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EMBEDDINGS OF $\mathrm{SL}(2,\mathbb{Z})$ INTO THE CREMONA GROUP

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Abstract. Geometric and dynamic properties of embeddings of $\mathrm{SL}(2,\mathbb{Z})$ into the Cremona group are studied. Infinitely many nonconjugate embeddings that preserve the type (i.e., that send elliptic, parabolic and hyperbolic elements onto elements of the same type) are provided. The existence of infinitely many nonconjugate elliptic, parabolic and hyperbolic embeddings is also shown. In particular, a group G of automorphisms of a smooth surface S obtained by blowing up 10 points of the complex projective plane is given. The group G is isomorphic to $\mathrm{SL}(2,\mathbb{Z})$, preserves an elliptic curve and all its elements of infinite order are hyperbolic.

1. Introduction

Our article is motivated by the following result on the embeddings of the groups $\mathrm{SL}(n,\mathbb{Z})$ into the group $\mathrm{Bir}(\mathbb{P}^2)$ of birational maps of $\mathbb{P}^2(\mathbb{C})$: the group $\mathrm{SL}(n,\mathbb{Z})$ does not embed into $\mathrm{Bir}(\mathbb{P}^2)$ as soon as $n \geq 4$ and $\mathrm{SL}(3,\mathbb{Z})$ only embeds linearly (i.e., in $\mathrm{Aut}(\mathbb{P}^2) = \mathrm{PGL}(3,\mathbb{C})$) into $\mathrm{Bir}(\mathbb{P}^2)$ up to conjugacy [Des, Theorem 1.4].

It is thus natural to look at the embeddings of $\mathrm{SL}(2,\mathbb{Z})$ into $\mathrm{Bir}(\mathbb{P}^2)$. As $\mathrm{SL}(2,\mathbb{Z})$ has almost the structure of a free group, it admits many embeddings of different types into $\mathrm{Bir}(\mathbb{P}^2)$, and it is not reasonable to look for a classification of *all* embeddings. We thus focus on embeddings having certain geometric properties; among them the most natural ones are the embeddings which *preserve the type* evoked by Favre in [Fav, Question 4].

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The elements of $\mathrm{SL}(2,\mathbb{Z})$ are classified into elliptic, parabolic and hyperbolic elements, with respect to their action on the hyperbolic upper-plane (or similarly to their trace; see Section 2.1). The Cremona group $\mathrm{Bir}(\mathbb{P}^2)$ naturally acts on a hyperbolic space of infinite dimension (see [Man, Can2]), so there is a notion of elliptic, parabolic and hyperbolic elements in this group; this classification can also be deduced from the growth rate of degrees of iterates (see [DiFa] and Section 2.3). Note that some authors prefer the term loxodromic elements instead of hyperbolic elements (see, for example, [And, Prop. 2.16]). A morphism from $\mathrm{SL}(2,\mathbb{Z})$ to $\mathrm{Bir}(\mathbb{P}^2)$ *preserves the type* if it sends elliptic, parabolic and hyperbolic elements of $\mathrm{SL}(2,\mathbb{Z})$ to elements of $\mathrm{Bir}(\mathbb{P}^2)$ of the same type. Up to now, the only known example is the classical embedding $\theta_s\colon \mathrm{SL}(2,\mathbb{Z}) \to \mathrm{Bir}(\mathbb{P}^2)$, which associates to a matrix $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ the birational map $\theta_s(M)$, given in affine coordinates by $(x,y) \dashrightarrow (x^a y^b, x^c y^d)$ (or written simply $(x^a y^b, x^c y^d)$). In this article, we provide infinitely many nonconjugate embeddings that preserve the type (Theorem 1 below).

Recall that the group $\mathrm{SL}(2,\mathbb{Z})$ is generated by the elements R and S given by

$$R = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \qquad \text{and} \qquad S = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Theorem 1 (see Section 3.1). *Let ε be a real positive number, and set*

$$\theta_\varepsilon(S) = (y, -x), \qquad \theta_\varepsilon(R) = \left(\frac{x+\varepsilon y}{\varepsilon + xy}, \varepsilon y\right).$$

Then θ_ε is an embedding of $\mathrm{SL}(2,\mathbb{Z})$ *into the Cremona group preserving the type.*

Furthermore, if ε and ε' are two real positive numbers such that $\varepsilon\varepsilon' \neq 1$, then $\theta_\varepsilon(\mathrm{SL}(2,\mathbb{Z}))$ and $\theta_{\varepsilon'}(\mathrm{SL}(2,\mathbb{Z}))$ are not conjugate in $\mathrm{Bir}(\mathbb{P}^2)$.

The standard embedding θ_s is conjugate to θ_1.

This family of embeddings is a first step in the classification of all embeddings of $\mathrm{SL}(2,\mathbb{Z})$ preserving the type. We do not know if other embeddings exist (except one special embedding θ_- described in Section 3.1 which is a "twist" of the standard embedding θ_s defined by: $\theta_-(S) = \theta_s(S) = (y, 1/x)$ and $\theta_-(R) = (xy, -y) \neq \theta_s(R) = (xy, y)$), in particular, if it is possible to find an embedding where the parabolic elements act by preserving elliptic fibrations.

Question 1.1. Does there exist an embedding of $\mathrm{SL}(2,\mathbb{Z})$ into $\mathrm{Bir}(\mathbb{P}^2)$ that preserves the type and which is not conjugate to θ_- or to some θ_ε?

The last two assertions of Theorem 1 yield to the following question:

Question 1.2. Is the embedding θ_- rigid, i.e., not extendable to a one parameter family of nonconjugate embeddings?

Note that some morphisms $\mathrm{SL}(2,\mathbb{Z}) \to \mathrm{Bir}(\mathbb{P}^2)$ preserving the type have been described ([Fav, p. 9], [CaLo] and [Gol]), but that these are not embedding, the central involution acting trivially. See Section 3.1 for more details.

One can also consider elliptic, parabolic and hyperbolic embeddings of $\mathrm{SL}(2,\mathbb{Z})$ into $\mathrm{Bir}(\mathbb{P}^2)$. An embedding θ of $\mathrm{SL}(2,\mathbb{Z})$ into the Cremona group is said to be *elliptic* if each element of $\mathrm{im}\,\theta$ is elliptic; θ is *parabolic* (respectively *hyperbolic*) if each element of infinite order of $\mathrm{im}\,\theta$ is parabolic (respectively hyperbolic).

In Sections 3.2, 3.3 and 3.4, we prove the existence of an infinite number of nonconjugate elliptic, parabolic and hyperbolic embeddings (see Propositions 3.7, 3.8, 3.9 and Corollary 3.11). It is possible to find many other such embeddings; we only give a simple way to construct infinitely many of each family.

One can then ask if it is possible to find an embedding of $\mathrm{SL}(2,\mathbb{Z})$ into the Cremona group which is *regularisable*, i.e., which comes from an embedding into the group of automorphisms of a projective rational surface. It is easy to construct elliptic embeddings which are regularisable (see Section 3.2). In Section 4, we give a way to construct infinitely many hyperbolic embeddings of $\mathrm{SL}(2,\mathbb{Z})$ into the Cremona group which are regularisable, and each of the groups constructed, moreover, preserves an elliptic curve (one fixing it pointwise). The existence of regularisable embeddings which preserve the type is still open (and should contain parabolic elements with quadratic growth of degree).

Note that the existence of hyperbolic automorphisms preserving an elliptic curve was not clear. In [Pan, Theorem 1.1], it was proved that a curve preserved by an hyperbolic element of $\mathrm{Bir}(\mathbb{P}^2)$ has geometric genus 0 or 1; examples of genus 0 (easy to create by blowing up) were provided, and the existence of genus 1 curves invariant was raised (see [Pan, p. 443]). The related question of the existence of curves of arithmetic genus 1 preserved by hyperbolic automorphisms of rational surfaces was also raised two years after in [DFS, p. 2987]. In [McM], the author constructs hyperbolic automorphisms of rational surfaces which correspond to Coxeter elements (any hyperbolic automorphism of a rational surface corresponds to an element of the Weyl group associated to the surface), that preserve a cuspidal (resp. nodal) curve. However, a general automorphism of a rational surface corresponding to a Coxeter element is hyperbolic but does not preserve any curve ([BeKi]).

The following statement yields the existence of a group of automorphisms preserving a (smooth) elliptic curve such that every nonperiodic element is hyperbolic. This is also possible with free groups (see [Can1, Remark 3.2] and [Bla1]), but the construction is harder with more complicated groups like $\mathrm{SL}(2,\mathbb{Z})$. The method that we describe in Section 4 should be useful to create other groups generated by elements of finite order.

Theorem 2. *There exist hyperbolic embeddings $\theta_{h,1}, \theta_{h,2}, \theta_{h,3}$ of $\mathrm{SL}(2,\mathbb{Z})$ into $\mathrm{Bir}(\mathbb{P}^2)$ such that:*

- *for each i, the group $\theta_{h,i}$ preserves a smooth cubic curve $\Gamma \subset \mathbb{P}^2$;*
- *the action of $\theta_{h,1}$ on Γ is trivial, the action of $\theta_{h,2}$ on Γ is generated by a translation of order 3 and the action of $\theta_{h,3}$ on Γ is generated by an automorphism of order 3 with fixed points;*
- *for $i = 1, 2, 3$, the blow up $X_i \to \mathbb{P}^2$ of respectively $12, 10, 10$ points of Γ conjugates $\theta_{h,i}(\mathrm{SL}(2,\mathbb{Z}))$ to a subgroup of automorphisms of X_i. The strict transform $\widetilde{\Gamma}$ of Γ on X_i is the only invariant curve; in particular the orbit of any element of $X_i \backslash \widetilde{\Gamma}$ is either finite or dense in the Zariski topology.*

Moreover, in cases $i = 1, 2$, we can choose Γ to be any smooth cubic curve, and this yields infinitely many hyperbolic embeddings of $\mathrm{SL}(2,\mathbb{Z})$ *into* $\mathrm{Bir}(\mathbb{P}^2)$*, up to conjugacy.*

Remark 1.3. In $\theta_{h,1}$, $\theta_{h,2}$, $\theta_{h,3}$, the letter h is not a parameter but only means "hyperbolic", to distinguish them from the other embeddings θ_s, θ_- and $\{\theta_\varepsilon\}_{\varepsilon\in\mathbb{R}}$, defined above.

It could be interesting to study more precisely the orbits of the action of the above groups, in particular to answer the following questions:

Question 1.4. Are the typical orbits of $\theta_{h,i}$ dense in the transcendental topology?

Question 1.5. Are there some finite orbits in $X_i\backslash\widetilde{\Gamma}$?

We finish this introduction by mentioning related results.

The statement of [Des, Theorem 1.4] for $\mathrm{SL}(3,\mathbb{Z})$ was generalised in [Can2], where it is proven that any finitely generated group having Kazhdan's property (T) only embeds linearly into $\mathrm{Bir}(\mathbb{P}^2)$ (up to conjugation).

Let us also mention [CaLa, Theorem A], which says that if a lattice Γ of a simple Lie group G embeds into the group $\mathrm{Aut}(\mathbb{C}^2)$, then G is isomorphic to $\mathrm{PSO}(1,n)$ or $\mathrm{PSU}(1,n)$ for some n. If the image of the embedding is not conjugate to a subgroup of the affine group, the only possibility is $G \simeq \mathrm{PSO}(1,2) \simeq \mathrm{PSL}(2,\mathbb{R})$, this latter case being intensively studied in [CaLa].

Note that our techniques heavily use the special structure of $\mathrm{SL}(2,\mathbb{Z})$, and one could ask similar questions for any lattice of $\mathrm{GL}(2,\mathbb{R})$ or $\mathrm{PGL}(2,\mathbb{R})$; the behaviour and results could be very different.

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2. Some reminders on $\mathrm{SL}(2,\mathbb{Z})$ and $\mathrm{Bir}(\mathbb{P}^2)$

2.1. About $\mathrm{SL}(2,\mathbb{Z})$

The division algorithm implies that the group $\mathrm{SL}(2,\mathbb{Z})$ is generated by the elements R and S given by

$$R = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \qquad \text{and} \qquad S = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Remark that R is of infinite order and S of order 4. The square of S generates the center of $\mathrm{SL}(2,\mathbb{Z})$. Moreover,

$$RS = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix} \qquad \text{and} \qquad SR = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$$

are conjugate by S and both have order 3.

A presentation of $\mathrm{SL}(2,\mathbb{Z})$ is given by

$$\langle R,\, S \mid S^4 = (RS)^3 = 1,\ S^2(RS) = (RS)S^2\rangle$$

(see, for example, [New, Chap. 8]). This implies that the quotient of $\mathrm{SL}(2,\mathbb{Z})$ by its center is a free product of $\mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}/3\mathbb{Z}$ generated by the classes $[S]$ of S and $[RS]$ of RS

$$\mathrm{PSL}(2,\mathbb{Z}) = \langle [S],\, [RS] \mid [S]^2 = [RS]^3 = 1\rangle.$$

2.2. Dynamic of elements of SL(2, ℤ)

Recall that the group $\mathrm{SL}(2,\mathbb{R})$ acts on the upper half plane

$$\mathbb{H} = \{x + \mathbf{i}y \in \mathbb{C} \mid x, y \in \mathbb{R}, y > 0\}$$

by Möbius transformations:

$$\mathrm{SL}(2,\mathbb{R}) \times \mathbb{H} \to \mathbb{H}, \qquad \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}, z\right) \mapsto \frac{az+b}{cz+d}\,.$$

The hyperbolic structure of $\mathbb{H}$ being preserved, this yields to a natural notion of *elliptic*, *parabolic*, and *hyperbolic* elements of $\mathrm{SL}(2,\mathbb{R})$, and thus to elements of $\mathrm{SL}(2,\mathbb{Z})$ (as in [Ive, II.8]).

If M is an element of $\mathrm{SL}(2,\mathbb{Z})$, we can be more precise and check the following easy observations:

- M is *elliptic* if and only if M has finite order;
- M is *parabolic* (respectively *hyperbolic*) if and only if M has infinite order and its trace is ± 2 (respectively $\neq \pm 2$).

Up to conjugacy the elliptic elements of $\mathrm{SL}(2,\mathbb{Z})$ are

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix};$$

in particular, an element of finite order is of order 2, 3, 4 or 6.

A parabolic element of $\mathrm{SL}(2,\mathbb{Z})$ is up to conjugacy one of the following:

$$\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} -1 & a \\ 0 & -1 \end{bmatrix}, \quad a \in \mathbb{Z}.$$

2.3. Cremona group and dynamic of its elements

Let us recall the following classical definitions:

Definition 2.1. A *rational map* of the projective plane into itself is a map of the following type:

$$f\colon \mathbb{P}^2(\mathbb{C}) \dashrightarrow \mathbb{P}^2(\mathbb{C}), \qquad (x:y:z) \dashrightarrow (f_0(x,y,z) : f_1(x,y,z) : f_2(x,y,z)),$$

where the f_i's are homogeneous polynomials of the same degree without common factor. The *degree* of f is by definition: $\deg f = \deg f_i$. A *birational map* f is a rational map that admits a rational inverse. We denote by $\mathrm{Bir}(\mathbb{P}^2)$ the group of birational maps of the projective plane into itself; $\mathrm{Bir}(\mathbb{P}^2)$ is also called the *Cremona group*.

The degree is not a birational invariant; if f and g are in $\mathrm{Bir}(\mathbb{P}^2)$, then in general $\deg(gfg^{-1}) \neq \deg f$. Nevertheless, there exist two strictly positive constants $a, b \in \mathbb{R}$ such that for all n the following holds:

$$a \deg f^n \leq \deg(gf^ng^{-1}) \leq b \deg f^n.$$

In other words, the degree growth is a birational invariant; so we introduce the following notion ([Fri], [RuSh]):

Definition 2.2. Let f be a birational map. The *first dynamical degree* of f is defined by

$$\lambda(f) = \lim(\deg f^n)^{1/n}.$$

There is a classification of birational maps of $\mathbb{P}^2$ up to birational conjugation.

Theorem 2.3 ([Giz], [DiFa]). *Let f be an element of* $\mathrm{Bir}(\mathbb{P}^2)$. *Up to birational conjugation, exactly one of the following holds:*

- *The sequence* $(\deg f^n)_{n\in\mathbb{N}}$ *is bounded, f is an automorphism on some projective rational surface and an iterate of f is an automorphism isotopic to the identity;*
- *the sequence* $(\deg f^n)_{n\in\mathbb{N}}$ *grows linearly, and f preserves a rational fibration; in this case f is not an automorphism on a projective surface;*
- *the sequence* $(\deg f^n)_{n\in\mathbb{N}}$ *grows quadratically, and and f is an automorphism preserving an elliptic fibration;*
- *the sequence* $(\deg f^n)_{n\in\mathbb{N}}$ *grows exponentially.*

In the second and third case, the invariant fibration is unique. In the first three cases $\lambda(f)$ is equal to 1, *in the last case $\lambda(f)$ is strictly greater than* 1.

Definition 2.4. Let f be a birational map of $\mathbb{P}^2$.

If the sequence $(\deg f^n)_{n\in\mathbb{N}}$ is bounded, f is said to be *elliptic.*

When $(\deg f^n)_{n\in\mathbb{N}}$ grows linearly or quadratically, we say that f is *parabolic.*

If $\lambda(f) > 1$, then f is an *hyperbolic map.*

As we said, the Cremona group acts naturally on an hyperbolic space of infinite dimension ([Man], [Can2]); we can say that a birational map is elliptic, resp. parabolic, resp. hyperbolic, if the corresponding isometry is elliptic, resp. parabolic, resp. hyperbolic ([GhHa, Chap. 8, §2]). This definition coincides with the previous one ([Can2]).

Examples 2.5. Any automorphism of $\mathbb{P}^2$ or of an Hirzebruch surface $\mathbb{F}_n$ and any birational map of finite order is elliptic.

The map $(x : y : z) \dashrightarrow (xy : yz : z^2)$ is parabolic.

A Hénon map (automorphism of $\mathbb{C}^2$)

$$(x, y) \mapsto (y, P(y) - \delta x), \qquad \delta \in \mathbb{C}^*,\ P \in \mathbb{C}[y],\ \deg P \geq 2$$

extends to a hyperbolic birational map of $\mathbb{P}^2$, of dynamical degree $\deg P$.

Definition 2.6. Let $\theta\colon \mathrm{SL}(2,\mathbb{Z}) \to \mathrm{Bir}(\mathbb{P}^2)$ be an embedding of $\mathrm{SL}(2,\mathbb{Z})$ into the Cremona group.

We say that θ *preserves the type* if θ sends elliptic (respectively parabolic, respectively hyperbolic) elements onto elliptic (respectively parabolic, respectively hyperbolic) maps.

We say that θ is *elliptic* if each element of $\mathrm{im}\,\theta$ is elliptic.

The morphism θ is *parabolic* (respectively *hyperbolic*) if each element of infinite order of $\mathrm{im}\,\theta$ is parabolic (respectively hyperbolic).

2.4. The central involution of $\mathrm{SL}(2,\mathbb{Z})$ and its image into $\mathrm{Bir}(\mathbb{P}^2)$

The element $S^2 \in \mathrm{SL}(2,\mathbb{Z})$ is an involution; therefore its image by any embedding $\theta\colon \mathrm{SL}(2,\mathbb{Z}) \to \mathrm{Bir}(\mathbb{P}^2)$ is a birational involution. As was proved by Bertini, we have the following classification:

Theorem 2.7 ([Ber])**.** *An element of order* 2 *of the Cremona group is up to conjugacy one of the following:*

- *an automorphism of* $\mathbb{P}^2$*;*
- *a de Jonquières involution* ι_{dJ} *of degree* $\nu \geq 2$*;*
- *a Bertini involution* ι_B*;*
- *a Geiser involution* ι_G*.*

Bayle and Beauville showed that the conjugacy classes of involutions in $\mathrm{Bir}(\mathbb{P}^2)$ are determined by the birational type of the curves of fixed points of positive genus ([BaBe]). More precisely, the set of conjugacy classes is parametrised by a disconnected algebraic variety whose connected components are respectively

- the moduli spaces of hyperelliptic curves of genus g (de Jonquières involutions);
- the moduli space of canonical curves of genus 3 (Geiser involutions);
- the moduli space of canonical curves of genus 4 with vanishing theta characteristic, isomorphic to a nonsingular intersection of a cubic surface and a quadratic cone in $\mathbb{P}^3(\mathbb{C})$ (Bertini involutions).

The image of S^2 can be neither a Geiser involution, nor a Bertini involution; more precisely, we have the following:

Lemma 2.8. *Let* θ *be an embedding of* $\mathrm{SL}(2,\mathbb{Z})$ *into the Cremona group. Up to birational conjugation, one of the following holds:*

- *The involution* $\theta(S^2)$ *is an automorphism of* $\mathbb{P}^2$*;*
- *the map* $\theta(S^2)$ *is a de Jonquières involution of degree* 3 *fixing* (*pointwise*) *an elliptic curve.*

Remark 2.9. The first case is satisfied by the examples of Sections 3.1, 3.2, and 3.3. The second case is also possible, for any elliptic curve (see Section 4).

Proof. Since S^2 commutes with $\mathrm{SL}(2,\mathbb{Z})$ the group $\mathrm{G} = \theta(\mathrm{SL}(2,\mathbb{Z}))$ is contained in the centraliser of the involution S^2. If $\theta(S^2)$ is a Bertini or Geiser involution, the centraliser of $\theta(S^2)$ is finite ([BPV2, Cor. 2.3.6]); as a consequence $\theta(S^2)$ is a de Jonquières involution.

Assume that $\theta(S^2)$ is not linearisable; then $\theta(S^2)$ fixes (pointwise) a unique irreducible curve Γ of genus ≥ 1. The group G preserves Γ and the action of G on Γ gives the exact sequence

$$1 \to \mathrm{G}' \to \mathrm{G} \to \mathrm{H} \to 1$$

where H is a subgroup of $\mathrm{Aut}(\Gamma)$, and G' contains $\theta(S^2)$ and fixes Γ. Since the genus of Γ is positive, H cannot be equal to $\mathrm{G}/\langle\theta(S^2)\rangle$, a free product of $\mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}/3\mathbb{Z}$. This implies that the normal subgroup G' of G strictly contains $\langle\theta(S^2)\rangle$ and thus that it is infinite and not abelian. In particular, the group of birational maps fixing (pointwise) Γ is infinite, and not abelian, thus Γ is of genus 1 (see [BPV1, Theorem 1.5]). □

3. Embeddings preserving the type and elliptic, parabolic embeddings

3.1. Embeddings preserving the type

Henceforth we will often denote by $(f_1(x,y,z):f_2(x,y,z):f_3(x,y,z))$ the map

$$(x:y:z) \dashrightarrow (f_1(x,y,z):f_2(x,y,z):f_3(x,y,z))$$

and by $(p(x,y),q(x,y))$ the birational map

$$(x,y) \dashrightarrow (p(x,y),q(x,y))$$

of $\mathbb{C}^2$.

Let us begin this section by a property satisfied by all embeddings of

$$\mathrm{SL}(2,\mathbb{Z}) \to \mathrm{Bir}(\mathbb{P}^2)$$

that preserve the type.

Lemma 3.1. *Let $\theta\colon \mathrm{SL}(2,\mathbb{Z}) \to \mathrm{Bir}(\mathbb{P}^2)$ be an embedding that preserves the type. Either for all parabolic matrices M, $\theta(M)$ preserves a unique rational fibration, or for all parabolic matrices M, $\theta(M)$ preserves a unique elliptic fibration.*

Proof. Let us recall that a parabolic element of $\mathrm{SL}(2,\mathbb{Z})$ is up to conjugacy one of the following:

$$T_a^+ = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}, \qquad T_a^- = \begin{bmatrix} -1 & a \\ 0 & -1 \end{bmatrix}, \quad a \in \mathbb{Z}.$$

For any $a \neq 0$, the image $\theta(T_a^+)$ of T_a^+ preserves a unique fibration on $\mathbb{P}^2$. Denote by $\mathcal{F}$ the fibration preserved by T_1^+, given by $F\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$. For any $a \neq 0$, T_a^+ and T_a^- commute with T_1^+ so the $\theta(T_a^+)$'s and the $\theta(T_a^-)$'s preserve the fibration $\mathcal{F}$ and $\mathcal{F}$ is the only fibration invariant by these elements.

Let M be a parabolic matrix. On the one hand M is conjugate to T_a^+ or T_a^- for some a via a matrix N_M and on the other hand parabolic maps preserve a unique fibration; thus $\theta(M)$ preserves the fibration given by $F\theta(N_M)^{-1}$. In particular, if F defines a rational (respectively elliptic) fibration, then $F\theta(N_M)^{-1}$ defines a rational (respectively elliptic) one. □

The standard embedding θ_s. The classical embedding

$$\theta_s\colon \mathrm{SL}(2,\mathbb{Z}) \to \mathrm{Bir}(\mathbb{P}^2), \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto (x^a y^b, x^c y^d)$$

preserves the type (see, for example, [Lin, Theorem 5.1]).

For any $M \in \mathrm{SL}(2,\mathbb{Z})$, if M is elliptic, $\theta_s(M)$ is, up to conjugacy, one of the following birational maps of finite order:

$$\left(\frac{1}{x},\frac{1}{y}\right), \qquad \left(y,\frac{1}{xy}\right), \qquad \left(y,\frac{1}{x}\right), \qquad \left(\frac{1}{y},x\right), \qquad \left(\frac{1}{x},xy\right).$$

If M is parabolic, $\theta_s(M^n)$ is, up to conjugacy, (xy^{na}, y), or $(y^{na}/x, 1/y)$ with a in $\mathbb{Z}$ so $\theta_s(M)$ is parabolic. If M is hyperbolic, M has two real eigenvalues μ and μ^{-1} such that $|\mu|^{-1} < 1 < |\mu|$ and $\lambda(\theta_s(M)) = |\mu| > 1$ and $\theta_s(M)$ is hyperbolic.

In [Fav, p. 9], a construction of a morphism $\mathrm{SL}(2,\mathbb{Z}) \to \mathrm{Bir}(\mathbb{P}^2)$ preserving the type was given, inspired from [CaLo] and [Gol]: the quotient of $\mathbb{P}^1 \times \mathbb{P}^1$ by the involution $(x, y) \mapsto (1/x, 1/y)$ is a rational (singular) cubic surface $C \subset \mathbb{P}^3$, called a *Cayley cubic surface.* Explicitly, we can assume (by a good choice of coordinates) that

$$C = \{(W : X : Y : Z) \in \mathbb{P}^3 \mid XYZ + WYZ + WXZ + WXY = 0\}$$

and that the quotient is given by

$$\mathbb{P}^1 \times \mathbb{P}^1 \to C,$$

$$(x, y) \mapsto \big((x-1)(x-y)(1+y) : (y-1)(y-x)(1+x) \\ : (xy+1)(x+1)(y+1) : (x-1)(y-1)(xy+1)\big).$$

The involution $(x, y) \mapsto (1/x, 1/y)$ being the center of $\theta_s(\mathrm{SL}(2,\mathbb{Z}))$, the quotient provides a morphism $\theta'_s \colon \mathrm{SL}(2,\mathbb{Z}) \to \mathrm{Bir}(C) \simeq \mathrm{Bir}(\mathbb{P}^2)$ whose kernel is generated by S^2. The morphism preserves the type, but is not an embedding. It is also possible to deform the construction in order to have similar actions on other cubic surfaces (see [CaLo]).

One first twisting of θ_s. We can "twist" the standard embedding θ_s in the following way:

Let $\theta_-(S) = \theta_s(S) = (y, 1/x)$ and $\theta_-(R) = (xy, -y) \neq \theta_s(R) = (xy, y)$. The map $\theta_-(RS) = \theta_-(R)\theta_-(S) = (y/x, -1/x)$ has order 3. Since $\theta_-(R)$ commutes with $\theta_-(S^2)$, the relations of $\mathrm{SL}(2,\mathbb{Z})$ are satisfied and θ_- is a morphism from $\mathrm{SL}(2,\mathbb{Z})$ into $\mathrm{Bir}(\mathbb{P}^2)$.

Proposition 3.2. *The map* $\theta_- \colon \mathrm{SL}(2,\mathbb{Z}) \to \mathrm{Bir}(\mathbb{P}^2)$ *is an embedding that preserves the type. The groups* $\theta_s(\mathrm{SL}(2,\mathbb{Z}))$ *and* $\theta_-(\mathrm{SL}(2,\mathbb{Z}))$ *are not conjugate in the Cremona group.*

Proof. For each $M \in \mathrm{SL}(2,\mathbb{Z})$, one has $\theta_-(M) = \alpha_M \circ \theta_s(M)$ where $\alpha_M = (\pm x, \pm y)$, and in particular $\theta_-(M)$ and $\theta_s(M)$ have the same degree. This observation implies that θ_- is an embedding, and that it preserves the type, since θ_s does.

We now prove the second assertion. Arguing on the contrary, suppose that $\theta_s(\mathrm{SL}(2,\mathbb{Z}))$ is conjugate to $\theta_-(\mathrm{SL}(2,\mathbb{Z}))$; then $\theta_s(R) = (xy, y)$ is conjugate to some parabolic element of $\theta_-(\mathrm{SL}(2,\mathbb{Z}))$, which has no root in the group. This implies that $\theta_s(R) = (xy, y)$ or its inverse is conjugate to $\theta_-(R) = (xy, -y)$ or $\theta_-(RS^2) = (1/xy, -1/y)$ in $\mathrm{Bir}(\mathbb{P}^2)$.

All these elements are parabolic elements of the Cremona group, and each of them preserves a unique rational fibration, which is $(x, y) \mapsto y$. Since $\theta_s(R)$ preserves any fibre and both $\theta_-(R)$, $\theta_-(RS^2)$ permute the fibres, neither $\theta_s(R)$ nor $\theta_s(R^{-1})$ is conjugate to $\theta_-(R)$ or $\theta_-(RS^2)$ in $\mathrm{Bir}(\mathbb{P}^2)$. □

The map θ_- yields a "new" embedding of $\mathrm{SL}(2,\mathbb{Z})$ preserving the type. However, this map is not very far from the first one, and remains in $(\mathbb{C}^*,\mathbb{C}^*)\rtimes\mathrm{SL}(2,\mathbb{Z})$. We construct now new ones, more interesting. Conjugating the elements $\theta_s(S)=(y,\frac{1}{x})$ and $\theta_s(R)=(xy,y)$ by the birational map $((x-1)/(x+1),(y-1)/(y+1))$, we get respectively $(y,-x)$ and $((x+y)/(xy+1),y)$.

More generally, we choose any $\varepsilon\in\mathbb{C}^*$, and set

$$\theta_\varepsilon(S)=(y,-x)\,,\qquad \theta_\varepsilon(R)=\left(\frac{x+\varepsilon y}{\varepsilon+xy},\varepsilon y\right).$$

The map $\theta_\varepsilon(R)$ commutes with $\theta_\varepsilon(S^2)=(-x,-y)$, and

$$\theta_\varepsilon(RS)=\left(\frac{y-\varepsilon x}{\varepsilon-xy},-\varepsilon x\right)$$

is of order 3, so θ_ε gives an homomorphism from $\mathrm{SL}(2,\mathbb{Z})$ to $\mathrm{Bir}(\mathbb{P}^2)$. The map θ_1 being conjugate to the standard embedding, we can view this family as a deformation of the standard embedding. We prove now some technical results to show that the family consists of embedding preserving the type when ε is a positive real number.

Lemma 3.3. *We view the following maps on $\mathbb{P}^1\times\mathbb{P}^1$ via the embedding $(x,y)\mapsto((x:1),(y:1))$.*

(i) *Writing* $R_1=\begin{bmatrix}1&1\\0&1\end{bmatrix}$, $R_2=\begin{bmatrix}1&0\\1&1\end{bmatrix}$, *both maps*

$$\theta_\varepsilon(R_1)=\left(\frac{x+\varepsilon y}{\varepsilon+xy},\varepsilon y\right)\quad\textit{and}\quad\theta_\varepsilon(R_2)=\left(\frac{x}{\varepsilon},\frac{\varepsilon(x+\varepsilon y)}{\varepsilon+xy}\right)$$

have exactly two base-points both belonging to $\mathbb{P}^1\times\mathbb{P}^1$ (no infinitely near point), and being $p_1=(\varepsilon,-1)$ and $p_2=(-\varepsilon,1)$ (or $((\varepsilon:1),(-1:1))$ and $((-\varepsilon:1),(1:1))$).

(ii) *Both maps*

$$\theta_\varepsilon(R_1)^{-1}=\left(\frac{\varepsilon(\varepsilon x-y)}{\varepsilon-xy},\frac{y}{\varepsilon}\right)\quad\textit{and}\quad\theta_\varepsilon(R_2)^{-1}=\left(\varepsilon x,\frac{y-\varepsilon x}{\varepsilon-xy}\right)$$

have exactly two base-points, being $q_1=(1,\varepsilon)$ and $q_2=(-1,-\varepsilon)$.

(iii) *If ε is a positive real number and $M=R_{i_k}\ldots R_{i_1}$, for $i_1,\ldots,i_k\in\{1,2\}$, the following hold:*

- *the points q_1 and q_2 are not base-points of $\theta_\varepsilon(M)$, and*

$$\theta_\varepsilon(M)(\{q_1,q_2\})\cap\{p_1,p_2\}=\varnothing;$$

- *the points p_1 and p_2 are not base-points of $\theta_\varepsilon(M^{-1})$, and*

$$\theta_\varepsilon(M^{-1})(\{p_1,p_2\})\cap\{q_1,q_2\}=\varnothing.$$

Proof. Parts (i) and (ii) follow from an easy calculation; it remains to prove (iii).

Let $U_+ \subset \mathbb{R}^2 \subset \mathbb{P}^1 \times \mathbb{P}^1$ (resp. $U_- \subset \mathbb{R}^2 \subset \mathbb{P}^1 \times \mathbb{P}^1$) be the subset of points (x, y) with $x,\ y \in \mathbb{R}$, $xy > 0$ (resp. $xy < 0$). When ε is a positive real number, $\{p_1, p_2\} \subset U_-$ and $\{q_1, q_2\} \subset U_+$, which implies that $\theta_\varepsilon(R_i)$ (resp. $\theta_\varepsilon(R_i^{-1})$) is defined at any point of U_+ (resp. of U_-), since $U_+ \cap U_- = \varnothing$.

Moreover, the explicit form of the four maps given in (i), (ii) shows that $\theta_\varepsilon(R_i)(U_+) \subset U_+$ and $\theta_\varepsilon(R_i^{-1})(U_-) \subset U_-$ for $i = 1, 2$. This yields the result. □

Recall that $\mathrm{Pic}(\mathbb{P}^1 \times \mathbb{P}^1) = \mathbb{Z}f_1 \oplus \mathbb{Z}f_2$, where f_i is the fibre of the projection on the ith factor. In particular, any curve on $\mathbb{P}^1 \times \mathbb{P}^1$ has a bidegree (d_1, d_2) and any element of $\mathrm{Bir}(\mathbb{P}^1 \times \mathbb{P}^1)$ has a quadridegree, which is given by the two bidegrees of the pull-backs of f_1 and f_2, or equivalently by the two bidegrees of the polynomials which define the map.

Remark that the dynamical degree of a birational map φ of $\mathbb{P}^1 \times \mathbb{P}^1$ is uniquely determined by the sequence of quadridegrees of φ^n.

Proposition 3.4. *If ε is a positive real number, the following hold:*

(i) *For any $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}(2, \mathbb{Z})$, the maps $\theta_\varepsilon(M)$ and $\theta_s(M)$ have the same quadridegree as birational maps of $\mathbb{P}^1 \times \mathbb{P}^1$, which is $(|a|, |b|, |c|, |d|)$.*

(ii) *The homomorphism θ_ε is an embedding of $\mathrm{SL}(2, \mathbb{Z})$ into the Cremona group that preserves the type.*

Proof. Observe first that (i) implies that the kernel of θ_ε is trivial (since $\theta_\varepsilon(S^2) = (-x, -y)$ is not trivial) so that θ_ε is an embedding, and also implies that the dynamical degree of $\theta_\varepsilon(M)$ and $\theta_s(M)$ are the same for any M. This shows that (i) implies (ii).

We now prove assertion (i). Since $\theta_s(S) = (y, 1/x)$ and $\theta_\varepsilon(S) = (y, -x)$ are automorphisms of $\mathbb{P}^1 \times \mathbb{P}^1$ having the same action on $\mathrm{Pic}(\mathbb{P}^1 \times \mathbb{P}^1)$, $\theta_\varepsilon(M)$ and $\theta_s(M)$ have the same quadridegree if and only if $\theta_\varepsilon(MS)$ and $\theta_s(MS)$ have the same quadridegree. The same holds when we multiply on the left: $\theta_\varepsilon(M)$ and $\theta_s(M)$ have the same quadridegree if and only if $\theta_\varepsilon(SM)$ and $\theta_s(SM)$ have the same quadridegree.

Recall that $\mathrm{SL}(2, \mathbb{Z})$ has the presentation $\langle R,\ RS \mid S^4 = (RS)^3 = 1,\ S^2(RS) = (RS)S^2 \rangle$. It suffices thus to prove that $\theta_\varepsilon(M)$ and $\theta_s(M)$ have the same quadridegree when $M = (RS)^{i_k} \cdots S(RS)^{i_2}S(RS)^{i_1}S$, for some $i_1, \ldots, i_k \in \{\pm 1\}$. For any index i_j equal to 1, we replace the S immediately after by S^{-1} (since S^2 commutes with all matrices), and obtain now a product of nonnegative powers of $(RS)S^{-1} = R$ and $(RS)^2S$. We will write $R_1 = R$ and $R_2 = (RS)^2S$, and have

$$R_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \qquad R_2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

It is thus sufficient to prove the following assertion:

($\star$) *If $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = R_{i_k} R_{i_{k-1}} \ldots R_{i_1}$, for some $i_1, \ldots, i_k \in \{1, 2\}$, then $a, b, c, d \geq 0$, and $\theta_s(M)$, $\theta_\varepsilon(M)$ have both quadridegree (a, b, c, d).*

We proceed now by induction on k. For $k=1$, Assertion $(\star)$ can be directly checked:

Both $\theta_s(R_1)=(xy,y)$ and $\theta_\varepsilon(R_1)=((x+\varepsilon y)/(\varepsilon+xy),\varepsilon y)$ have quadridegree $(1,1,0,1)$. Both $\theta_s(R_2)=(x,xy)$ and $\theta_\varepsilon(R_2)=(x/\varepsilon,\varepsilon(x+\varepsilon y)/(\varepsilon+xy))$ have quadridegree $(1,0,1,1)$.

Now, assume that $(\star)$ is true for $M=\begin{bmatrix} a & b\\ c & d\end{bmatrix}$, and let us prove it for $R_1M=\begin{bmatrix} a+c & b+d\\ c & d\end{bmatrix}$ and $R_2M=\begin{bmatrix} a & b\\ a+c & b+d\end{bmatrix}$. By induction hypothesis one has

$$\theta_\varepsilon(M)=((x_1:x_2),(y_1:y_2))\dashrightarrow((P_1:P_2),(P_3:P_4)),$$

where P_1, P_2, P_3, $P_4\in\mathbb{C}[x_1,x_2,y_1,y_2]$ are bihomogeneous polynomials, of bidegree (a,b), (a,b), (c,d), (c,d).

We thus have:

$$\begin{aligned}
&\theta_\varepsilon(R_1)\theta_\varepsilon(M)=\theta_\varepsilon(R_1M)=\\
&\qquad((x_1:x_2),(y_1:y_2))\dashrightarrow((P_1P_4+\varepsilon P_2P_3:\varepsilon P_2P_4+P_1P_3),(\varepsilon P_3:P_4)),\\
&\theta_\varepsilon(R_2)\theta_\varepsilon(M)=\theta_\varepsilon(R_2M)=\\
&\qquad((x_1:x_2),(y_1:y_2))\dashrightarrow((P_1:\varepsilon P_2),(\varepsilon(P_1P_4+\varepsilon P_2P_3):\varepsilon P_2P_4+P_1P_3)).
\end{aligned}$$

To prove $(\star)$ for R_1M and R_2M, it suffices to show that the polynomials $P_1P_4+\varepsilon P_2P_3$ and $\varepsilon P_2P_4+P_1P_3$ have no common component. Suppose the converse for contradiction, and denote by $h\in\mathbb{C}[x_1,x_2,y_1,y_2]$ the common component. The polynomial h corresponds to a curve of $\mathbb{P}^1\times\mathbb{P}^1$ that is contracted by $\theta_\varepsilon(M)$ onto a base-point of $\theta_\varepsilon(R_1)$ or $\theta_\varepsilon(R_2)$, i.e., onto $p_1=(\varepsilon,-1)$ or $p_2=(-\varepsilon,1)$ (Lemma 3.3). But this condition means that $(\theta_\varepsilon(M))^{-1}$ has a base-point at p_1 or p_2. We proved in Lemma 3.3 that this is impossible when ε is a positive real number. □

We now show that this construction yields infinitely many conjugacy classes of embeddings of $\mathrm{SL}(2,\mathbb{Z})$ into the Cremona group that preserve the type.

Proposition 3.5. *If ε and ε' are two real positive numbers with $\varepsilon\varepsilon'\neq 1$, the two groups $\theta_\varepsilon(\mathrm{SL}(2,\mathbb{Z}))$ and $\theta_{\varepsilon'}(\mathrm{SL}(2,\mathbb{Z}))$ are not conjugate in the Cremona group.*

The standard embedding θ_s is conjugate to θ_1, but $\theta_-(\mathrm{SL}(2,\mathbb{Z}))$ is not conjugate to $\theta_\varepsilon(\mathrm{SL}(2,\mathbb{Z}))$ for any positive $\varepsilon\in\mathbb{R}$.

Proof. The proof is similar to the one of Proposition 3.2. Assume, for contradiction, that $\theta_\varepsilon(\mathrm{SL}(2,\mathbb{Z}))$ is conjugate to $\theta_{\varepsilon'}(\mathrm{SL}(2,\mathbb{Z}))$; then $\theta_\varepsilon(R)=((x+\varepsilon y)/(\varepsilon+xy),\varepsilon y)$ is conjugate to some parabolic element of $\theta_{\varepsilon'}(\mathrm{SL}(2,\mathbb{Z}))$, which has no root in the group. This implies that $\theta_\varepsilon(R)=((x+\varepsilon y)/(\varepsilon+xy),\varepsilon y)$ or its inverse is conjugate to $\theta_{\varepsilon'}(R)=((x+\varepsilon' y)/(\varepsilon'+xy),\varepsilon' y)$ or to $\theta_{\varepsilon'}(RS^2)=((-x-\varepsilon' y)/(\varepsilon'+xy),-\varepsilon' y)$ in $\mathrm{Bir}(\mathbb{P}^2)$.

These elements are parabolic elements of the Cremona group, and each of them preserves a unique rational fibration, which is $(x,y)\mapsto y$. The action on the basis being different up to conjugacy (since $\varepsilon\varepsilon'\neq\pm 1$), neither $\theta_\varepsilon(R)$ nor its inverse is conjugate to $\theta_{\varepsilon'}(R)$ or $\theta_{\varepsilon'}(RS^2)$ in $\mathrm{Bir}(\mathbb{P}^2)$.

It remains to show that $\theta_-(\mathrm{SL}(2,\mathbb{Z}))$ is not conjugate to $\theta_\varepsilon(\mathrm{SL}(2,\mathbb{Z}))$ for any positive $\varepsilon \in \mathbb{R}$. Every parabolic element of $\theta_-(\mathrm{SL}(2,\mathbb{Z}))$ without root is conjugate to $\theta_-(R) = (xy, -y)$, $\theta_-(RS^2) = (1/xy, -1/y)$ or their inverses, and acts thus nontrivially on the basis of the unique fibration preserved, with an action of order 2. We get the result by observing that $\theta_\varepsilon(\mathrm{SL}(2,\mathbb{Z}))$ contains $\theta_\varepsilon(R) = ((x+\varepsilon y)/(\varepsilon + xy), \varepsilon y)$, which is parabolic, without root and acting on the basis with an action which has not order 2. □

Note that in all our examples of embeddings preserving the type, the parabolic elements have a linear degree growth. One can then ask the following question (which could yield a positive answer to Question 1.1).

Question 3.6. Does there exist an embedding of $\mathrm{SL}(2,\mathbb{Z})$ into $\mathrm{Bir}(\mathbb{P}^2)$ that preserves the type and such that the degree growth of parabolic elements is quadratic?

3.2. Elliptic embeddings

The simplest elliptic embedding is given by

$$\theta_e \colon \mathrm{SL}(2,\mathbb{Z}) \to \mathrm{Bir}(\mathbb{P}^2), \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto (ax+by : cx+dy : z).$$

We now generalise this embedding. Choose $n \in \mathbb{N}$ and let $\chi \colon \mathrm{SL}(2,\mathbb{Z}) \to \mathbb{C}^*$ be a character such that $\chi\left(\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\right) \neq (-1)^n$. For simplicity, we choose χ such that $\chi(RS) = 1$, and such that $\chi(S)$ is equal to 1 if n is odd and to $\mathbf{i}$ if n is even. Then we define $\theta_n \colon \mathrm{SL}(2,\mathbb{Z}) \to \mathrm{Bir}(\mathbb{P}^2)$ by

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \left(\frac{ax+b}{cx+d}, \frac{\chi(M)y}{(cx+d)^n} \right).$$

The action on the first component and the fact that $\theta_n(S^2) \neq 1$ imply that θ_n is an embedding. The degree of all elements being bounded, the embeddings are elliptic.

Proposition 3.7. *For any $n \in \mathbb{N}$, the group $\theta_n(\mathrm{SL}(2,\mathbb{Z}))$ is conjugate to a subgroup of $\mathrm{Aut}(\mathbb{F}_n)$, where $\mathbb{F}_n$ is the nth Hirzebruch surface.*

The groups $\theta_m(\mathrm{SL}(2,\mathbb{Z}))$ and $\theta_n(\mathrm{SL}(2,\mathbb{Z}))$ are conjugate in the Cremona group if and only if $m = n$.

Proof. If $n = 0$, the embedding $(x,y) \mapsto ((x:1),(y:1))$ of $\mathbb{C}^2$ into $\mathbb{P}^1 \times \mathbb{P}^1 = \mathbb{F}_0$ conjugates $\theta_0(\mathrm{SL}(2,\mathbb{Z}))$ to a subgroup of $\mathrm{Aut}(\mathbb{F}_0)$.

For $n \geq 1$, recall that the weighted projective space $\mathbb{P}(1,1,n)$ is equal to

$$\mathbb{P}(1,1,n) = \left\{ (x_1,x_2,z) \in \mathbb{C}^3 \backslash \{0\} \mid (x_1,x_2,z) \sim (\mu x_1, \mu x_2, \mu^n z),\ \mu \in \mathbb{C}^* \right\}.$$

The surface $\mathbb{P}(1,1,1)$ is equal to $\mathbb{P}^2$, and the surfaces $\mathbb{P}(1,1,n)$ for $n \geq 2$ have one singular point, which is $(0:0:1)$.

For any $n \geq 1$, the embedding $(x,y) \mapsto (x:y:1)$ of $\mathbb{C}^2$ into $\mathbb{P}(1,1,n)$ conjugates $\theta_n(\mathrm{SL}(2,\mathbb{Z}))$ to a subgroup of $\mathrm{Aut}(\mathbb{P}(1,1,n))$ that fixes the point $(0:0:1)$.

The blow up of this fixed point gives the Hirzebruch surface $\mathbb{F}_n$, and conjugates thus $\theta_n(\mathrm{SL}(2,\mathbb{Z}))$ to a subgroup of $\mathrm{Aut}(\mathbb{F}_n)$.

In all cases $n \geq 0$, the group preserves the fibration $\mathbb{F}_n \to \mathbb{P}^1$ corresponding to $(x, y) \mapsto x$. The action on the basis of the fibration corresponds to the standard homomorphism $\mathrm{SL}(2,\mathbb{Z}) \to \mathrm{PSL}(2,\mathbb{Z}) \subset \mathrm{PGL}(2,\mathbb{C}) = \mathrm{Aut}(\mathbb{P}^1)$. This action has no orbit of finite size on $\mathbb{P}^1$. In particular, there is no orbit of finite size on $\mathbb{F}_n$. This shows that the subgroup of $\mathrm{Aut}(\mathbb{F}_n)$ corresponding to $\theta_n(\mathrm{SL}(2,\mathbb{Z}))$ is birationally rigid for $n \neq 1$, i.e., that it is not conjugate to any group of automorphisms of any other smooth projective surface. This shows that $\theta_m(\mathrm{SL}(2,\mathbb{Z}))$ and $\theta_n(\mathrm{SL}(2,\mathbb{Z}))$ are conjugate in the Cremona group only when $m = n$. □

3.3. Parabolic embeddings

Recall that the morphism θ_0 defined in Section 3.2 can also be viewed as follows: $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto ((ax+b)/(cx+d), \chi(M)y)$; it preserves the fibration $(x, y) \mapsto x$. Remembering that $\chi(S) = \mathbf{i}$ and $\chi(RS) = 1$ we have

$$\theta_0(S) = \left(-\frac{1}{x}, \mathbf{i}y\right) \quad \text{and} \quad \theta_0(RS) = \left(\frac{x-1}{x}, y\right).$$

We will "twist" θ_0 in order to construct parabolic embeddings. Recall that $\mathrm{SL}(2,\mathbb{Z})$ acts via θ_0 on the projective line; the element $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ acts as $x \dashrightarrow (ax+b)/(cx+d)$. The group is countable so a very general point of the line has no isotropy. Let $P \in \mathbb{C}(x)$ be a rational function with m simple poles and m simple zeroes, where $m > 0$, and such that the $2m$ corresponding points of $\mathbb{C}$ are all on different orbits under the action of $\mathrm{SL}(2,\mathbb{Z})$ and have no isotropy. We denote by $\varphi_P = (x, y \cdot P(x))$ the associated birational map; it preserves the fibration and commutes with $\theta_0(S^2) = (x, -y)$.

We choose

$$\theta_P(S) = \theta_0(S) = \left(-\frac{1}{x}, \mathbf{i}y\right) \quad \text{and} \quad \theta_P(RS) = \varphi_P \circ \theta_0(RS) \circ \varphi_P^{-1},$$

therefore

$$\theta_P(S) = \left(-\frac{1}{x}, \mathbf{i}y\right) \quad \text{and} \quad \theta_P(RS) = \left(\frac{x-1}{x}, y \cdot \frac{P((x-1)/x)}{P(x)}\right).$$

The maps φ_P and $\theta_P(S^2)$ commute, so $\theta_P(RS)$ and $\theta_P(S^2)$ commute too. Then, by definition of $\theta_P(S)$ and $\theta_P(RS)$ there is a unique morphism $\theta_P \colon \mathrm{SL}(2,\mathbb{Z}) \to \mathrm{Bir}(\mathbb{C}^2)$.

Proposition 3.8. *The morphism θ_P is a parabolic embedding for any $P \in \mathbb{C}(x)$.*

Proof. The action on the basis of the fibration and the fact that $\theta_P(S^2) \neq \mathrm{id}$ imply that θ_P is an embedding. It remains to show that any element of infinite order is sent onto a parabolic element.

Writing $\alpha = \theta_P(RS)$ and $\beta = \theta_P(S)$, it suffices to show that h or $h\beta^2$ is parabolic, where

$$h = \beta\alpha^{i_n}\beta\ldots\alpha^{i_2}\beta\alpha^{i_1}, \qquad n \geq 1 \quad \text{and} \quad i_1,\ldots,i_n \in \{-1,1\}.$$

We view our maps acting on $\mathbb{P}^1 \times \mathbb{P}^1$. The fibration given by the projection on the first factor is preserved by h, which is thus either parabolic or elliptic. The first possibility occurs if the sequence of the number of base-points of h^k grows linearly, and the second if the sequence is bounded.

Let $p \in \mathbb{C}$ be a pole or a zero of P. Let $F_0 \subset \mathbb{P}^1 \times \mathbb{P}^1$ be the fibre of $(p:1)$ and let $\Sigma \subset \mathbb{P}^1 \times \mathbb{P}^1$ be the (countable) union of fibres of points that belong to the orbit of $(p:1)$ under the action of $\mathrm{SL}(2,\mathbb{Z})$.

Recall that $\theta_0(RS)$ is an automorphism of $\mathbb{P}^1 \times \mathbb{P}^1$. Set $F_1 = \theta_0(RS)(F_0)$ and $F_2 = \theta_0(RS)(F_1)$; remark that $F_0 = \theta_0(RS)(F_2)$. Then φ_P and its inverse contract F_0 on a point of F_0 but send isomorphically F_1 and F_2 onto themselves. The map α is the conjugate of $\theta_0(RS)$ by φ_P, so it contracts F_0 and F_2 on points lying respectively on F_1 and F_0, but sends isomorphically F_1 onto F_2 and doesn't contract any other fibre contained in Σ. Similarly α^{-1} contracts F_0 and F_1 on points lying on F_2 and F_0 and contracts neither F_2, nor any other fiber of Σ.

Each fibre is preserved by β^2, but β and β^3 send F_0, F_1, F_2 onto three other fibres contained in Σ. Then $\alpha^{\pm 1}\beta$ and $\alpha^{\pm 1}\beta^3$ send isomorphically F_0 onto a fibre contained in $\Sigma \setminus \{F_i\}$. By induction on n, we obtain that for any $k < 0$, h^k and $(h\beta^2)^k$ send isomorphically F_0 onto a curve in $\Sigma \setminus \{F_i\}$.

Then we note that α and α^{-1} contract F_0 on a point contained in one of the F_i, a point that β sends to another point not contained in the F_i's. So, by induction on n, for any $k > 0$ both h^k and $(h\beta^2)^k$ contract F_0 on a point not contained in the F_i's and for which the fibre belongs to Σ.

For each integer $k > 0$, the fibre F_0 is contracted by h^k and by $(h\beta^2)^k = h^k(\beta^{2k})$ on a point of Σ. Moreover, for each integer $k < 0$, F_0 is sent isomorphically by h^k onto a fibre contained in Σ. Set $F_i' = h^{-i}(F_0)$ for all $i > 0$; we obtain that h^k and $(h\beta^2)^k$ contract F_0 and $F_1',\ldots,F_k'$ for each integer $k > 0$. This means that the number of base-points of h^k and $(h\beta^2)^k$ is at least equal to k. As h and $h\beta^2$ preserve the fibration, they are parabolic. □

Proposition 3.9. *When P varies, we obtain infinitely many parabolic embeddings.*

Proof. Let $P, Q \in \mathbb{C}(x)$, and suppose that $\theta_P(\mathrm{SL}(2,\mathbb{Z}))$ is conjugate to $\theta_Q(\mathrm{SL}(2,\mathbb{Z}))$ by some birational map φ of $\mathbb{P}^1 \times \mathbb{P}^1$. Then φ preserves the fibration $(x,y) \mapsto x$, which is the unique fibration preserved by the two groups. Its action on the basis of the fibration is an element $\psi \in \mathrm{PGL}(2,\mathbb{C})$ that normalises $\mathrm{PSL}(2,\mathbb{Z}) \subset \mathrm{PSL}(2,\mathbb{C}) = \mathrm{PGL}(2,\mathbb{C})$. This means that $\psi \in \mathrm{PSL}(2,\mathbb{Z})$. Replacing φ by its product with an element of $\theta_Q(\mathrm{SL}(2,\mathbb{Z}))$, we can thus assume that φ acts trivially on the basis.

This means that φ is equal to $(x, (a(x)y + b(x))/(c(x)y + d(x)))$ for some $a, b, c, d \in \mathbb{C}(x)$, $ad - bc \neq 0$. Since φ conjugates $\theta_P(S) = \theta_Q(S) = (-1/x, \mathbf{i}y)$ to itself or its inverse, the map φ is equal to $(x, a(x)y^{\pm 1})$ where $a \in \mathbb{C}(x)$, $a(-1/x) = \pm a(x)$.

The map φ conjugates $\theta_P(RS) = ((x-1)/x, y \cdot P((x-1)/x)/P(x))$ to

$$\theta_Q(RS) = \left(\frac{x-1}{x}, y \cdot \frac{Q((x-1)/x)}{Q(x)}\right)$$

or to

$$\theta_Q(RS^3) = \left(\frac{(x-1)}{x}, -y\cdot\frac{Q((x-1)/x)}{Q(x)}\right)$$

in $\mathrm{Bir}(\mathbb{P}^1\times\mathbb{P}^1)$. Assume that

$$\varphi = (x, a(x)y) \qquad \text{where} \quad a\in\mathbb{C}(x),\ a\left(-\frac{1}{x}\right) = a(x);$$

then

$$\varphi\theta_P(RS)\varphi^{-1} = \left(\frac{x-1}{x}, y\cdot\frac{a\left((x-1)/x\right)P\left((x-1)/x\right)}{a(x)P(x)}\right).$$

Thus $\varphi\theta_P(RS)\varphi^{-1} = \theta_Q(RS)$, resp. $\theta_Q(RS^3)$, if and only if

$$\frac{a\left((x-1)/x\right)}{a(x)} = \frac{P(x)Q((x-1)/x)}{Q(x)P\left((x-1)/x\right)}, \quad \text{resp.} \quad \frac{a\left((x-1)/x\right)}{a(x)} = -\frac{P(x)Q((x-1)/x)}{Q(x)P\left((x-1)/x\right)}$$

and since $a(x)$ is invariant under the homography $x\mapsto -1/x$, the same holds for

$$\frac{P(x)Q((x-1)/x)}{Q(x)P\left((x-1)/x\right)}.$$

This implies, in both cases, the following condition on P and Q:

$$\frac{P(x)P(1+x)}{P\left(-1/x\right)P\left((x-1)/x\right)} = \frac{Q(x)Q(1+x)}{Q\left(-1/x\right)Q\left((x-1)/x\right)}.$$

We get the same formula when φ is equal to $(x, a(x)y^{-1})$ where $a\in\mathbb{C}(x)$, $a(-1/x) = -a(x)$. When P varies, we thus obtain infinitely many parabolic embeddings. □

3.4. Hyperbolic embeddings

In this section, we "twist" the standard elliptic embedding θ_e defined in Section 3.2 to get many hyperbolic embeddings of $\mathrm{SL}(2,\mathbb{Z})$ into $\mathrm{Bir}(\mathbb{P}^2)$. Recall that θ_e is given by

$$\theta_e\colon \mathrm{SL}(2,\mathbb{Z})\to\mathrm{Bir}(\mathbb{P}^2), \qquad \begin{bmatrix} a & b\\ c & d\end{bmatrix}\mapsto (ax+by:cx+dy:z).$$

The group $\theta_e(\mathrm{SL}(2,\mathbb{Z}))$ preserves the line L_z of equation $z=0$, and acts on it via the natural maps $\mathrm{SL}(2,\mathbb{Z})\to\mathrm{PSL}(2,\mathbb{Z})\subset\mathrm{PSL}(2,\mathbb{C}) = \mathrm{Aut}(L_z)$.

We choose $\mu\in\mathbb{C}^*$ such that the point $p=(\mu:1:0)\in L_z$ has a trivial isotropy group under the action of $\mathrm{PSL}(2,\mathbb{Z})$, fix an even integer $k>0$, and then define a morphism $\theta_k\colon \mathrm{SL}(2,\mathbb{Z})\to\mathrm{Bir}(\mathbb{P}^2)$ in the following way:

$$\theta_k(S) = \theta_e(S) = (y:-x:z),$$
$$\theta_k(RS) = \psi\theta_e(RS)\psi^{-1}$$

where ψ is the conjugation of $\psi' = (x^k : yx^{k-1}+z^k : zx^{k-1})$ by $(x+\mu y:y:z)$.

Note that ψ' restricts to an automorphism of the affine plane where $x\neq 0$, commutes with $\theta_e(S^2) = (x:y:-z)$ and acts trivially on L_z. Since ψ commutes with $\theta_e(S^2) = \theta_k(S^2)$, the element $\theta_k(RS)$ commutes with $\theta_k(S^2)$, and θ_k is thus a well-defined morphism. The fact that ψ preserves L_z and acts trivially on it implies that the action of θ_e and θ_k on L_z are the same, so θ_k is an embedding.

Lemma 3.10. *Let m be a positive integer, and let $a_1,\dots,a_m,b_1,\dots,b_m \in \{\pm 1\}$. The birational map*

$$\theta_k(S^{b_m}(RS)^{a_m}\cdots S^{b_1}(RS)^{a_1})$$

has degree k^{2m} and exactly $2m$ proper base-points, all lying on L_z, which are

$$p,\ ((RS)^{a_1})^{-1}(p),\ (S^{b_1}(RS)^{a_1})^{-1}(p),\ ((RS)^{a_2}S^{b_1}(RS)^{a_1})^{-1}(p),\dots,$$
$$((RS)^{a_m}\cdots S^{b_1}(RS)^{a_1})^{-1}(p),\ (S^{b_m}(RS)^{a_m}\cdots S^{b_1}(RS)^{a_1})^{-1}(p),$$

where the action of $R, RS \in \mathrm{SL}(2,\mathbb{Z})$ on L_z is here the action via θ_e or θ_k.

Proof. The birational map ψ has degree k and has an unique proper base-point which is $p=(\mu:1:0)\in L_z$; the same is true for ψ^{-1}. Moreover, both maps fix any other point of L_z.

Since $\theta_e(RS)^{a_1}$ is an automorphism of $\mathbb{P}^2$ that moves the point p onto another point of L_z, the map $\theta_k((RS)^{a_1})=\psi\theta_e(RS)^{a_1}\psi^{-1}$ has degree k^2 and exactly two proper base-points, which are p and $\psi\theta_e(RS)^{-a_1}(p)=((RS)^{a_1})^{-1}(p)$. The map $\theta_k(S)$ being an automorphism of $\mathbb{P}^2$, $\theta_k(S^{b_1}(RS)^{a_1})$ has also degree k^2 and two proper base-points, which are p and $((RS)^{a_1})^{-1}(p)$. This gives the result for $m=1$.

Proceeding by induction on $m>1$, we assume that $\theta_k(S^{b_m}(RS)^{a_m}\cdots S^{b_2}(RS)^{a_2})$ has degree k^{2m-2} and exactly $2m-2$ proper base-points, all lying on L_z, which are

$$p,\ ((RS)^{a_2})^{-1}(p),\ (S^{b_2}(RS)^{a_2})^{-1}(p),\dots,(S^{b_m}(RS)^{a_m}\cdots S^{b_2}(RS)^{a_2})^{-1}(p).$$

The map $\theta_k(S^{b_1}(RS)^{a_1})^{-1}=\theta_k((RS)^{-a_1})\theta_k(S^{-b_1})$ has degree k^2 and two proper base-points, which are $S^{b_1}(p)$ and $S^{b_1}(RS)^{a_1}(p)$. These two points being distinct from the $2m-2$ points above, the map $\theta_k(S^{b_m}(RS)^{a_m}\cdots S^{b_1}(RS)^{a_1})$ has degree $k^2\cdot k^{2m-2}=k^{2m}$, and its proper base-points are the 2 proper base-points of $\theta_k(S^{b_1}(RS)^{a_1})$ and the image by $(S^{b_1}(RS)^{a_1})^{-1}$ of the base-points of $\theta_k(S^{b_m}(RS)^{a_m}\cdots S^{b_2}(RS)^{a_2})$. This gives the result. □

As a corollary, we get infinitely many hyperbolic embeddings of $\mathrm{SL}(2,\mathbb{Z})$ into the Cremona group.

Corollary 3.11. *Let m be a positive integer, and let $a_1,\dots,a_m,b_1,\dots,b_m \in \{\pm 1\}$. The birational map*

$$\theta_k(S^{b_m}(RS)^{a_m}\cdots S^{b_1}(RS)^{a_1})$$

has dynamical degree k^{2m}. In particular, the map θ_k is an hyperbolic embedding and the set of all dynamical degrees of $\theta_k(\mathrm{SL}(2,\mathbb{Z}))$ is $\{1,k^2,k^4,k^6,\dots\}$.

Proof. Any element of infinite order of $\mathrm{SL}(2,\mathbb{Z})$ is conjugate to $g=S^{b_m}(RS)^{a_m}\cdots\cdots S^{b_1}(RS)^{a_1}$ for some $a_1,\dots,a_m,b_1,\dots,b_m\in\{\pm1\}$. Lemma 3.10 implies that the degree of $\theta_k(g^r)$ is equal to k^{2mr}. The dynamical degree of $\theta_k(g)$ is therefore equal to k^{2m}. □

4. Description of hyperbolic embeddings for which the central element fixes (pointwise) an elliptic curve

4.1. Outline of the construction and notation

In this section, we show a general way of constructing embeddings of $\mathrm{SL}(2,\mathbb{Z})$ into the Cremona group where the central involution fixes pointwise an elliptic curve. Recall that all conjugacy classes of elements of order 4 or 6 in $\mathrm{Bir}(\mathbb{P}^2)$ have been classified (see [Bla3]). Many of them can act on del Pezzo surfaces of degree 1, 2, 3 or 4.

In order to create our embedding, we will define del Pezzo surfaces X, Y of degree ≤ 4, and automorphisms $\alpha \in \mathrm{Aut}(X)$, $\beta \in \mathrm{Aut}(Y)$ of order respectively 6 and 4, so that α^3 and β^2 fix pointwise an elliptic curve, and that $\mathrm{Pic}(X)^\alpha$, $\mathrm{Pic}(Y)^\beta$ both have rank 1. Note that we say that a curve is *fixed* by a birational map if it is pointwise fixed, and say that it is *invariant* or *preserved* if the map induces a birational action (trivial or not) on the curve. Contracting (-1)-curves invariant by these involutions (but not by α, β, which act minimally on X and Y), we obtain birational morphisms $X \to X_4$ and $Y \to Y_4$, where X_4, Y_4 are del Pezzo surfaces on which α^3 and β^2 act minimally. Lemma 4.1 below shows that X_4 and Y_4 are del Pezzo surfaces of degree 4 and both $\mathrm{Pic}(X_4)^{\alpha^3}$ and $\mathrm{Pic}(Y_4)^{\beta^2}$ have rank 2 and are generated by the fibres of two conic bundles on X_4 and Y_4. Choosing a birational map $X_4 \dashrightarrow Y_4$ conjugating α^3 to β^2 (which exists if and only if the elliptic curves are isomorphic), which is general enough, we should obtain an embedding of $\mathrm{SL}(2,\mathbb{Z})$ such that any element of infinite order is hyperbolic.

In order to prove that there is no other relation in the group generated by α and β and that all elements of infinite order are hyperbolic, we describe the morphisms $X \to X_4$ and $Y \to Y_4$ and the action of α and β on $\mathrm{Pic}(X)^{\alpha^3}$ and $\mathrm{Pic}(Y)^{\beta^2}$ (which are generated by the fibres of the two conic bundles on X_4, and Y_4 and by the exceptional curves obtained by blowing up points on the elliptic curves fixed), and then observe that the composition of the elements does what is expected.

4.2. Technical results on automorphisms of del Pezzo surfaces of degree 4

Recall some classical facts about del Pezzo surfaces, that the reader can find in [Dem] (see also [Man]). A del Pezzo surface is a smooth projective surface Z such that the anti-canonical divisor $-K_Z$ is ample. These are $\mathbb{P}^1 \times \mathbb{P}^1$, $\mathbb{P}^2$ or $\mathbb{P}^2$ blown up at $1 \leq r \leq 8$ points in general position (no three of them are collinear, no six are on the same conic, no eight are on the same cubic singular at one of the 8 points). The degree of a del Pezzo surface Z is $(K_Z)^2$, which is 8 for $\mathbb{P}^1 \times \mathbb{P}^1$, 9 for $\mathbb{P}^2$ and $9 - r$ for the blow up of $\mathbb{P}^2$ at r points.

Any del Pezzo surface Z contains a finite number of (-1)-curves (smooth curves isomorphic to $\mathbb{P}^1$ and of self-intersection -1); each of these can be contracted to obtain another del Pezzo surface of degree $(K_Z)^2 + 1$. These are, moreover, the only irreducible curves of Z of negative self-intersection. If Z is not $\mathbb{P}^2$, there is a finite number of conic bundles $Z \to \mathbb{P}^1$ (up to automorphism of $\mathbb{P}^1$), and each of them has exactly $8 - (K_Z)^2$ singular fibres. This latter fact can be found by contracting one component in each singular fibre, which is the union of two (-1)-curves, obtaining a line bundle on a del Pezzo surface, isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ or

$\mathbb{F}_1$ and having degree 8.

Lemma 4.1. *Let Z be a del Pezzo surface, and let $\sigma \in \mathrm{Aut}(Z)$ be an involution that fixes (pointwise) an elliptic curve. Denote by $\eta\colon Z \to Z_4$ any $<\sigma>$-invariant birational morphism such that the action on Z_4 is minimal. Then Z_4 is a del Pezzo surface of degree 4, and $\mathrm{Pic}(Z_4)^\sigma = \mathbb{Z}f_1 \oplus \mathbb{Z}f_2$, where f_1, f_2 correspond to the fibres of the two conic bundles $\pi_1, \pi_2\colon Z_4 \to \mathbb{P}^1$ (defined up to automorphism of $\mathbb{P}^1$) that are invariant by σ. Moreover,*

$$f_1 + f_2 = -K_{Z_4}, \quad f_1 \cdot f_2 = 2, \quad \textit{and}$$
$$\mathrm{Pic}(Z)^\sigma = \mathbb{Z}\eta^*(f_1) \oplus \mathbb{Z}\eta^*(f_2) \oplus \mathbb{Z}E_1 \oplus \cdots \oplus \mathbb{Z}E_r$$

where $E_1, \ldots, E_r$ are the r irreducible curves contracted by η (in particular, η only contracts invariant (-1)-curves).

Proof. Since Z is a del Pezzo surface, Z_4 is also a del Pezzo surface. As σ acts minimally on Z_4 and fixes an elliptic curve, we have the following situation ([BaBe, Theorem 1.4]): there exists a conic bundle $\pi_1\colon Z_4 \to \mathbb{P}^1$ such that $\pi_1\sigma = \pi_1$, σ induces a nontrivial involution on each smooth fibre of π_1, and exchanges the two components of each singular fibre, which meet at one point. The restriction of π_1 to the elliptic curve is a double covering ramified over 4 points, which implies that there are four singular fibres. The surface Z_4 is thus the blow up of four points on $\mathbb{F}_1$ or $\mathbb{P}^1 \times \mathbb{P}^1$, and has therefore degree 4. The fact that there are exactly two conic bundles $\pi_1, \pi_2\colon Z_4 \to \mathbb{P}^2$ invariant by σ, that $\mathrm{Pic}(Z_4)^\sigma$ is generated by the two fibres, that $f_1 + f_2 = -K_{Z_4}$ and that $f_1 \cdot f_2 = 2$ can be checked in [Bla2, Lemma 9.11].

It remains to observe that all points blown up by η are fixed by σ. If η blows up an orbit of at least two points of Z_4 invariant by σ, the points would be on the same fibre of π_1. The transform of this fibre on Z would then contain a curve isomorphic to $\mathbb{P}^1$ and having self-intersection ≤ -2; this is impossible on a del Pezzo surface. □

Lemma 4.2. *For $i = 1, 2$, let X_i be a projective smooth surface, with $K_{X_i}^2 = 4$, and let $\sigma_i \in \mathrm{Aut}(X_i)$ be an involution which fixes an elliptic curve $\Gamma_i \subset X_i$. Let $\pi_i\colon X_i \to \mathbb{P}^1$ be a conic bundle such that $\pi_i\sigma_i = \pi_i$ and let $F_i, G_i \subset X_i$ be two sections of π_i of self-intersection -1, intersecting transversally into one point.*

Then, X_1, X_2 are del Pezzo surfaces of degree 4 and the following assertions are equivalent:

1. *There exists an isomorphism $\varphi\colon X_1 \to X_2$ which conjugates σ_1 to σ_2, sends F_1, resp. G_1 onto F_2, resp. G_2 and such that $\pi_2\varphi = \pi_1$.*
2. *The points of $\mathbb{P}^1$ whose fibres by π_i are singular are the same for $i = 1, 2$, and $\pi_1(F_1 \cap G_1) = \pi_2(F_2 \cap G_2)$.*

Proof. For $i = 1, 2$, we denote by $\eta_i\colon X_i \to \mathbb{F}_1$ the birational morphism that contracts, in each singular fibre of π_i, the (-1)-curve that does not intersect F_i. The curve $\eta_i(F_i)$ is equal to the exceptional section E of the line bundle $\pi\colon \mathbb{F}_1 \to \mathbb{P}^1$, with $\pi = \pi_i\eta_i^{-1}$. Since $\eta_i(G_i)$ intersects E into exactly one point, it is a section of self-intersection 3. In particular, the four points blown up by η_i lie on $\eta_i(G_i)$.

Contracting E onto a point of $\mathbb{P}^2$, $\eta_i(G_i)$ becomes a conic of $\mathbb{P}^2$ passing through the five points blown up by the birational morphism $X_i \to \mathbb{P}^2$; this implies that no three of them are collinear and thus that X_i is a del Pezzo surface of degree 4.

It is clear that the first assertion implies the second one. It remains to prove the converse. The second assertion implies that $\eta_1(G_1) \cap E = \eta_2(G_2) \cap E$, and this yields the existence of an automorphism of $\mathbb{F}_1$ that sends $\eta_1(G_1)$ onto $\eta_2(G_2)$ and that preserves any fibre of π. We can thus assume that $\eta_1(G_1) = \eta_2(G_2)$, which implies that the four points blown up by η_1 and η_2 are the same. The isomorphism φ can be chosen as $\varphi = \eta_2^{-1} \circ \eta_1$. The map φ conjugates σ_1 to σ_2 because, for each i, σ_i is the unique involution that preserves any fibre of π_i and exchanges the two components of each singular fibre (see, for example, [Bla2, Lemma 9.11]). □

4.3. Actions on the Picard groups of α and β

We now describe the actions of α and β on $\mathrm{Pic}(X)$ and $\mathrm{Pic}(Y)$.

Proposition 4.3. *Let X be a del Pezzo surface of degree $(K_X)^2 < 4$, and let $\alpha \in \mathrm{Aut}(X)$ be an automorphism of order 6 such that $\mathrm{Pic}(X)^\alpha = \mathbb{Z}K_X$ and such that α^3 fixes pointwise an elliptic curve. Let $\eta_X \colon X \to X_4$ be a birational morphism, so that α^3 acts minimally on X_4, and let $f_1, f_2 \in \mathrm{Pic}(X)$ be the divisors corresponding to the two conic bundles on X_4 which are invariant by α^3 (see Lemma 4.1). Then, one of the following occurs:*

(i) *$(K_X)^2 = 3$, η_X contracts a curve E_1, and α, α^2 act on $\mathrm{Pic}(X)^{\alpha^3}$ as*

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ -2 & 0 & -1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & -2 & -1 \end{bmatrix}$$

relative to the basis (f_1, f_2, E_1) (up to an exchange of f_1, f_2).

(ii) *$(K_X)^2 = 1$, η_X contracts E_1, E_2, E_3, and α, α^2 act on $\mathrm{Pic}(X)^{\alpha^3}$ as*

$$\begin{bmatrix} 1 & 3 & 1 & 1 & 1 \\ 3 & 4 & 2 & 2 & 2 \\ -2 & -4 & -2 & -2 & -1 \\ -2 & -4 & -1 & -2 & -2 \\ -2 & -4 & -2 & -1 & -2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 4 & 3 & 2 & 2 & 2 \\ 3 & 1 & 1 & 1 & 1 \\ -4 & -2 & -2 & -1 & -2 \\ -4 & -2 & -2 & -2 & -1 \\ -4 & -2 & -1 & -2 & -2 \end{bmatrix}$$

relative to the basis $(f_1, f_2, E_1, E_2, E_3)$ (up to a good choice of E_1, E_2, E_3 and an exchange of f_1, f_2).

Proof. Let $E \subset X$ be any (-1)-curve invariant by α^3. The divisor $E+\alpha(E)+\alpha^2(E)$ is invariant by α and thus equivalent to sK_X for some integer s. Computing the intersection with K_X and the self-intersection, we obtain $-3 = s(K_X)^2$ and $-3 + 6(E \cdot \alpha(E)) = s^2(K_X)^2$. This gives two possibilities:

(i) $(K_X)^2 = 3,\ s = -1,\ E \cdot \alpha(E) = 1;$
(ii) $(K_X)^2 = 1,\ s = -3,\ E \cdot \alpha(E) = 2.$

In case (i), η_X is given by the choice of one (-1)-curve E_1 invariant by α^3. Since $E_1 \cdot \alpha(E_1) = 1$, the divisor $E_1 + \alpha(E_1)$ corresponds to a conic bundle on X and on X_4. Up to renumbering, we can say that $f_1 = E_1 + \alpha(E_1)$ and $f_2 = E_1 + \alpha^2(E_1)$. This means that $\alpha(E_1) = f_1 - E_1$, $\alpha^2(E_1) = f_2 - E_1$, $\alpha(f_1) = f_1 + f_2 - 2E_1$ and $\alpha(f_2) = f_1$.

In case (ii), there are three curves E_1, E_2, E_3 contracted by η_X. We first choose E_1, and then choose $E_2' = \iota_B(\alpha(E_1)) = -2K_X - \alpha(E_1)$ (where ι_B is the Bertini involution of the surface). Since E_2' does not intersect E_1, we can contract E_1, E_2', and another curve E_3 to obtain an α^3-equivariant birational morphism $X \to X_4'$, where X_4' is a del Pezzo surface of degree 4. This choice gives us two conic bundles f_1', f_2' on X_4', which we also see on X_4, invariant by α^3. We now compute $\alpha(E_3)$. We have $\alpha(E_3) \cdot E_3 = 2$,

$$\begin{aligned}
\alpha(E_3) \cdot E_1 &= E_3 \cdot \alpha^2(E_1) = E_3 \cdot (-3K_X - E_1 - \alpha(E_1)) \\
&= E_3 \cdot (-K_X - E_1 + E_2') = 1, \\
\alpha(E_3) \cdot E_2' &= E_3 \cdot \alpha^2(E_2') = E_3 \cdot (-2K_X - E_1) = 2.
\end{aligned}$$

This implies that $\alpha(E_3) = af_1' + bf_2' - E_1 - 2E_2' - 2E_3$, for some integers a, b. Computing the intersection with $-K_X$ we find $1 = 2a+2b-1-2-2 = 2(a+b)-5$, which means that $a+b = 3$. Computing the self-intersection, we obtain that $-1 = 2ab-1-4-4 = 4ab-9$, so $ab = 2$. Up to an exchange of f_1', f_2', we can assume that $a = 1, b = 2$, and obtain that $\alpha(E_3) = f_1'+2f_2'-E_1-2E_2'-2E_3 = -2K_X-(f_1'-E_1)$.

We now call E_2 the (-1)-curve $f_1' - E_2'$, which does not intersect E_1 or E_3. We take $f_1 = f_1'$ and $f_2 = f_1' + f_2' - 2E_2'$, so that f_1, f_2 are conic bundles, with intersection 2, and $-K_X = f_1 + f_2 - E_1 - E_2 - E_3$. The contraction of E_1, E_2, E_3 is a α^3-equivariant birational morphism $X \to X_4$ and f_1, f_2 correspond to the two conic bundles of X_4 invariant by α^3. With this choice, we can compute

$$\begin{aligned}
\alpha(E_1) &= \iota_B(E_2') = \iota_B(f_1 - E_2) = -2K_X - (f_1 - E_2), \\
\alpha^2(E_1) &= -3K_X - \alpha(E_1) - E_1 = -K_X - (f_1 - E_2) - E_1 \\
&= -2K_X - (f_2 - E_3), \\
\alpha(E_3) &= -2K_X - (f_1' - E_1) = -2K_X - (f_1 - E_1), \\
\alpha^2(E_3) &= -3K_X - \alpha(E_3) - E_3 = -K_X - (f_1 - E_1) - E_3 \\
&= -2K_X - (f_1 - E_2).
\end{aligned}$$

This yields the equalities $f_1 = -2K_X + E_1 - \alpha(E_3)$, $f_2 = 2K_X + E_3 - \alpha^2(E_1)$, $E_2 = \alpha^2(E_1) + 2K_X - f_1$, and a straightforward computation gives, with the four equations above, $\alpha^i(f_j)$ and $\alpha^i(E_2)$ for $i, j = 1, 2$. □

Proposition 4.4. *Let Y be a del Pezzo surface of degree $(K_Y)^2 < 4$, and let $\beta \in \mathrm{Aut}(Y)$ be an automorphism of order 4 such that $\mathrm{Pic}(Y)^\beta = \mathbb{Z}K_Y$ and that β^2 fixes pointwise an elliptic curve. Let $\eta_Y \colon Y \to Y_4$ be a birational morphism, so that β^2 acts minimally on Y_4, and let f_1, $f_2 \in \mathrm{Pic}(Y)$ be the divisors corresponding to the two conic bundles on Y_4 that are invariant by β^2 (see Lemma 4.1). Then, one of the following occurs:*

(i) $(K_Y)^2 = 2$, η_Y *contracts two curves* E_1, E_2 *and* β *acts on* $\mathrm{Pic}(Y)^{\beta^2}$ *as*

$$\begin{bmatrix} 1 & 2 & 1 & 1 \\ 2 & 1 & 1 & 1 \\ -2 & -2 & -2 & -1 \\ -2 & -2 & -1 & -2 \end{bmatrix}$$

relative to the basis (f_1, f_2, E_1, E_2).

(ii) $(K_Y)^2 = 1$, η_Y *contracts* E_1, E_2, E_3, *and* β *acts on* $\mathrm{Pic}(Y)^{\beta^2}$ *as*

$$\begin{bmatrix} 3 & 4 & 2 & 2 & 2 \\ 4 & 3 & 2 & 2 & 2 \\ -3 & -3 & -3 & -2 & -2 \\ -3 & -3 & -2 & -3 & -2 \\ -3 & -3 & -2 & -2 & -3 \end{bmatrix}$$

relative to the basis $(f_1, f_2, E_1, E_2, E_3)$.

Remark 4.5. The second case, numerically possible, does not exist (see [DoIs] or [Bla3]).

Proof. Let $E \subset Y$ be any (-1)-curve invariant by β^2. The divisor $E + \beta(E)$ is invariant by β and thus equivalent to sK_Y for some integer s. Computing the intersection with K_Y and the self-intersection, we obtain $-2 = s(K_Y)^2$ and $-2 + 2(E \cdot \beta(E)) = s^2(K_Y)^2$. This gives two possibilities:

$$\begin{array}{ll} \text{(i)} & (K_Y)^2 = 2,\ s = -1,\ E \cdot \beta(E) = 2, \\ \text{(ii)} & (K_Y)^2 = 1,\ s = -2,\ E \cdot \beta(E) = 3. \end{array}$$

In case (i), there are two curves E_1, E_2 contracted by η_Y, and $\beta(E_i) = -K_Y - E_i$ for $i = 1, 2$. Moreover, $f_i - E_1$ is also a (-1)-curve for $i = 1, 2$, so $\beta(f_i) = \beta(E_1) + \beta(f_i - E_1) = -K_Y - E_1 - K_Y - (f_i - E_1) = -2K_Y - f_i$.

In case (ii), there are three curves E_1, E_2, E_3 contracted by η_Y, and $\beta(E_i) = -2K_Y - E_i$ for $i = 1, 2, 3$. As before, we find $\beta(f_i) = -4K_Y - f_i$. □

4.4. Automorphisms of del Pezzo surfaces of order 6, resp. 4,— description of α and β

Automorphisms of del Pezzo surfaces of order 6. We now give explicit possibilities for the automorphism $\alpha \in \mathrm{Aut}(X)$ of order 6.

Case I.

$$X = \{(w : x : y : z) \in \mathbb{P}(3, 1, 1, 2) \mid w^2 = z^3 + \mu x z^4 + x^6 + y^6\},$$
$$\alpha((w : x : y : z)) = (w : x : -\omega y : z)$$

for some general $\mu \in \mathbb{C}$ so that the surface is smooth and where $\omega = e^{2\mathbf{i}\pi/3}$. The surface is a del Pezzo surface of degree 1, and α fixes pointwise the elliptic curve given by $y = 0$. When μ varies, all possible elliptic curves are obtained. The rank of $\mathrm{Pic}(X)^\alpha$ is 1 (see [DoIs, Cor. 6.11]).

Case II.

$$X = \{(w : x : y : z) \in \mathbb{P}^3 \mid wx^2 + w^3 + y^3 + z^3 + \mu wyz = 0\},$$
$$\alpha((w : x : y : z)) = (w : -x : \omega y : \omega^2 z),$$

where $\mu \in \mathbb{C}$ is such that the cubic surface is smooth. The surface is a del Pezzo surface of degree 3, α^3 fixes pointwise the elliptic curve given by $x = 0$, and α acts on this via a translation of order 3. When μ varies, all possible elliptic curves are obtained. The rank of $\mathrm{Pic}(X)^\alpha$ is 1 (see [DoIs, p. 79]).

Case III.

$$X = \{(w : x : y : z) \in \mathbb{P}^3 \mid w^3 + x^3 + y^3 + (x + \mu y)z^2 = 0\},$$
$$\alpha((w : x : y : z)) = (\omega w : x : y : -z),$$

where $\mu \in \mathbb{C}$ is such that the cubic surface is smooth. The surface is a del Pezzo surface of degree 3, α^3 fixes pointwise the elliptic curve given by $z = 0$, and α acts on it via an automorphism of order 3 with 3 fixed points. When μ varies, the birational class of α changes (because the isomorphism class of the curve fixed by α^2 changes) but not the isomorphism class of the elliptic curve fixed by α^3. The rank of $\mathrm{Pic}(X)^\alpha$ is 1 (see [DoIs, p. 79]).

Automorphisms of del Pezzo surfaces of order 4. We now give explicit possibilities for the automorphism $\beta \in \mathrm{Aut}(Y)$ of order 4.

$$Y = \left\{(w : x : y : z) \in \mathbb{P}(2, 1, 1, 1) \;\middle|\; w^2 - x^4 = \prod_{i=1}^{4} yz(y + z)(y + \mu z) = 0\right\},$$
$$\beta((w : x : y : z)) = (w : \mathbf{i}x : y : z),$$

where $\mu \in \mathbb{C}\backslash\{0, 1\}$. The surface is a del Pezzo surface of degree 2 and β fixes pointwise the elliptic curve given by $x = 0$. When μ varies, all possible elliptic curves are obtained. The rank of $\mathrm{Pic}(Y)^\beta$ is 1 (see [DoIs, last line of p. 67] or [Bla3]).

There are other possibilities of automorphisms β of order 4 of rational surfaces Y such that β^2 fixes an elliptic curve, but none for which the rank of $\mathrm{Pic}(Y)^\beta$ is 1 (see [Bla3]).

4.5. The map $X_4 \dashrightarrow Y_4$ that conjugates α^3 to β^2

We now fix $\alpha \in \mathrm{Aut}(X)$, $\beta \in \mathrm{Aut}(Y)$, automorphisms of order 6 and 4 respectively, which act minimally on del Pezzo surfaces X and Y, so that α^3 and β^2 fix (pointwise) elliptic curves $\Gamma_X \subset X$ and $\Gamma_Y \subset Y$, which are isomorphic (as abstract curves).

We denote by $\eta_X : X \to X_4$ and $\eta_Y : Y \to Y_4$ two birational morphisms to del Pezzo surfaces of degree 4, so that α^3 and β^2 act minimally on X_4 and Y_4 respectively. We denote by f_1, $f_2 \in \mathrm{Pic}(X_4) \subset \mathrm{Pic}(X)$, respectively by f_1', $f_2' \in \mathrm{Pic}(Y_4) \subset \mathrm{Pic}(Y)$, the two divisors corresponding to the two conic bundles invariant by α^3, respectively by β^2.

We will choose two points q_1, $q_2 \in \eta_X(\Gamma_X) \subset X_4$, and denote by $\tau : Z_4 \to X_4$ the blow up of these two points.

Lemma 4.6. *For some good choice of q_1, q_2, there exists a birational morphism $\tau'\colon Z_4 \to Y_4$ satisfying the following properties:*

1. *the morphism τ' is the contraction of two disjoints (-1)-curves onto two points q_1' and q_2' of $\eta_Y(\Gamma_Y)$. The two contracted curves are respectively the strict transforms of the curves equivalent to f_1 passing through q_1 and q_2;*
2. *the map $\varphi = \tau'\tau^{-1}$ conjugates α^3 to β^2 (i.e., $\varphi\alpha^3 = \beta^2\varphi$);*
3. *neither q_1 nor q_2 is blown up by η_X, and neither q_1' nor q_2' is blown up by η_Y;*
4. *identifying f_1, f_2 with $\tau^*(f_1), \tau^*(f_2) \in \operatorname{Pic}(Z_4)$ and f_1', f_2' with $\tau'^*(f_1'), \tau'^*(f_2') \in \operatorname{Pic}(Z_4)$, we have the following relations in $\operatorname{Pic}(Z_4)$:*

$$
\begin{aligned}
f_1 &= f_1', & f_1' &= f_1,\\
f_2 &= f_2' + 2f_1' - 2E_{\tau'}, & f_2' &= f_2 + 2f_1 - 2E_\tau,\\
E_\tau &= 2f_1' - E_{\tau'}, & E_{\tau'} &= 2f_1 - E_\tau,
\end{aligned}
$$

where E_τ, $E_{\tau'} \in \operatorname{Pic}(Z_4)$ correspond to the exceptional divisors of τ and τ' respectively, which are the sum of two exceptional curves.

Proof. Denote by $\pi\colon X_4 \to \mathbb{P}^1$ and $\pi'\colon Y_4 \to \mathbb{P}^1$ the morphisms whose fibres are f_1 and f_1' respectively. As was already observed in the proof of Lemma 4.1, both π, π' are conic bundles, with four singular fibres, and the four singular fibres correspond to the four branch points of the double coverings $\pi\colon \eta_X(\Gamma_X) \to \mathbb{P}^1$ and $\pi'\colon \eta_Y(\Gamma_Y) \to \mathbb{P}^1$. Since Γ_X and Γ_Y are isomorphic elliptic curves, we can assume that the four points are the same for both morphisms. Denote by $\Delta \subset \mathbb{P}^1$ the union of the image by π of the points blown up by η_X, the image by π' of the points blown up by η_Y, and the points corresponding to singular fibres of π (or π').

We define a closed subset $V \subset \Gamma_X \times \Gamma_X$ consisting of pairs (q_1, q_2) that we "do not want", and denote by U its complement. The closed subset V is the union of the pairs (q_1, q_2) such that $\pi(q_1)$ or $\pi(q_2)$ belongs to Δ. Observe that V is a finite union of curves of $\Gamma_X \times \Gamma_X$ (of bidegree $(0,1)$ or $(1,0)$).

Choosing $(q_1, q_2) \in U$, such that q_1, q_2 are on distinct fibres of π, we can define a birational morphism $\tau'\colon Z_4 \to W$ which contracts the strict transforms of the fibres of π which pass through q_1 and q_2. The map $\varphi = \tau'\tau^{-1}$ conjugates α^3 to a biregular automorphism of W, which preserves any fibre of the conic bundle $\pi_W = \pi\varphi^{-1}$. In fact, φ is a sequence of two elementary links of conic bundles. It remains to show that for a good choice of $(q_1, q_2) \in U$, the triplet $(W, \pi_W, \varphi\alpha^3\varphi^{-1})$ is isomorphic to (Y, π', β^2), using Lemma 4.2.

Let $E_1 \subset X_4$ be a (-1)-curve which is a section of π; we fix a birational morphism $\mu_X\colon X_4 \to \mathbb{P}^2$ which contracts E_1 and all (-1)-curves lying on fibres of π that do not intersect E_1, which we call $E_2, \dots, E_5$. The fibres of π correspond to lines of $\mathbb{P}^2$ passing through the point $p_1 = \mu_X(E_1)$, the curves equivalent to f_2 correspond to conics passing through $p_2 = \mu_X(E_2), \dots, p_5 = \mu_X(E_5)$. For any pair (q_1, q_2), we denote by $C \subset X_4$ (respectively $D \subset X_4$) the strict transform of the conic of $\mathbb{P}^2$ passing through p_1, p_2, p_3, q_1, q_2 (respectively p_1, p_4, p_5, q_1, q_2), and denote by $C', D' \subset W$ their strict transforms by φ. The curves C, D are sections of π and intersect into three points: q_1, q_2, $r \in X_4$. The curves C', D'

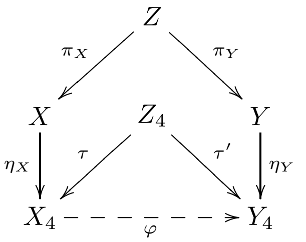
are sections of π_W of self-intersection -1, and intersect into one point, which is $\varphi(r) \in W$. The isomorphism class of the triplet $(W, \pi_W, \varphi\alpha^3\varphi^{-1})$ is given by $\pi_W(\varphi(r)) \in \mathbb{P}^1$ (Lemma 4.2), equal to $\pi(r) \in \mathbb{P}^1$. Fixing q_1, and choosing one of the two possibilities for r, on the fibre given by the isomorphism class of (Y, π_Y, β^2), the curves C, D can be chosen as the conics passing respectively through p_1, p_2, p_3, q_1, r and p_1, p_4, p_5, q_1, r, so q_2 is uniquely defined. This gives us two irreducible curves V_1, V_2 of bidegree $(1, 1)$ in $\Gamma_X \times \Gamma_X$, which are thus not contained in V. Choosing a general point of $V_1 \cap U$, the triplet $(W, \pi_W, \varphi\alpha^3\varphi^{-1})$ is isomorphic to (Y, π_Y, β^2).

The fact that η_X does not blow up q_1 or q_2 and that η_Y does not blow up q_1' or q_2' is given by the fact that $\pi(q_i) = \pi'(q_i') \notin \Delta$ for $i = 1, 2$.

It remains to show the relations in $\mathrm{Pic}(Z_4)$. The equalities $f_1 = f_1'$ and $E_\tau + E_{\tau'} = 2f_1$ are given by the construction of τ, τ'. The adjunction formula, and the fact that $-K_{X_4} = f_1 + f_2$, $-K_{Y_4} = f_1' + f_2'$ yields $-K_{Z_4} = f_1 + f_2 - E_\tau = f_1' + f_2' - E_{\tau'}$ and the remaining equalities. □

4.6. The hyperbolic embeddings

Now we have the map $\varphi\colon X_4 \dashrightarrow Y_4$ constructed in Section 4.5 above, which conjugates α^3 to β^2, and the group generated by α and β is a subgroup of the Cremona group, which is isomorphic to $\mathrm{SL}(2, \mathbb{Z})$ if and only if there is no other relation than the obvious $1 = \alpha^6 = \beta^4 = \alpha^3\beta^2$ which arise by construction. We compute the action of α, β on $\mathrm{Pic}(X)$, $\mathrm{Pic}(Y)$, and on a surface Z which dominates X, Y, where both α, β act. This surface exists if the group generated by the action of both maps on the elliptic curve fixed by α^3 and β^2 is a finite subgroup of automorphisms of the curve (which is true, for example, when either α or β fixes the curve); and if it does not exist, we can also compute the action on the limit of the Picard groups obtained.



(1)

Proposition 4.7. *For $j = 1, 2, 3$, choose $\alpha \in \mathrm{Aut}(X)$ as an automorphism of order 6 of a del Pezzo surface X, which is respectively given in Case* I, II *or* III *of Section* 4.4, *such that α^3 fixes pointwise an elliptic curve Γ_X, and choose β as an automorphism of order* 4 *of a del Pezzo surface Y of degree* 2, *which fixes pointwise an elliptic curve Γ_Y isomorphic to Γ_X, (which implies that α^3 and β^2 are conjugate). This yields, with the above construction, a hyperbolic embedding $\theta_{h,j}\colon \mathrm{SL}(2, \mathbb{Z}) \subset \mathrm{Aut}(Z) \subset \mathrm{Bir}(Z) \simeq \mathrm{Bir}(\mathbb{P}^2)$ which preserves an elliptic curve Γ isomorphic to Γ_X and Γ_Y.*

The surface Z is obtained by blowing up respectively 12, 10 *and* 10 *points on a smooth cubic curve of $\mathbb{P}^2$ isomorphic to Γ, and the action of $\theta_{h,i}(\mathrm{SL}(2, \mathbb{Z}))$ on Γ*

is respectively the identity, a translation of order 3 *and an automorphism of order* 3 *with fixed point. There is no curve of Z distinct from Γ which is invariant by* $\theta_{h,i}(\mathrm{SL}(2,\mathbb{Z}))$*. The curve Γ can be chosen to be any elliptic curve for $j = 1, 2$.*

Proof. In case $j = 1$, we take $(f_1, f_2, E_1, E_2, E_3)$ as a basis of $\mathrm{Pic}(X)^{\alpha^3}$, where E_1, E_2, E_3 are the three curves contracted by η_X, and f_1, f_2 correspond to the fibres of the two conic bundles invariant by α^3 on X_4. Applying Proposition 4.3, α preserves the submodule generated by f_1, f_2, E, where $E = E_1 + E_2 + E_3$ is the divisor contracted by η_X, and its action relative to this basis is

$$\begin{bmatrix} 1 & 3 & 3 \\ 3 & 4 & 6 \\ -2 & -4 & -5 \end{bmatrix}.$$

In cases $j = 2, 3$, we take (f_1, f_2, E) as a basis of $\mathrm{Pic}(X)^{\alpha^3}$, where $E = E_1$ is the (irreducible) divisor contracted by η_X, and f_1, f_2 correspond to the fibres of the two conic bundles invariant by α^3 on X_4. Applying Proposition 4.3, the action of α on $\mathrm{Pic}(X)^{\alpha^3}$ relative to this basis is

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & -2 & -1 \end{bmatrix}$$

(for a good choice of f_1, f_2, E).

In each of the three cases, we take (f_1', f_2', E_1', E_2') as a basis of $\mathrm{Pic}(Y)^{\beta^2}$, where E_1', E_2' are the divisors contracted by η_Y, and f_1', f_2' correspond to the fibres of the two conic bundles invariant by β^2 on Y_4. Applying Proposition 4.4, β preserves the submodule generated by f_1', f_2', E', where $E' = E_1' + E_2'$ is the divisor contracted by η_Y and its action relative to this basis is

$$\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ -2 & -2 & -3 \end{bmatrix}.$$

We denote by $\pi_X \colon Z \to X$ the blow up of the points corresponding to the points blown up by τ and η_Y (see Diagram (1)), and denote again their exceptional divisors by E_τ and E'. Similarly, we denote by $\pi_Y \colon Z \to Y$ the blow up of the points corresponding to the two points blown up by τ' and η_X, and denote again their exceptional divisors by $E_{\tau'}$ and E. Since X_4 and Y_4 are del Pezzo surfaces of degree 4, they are obtained by blowing up 5 points of $\mathbb{P}^2$, all lying on the smooth cubic that is the image of Γ_X or Γ_Y. This implies that Z is the blow up of 12 points of $\mathbb{P}^2$ if $i = 1$ and of 10 points of $\mathbb{P}^2$ if $i = 2, 3$, all points belonging to the smooth cubic curve. Moreover, both α and β lift to automorphisms of Z.

We denote by the same name the pull-backs of the divisors f_1, f_2, E, E', E_τ on Z. Recall that E_τ is the sum of two (-1)-curves. The action of α in case $j = 1$, α in case $j \in \{2, 3\}$ and β in each case on the subvectorspace W of $\mathrm{Pic}(Z) \otimes \mathbb{R}$

generated by $(f_1, f_2, E, E', E_\tau)$ are respectively

$$\begin{bmatrix} 1 & 3 & 3 & 0 & 0 \\ 3 & 4 & 6 & 0 & 0 \\ -2 & -4 & -5 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & -2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 5 & 10 & 0 & 6 & 8 \\ 2 & 5 & 0 & 2 & 4 \\ 0 & 0 & 1 & 0 & 0 \\ -2 & -6 & 0 & -3 & -4 \\ -4 & -8 & 0 & -4 & -7 \end{bmatrix}$$

relative to this basis. The first two matrices are obtained because α fixes the curve Γ_X, and because E', E_τ correspond to points of Γ_X which are not blown up by η_X (Lemma 4.6). The third matrix is obtained by applying again Lemma 4.6, which yields the equations $f_1 = f_1'$, $f_2 = f_2' + 2f_1' - 2E_{\tau'}$, $E_\tau = 2f_1' - E_{\tau'}$. One easily checks that the only elements of W which are fixed by α and β are the multiples of the canonical divisor, corresponding to $[1, 1, -1, -1, -1]$. This implies that any curve $C \subset Z$ invariant by the group is a multiple of the elliptic curve $\Gamma_Z \subset Z$ (strict transform of Γ_X and Γ_Y). This curve having negative self-intersection, C has to be equal to Γ_Z.

By construction, we have $\alpha^6 = \beta^4 = 1$ and $\beta^2 = \alpha^3$. We have to prove that no other relation holds, and that any element of infinite order corresponds to a hyperbolic element of $\mathrm{Aut}(Z)$. Writing $\rho_1 = \alpha\beta$ and $\rho_2 = \alpha^2\beta$, this corresponds to show that for any sequence $(i_1, \dots, i_n)$ with $i_k \in \{1, 2\}$, the element $\rho_{i_n} \cdot \dots \cdot \rho_{i_1}$ is a hyperbolic element of $\mathrm{Aut}(Z)$.

To show this, we look at the action of α, β on the orthogonal $W_0 = K^\perp$ of the canonical divisor $K \in W \subset \mathrm{Pic}(Z)$ in W. We choose a basis of W_0, made of orthogonal eigenvectors of β.

If $j = 1$, the basis is

$$[1, 0, 0, -1, 0],\ [2, 1, 0, -1, -2],\ [3, 1, -2, -1, -2],\ [4, 2, -2, -2, -3],$$

which has signature $\langle -2, -2, -2, 2\rangle$ and the actions of α, α^2, β relative to it are respectively

$$\begin{bmatrix} 0 & -1 & -2 & -2 \\ -2 & -2 & -3 & -4 \\ -1 & 0 & -2 & -2 \\ 2 & 2 & 4 & 5 \end{bmatrix}, \quad \begin{bmatrix} 0 & -2 & -1 & -2 \\ -1 & -2 & 0 & -2 \\ -2 & -3 & -2 & -4 \\ 2 & 4 & 2 & 5 \end{bmatrix}, \quad \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

We denote by H the fourth basis vector, which is the only one with a positive square, and compute by induction on n the vector $H_n = \rho_{i_n} \cdot \dots \cdot \rho_{i_1}(H)$ for $n \geq 0$ (with $H_0 = H$). Writing $H_n = \begin{bmatrix} -a_n \\ -b_n \\ -c_n \\ \ell_n \end{bmatrix}$, we prove by induction on n the following inequalities:

$$\begin{aligned} a_n,\ b_n,\ c_n,\ \ell_n &\geq 0, \\ \ell_n &> \frac{6}{5} c_n, \\ \ell_n &> 2a_n, \\ \ell_n &\geq \left(\frac{5}{3}\right)^n, \end{aligned} \tag{2}$$

where the last one will yield the result, implying that $\rho_{i_k}\cdot\dots\cdot\rho_{i_1}$ is a hyperbolic element of $\mathrm{Aut}(Z)$ of dynamical degree $\geq (5/3)^k$.

Note that (2) is easily checked for $n=0$ since $\ell_0=1$, $a_0=b_0=c_0=0$. We assume the result true for n and prove it for $n+1$. We have $H_{n+1}=\rho_{i_{n+1}}(H_n)=\alpha^{i_{n+1}}\beta(H_n)$, which is equal to

$$\begin{bmatrix} -b_n+2c_n-2\ell_n\\ -2a_n-2b_n+3c_n-4\ell_n\\ -a_n+2c_n-2\ell_n\\ 2a_n+2b_n-4c_n+5\ell_n\end{bmatrix} \quad\text{or}\quad \begin{bmatrix} -2b_n+c_n-2\ell_n\\ -a_n-2b_n-2\ell_n\\ -2a_n-3b_n+2c_n-4\ell_n\\ 2a_n+4b_n-2c_n+5\ell_n\end{bmatrix}.$$

We deduce the inequalities a_{n+1}, b_{n+1}, c_{n+1}, $\ell_{n+1}\geq 0$ directly from a_n, $b_n\geq 0$ and $\ell_n\geq c_n\geq 0$. Computing $\ell_{n+1}-2a_{n+1}=\ell_n+2a_{2n}$, we obtain $\ell_{n+1}>2a_{n+1}$. We then compute $5\ell_{n+1}-6c_{n+1}$ to see that it is positive, and obtain either $13\ell_n-8c_n+4a_n+10b_n>(13-8\cdot 5/6)\ell_n+4a_n+10b_n>0$ or $\ell_n+2c_n+2b_n-2a_n>0$. To get (2), it remains to see that $\ell_{n+1}\geq 5\ell_n-4c_n=(5/3)\ell_n+4((5/6)\ell_n-c_n)>(5/3)\ell_n\geq(5/3)^{n+1}$.

For $j=2,3$, the situation is similar, with other data. The basis is now $[1,0,0,-1,0]$, $[2,1,0,-1,-2]$, $[8,2,-2,-2,-5]$, $[9,3,-2,-3,-6]$, which has signature $\langle -2,-2,-6,6\rangle$ and the actions of α,α^2,β relative to it are respectively

$$\begin{bmatrix} -2&-9&-18&-24\\ -6&-20&-36&-51\\ -6&-18&-35&-48\\ 7&22&42&58\end{bmatrix},\quad \begin{bmatrix} -2&-6&-18&-21\\ -9&-20&-54&-66\\ -6&-12&-35&-42\\ 8&17&48&58\end{bmatrix},\quad \begin{bmatrix} -1&0&0&0\\ 0&-1&0&0\\ 0&0&1&0\\ 0&0&0&1\end{bmatrix}.$$

We again denote by H the fourth basis vector, which is the only one with a positive square, and compute by induction on n the vector $H_n=\rho_{i_n}\cdots\rho_{i_1}(H)$ for $n\geq 0$ (with $H_0=H$). Writing $H_n=\begin{bmatrix}-a_n\\ -b_n\\ -c_n\\ \ell_n\end{bmatrix}$, we prove by induction on n the following inequalities:

$$\begin{aligned} a_n,b_n,c_n,\ell_n&\geq 0,\\ \ell_n&>c_n,\\ \ell_n&\geq 10^n,\end{aligned}\tag{3}$$

where the last one will yield the result, implying that $\rho_{i_k}\cdots\rho_{i_1}$ is a hyperbolic element of $\mathrm{Aut}(Z)$ of dynamical degree $\geq 10^k$.

Again, (3) is easily checked for $n=0$ since $\ell_0=1$, $a_0=b_0=c_0=0$. We assume the result true for n and prove it for $n+1$. We have $H_{n+1}=\rho_{i_{n+1}}(H_n)=\alpha^{i_{n+1}}\beta(H_n)$, which is equal to

$$\begin{bmatrix} -2a_n-9b_n+18c_n-24\ell_n\\ -6a_n-20b_n+36c_n-51\ell_n\\ -6a_n-18b_n+35c_n-48\ell_n\\ 7a_n+22b_n-42c_n+58\ell_n\end{bmatrix} \quad\text{or}\quad \begin{bmatrix} -2a_n-6b_n+18c_n-21\ell_n\\ -9a_n-20b_n+54c_n-66\ell_n\\ -6a_n-12b_n+35c_n-42\ell_n\\ 8a_n+17b_n-48c_n+58\ell_n\end{bmatrix}.$$

We deduce the inequalities a_{n+1}, b_{n+1}, c_{n+1}, $\ell_{n+1} \geq 0$ directly from a_n, $b_n \geq 0$ and $\ell_n \geq c_n \geq 0$. Since $\ell_{n+1} - c_{n+1}$ is either equal to $a_n + 4b_n - 7b_n + 10\ell_n$ or to $2a_n + 5b_n - 13c_n + 16\ell_n$, it is positive. To get (3), it remains to see that

$$\ell_{n+1} \geq 58\ell_n - 48c_n = 10\ell_n + 48(\ell_n - c_n) \geq 10\ell_n \geq (10)^{n+1}. \qquad \square$$

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