}\int\limits_{S_0} |\nabla w|^2\,dx.$$

This completes the proof by setting $C = \frac{1}{r}$ and $\alpha = -\ln r$.

The theorem below will play an important role in the following.

Theorem 2.3. *Let u_ε and u_0 be the solutions to* (1.7) *and* (1.2), *respectively. Then under the assumption* (2.4) *there exist two constants C and $\alpha > 0$ independent of ε, such that*

$$\frac{3}{4}\int\limits_{\Omega}\big(\varepsilon^2(\partial_{X_1}(u_\varepsilon-u_0-w_\varepsilon))^2 + |\nabla_{X_2}(u_\varepsilon-u_0-w_\varepsilon)|^2\big)\,dx$$
$$\le Ce^{-\frac{\alpha}{\varepsilon}}\int\limits_{S_0}|\nabla w|^2\,dx - \varepsilon^2\int\limits_{\Omega}\partial_{X_1}u_0\,\partial_{X_1}(u_\varepsilon-u_0-w_\varepsilon)\,dx. \quad (2.15)$$

Proof. Subtracting (2.6) from (2.2), we obtain

$$\int\limits_{\Omega}\big(\varepsilon^2\partial_{X_1}(u_\varepsilon-u_0)\partial_{X_1}v + \nabla_{X_2}(u_\varepsilon-u_0)\cdot\nabla_{X_2}v\big)\,dx = -\varepsilon^2\int\limits_{\Omega}\partial_{X_1}u_0\,\partial_{X_1}v\,dx \qquad \forall\, v\in V. \quad (2.16)$$

Since $u_\varepsilon - u_0 - w_\varepsilon \in V$, we get

$$\int\limits_{\Omega}\big(\varepsilon^2\partial_{X_1}(u_\varepsilon-u_0)\partial_{X_1}(u_\varepsilon-u_0-w_\varepsilon) + \nabla_{X_2}(u_\varepsilon-u_0)\cdot\nabla_{X_2}(u_\varepsilon-u_0-w_\varepsilon)\big)\,dx$$
$$= -\varepsilon^2\int\limits_{\Omega}\partial_{X_1}u_0\,\partial_{X_1}(u_\varepsilon-u_0-w_\varepsilon)\,dx.$$

According to Lemma 2.1, the identity above can be written as

$$\int_\Omega \big(\varepsilon^2(\partial_{X_1}(u_\varepsilon - u_0 - w_\varepsilon))^2 + |\nabla_{X_2}(u_\varepsilon - u_0 - w_\varepsilon)|^2\big)\,dx$$
$$= -\int_\Omega \big(\varepsilon^2\partial_{X_1}w_\varepsilon\,\partial_{X_1}(u_\varepsilon - u_0 - w_\varepsilon) + \nabla_{X_2}w_\varepsilon\cdot\nabla_{X_2}(u_\varepsilon - u_0 - w_\varepsilon)\big)\,dx$$
$$-\,\varepsilon^2\int_\Omega \partial_{X_1}u_0\,\partial_{X_1}(u_\varepsilon - u_0 - w_\varepsilon)\,dx$$
$$= \int_{\Omega_-} \big(\varepsilon^2\partial_{X_1}w_\varepsilon\,\partial_{X_1}(\widehat{u_\varepsilon - u_0 - w_\varepsilon}) + \nabla_{X_2}w_\varepsilon\cdot\nabla_{X_2}(\widehat{u_\varepsilon - u_0 - w_\varepsilon})\big)\,dx$$
$$-\,\varepsilon^2\int_\Omega \partial_{X_1}u_0\,\partial_{X_1}(u_\varepsilon - u_0 - w_\varepsilon)\,dx. \tag{2.17}$$

We separately estimate the integral over Ω_- using the Cauchy–Schwarz and Young's inequalities. We derive

$$\int_{\Omega_-} \big(\varepsilon^2\partial_{X_1}w_\varepsilon\,\partial_{X_1}(\widehat{u_\varepsilon - u_0 - w_\varepsilon}) + \nabla_{X_2}w_\varepsilon\cdot\nabla_{X_2}(\widehat{u_\varepsilon - u_0 - w_\varepsilon})\big)\,dx$$
$$\le \Bigg(\int_{\Omega_-}\big(\varepsilon^2(\partial_{X_1}w_\varepsilon)^2 + |\nabla_{X_2}w_\varepsilon|^2\big)\,dx\Bigg)^{1/2}$$
$$\times\Bigg(\int_{\Omega_-}\big(\varepsilon^2(\partial_{X_1}(\widehat{u_\varepsilon - u_0 - w_\varepsilon}))^2 + |\nabla_{X_2}(\widehat{u_\varepsilon - u_0 - w_\varepsilon})|^2\big)\,dx\Bigg)^{1/2}$$
$$\le \int_{\Omega_-}\big(\varepsilon^2(\partial_{X_1}w_\varepsilon)^2 + |\nabla_{X_2}w_\varepsilon|^2\big)\,dx + \frac{1}{4}\int_\Omega\big(\varepsilon^2|\partial_{X_1}(u_\varepsilon - u_0 - w_\varepsilon)|^2 + |\nabla_{X_2}(u_\varepsilon - u_0 - w_\varepsilon)|^2\big)\,dx.$$

Going back to (2.17), we find that

$$\frac{3}{4}\int_\Omega \big(\varepsilon^2(\partial_{X_1}(u_\varepsilon - u_0 - w_\varepsilon))^2 + |\nabla_{X_2}(u_\varepsilon - u_0 - w_\varepsilon)|^2\big)\,dx$$
$$\le \int_{\Omega_-}\big(\varepsilon^2(\partial_{X_1}w_\varepsilon)^2 + |\nabla_{X_2}w_\varepsilon|^2\big)\,dx - \varepsilon^2\int_\Omega \partial_{X_1}u_0\,\partial_{X_1}(u_\varepsilon - u_0 - w_\varepsilon)\,dx.$$

Making the change of variable $X_1 \to \frac{1-X_1}{\varepsilon}$ in the first integral of the second line, we get

$$\frac{3}{4}\int_\Omega \big(\varepsilon^2(\partial_{X_1}(u_\varepsilon - u_0 - w_\varepsilon))^2 + |\nabla_{X_2}(u_\varepsilon - u_0 - w_\varepsilon)|^2\big)\,dx$$
$$\le \varepsilon\int_{\Omega_{2/\varepsilon}\setminus\Omega_{1/\varepsilon}} |\nabla w|^2\,dx - \varepsilon^2\int_\Omega \partial_{X_1}u_0\,\partial_{X_1}(u_\varepsilon - u_0 - w_\varepsilon)\,dx$$
$$\le \varepsilon\int_{S_{1/\varepsilon}} |\nabla w|^2\,dx - \varepsilon^2\int_\Omega \partial_{X_1}u_0\,\partial_{X_1}(u_\varepsilon - u_0 - w_\varepsilon)\,dx. \tag{2.18}$$

Combining (2.13) and (2.18) leads to the basic inequality (2.15). This completes the proof of the theorem.

2.2. Convergence results. As a first application of Theorem 2.3 we have

Theorem 2.4. *The solution* u_0 *is a strong limit of the sequence* $u_\varepsilon - w_\varepsilon$ *in* $H^1(\Omega)$ *and the following error estimate is valid*:

$$|u_\varepsilon - u_0 - w_\varepsilon|_{L^2(\Omega)}, |\nabla_{X_2}(u_\varepsilon - u_0 - w_\varepsilon)|_{L^2(\Omega)} = o(\varepsilon),$$
$$|\partial_{X_1}(u_\varepsilon - u_0 - w_\varepsilon)|_{L^2(\Omega)} = o(1).$$

Proof. Applying the Cauchy–Schwarz inequality to the last term of (2.15), we derive

$$\frac{3}{4}\int\limits_\Omega \big(\varepsilon^2(\partial_{X_1}(u_\varepsilon - u_0 - w_\varepsilon))^2 + |\nabla_{X_2}(u_\varepsilon - u_0 - w_\varepsilon)|^2\big)\,dx$$
$$\le Ce^{-\frac{\alpha}{\varepsilon}}\int\limits_{S_0}|\nabla w|^2\,dx + \varepsilon^2|\partial_{X_1}u_0|_{L^2(\Omega)}|\partial_{X_1}(u_\varepsilon - u_0 - w_\varepsilon)|_{L^2(\Omega)}.$$

Then by Young's inequality we get for some constant C

$$\varepsilon^2|\partial_{X_1}(u_\varepsilon - u_0 - w_\varepsilon)|^2_{L^2(\Omega)} + |\nabla_{X_2}(u_\varepsilon - u_0 - w_\varepsilon)|^2_{L^2(\Omega)} \le Ce^{-\frac{\alpha}{\varepsilon}}\int\limits_{S_0}|\nabla w|^2\,dx + C\varepsilon^2|\partial_{X_1}u_0|^2_{L^2(\Omega)}.$$

This estimate shows in particular that

$$|\nabla_{X_2}(u_\varepsilon - u_0 - w_\varepsilon)|_{L^2(\Omega)} = O(\varepsilon) \tag{2.19}$$

since $e^{-\frac{\alpha}{\varepsilon}} = o(\varepsilon^2)$. At the same time we have proved the boundedness of $|\partial_{X_1}(u_\varepsilon - u_0 - w_\varepsilon)|_{L^2(\Omega)}$. This allows us to extract a weakly convergent subsequence of $\partial_{X_1}(u_\varepsilon - u_0 - w_\varepsilon)$ in $L^2(\Omega)$ and according to (2.19) it follows that the whole sequence converges weakly to 0, i.e.

$$\partial_{X_1}(u_\varepsilon - u_0 - w_\varepsilon) \rightharpoonup 0 \qquad \text{in } L^2(\Omega).$$

Going back to (2.15), using the fact that $e^{-\frac{\alpha}{\varepsilon}} = o(\varepsilon^2)$ and the weak convergences above, we obtain

$$|\nabla_{X_2}(u_\varepsilon - u_0 - w_\varepsilon)|_{L^2(\Omega)} = o(\varepsilon), \qquad |\partial_{X_1}(u_\varepsilon - u_0 - w_\varepsilon)|_{L^2(\Omega)} = o(1).$$

Finally, using the Poincaré inequality in the direction X_2, with the help of the estimates above we complete the proof of the theorem.

We can improve the rate of convergence above if we assume more smoothness of f as in the following theorem.

Theorem 2.5. *Under the assumptions of Theorem* 2.3 *and if*

$$\partial^2_{X_1}u_0 \in L^2(\Omega) \qquad \textit{and} \qquad \partial_{X_1}u_0 = 0 \quad \textit{on } \{0\}\times\omega, \tag{2.20}$$

we have

$$|u_\varepsilon - u_0 - w_\varepsilon|_{L^2(\Omega)}, |\nabla_{X_2}(u_\varepsilon - u_0 - w_\varepsilon)|_{L^2(\Omega)} = O(\varepsilon^2),$$
$$|\partial_{X_1}(u_\varepsilon - u_0 - w_\varepsilon)|_{L^2(\Omega)} = O(\varepsilon)$$

as $\varepsilon \to 0$.

Remark 2.6. (i) The second hypothesis in (2.20) means that for a.e. $X_2 \in \omega$ we have

$$\partial_{X_1} u_0(0, X_2) = 0.$$

(ii) For instance, if f is smooth enough, we can show that the hypotheses

$$\partial^2_{X_1} f \in L^2(\Omega) \qquad \text{and} \qquad \partial_{X_1} f = 0 \quad \text{on } \{0\} \times \omega$$

imply (2.20) using the representation formula

$$u_0(x) = \int_\omega f(X_1, y) G(X_2, y)\, dy,$$

where G is the Green function (see [7]).

Proof of Theorem 2.5. Integrating by parts the last integral of (2.15), we get

$$\begin{aligned}
&\frac{3}{4} \int_\Omega \big(\varepsilon^2 (\partial_{X_1}(u_\varepsilon - u_0 - w_\varepsilon))^2 + |\nabla_{X_2}(u_\varepsilon - u_0 - w_\varepsilon)|^2\big)\, dx \\
&\qquad \le C e^{-\frac{\alpha}{\varepsilon}} \int_{S_0} |\nabla w|^2\, dx + \varepsilon^2 \int_\Omega \partial^2_{X_1} u_0 (u_\varepsilon - u_0 - w_\varepsilon)\, dx \\
&\qquad\quad + \varepsilon^2 \int_\omega \partial_{X_1} u_0(0, X_2)(u_\varepsilon - u_0 - w_\varepsilon)(0, X_2)\, dX_2 \\
&\qquad = C e^{-\frac{\alpha}{\varepsilon}} \int_{S_0} |\nabla w|^2\, dx + \varepsilon^2 \int_\Omega \partial^2_{X_1} u_0 (u_\varepsilon - u_0 - w_\varepsilon)\, dx
\end{aligned}$$

($u_\varepsilon - u_0 - w_\varepsilon \in V$ and $\partial_{X_1} u_0 = 0$ on $\{0\} \times \omega$). By the Cauchy–Schwarz and Young's inequalities it follows that

$$\begin{aligned}
&\frac{3}{4} \int_\Omega \big(\varepsilon^2 (\partial_{X_1}(u_\varepsilon - u_0 - w_\varepsilon))^2 + |\nabla_{X_2}(u_\varepsilon - u_0 - w_\varepsilon)|^2\big)\, dx \\
&\qquad \le C e^{-\frac{\alpha}{\varepsilon}} \int_{S_0} |\nabla w|^2\, dx + \varepsilon^2 |\partial^2_{X_1} u_0|_{L^2(\Omega)} |u_\varepsilon - u_0 - w_\varepsilon|_{L^2(\Omega)} \\
&\qquad \le C e^{-\frac{\alpha}{\varepsilon}} \int_{S_0} |\nabla w|^2\, dx + \mu \varepsilon^4 |\partial^2_{X_1} u_0|^2_{L^2(\Omega)} + \frac{1}{4\mu} |u_\varepsilon - u_0 - w_\varepsilon|^2_{L^2(\Omega)} \\
&\qquad \le C e^{-\frac{\alpha}{\varepsilon}} \int_{S_0} |\nabla w|^2\, dx + \mu \varepsilon^4 |\partial^2_{X_1} u_0|^2_{L^2(\Omega)} + \frac{C_\omega}{4\mu} |\nabla_{X_2}(u_\varepsilon - u_0 - w_\varepsilon)|^2_{L^2(\Omega)},
\end{aligned}$$

where C_ω is the Poincaré inequality constant. Choosing $\mu = C_\omega$ and since $e^{-\frac{\alpha}{\varepsilon}} = o(\varepsilon^4)$, we are ending up with

$$\varepsilon^2 \int_\Omega \big((\partial_{X_1}(u_\varepsilon - u_0 - w_\varepsilon))^2 + |\nabla_{X_2}(u_\varepsilon - u_0 - w_\varepsilon)|^2\big)\, dx \le C \varepsilon^4.$$

Applying the Poincaré inequality to $u_\varepsilon - u_0 - w_\varepsilon \in V$, we complete the proof of the theorem.

Thanks to Theorem 2.3, if we assume that f is independent of X_1, we get an exponential rate of convergence. This is the following theorem.

Theorem 2.7. *Under the assumptions above and if in addition f is independent of X_1, we have an exponential convergence of $u_\varepsilon - w_\varepsilon$ to u_0 in the whole domain Ω, i.e. there exist positive constants C and α independent of ε such that*

$$\int_\Omega |\nabla(u_\varepsilon - u_0 - w_\varepsilon)|^2\,dx \le Ce^{-\frac{\alpha}{\varepsilon}} \int_{S_0} |\nabla w|^2\,dx.$$

Proof. This is an immediate consequence of (2.15).

3. PROBLEMS IN DOMAINS BECOMING UNBOUNDED

Let Ω_ℓ^m be a bounded open subset of $\mathbb{R}^{m+p}$ defined by

$$\Omega_\ell^m = \begin{cases} (0,\ell)^m \times \omega & \text{if } \ell > 0, \\ (\ell,0)^m \times \omega & \text{if } \ell < 0, \end{cases}$$

where $m, p > 0$ are integers and ω is a bounded open subset of $\mathbb{R}^p$. For simplicity we drop the index 1 in Ω_ℓ^1 and Ω_1; i.e., to be consistent with our notation of Section 1, we set

$$\Omega_\ell^1 := \Omega_\ell, \qquad \Omega_1 := \Omega.$$

The points in $\mathbb{R}^{m+p}$ will be denoted by $x = (X_1, X_2) = (x_1, \dots, x_m, x_1', \dots, x_p')$ with

$$X_1 = (x_1, \dots, x_m), \qquad X_2 = (x_1', \dots, x_p').$$

With this notation we set

$$\nabla u = \big(\partial_{x_1} u, \dots, \partial_{x_m} u, \partial_{x_1'} u, \dots, \partial_{x_p'} u\big)^{\mathrm{T}} = \begin{pmatrix} \nabla_{X_1} u \\ \nabla_{X_2} u \end{pmatrix},$$

where

$$\nabla_{X_1} u = \big(\partial_{x_1} u, \dots, \partial_{x_m} u\big)^{\mathrm{T}}, \qquad \nabla_{X_2} u = \big(\partial_{x_1'} u, \dots, \partial_{x_p'} u\big)^{\mathrm{T}}.$$

We divide the boundary Γ_ℓ^m of Ω_ℓ^m into two parts $\mathcal{D}_\ell^m$ and $\mathcal{N}_\ell^m$ such that

$$\mathcal{N}_\ell^m = \bigcup_{i=1,\dots,m} \{x_i = 0\} \cap \overline{\Omega}_\ell^m, \qquad \mathcal{D}_\ell^m = \Gamma_\ell^m \setminus \mathcal{N}_\ell^m.$$

We also set

$$\mathcal{N}_\ell^1 := \mathcal{N}_\ell, \qquad \mathcal{N}_1 := \mathcal{N}, \qquad \mathcal{D}_\ell^1 := \mathcal{D}_\ell, \qquad \mathcal{D}_1 := \mathcal{D}.$$

In this section we deal with the asymptotic behaviour, when $\ell \to +\infty$, of the solution u_ℓ^m to the Laplace boundary value problem

$$\begin{cases} -\Delta u_\ell^m = f & \text{in } \Omega_\ell^m, \\ u_\ell^m = 0 & \text{on } \mathcal{D}_\ell^m, \\ \partial_\eta u_\ell^m = 0 & \text{on } \mathcal{N}_\ell^m, \end{cases} \tag{3.1}$$

where f is independent of X_1, i.e.

$$f(x) = f(X_2) \in L^2(\omega).$$

Here ∂_η denotes the derivative along the outward normal to the boundary Γ_ℓ^m. The existence of a weak solution u_ℓ^m of (3.1) is ensured by the Lax–Milgram theorem in the space

$$V_\ell^m = \big\{v \in H^1(\Omega_\ell^m) \mid v = 0 \text{ on } \mathcal{D}_\ell^m\big\}.$$

Thanks to Lemma 3.1 below and [6, Theorem 1.1] it follows that u_ℓ^m converges towards the solution u_0 to (1.2), as $\ell \to +\infty$, in $H^1(\Omega_{\ell_0}^m)$ where $\ell_0 < \ell$ is a constant. More precisely, we have in fact

$$\int\limits_{\Omega_{\ell/2}^m} |\nabla(u_\ell^m - u_0)|^2\,dx \le Ce^{-\alpha\ell}, \tag{3.2}$$

where C and α are positive constants independent of ℓ. In this section we are interested in the asymptotic behaviour of u_ℓ^m in the neighbourhood of $\mathcal{D}_\ell^m$. We start with the case $m = 1$ in the following subsection and next we consider the general case.

3.1. Domains becoming unbounded in one direction.

3.1.1. *Mixed boundary value problems.* We consider here the special case $m = 1$. By making the change of variable

$$X_1 \to \frac{1}{\varepsilon}X_1, \tag{3.3}$$

where $\varepsilon = \frac{1}{\ell}$, we deduce that u_ℓ^1 is a solution of (3.1) if and only if the function

$$\Omega \to \mathbb{R}, \qquad x \to u_\ell^1\left(\frac{1}{\varepsilon}X_1, X_2\right)$$

is a solution of (1.7). Then we set

$$w_\ell(X_1, X_2) := w_\varepsilon\left(\frac{1}{\ell}X_1, X_2\right) = w(\ell - X_1, X_2),$$

where w is a solution of (2.10). The following theorem is a direct consequence of Theorem 2.7 and (3.3).

Theorem 3.1. *We have the convergence $u_\ell^1 - w_\ell \to u_0$ on the whole domain Ω_ℓ, i.e. in $H_0^1(\Omega_\ell)$, and the following estimate is true*:

$$\int\limits_{\Omega_\ell} |\nabla(u_\ell^1 - u_0 - w_\ell)|^2\,dx \le Ce^{-\alpha\ell} \int\limits_{S_0} |\nabla w|^2\,dx, \tag{3.4}$$

where C and α are positive constants independent of ℓ.

Remark 3.2. Estimate (3.2) is a corollary of Theorem 3.1. Indeed, we have

$$\begin{aligned}\int\limits_{\Omega_{\ell/2}} |\nabla(u_\ell^1 - u_0)|^2\,dx &\le 2\int\limits_{\Omega_{\ell/2}} |\nabla(u_\ell^1 - u_0 - w_\ell)|^2\,dx + 2\int\limits_{\Omega_{\ell/2}} |\nabla w_\ell|^2\,dx \\ &\le Ce^{-\alpha\ell}\int\limits_{S_0} |\nabla w|^2\,dx + 2\int\limits_{S_{\ell/2}} |\nabla w|^2\,dx,\end{aligned}$$

by a change of variable. The last integral converges towards 0 at an exponential rate by Lemma 2.2, which shows (3.2).

Remark 3.3. For any $a > 0$,

$$\int\limits_{\Omega_{\ell-a}} |\nabla(u_\ell^1 - u_0)|^2\,dx \nrightarrow 0.$$

To show this, one notices that

$$\int_{\Omega_{\ell-a}} |\nabla(u_\ell^1 - u_0)|^2\,dx = \int_{\Omega_{\ell-a}} |\nabla(u_\ell^1 - u_0 - w_\ell) + \nabla w_\ell|^2\,dx$$
$$\geq \frac{1}{2}\int_{\Omega_{\ell-a}} |\nabla w_\ell|^2\,dx - \int_{\Omega_{\ell-a}} |\nabla(u_\ell^1 - u_0 - w_\ell)|^2\,dx$$
$$= \frac{1}{2}\int_{S_a} |\nabla w|^2\,dx + o(1).$$

Since w is a harmonic function, one has for every a

$$\int_{S_a} |\nabla w|^2\,dx > 0.$$

Then the convergence of u_ℓ^1 towards u_0 may not occur in $H^1(\Omega)$.

3.1.2. *Dirichlet boundary value problems.* Let us consider in $\mathcal{O}_\ell = (-\ell,\ell)\times\omega$ the Dirichlet boundary value problem

$$\begin{cases} -\Delta U_\ell = f & \text{in } \mathcal{O}_\ell, \\ U_\ell = 0 & \text{on } \partial\mathcal{O}_\ell. \end{cases}$$

It is clear that U_ℓ is a unique function of $H_0^1(\mathcal{O}_\ell)$ satisfying

$$\int_{\mathcal{O}_\ell} \nabla U_\ell\cdot\nabla v\,dx = \int_{\mathcal{O}_\ell} fv\,dx \qquad \forall v\in H_0^1(\mathcal{O}_\ell). \tag{3.5}$$

The following lemma summarizes some useful properties of the solution U_ℓ.

Lemma 3.4. *Under the previous assumptions, we have*

- $U_\ell(-X_1,X_2) = U_\ell(X_1,X_2)$ *for a.e.* $x\in\mathcal{O}_\ell$;
- *the restriction of* U_ℓ *to* Ω_ℓ *is a unique solution to*

$$\begin{cases} U_\ell\in V_\ell, \\ \displaystyle\int_{\Omega_\ell} \nabla U_\ell\cdot\nabla v\,dx = \int_{\Omega_\ell} fv\,dx \qquad \forall\, v\in V_\ell. \end{cases}$$

Proof. For $v\in H_0^1(\mathcal{O}_\ell)$ denote by $\widetilde{v}$ the function defined by $\widetilde{v}(X_1,X_2) = v(-X_1,X_2)$. It is clear that $\widetilde{v}\in H_0^1(\mathcal{O}_\ell)$, and if we make the change of variable $\overline{X_1} = -X_1$ in (3.5), we derive

$$\int_{\mathcal{O}_\ell} \nabla\widetilde{U}_\ell\cdot\nabla v\,dx = \int_{\mathcal{O}_\ell} \nabla\widetilde{U}_\ell\cdot\nabla\widetilde{\widetilde{v}}\,dx$$
$$= \int_{\mathcal{O}_\ell} \big\{-\partial_{\overline{X_1}}U_\ell(-\partial_{\overline{X_1}}\widetilde{v}) + \nabla_{X_2}U_\ell\cdot\nabla_{X_2}\widetilde{v}\big\}(\overline{X_1},X_2)\,dx$$
$$= \int_{\mathcal{O}_\ell} \nabla U_\ell\cdot\nabla\widetilde{v}\,dx = \int_{\mathcal{O}_\ell} f\widetilde{v}\,dx = \int_{\mathcal{O}_\ell} fv\,dx,$$

since f is independent of X_1. This means that $\widetilde{U}_\ell$ is also a weak solution to (3.5) and by uniqueness of the solution we deduce the first point of the lemma. For $v \in V$ we can easily check that $\widehat{v}$ defined by (2.11) belongs to $H_0^1(\mathcal{O}_\ell)$. Moreover, we have

$$\int_{\mathcal{O}_\ell} \nabla U_\ell \cdot \nabla \widehat{v}\, dx = \int_{\Omega_\ell} \nabla U_\ell \cdot \nabla \widehat{v}\, dx + \int_{\Omega_{-\ell}} \nabla U_\ell \cdot \nabla \widehat{v}\, dx.$$

Thanks to the first point, the last integral can be written as ($\overline{X_1} = -X_1$)

$$\int_{\Omega_{-\ell}} \nabla U_\ell \cdot \nabla \widehat{v}\, dx = \int_{\Omega_{-\ell}} \big(-\partial_{X_1} U_\ell(-\overline{X_1}, X_2)\partial_{\overline{X_1}} v(\overline{X_1}, X_2) + \nabla_{X_2} U_\ell \cdot \nabla_{X_2} \widehat{v}\big)\, dx = \int_{\Omega_\ell} \nabla U_\ell \cdot \nabla v\, dx.$$

Thus we have for every $v \in V_\ell$

$$\int_{\mathcal{O}_\ell} \nabla U_\ell \cdot \nabla \widehat{v}\, dx = 2\int_{\Omega_\ell} \nabla U_\ell \cdot \nabla v\, dx. \tag{3.6}$$

Also by (3.5) we have

$$\int_{\mathcal{O}_\ell} \nabla U_\ell \cdot \nabla \widehat{v}\, dx = \int_{\mathcal{O}_\ell} f\widehat{v}\, dx = 2\int_{\Omega_\ell} fv\, dx. \tag{3.7}$$

Combining (3.6) and (3.7) shows the second point.

As a consequence of the second point of Lemma 3.4, it follows that

$$U_\ell = u_\ell^1 \qquad \text{on } \Omega_\ell.$$

Then, thanks to Theorem 3.1 and the first point of Lemma 3.4 we can state

Theorem 3.5. *There exist positive constants C and α independent of ℓ such that*

$$\int_{\mathcal{O}_\ell} |\nabla(U_\ell - u_0 - \widehat{w}_\ell)|^2\, dx \le Ce^{-\alpha\ell} \int_{S_0} |\nabla w|^2\, dx.$$

3.2. More general domains. For $m = 1$, we defined in the previous subsection a corrector $w_\ell^1 := w_\ell$ satisfying (3.4). In order to construct a corrector for $m = 2$, we introduce a function $w^2 \in H_0^1((0, +\infty) \times \Omega_\ell^1)$ as a solution to

$$\begin{cases} \Delta w^2 = 0 \quad \text{in } (0, +\infty) \times \Omega_\ell^1, \\ w^2 = -u_0 - w_\ell^1 \quad \text{on } \{0\} \times \Omega_\ell^1, \qquad w^2 = 0 \quad \text{on } (0, +\infty) \times \partial\Omega_\ell^1. \end{cases}$$

The existence of w^2 is ensured by the Lax–Milgram theorem. The corrector candidate in this case is $w_\ell^1 + w_\ell^2$ where w_ℓ^2 is given by

$$w_\ell^2(x_1, x_2, X_2) = w^2(\ell - x_1, x_2, X_2).$$

Instead of showing this only for the case $m = 2$, we construct by induction for $i = 2, \dots, m$ functions $w_\ell^i\colon S_0^i \to \mathbb{R}$ defined as follows. For a solution u to

$$\begin{cases} u \in H_0^1(S_0^i), \\ \displaystyle\int_{S_0^i} \nabla u \cdot \nabla v\, dx = \int_{S_0^i} \nabla \left[\rho(x_1)\left(u_0 + \sum_{j=1}^{i-1} w_\ell^j\right)\right] \cdot \nabla v\, dx \qquad \forall\, v \in H_0^1(S_0^i), \end{cases} \tag{3.8}$$

where $S_a^i = (a, +\infty) \times \Omega_\ell^{i-1}$ ($a \in \mathbb{R}$), we set

$$w^i(x_1, \dots, x_i, X_2) = u(x_1, \dots, x_i, X_2) - \rho(x_1)\left(u_0(X_2) + \sum_{j=1}^{i-1} w_\ell^j(x_{i-j+1}, \dots, x_i, X_2)\right) \tag{3.9}$$

and denote by w_ℓ^i the function defined as

$$w_\ell^i(x) = w^i(\ell - x_1, x_2, \dots, x_i, X_2).$$

Then we have

Theorem 3.6. *Under the assumptions above, the difference* $u_\ell^m - \sum_{j=1}^m w_\ell^j$ *converges towards* u_0 *on the whole domain* Ω_ℓ^m, *i.e. in* $H_0^1(\Omega_\ell^m)$, *and there exist positive constants* C *and* α *independent of* ℓ *such that*

$$\int_{\Omega_\ell^m} \left|\nabla\left(u_\ell^m - u_0 - \sum_{j=1}^m w_\ell^j\right)\right|^2 dx \le Ce^{-\alpha\ell}. \tag{3.10}$$

Proof. In order to check that $\sum_{j=1}^m w_\ell^j(x_{m-j+1}, \dots, x_m, X_2)$ is a corrector corresponding to problem (3.1) and satisfying (3.10), we will argue by induction. According to Theorem 2.4 the statement holds when $m = 1$; then we assume that $\sum_{j=1}^{m-1} w_\ell^j(x_{m-j+1}, \dots, x_m, X_2)$ is a corrector satisfying

$$\int_{\Omega_\ell^{m-1}} \left|\nabla\left(u_\ell^{m-1} - u_0 - \sum_{j=1}^{m-1} w_\ell^j\right)\right|^2 dx \le Ce^{-\alpha\ell}, \tag{3.11}$$

where C and α are some positive constants independent of ℓ. In the following we show the same estimate for m. Let us introduce a function $\overline{w}_\ell^m$ defined as below. For a solution $\overline{u}$ to

$$\begin{cases} \overline{u} \in H_0^1(S_0^m), \\ \displaystyle\int_{S_0^m} \nabla\overline{u} \cdot \nabla v\, dx = \int_{S_0^m} \nabla\big(\rho(x_1)u_\ell^{m-1}\big) \cdot \nabla v\, dx \qquad \forall\, v \in H_0^1(S_0^m), \end{cases} \tag{3.12}$$

we set

$$\overline{w}(x) = \overline{u}(x) - \rho(x_1)u_\ell^{m-1}(x_2, \dots, x_m, X_2) \tag{3.13}$$

($\overline{w}$ depends on ℓ) and denote by $\overline{w}_\ell^m$ the function defined as

$$\overline{w}_\ell^m(x) = \overline{w}(\ell - x_1, x_2, \dots, x_m, X_2). \tag{3.14}$$

Then we have

Lemma 3.7. *For any* $\ell > 0$, *there exist constants* $C > 0$ *and* $\alpha' > 0$ *such that*

$$\int_{\Omega_\ell^m} |\nabla(u_\ell^m - u_\ell^{m-1} - \overline{w}_\ell^m)|^2\, dx \le Ce^{-\alpha'\ell}. \tag{3.15}$$

Proof. Without loss of generality, we assume that $\ell > 1$. Arguing as in the previous section and replacing ω by Ω_ℓ^{m-1}, we can show an estimate similar to (3.4), i.e.

$$\int_{\Omega_\ell^m} |\nabla(u_\ell^m - u_\ell^{m-1} - \overline{w}_\ell^m)|^2\, dx \le Ce^{-\alpha\ell} \int_{S_0^m} |\nabla\overline{w}|^2\, dx. \tag{3.16}$$

(We use the fact that Ω_ℓ^{m-1} is bounded in the direction X_2 to get a Poincaré constant independent of ℓ.) We have now to estimate the last integral in (3.16). By using, in (3.12), $v = \overline{u}$ we obtain easily

$$|\nabla \overline{u}|^2_{L^2(S_0^m)} = \int\limits_{S_{\ell-1}^m \setminus S_\ell^m} \nabla\big(\rho(x_1)u_\ell^{m-1}\big) \cdot \nabla \overline{u}\, dx \le C|\nabla u_\ell^{m-1}|_{L^2(\Omega_\ell^{m-1})} |\nabla \overline{u}|_{L^2(S_0^m)},$$

whence

$$|\nabla \overline{u}|_{L^2(S_0^m)} \le C|\nabla u_\ell^{m-1}|_{L^2(\Omega_\ell^{m-1})}.$$

Then by (3.13) we derive

$$|\nabla \overline{w}|_{L^2(S_0^m)} \le |\nabla \overline{u}|_{L^2(S_0^m)} + |\nabla(\rho(x_1)u_\ell^{m-1})|_{L^2(S_0^m)} \le C|\nabla u_\ell^{m-1}|_{L^2(\Omega_\ell^{m-1})},$$

where C is independent of ℓ. Next, taking in the weak formulation of (3.1), written for $m-1$, $v = u_\ell^{m-1}$ yields

$$|\nabla u_\ell^{m-1}|_{L^2(\Omega_\ell^{m-1})} \le C|f|^2_{L^2(\Omega_\ell^{m-1})} = C\ell^{m-1}|f|^2_{L^2(\omega)},$$

since f is independent of X_1. Thus, it follows that

$$|\nabla \overline{w}|_{L^2(S_0^m)} \le C\ell^{m-1}|f|^2_{L^2(\omega)}. \tag{3.17}$$

Going back to (3.16), we have

$$\int\limits_{\Omega_\ell^m} |\nabla(u_\ell^m - u_\ell^{m-1} - \overline{w}_\ell^m)|^2\, dx \le C\ell^{m-1}|f|^2_{L^2(\omega)} e^{-\alpha\ell}.$$

Since $\ell \to +\infty$, there exist constants $0 < \alpha' < \alpha$ and $C > 0$ such that

$$\int\limits_{\Omega_\ell^m} |\nabla(u_\ell^m - u_\ell^{m-1} - \overline{w}_\ell^m)|^2\, dx \le Ce^{-\alpha'\ell}.$$

This completes the proof of the lemma.

We now return to the proof of the theorem. The integral in (3.10) can be estimated as

$$\int\limits_{\Omega_\ell^m} \left|\nabla\left(u_\ell^m - u_0 - \sum_{j=1}^m w_\ell^j\right)\right|^2 dx \le 3\int\limits_{\Omega_\ell^m} |\nabla(u_\ell^m - u_\ell^{m-1} - \overline{w}_\ell^m)|^2\, dx$$
$$+ 3\int\limits_{\Omega_\ell^m} \left|\nabla\left(u_\ell^{m-1} - \sum_{j=1}^{m-1} w_\ell^j - u_0\right)\right|^2 dx + 3\int\limits_{\Omega_\ell^m} |\nabla(\overline{w}_\ell^m - w_\ell^m)|^2\, dx.$$

The exponential convergences to 0 of the first and the second integral of the right-hand side are given by (3.15) and the induction hypothesis (3.11), respectively. Then it remains to show the same rate of convergence for the last integral to complete the proof. First, we estimate the difference between $\overline{w}$ and w^m, defined in (3.13) and (3.9), respectively, as

$$|\nabla(\overline{w} - w^m)|_{L^2(S_0^m)} \le |\nabla(\overline{u} - u)|_{L^2(S_0^m)} + \left|\nabla\left[\rho(x_1)\left(u_\ell^{m-1} - u_0 - \sum_{j=1}^{m-1} w_\ell^j\right)\right]\right|_{L^2(S_0^m)}. \tag{3.18}$$

We estimate the last term in the inequality above using the Poincaré inequality and the induction hypothesis (3.11); then we have

$$\left|\nabla\left[\rho(x_1)\left(u_\ell^{m-1}-u_0-\sum_{j=1}^{m-1}w_\ell^j\right)\right]\right|_{L^2(S_0^m)}\le C\left|\nabla\left(u_\ell^{m-1}-u_0-\sum_{j=1}^{m-1}w_\ell^j\right)\right|_{L^2(\Omega_\ell^{m-1})}$$
$$\le Ce^{-\alpha\ell}. \tag{3.19}$$

For the first term of the right-hand side of (3.18), we compare (3.12) and (3.8) for $i=m-1$ and, taking $v=\overline{u}-u\in H_0^1(S_0^m)$ as a test function, obtain

$$|\nabla(\overline{u}-u)|^2_{L^2(S_0^m)}\le\left|\nabla\left[\rho(x_1)\left(u_\ell^{m-1}-u_0-\sum_{j=1}^{m-1}w_\ell^j\right)\right]\right|_{L^2(S_0^m)}|\nabla(\overline{u}-u)|_{L^2(S_0^m)}.$$

Applying (3.19) here and in (3.18), we get

$$|\nabla(\overline{w}-w^m)|_{L^2(S_0^m)}\le Ce^{-\alpha\ell}. \tag{3.20}$$

Finally, the change of variable $x_1\to\ell-x_1$ and (3.20) lead to

$$|\nabla(\overline{w}_\ell^m-w_\ell^m)|_{L^2(\Omega_\ell^m)}=|\nabla(\overline{w}-w^m)|_{L^2((0,2\ell)\times\Omega_\ell^{m-1})}$$
$$\le|\nabla(\overline{w}-w^m)|_{L^2(S_0^m)}$$
$$\le Ce^{-\alpha\ell}.$$

This completes the proof.

Remark 3.8. As in Theorem 3.5, using symmetries, we can construct correctors for the Laplace equation defined in $(-\ell,\ell)^m\times\omega$ coupled with the homogeneous Dirichlet boundary conditions.

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