# On convergence in mixed integer programming 

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#### Abstract

Let $P \subseteq \mathbb{R}^{m+n}$ be a rational polyhedron, and let $P_{I}$ be the convex hull of $P \cap\left(\mathbb{Z}^{m} \times \mathbb{R}^{n}\right)$. We define the integral lattice-free closure of $P$ as the set obtained from $P$ by adding all inequalities obtained from disjunctions associated with integral lattice-free polyhedra in $\mathbb{R}^{m}$. We show that the integral lattice-free closure of $P$ is again a polyhedron, and that repeatedly taking the integral lattice-free closure of $P$ gives $P_{I}$ after a finite number of iterations. Such results can be seen as a mixed integer analogue of theorems by Chvátal and Schrijver for the pure integer case. One ingredient of our proof is an extension of a result by Owen and Mehrotra. In fact, we prove that for each rational polyhedron $P$, the split closures of $P$ yield in the limit the set $P_{I}$.


Keywords Convergence • Cutting planes • Disjunctive programming • Lattice-free polyhedra • Mixed integer programming • Split cuts

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## 1 Introduction

Cutting plane techniques have been one of the prominent topics in the theory of integer and mixed integer programming. A fundamental result in the theory of cutting planes was shown by Chvátal [4] and Schrijver [13], and was inspired by Gomory's [6] early work. To state such result we recall the notion of Gomory cuts. An inequality cx $\leq\lfloor\gamma\rfloor$ is a Gomory cut for $P \subseteq \mathbb{R}^{m}$ if $c \in \mathbb{Z}^{m}$ and if $c x \leq \gamma$ is valid for $P$. The Chvátal

[^0]closure $P^{\prime}$ of $P$ is the set of all vectors that satisfy every Gomory cut for $P$. We denote by $P^{(i)}, i \in \mathbb{N}$, the $i$-th Chvátal closure of $P$, i.e. $P^{(i)}:=\left(P^{(i-1)}\right)^{\prime}$, where $P^{(0)}:=P$. We recall that a polyhedron is called rational if it can be described by a finite system of linear inequalities with rational data. A series of results of Chvátal and Schrijver gives the following theorem, where we denote by $P_{I}$ the convex hull of $P \cap \mathbb{Z}^{m}$.

Theorem 1 For each rational polyhedron $P$, then
(i) $P^{\prime}$ is again a rational polyhedron,
(ii) $P^{(k)}=P_{I}$ for some integer $k$.

In a mixed integer programming problem, only some of the variables are restricted to integer values. Then the set of feasible solutions to such a problem attains the form

$$
\left\{(x, y) \in P: x \in \mathbb{Z}^{m}\right\}
$$

where $P$ is a polyhedron in $\mathbb{R}^{m+n}$. Note that, with a slight abuse of notation we write column vectors in $\mathbb{R}^{m+n}$ in the form $(x, y)$. The vectors $(x, y) \in \mathbb{R}^{m+n}$ such that $x \in \mathbb{Z}^{m}$ are called $x$-integral. We denote by $P_{I}$ the convex hull of the $x$-integral vectors in $P$, and we say that $P$ is $x$-integral if $P=P_{I}$. It is well-known that a rational polyhedron $P$ is $x$-integral if and only if each minimal face of $P$ contains $x$-integral vectors. Moreover, if $P$ is rational then $P_{I}$ is a rational polyhedron (see [9], [14, Sect. 16.7]). In the pure integer setting, i.e. when $n=0$, we call vectors and polyhedra simply integral, instead of $x$-integral.

An inequality $c x+d y \leq \gamma$ is a split cut for $P \subseteq \mathbb{R}^{m+n}$ if there exists a vector $a \in \mathbb{Z}^{m}$ and an integer $\beta$ such that $c x+d y \leq \gamma$ is valid for both

$$
\{(x, y) \in P: a x \leq \beta\} \text { and }\{(x, y) \in P: a x \geq \beta+1\} .
$$

The split closure of $P$ is defined as the set of all vectors that satisfy every split cut for $P$. Given $P \subseteq \mathbb{R}^{m+n}$, we denote by $\mathcal{S}(P)$ its split closure. Moreover, for every $i \in \mathbb{N}$, we denote by $\mathcal{S}^{i}(P)$ the $i$-th split closure of $P$, i.e. $\mathcal{S}^{i}(P):=\mathcal{S}\left(\mathcal{S}^{i-1}(P)\right)$, where $\mathcal{S}^{0}(P):=P$. Cook et al. [5] proved that the split closure of a rational polyhedron $P \subseteq \mathbb{R}^{m+n}$ is again a rational polyhedron. In the general mixed integer case, Cook et al. showed that determining split closures does not suffice to generate $P_{I}$ in a finite number of iterations. The reason is that, even if $\mathcal{S}^{i}(P) \neq P_{I}$ implies that $\mathcal{S}^{i+1}(P) \subset \mathcal{S}^{i}(P)$, the difference between $\mathcal{S}^{i}(P)$ and $\mathcal{S}^{i+1}(P)$ may become arbitrarily small as $i$ grows.

Cook et al. also showed that, given a rational polyhedron $P, P_{I}$ can be generated in a finite number of iterations by combining split closures with certain rounding cuts that correspond to a fixed discretization of the continuous variables based on the original system defining $P$. However, it remained a challenge for many years to design finite cutting plane algorithms that directly work in the original mixed integer setting, without discretizing the continuous variables, and without remembering the original system.

Split closures alone do not lead to finite convergence to $P_{I}$ even if $P$ is bounded. In this special case, however, Owen and Mehrotra [10] showed that the sequence of split
closures does yield in the limit the set $P_{I}$. In Sect. 2 we extend the result of Owen and Mehrotra from polytopes to rational polyhedra. That is, we prove that for each rational polyhedron $P$, the repeated computation of the split closures yields in the limit the set $P_{I}$. This result is a backbone of the main theorem of our paper, which can be seen as an analogue of Theorem 1 in mixed integer programming. In order to state it precisely, we next introduce the notion of lattice-free polyhedra.

A polyhedron $L \subseteq \mathbb{R}^{m}$ is said to be lattice-free if $\operatorname{relint}(L) \cap \mathbb{Z}^{m}=\emptyset$. (We recall that relint $L$ denotes the relative interior of a polyhedron $L$, which is the set of points $x$ for which there exists a ball centered in $x$ whose intersection with the affine hull of $L$ is contained in $L$.) An inequality $c x+d y \leq \gamma$ is an integral lattice-free cut for $P \subseteq \mathbb{R}^{m+n}$ if there exists an integral lattice-free polyhedron $\left\{x \in \mathbb{R}^{m}: a_{j} x \leq \beta_{j}, j \in J\right\}$ such that $c x+d y \leq \gamma$ is valid for every set

$$
\left\{(x, y) \in P: a_{j} x \geq \beta_{j}\right\}, \quad j \in J .
$$

It is easy to see that an integral lattice-free cut is satisfied by all $x$-integral vectors in $P$. Clearly, every split cut for $P$ is also an integral lattice-free cut for $P$. The integral lattice-free closure of $P$ is defined as the set of all vectors that satisfy every integral lattice-free cut for $P$. Given $P \subseteq \mathbb{R}^{m+n}$, we denote by $\mathcal{L}(P)$ its integral lattice-free closure. Moreover, for every $i \in \mathbb{N}$, we denote by $\mathcal{L}^{i}(P)$ the $i$-th integral lattice-free closure of $P$, i.e. $\mathcal{L}^{i}(P):=\mathcal{L}\left(\mathcal{L}^{i-1}(P)\right)$, where $\mathcal{L}^{0}(P):=P$.

We are now ready to state the main result of our paper. In Sect. 3 we prove that, if $P$ is a rational polyhedron, then repeatedly taking the integral lattice-free closure of $P$ gives $P_{I}$ after a finite number of iterations. Moreover, we prove that the integral lattice-free closure of a rational polyhedron $P$ is again a rational polyhedron. We also show that in general it is not true that the integral lattice-free closure of $P$ equals $P_{I}$.

From now on in this paper, if not explicitly stated, we work with rational spaces, rather than real ones. In particular, any matrix, any vector, and any polyhedron is supposed to be rational. Moreover, given sets $P$ and $Q$, we define $P+Q:=\{p+q$ : $p \in P, q \in Q\}$. Finally, we denote by $\mathbb{N}$ the set of nonnegative integers, and by $B$ be the unit ball.

## 2 Infinite convergence

Let $\left\{\tilde{P}, P^{i}: i \in \mathbb{N}\right\}$ be a family of closed sets such that $\tilde{P} \subseteq P^{i+1} \subseteq P^{i}$ for every $i \in \mathbb{N}$. We say that the sequence $\left\{P^{i}: i \in \mathbb{N}\right\}$ (Hausdorff) converges to $\tilde{P}$, and we write $\lim _{i \rightarrow \infty} P^{i}=\tilde{P}$, if for every $\epsilon>0$, there exists $k \in \mathbb{N}$ such that $P^{k} \subseteq \tilde{P}+\epsilon B$. The given definition of convergence is based on the well-known Hausdorff distance, see [12, Sect. 3] for more details. Note that, if $\tilde{P}=\emptyset$, then it follows by definition that there exists $k \in \mathbb{N}$ such that $P^{i}=\emptyset$ for all $i \geq k$. It is a well-known fact that if the sequence $\left\{P^{i}: i \in \mathbb{N}\right\}$ converges, then $\lim _{i \rightarrow \infty} P^{i}=\bigcap_{i \in \mathbb{N}} P^{i}$ (see for example [12, Theorem 2, Proposition 2]).

The main result of this section is that $\lim _{i \rightarrow \infty} \mathcal{S}^{i}(P)=P_{I}$, for each polyhedron $P$. In the remainder of this section, we develop the proof of such statement. Note that in the special case when $P$ is a polytope, this result has been shown by Owen
and Mehrotra [10]. We apply their overall proof strategy. However, several technical results are necessary to provide the proof for general polyhedra. We now give some easy observations about Hausdorff convergence that we need later, only sketching their proofs.

Given a set of vectors $P$, in what follows we denote with conv.hull $P$, lin.hull $P$, aff.hull $P$ respectively the convex hull, the linear hull, and the affine hull of $P$. We refer to [14] for the standard definitions of such concepts. We also denote with cl $P$ the topological closure of $P$.
Observation 1 Let $\left\{\tilde{P}, P^{i}, Q^{i}: i \in \mathbb{N}\right\}$ be a family of closed sets with $\tilde{P} \subseteq P^{i+1} \subseteq$ $P^{i}, \tilde{P} \subseteq Q^{i+1} \subseteq Q^{i}, Q^{i} \subseteq P^{i}$ for every $i \in \mathbb{N}, \lim _{i \rightarrow \infty} P^{i}=\tilde{P}$. Then

$$
\lim _{i \rightarrow \infty} Q^{i}=\tilde{P}
$$

Proof Let $\epsilon>0$. Since $\lim _{i \rightarrow \infty} P^{i}=\tilde{P}$, there exists $k \in \mathbb{N}$ such that $P^{k} \subseteq \tilde{P}+\epsilon B$. This implies that $Q^{k} \subseteq P^{k} \subseteq \tilde{P}+\epsilon B$.
Observation 2 Let $\left\{\tilde{P}, \tilde{Q}, P^{i}, Q^{i}: i \in \mathbb{N}\right\}$ be a family of closed sets with $\tilde{P} \subseteq$ $P^{i+1} \subseteq P^{i}, \tilde{Q} \subseteq Q^{i+1} \subseteq Q^{i}$ for every $i \in \mathbb{N}, \lim _{i \rightarrow \infty} P^{i}=\tilde{P}, \lim _{i \rightarrow \infty} Q^{i}=\overline{\tilde{Q}}$. Then

$$
\lim _{i \rightarrow \infty}\left(P^{i} \cup Q^{i}\right)=\tilde{P} \cup \tilde{Q}
$$

Proof Clearly $\left\{\tilde{P} \cup \tilde{Q}, P^{i} \cup Q^{i}: i \in \mathbb{N}\right\}$ is a family of closed sets with $\tilde{P} \cup \tilde{Q} \subseteq$ $P^{i+1} \cup Q^{i+1} \subseteq P^{i} \cup Q^{i}$ for every ${\underset{\sim}{P}} \in \mathbb{N}$.

Let $\epsilon>0$. Since $\lim _{i \rightarrow \infty} P^{i}=\tilde{P}$, and $\lim _{i \rightarrow \infty} Q^{i}=\tilde{Q}$, there exists $k \in \mathbb{N}$ such that $P^{k} \subseteq \tilde{P}+\epsilon B$, and $Q^{k} \subseteq \tilde{Q}+\epsilon B$. This implies that $P^{k} \cup Q^{k} \subseteq(\tilde{P}+\epsilon B) \cup$ $(\tilde{Q}+\epsilon B)=(\tilde{P} \cup \tilde{Q})+\epsilon B$.
Observation 3 Let $\left\{\tilde{P}, P^{i}: i \in \mathbb{N}\right\}$ be a family of closed sets with $\tilde{P} \subseteq P^{i+1} \subseteq P^{i}$ for every $i \in \mathbb{N}, \lim _{i \rightarrow \infty} P^{i}=\tilde{P}$. Then

$$
\lim _{i \rightarrow \infty} \text { cl conv.hull } P^{i}=\text { cl conv.hull } \tilde{P}
$$

Proof Clearly $\left\{\right.$ cl conv.hull $\tilde{P}$, cl conv.hull $\left.P^{i}: i \in \mathbb{N}\right\}$ is a family of closed sets with cl conv.hull $\tilde{P} \subseteq$ cl conv.hull $P^{i+1} \subseteq$ cl conv.hull $P^{i}$ for every $i \in \mathbb{N}$.

Let $\epsilon>0$. Since $\lim _{i \rightarrow \infty} P^{i}=\tilde{P}$, there exists $k \in \mathbb{N}$ such that $P^{k} \subseteq \tilde{P}+\epsilon B$. This implies that cl conv.hull $P^{k} \subseteq$ cl conv.hull $(\tilde{P}+\epsilon B)$. Since $\epsilon B$ is convex, it is easy to verify that conv.hull $(\tilde{P}+\epsilon B)=($ conv.hull $\tilde{P})+\epsilon B$. Since moreover $\epsilon B$ is compact, it can be checked that cl conv.hull $(\tilde{P}+\epsilon B)=\operatorname{cl}\left(\left(\right.\right.$ conv.hull $\left.\left._{\tilde{P}}\right)+\epsilon B\right)=$ $($ cl conv.hull $\tilde{P})+\epsilon B$. Hence cl conv.hull $P^{k} \subseteq($ cl conv.hull $\tilde{P})+\epsilon B$.

Observation 4 Let $\left\{\tilde{P}, \tilde{Q}, P^{i}, Q^{i}: i \in \mathbb{N}\right\}$ be a family of closed sets with $\tilde{P} \subseteq$ $P^{i+1} \subseteq P^{i}, \tilde{Q} \subseteq Q^{i+1} \subseteq Q^{i}, P^{i} \subseteq Q^{i}$ for every $i \in \mathbb{N}, \lim _{i \rightarrow \infty} P^{i}=\tilde{P}, \lim _{i \rightarrow \infty}$ $Q^{i}=\overline{\tilde{Q}}$. Then

$$
\tilde{P} \subseteq \tilde{Q} .
$$

Proof Since $P^{i} \subseteq Q^{i}$ for every $i \in \mathbb{N}$, it follows that $\bigcap_{i \in \mathbb{N}} P^{i} \subseteq \bigcap_{i \in \mathbb{N}} Q^{i}$. The statement follows since $\tilde{P}=\bigcap_{i \in \mathbb{N}} P^{i}$, and $\tilde{Q}=\bigcap_{i \in \mathbb{N}} Q^{i}$.

Observation 5 Let $\left\{\tilde{P}, P^{i}: i \in \mathbb{N}\right\}$ be a family of closed sets with $\tilde{P} \subseteq P^{i+1} \subseteq P^{i}$ for every $i \in \mathbb{N}$, where $\tilde{P}$ is the polyhedron defined by the finite system $c_{j} z \leq \gamma_{j}, j \in J$. Then $\lim _{i \rightarrow \infty} P^{i}=\tilde{P}$ if and only if for every $j \in J, \epsilon>0$, there exists $k \in \mathbb{N}$ such that $c_{j} z \leq \gamma_{j}+\epsilon$ is valid for $P^{k}$.

Proof At first we prove necessity of the condition. By hypothesis, for every $\delta>0$, there exists $k \in \mathbb{N}$ such that $P^{k} \subseteq \tilde{P}+\delta B$. Let $j \in J$. Since $\tilde{P}$ is a polyhedron, it can be seen that there exists $\lambda>0$, independent on $\delta$, such that $c_{j} z \leq \gamma_{j}+\lambda \delta$ is valid for $P^{k}$. Now let $\epsilon>0$. The statement follows by choosing $\delta$ such that $\lambda \delta \leq \epsilon$.

We now prove sufficiency of the condition. Since $J$ is finite, it follows by hypothesis that for every $\delta>0$, there exists $k \in \mathbb{N}$ such that $c_{j} z \leq \gamma_{j}+\delta$ is valid for $P^{k}$ for every $j \in J$. If $\tilde{P}=\emptyset$, there exists $\delta>0$ such that the system $c_{j} z \leq \gamma_{j}+\delta, j \in J$, is infeasible, and so in this case $P^{k}=\emptyset$, and we are done. Otherwise, since $\tilde{P}$ is a polyhedron, it can be seen that there exists $\lambda>0$, independent on $\delta$, such that $P^{k} \subseteq \tilde{P}+\lambda \delta B$. Let $\epsilon>0$. The statement follows by choosing $\delta$ such that $\lambda \delta \leq \epsilon$.

Observation 6 Let $\left\{\tilde{P}, \tilde{Q}, P^{i}, Q^{i}: i \in \mathbb{N}\right\}$ be a family of closed sets with $\tilde{P} \subseteq$ $P^{i+1} \subseteq P^{i}, \tilde{Q} \subseteq Q^{i+1} \subseteq Q^{i}$ for every $i \in \mathbb{N}, \lim _{i \rightarrow \infty} P^{i}=\tilde{P}, \lim _{i \rightarrow \infty} Q^{i}=\overline{\tilde{Q}}$, and where $\tilde{P}, \tilde{Q}$ are polyhedra. Then

$$
\lim _{i \rightarrow \infty}\left(P^{i} \cap Q^{i}\right)=\tilde{P} \cap \tilde{Q}
$$

Proof Clearly $\left\{\tilde{P} \cap \tilde{Q}, P^{i} \cap Q^{i}: i \in \mathbb{N}\right\}$ is a family of closed sets with $\tilde{P} \cap \tilde{Q} \subseteq$ $P^{i+1} \cap Q^{i+1} \subseteq P^{i} \cap Q^{i}$ for every $i \in \mathbb{N}$.

Let $c_{j}^{P} z \leq \gamma_{j}^{P}, j \in J^{P}$, be a finite system defining $\tilde{P}$, and let $c_{j}^{Q} z \leq \gamma_{j}^{Q}, j \in J^{Q}$, be a finite system defining $\tilde{Q}$. Since $\lim _{i \rightarrow \infty} P^{i}=\tilde{P}, \lim _{i \rightarrow \infty} Q^{i}=\tilde{Q}$, it follows by Observation 5 that for every $j \in J^{P}, \epsilon>0$, there exists $k \in \mathbb{N}$ such that $c_{j}^{P} z \leq \gamma_{j}^{P}+\epsilon$ is valid for $P^{k}$, and that for every $j \in J^{Q}, \epsilon>0$, there exists $k \in \mathbb{N}$ such that $c_{j}^{Q} z \leq \gamma_{j}^{Q}+\epsilon$ is valid for $Q^{k}$. Since the polyhedron $\tilde{P} \cap \tilde{Q}$ is defined by the finite system $c_{j}^{P} z \leq \gamma_{j}^{P}, j \in J^{P}, c_{j}^{Q} z \leq \gamma_{j}^{Q}, j \in J^{Q}$, it follows by Observation 5 that $\lim _{i \rightarrow \infty}\left(P^{i} \cap Q^{i}\right)=\tilde{P} \cap \tilde{Q}$.

One important ingredient of our proof is the following result about existence of a hyperplane which preserves mixed integrality under projection along a vector. To make this precise, we introduce the following notation. Given a nonzero vector $v \in \mathbb{Q}^{m+n}$, a subspace $H$ of $\mathbb{R}^{m+n}$ of dimension $m+n-1$ such that $v \notin H$, and a set $W$ in $\mathbb{R}^{m+n}$, we denote by $\operatorname{proj}_{v, H} W$ the projection of $W$ to the subspace $H$ along the direction $v$, i.e. $\operatorname{proj}_{v, H} W=\{z \in H: \exists \lambda \in \mathbb{R}, z+\lambda v \in W\}$. Moreover we denote with $\operatorname{proj}_{x} W$ the orthogonal projection of $W$ onto the space of the $x$-variables, i.e. $\operatorname{proj}_{x} W=\left\{x \in \mathbb{R}^{m}: \exists y \in \mathbb{R}^{n},(x, y) \in W\right\}$. Given a nonzero vector $v \in \mathbb{Q}^{m+n}$, and a subspace $H$ of $\mathbb{R}^{m+n}$ of dimension $m+n-1$, we say that $H$ is mixed integer
invariant under projection along $v$, if $v \notin H$, and if $\operatorname{proj}_{v, H} w \in \mathbb{Z}^{m} \times \mathbb{R}^{n}$ for every vector $w \in \mathbb{Z}^{m} \times \mathbb{R}^{n}$.

Lemma 7 (Mixed integer invariance under projection) Let $v$ be a nonzero vector in $\mathbb{Q}^{m+n}$. Then there exists a subspace $H$ of $\mathbb{R}^{m+n}$ that is mixed integer invariant under projection along $v$.

Proof Let $v_{x}:=\operatorname{proj}_{x} v$. If $v_{x}=0$, then the result follows trivially by taking any subspace $H$ of $\mathbb{R}^{m+n}$ of dimension $m+n-1$ such that $v \notin H$. So we now assume $v_{x} \neq 0$. By scaling, we can assume that $v_{x}$ is integral with $\operatorname{gcd}\left(v_{1}, \ldots, v_{m}\right)=1$. Then it is well-known (see for example [7, Theorem 5 on page 21]) that there exists a lattice basis $V$ of $\mathbb{Z}^{m}$ containing $v_{x}$. Let $\bar{H}=\left\{x \in \mathbb{R}^{m}: h x=0\right\}$ be the subspace of dimension $m-1$ of $\mathbb{R}^{m}$ spanned by the vectors in $V \backslash\left\{v_{x}\right\}$. Clearly $v_{x} \notin \bar{H}$, and since $V$ is a basis of $\mathbb{Z}^{m}$, it follows that $\operatorname{proj}_{v_{x}}, \bar{H} \bar{w} \in \mathbb{Z}^{m}$ for every vector $\bar{w} \in \mathbb{Z}^{m}$.

Now let $H=\left\{(x, y) \in \mathbb{R}^{m+n}: h x=0\right\}$. Clearly, the dimension of $H$ is $m+$ $n-1$. Since $v_{x} \notin \bar{H}$, then $h v_{x} \neq 0$, hence $v \notin H$. Let $w \in \mathbb{Z}^{m} \times \mathbb{R}^{n}$, and let $w_{x}:=\operatorname{proj}_{x} w$. We now show that $\operatorname{proj}_{x}\left(\operatorname{proj}_{v, H} w\right)=\operatorname{proj}_{v_{x}, \bar{H}}\left(w_{x}\right)$. Notice that $\operatorname{proj}_{v, H} w=w+\lambda v$, where $\lambda$ is a scalar with $h\left(w_{x}+\lambda v_{x}\right)=0$. This implies that $\operatorname{proj}_{x}\left(\operatorname{proj}_{v, H} w\right)=w_{x}+\lambda v_{x}$. On the other hand, $\operatorname{proj}_{v_{x}, \bar{H}}\left(w_{x}\right)=w_{x}+\mu v_{x}$ for a scalar $\mu$ with $h\left(w_{x}+\mu v_{x}\right)=0$. Hence $\operatorname{proj}_{x}\left(\operatorname{proj}_{v, H} w\right)=\operatorname{proj}_{v_{x}, \bar{H}}\left(w_{x}\right)$. Since $w_{x} \in \mathbb{Z}^{m}$, it follows from the first part of the proof that $\operatorname{proj}_{v_{x}, \bar{H}}\left(w_{x}\right) \in \mathbb{Z}^{m}$, thus $\operatorname{proj}_{x}\left(\operatorname{proj}_{v, H} w\right) \in \mathbb{Z}^{m}$, which completes the proof.

The next two lemmas establish important properties of the subspace $H$ that is mixed integer invariant under projection along $v$. Their proofs are given in Sect. 4. We recall that the characteristic cone of a set $P$ is char.cone $P:=\{w: z+w \in P$ for all $z \in P\}$, and the lineality space of $P$ is lin.space $P:=($ char.cone $P) \cap(-$ char.cone $P)$. A set $P$ is called pointed if lin.space $P$ has dimension zero.

Lemma 8 Let $P$ be a polyhedron in $\mathbb{R}^{m+n}$, let $v \in \mathbb{Q}^{m+n}$ be a nonzero vector, and let $H$ be a subspace that is mixed integer invariant under projection along $v$. Then

$$
\operatorname{proj}_{v, H} \mathcal{S}^{i}(P) \subseteq \mathcal{S}^{i}(\underset{v, H}{\operatorname{proj}} P) \quad \text { for every } i \in \mathbb{N} .
$$

Lemma 9 Let $P$ be an unbounded polyhedron in $\mathbb{R}^{m+n}$, let $v$ be a nonzero vector in char.cone $P$, and let $H$ be a subspace that is mixed integer invariant under projection along $v$. Then

$$
\operatorname{proj}_{v, H}\left(P_{I}\right)=(\underset{v, H}{(\operatorname{proj}} P)_{I} .
$$

We are now prepared to prove the main result of this section.
Theorem 2 For each rational polyhedron $P, \lim _{i \rightarrow \infty} \mathcal{S}^{i}(P)=P_{I}$.
Proof Let $P \subseteq \mathbb{R}^{m+n}$ be a polyhedron, and notice that $P_{I} \subseteq \mathcal{S}^{i+1}(P) \subseteq \mathcal{S}^{i}(P)$ for every $i \in \mathbb{N}$. Moreover by Cook et al. [5], $\left\{P_{I}, \mathcal{S}^{i}(P): i \in \mathbb{N}\right\}$ is a family of polyhedra, thus of closed sets. The proof is by induction on the dimension of $P$, the base cases $\operatorname{dim} P=-1$ (i.e. $P=\emptyset$ ) and $\operatorname{dim} P=0$ (i.e. $P$ is a singleton) being
trivial. To prove the inductive step, let $\operatorname{dim} P \geq 1$, and assume that the statement is true for every polyhedron of dimension strictly smaller than $\operatorname{dim} P$. If $P$ is bounded, then the result follows from Owen and Mehrotra [10]. Thus, from now on, we assume that $P$ is unbounded.

In the remainder of the proof, we use the following fact (that follows from [14, Theorem 16.1]). If $P_{I}$ is nonempty, then

$$
\text { char.cone }\left(P_{I}\right)=\text { char.cone } \mathcal{S}^{i}(P)=\text { char.cone } P \quad \text { for every } i \in \mathbb{N} .
$$

Claim 1 If lin.space $P \neq\{0\}$, then $\lim _{i \rightarrow \infty} \mathcal{S}^{i}(P)=P_{I}$.
Proof of Claim Let $v$ be a nonzero vector in lin.space $P$. By Lemma 7 there exists a subspace $H$ of $\mathbb{R}^{m+n}$ that is mixed integer invariant under projection along $v$. Let $\bar{P}=$ $\operatorname{proj}_{v, H} P=P \cap H$. Since $v$ is a nonzero vector in lin.space $P$, then $\operatorname{dim} \bar{P}<\operatorname{dim} P$. Thus by the hypothesis of the induction the sequence $\left\{\mathcal{S}^{i}(\bar{P}): i \in \mathbb{N}\right\}$ converges to $\bar{P}_{I}$. Clearly the sequence $\left\{\mathcal{S}^{i}(\bar{P})+\right.$ lin.hull $\left.\{v\}: i \in \mathbb{N}\right\}$ converges to $\bar{P}_{I}+$ lin.hull $\{v\}$. By Lemma 9, $\operatorname{proj}_{v, H}\left(P_{I}\right)=\bar{P}_{I}$, and, since $v \in \operatorname{lin}$.space $\left(P_{I}\right)$ if $P_{I} \neq \emptyset$, it follows that $P_{I}=\bar{P}_{I}+\operatorname{lin}$.hull $\{v\}$. Hence the sequence $\left\{\mathcal{S}^{i}(\bar{P})+\operatorname{lin} . h u l l\{v\}: i \in \mathbb{N}\right\}$ converges to $P_{I}$. By Lemma $8, \operatorname{proj}_{v, H} \mathcal{S}^{i}(P) \subseteq \mathcal{S}^{i}(\bar{P})$ for every $i \in \mathbb{N}$, which implies that $\mathcal{S}^{i}(P) \subseteq \mathcal{S}^{i}(\bar{P})+$ lin.hull $\{v\}$ for every $i \in \mathbb{N}$. Since moreover $P_{I} \subseteq \mathcal{S}^{i}(P)$ for every $i \in \mathbb{N}$, it follows by Observation 1 that $\lim _{i \rightarrow \infty} \mathcal{S}^{i}(P)=P_{I}$.

Thus, from now on, we can assume that $P$ is unbounded and pointed.
Claim 2 If $P_{I}=\emptyset$, then $\lim _{i \rightarrow \infty} \mathcal{S}^{i}(P)=\emptyset$.
Proof of Claim Let $v \in$ char.cone $P$, and let $P^{\prime}=P+\operatorname{lin}$.hull $\{v\}$. Clearly $P \subseteq$ $P^{\prime}, \operatorname{dim} P^{\prime}=\operatorname{dim} P$ and lin.space $P^{\prime} \neq \emptyset$. Moreover it is straightforward to verify that $P_{I}^{\prime}=P_{I}=\emptyset$. By Claim 1, $\lim _{i \rightarrow \infty} \mathcal{S}^{i}\left(P^{\prime}\right)=\emptyset$. Since $P \subseteq P^{\prime}$, then $\mathcal{S}^{i}(P) \subseteq \mathcal{S}^{i}\left(P^{\prime}\right)$ for every $i \in \mathbb{N}$. Thus by Observation $1, \lim _{i \rightarrow \infty} \mathcal{S}^{i}(P)=\emptyset$.

From now on we can assume that $P$ is unbounded, pointed, and such that $P_{I} \neq \emptyset$. Notice that this implies that $\mathcal{S}^{i}(P) \neq \emptyset$ for all $i \in \mathbb{N}$.

Claim 3 There exists a closed convex set $\tilde{P}$, with $\tilde{P} \supseteq P_{I}$ and char.cone $\tilde{P}=$ char.cone $P_{I}$, such that $\lim _{i \rightarrow \infty} \mathcal{S}^{i}(P)=\tilde{P}$. Moreover for any half-space $Q, \lim _{i \rightarrow \infty}$ $\left(\mathcal{S}^{i}(P) \cap Q\right)=\tilde{P} \cap Q$.

Proof of Claim Since $P$ is pointed and $P_{I} \neq \emptyset$, it follows that also $P_{I}$ is pointed. Hence it is easy to see that there exists a half-space

$$
R^{\geq}=\left\{z \in \mathbb{R}^{m+n}: p z \geq \rho\right\}
$$

that contains every vertex of $P_{I}$, and such that $P \cap R^{\geq}$is bounded. Since $\left\{\mathcal{S}^{i}(P): i \in\right.$ $\mathbb{N}\}$ is a sequence of polyhedra, it follows that $\left\{\mathcal{S}^{i}(P) \cap R^{\geq}: i \in \mathbb{N}\right\}$ is a sequence of compact sets such that $\mathcal{S}^{i+1}(P) \cap R^{\geq} \subseteq \mathcal{S}^{i}(P) \cap R^{\geq}$for every $i \in \mathbb{N}$. Hence it follows from the Blaschke selection theorem [3] that $\left\{\mathcal{S}^{i}(P) \cap R^{\geq}: i \in \mathbb{N}\right\}$ converges to a set $\hat{P}$. It follows that $\hat{P}:=\bigcap_{i \in \mathbb{N}}\left(\mathcal{S}^{i}(P) \cap R^{\geq}\right)$, hence $\hat{P}$ is convex and compact as it is the intersection of convex compact sets.

Similarly also the sequence of compact sets $\left\{\mathcal{S}^{i}(P) \cap R^{\geq} \cap Q: i \in \mathbb{N}\right\}$ converges, and it converges to the set $\bigcap_{i \in \mathbb{N}}\left(\mathcal{S}^{i}(P) \cap R^{\geq} \cap Q\right)=\hat{P} \cap Q$.

Now let

$$
R_{1}:=\left\{z \in \mathbb{R}^{m+n}: p z \leq \rho\right\} .
$$

We now show that $\lim _{i \rightarrow \infty}\left(\mathcal{S}^{i}(P) \cap R_{1}\right)=P_{I} \cap R_{1}$. Notice that, since all the vertices of $P_{I}$ are contained in $R_{2}$, an irredundant inequality description of $P_{I} \cap R_{1}$ is given by the inequality $p z \leq \rho$, by a system of inequalities defining the affine hull of $P_{I}$, and by the inequalities defining the unbounded facets of $P_{I}$. Note that, for each inequality of such system different from $p z \leq \rho$, the corresponding supporting hyperplane contains an unbounded face of $P_{I}$. By Observation 5, to prove that $\lim _{i \rightarrow \infty}\left(\mathcal{S}^{i}(P) \cap R_{1}\right)=P_{I} \cap R_{1}$, we only need to prove that for every inequality $c z \leq \gamma$ of such irredundant system defining $P_{I} \cap R_{1}$, and for every $\epsilon>0$, there exists $k \in \mathbb{N}$ such that $c z \leq \gamma+\epsilon$ is valid for $\mathcal{S}^{k}(P) \cap R_{1}$.

By definition of $R_{1}$, such property is trivially valid for the constraint $p z \leq \rho$, hence we can assume that $c z \leq \gamma$ is valid for $P_{I}$, and $\left\{z \in P_{I}: c z=\gamma\right\}$ is unbounded. Since $\mathcal{S}^{i}(P) \cap R_{1} \subseteq \mathcal{S}^{i}(P)$ for every $i \in \mathbb{N}$, we only need to verify that, for every $\epsilon>0$, there exists $k \in \mathbb{N}$ such that $c z \leq \gamma+\epsilon$ is valid for $\mathcal{S}^{k}(P)$. Now let $\epsilon>0$, and let $v$ be a nonzero vector in char.cone $\left\{z \in P_{I}: c z=\gamma\right\}$. Clearly $v \in \operatorname{char}$.cone $(P)$. By Lemma 7 there exists a subspace $H$ of $\mathbb{R}^{m+n}$ that is mixed integer invariant under projection along $v$. Let $\bar{P}=\operatorname{proj}_{v, H} P$. Since $v$ is a nonzero vector in char.cone $(P)$, then $\operatorname{dim}(\bar{P})<\operatorname{dim}(P)$. Thus by induction the sequence $\left\{\mathcal{S}^{i}(\bar{P}): i \in \mathbb{N}\right\}$ converges to $\bar{P}_{I}$. Since $c z \leq \gamma$ is valid for $P_{I}$, and since $c v=0$, it follows that $c z \leq \gamma$ is also valid for $\operatorname{proj}_{v, H}\left(P_{I}\right)$. By Lemma 9, $\operatorname{proj}_{v, H}\left(P_{I}\right)=\bar{P}_{I}$, hence $c z \leq \gamma$ is valid for $\bar{P}_{I}$. Since $\left\{\mathcal{S}^{i}(\bar{P}): i \in \mathbb{N}\right\}$ converges to $\bar{P}_{I}$, there exists $k \in \mathbb{N}$ such that $c z \leq \gamma+\epsilon$ is valid for $\mathcal{S}^{k}(\bar{P})$. By Lemma $8, \operatorname{proj}_{v, H} \mathcal{S}^{k}(P) \subseteq \mathcal{S}^{k}(\bar{P})$, thus $c z \leq \gamma+\epsilon$ is valid for $\operatorname{proj}_{v, H} \mathcal{S}^{k}(P)$. Finally, since $c v=0$, it follows that $c z \leq \gamma+\epsilon$ is valid for $\mathcal{S}^{k}(P)$. Hence we showed that $\lim _{i \rightarrow \infty}\left(\mathcal{S}^{i}(P) \cap R_{1}\right)=P_{I} \cap R_{1}$.

It follows from Observation 6 that the sequence of polyhedra $\left\{\mathcal{S}^{i}(P) \cap R_{1} \cap Q\right.$ : $i \in \mathbb{N}\}$ converges to the polyhedron $P_{I} \cap R_{1} \cap Q$.

Since $\lim _{i \rightarrow \infty}\left(\mathcal{S}^{i}(P) \cap R_{2}\right)=\hat{P}$ and $\lim _{i \rightarrow \infty}\left(\mathcal{S}^{i}(P) \cap R_{1}\right)=P_{I} \cap R_{1}$, then by Observation $2, \lim _{\tilde{P} \rightarrow \infty} \mathcal{S}^{i}(P)=\hat{P} \cup\left(P_{I} \cap R_{1}\right)=: \tilde{P}$. It follows that $\tilde{P}=$ $\bigcap_{i \in \mathbb{N}} \mathcal{S}^{i}(P)$. Since $\tilde{P}$ is the intersection of closed convex sets, it is itself closed and convex. Since moreover $P_{I} \subseteq \mathcal{S}^{i}(P)$ for every $i \in \mathbb{N}$, it follows that $P_{I} \subseteq \tilde{P}$. In particular it follows that $\tilde{P}=\hat{P} \cup P_{I}$, where $\hat{P}$ is convex and compact. This implies that char.cone $\tilde{P} \subseteq$ char. cone $P_{I}$. Moreover, since $\tilde{P}$ is convex and closed, it follows easily that char.cone $\tilde{P} \supseteq$ char.cone $P_{I}$.

Moreover, since $\lim _{i \rightarrow \infty}\left(\mathcal{S}^{i}(P) \cap R_{2} \cap Q\right)=\hat{P} \cap Q$ and $\lim _{i \rightarrow \infty}\left(\mathcal{S}^{i}(P) \cap R_{1} \cap Q\right)=$ $P_{I} \cap R_{1} \cap Q$, then by Observation $2, \lim _{i \rightarrow \infty}\left(\mathcal{S}^{i}(P) \cap Q\right)=(\hat{P} \cap Q) \cup\left(P_{I} \cap R_{1} \cap Q\right)=$ $\left(\hat{P} \cup\left(P_{I} \cap R_{1}\right)\right) \cap Q=\tilde{P} \cap Q$.
Claim $4 \mathcal{S}(\tilde{P})=\tilde{P}$.
Proof of Claim Clearly $\mathcal{S}(\tilde{P}) \subseteq \tilde{P}$, hence now we prove the opposite inclusion. Let $a \in \mathbb{Z}^{m}, \beta \in \mathbb{Z}$, and let $Q_{1}:=\left\{(x, y) \in \mathbb{R}^{m+n}: a x \leq \beta\right\}, Q_{2}:=\left\{(x, y) \in \mathbb{R}^{m+n}:\right.$ $a x \geq \beta+1\}$. Then, by definition of split closure,

$$
\mathcal{S}^{i+1}(P) \subseteq \text { cl conv.hull }\left(\left(\mathcal{S}^{i}(P) \cap Q_{1}\right) \cup\left(\mathcal{S}^{i}(P) \cap Q_{2}\right)\right)
$$

By Claim 3, $\lim _{i \rightarrow \infty}\left(\mathcal{S}^{i}(P) \cap Q_{1}\right)=\tilde{P} \cap Q_{1}$, and $\lim _{i \rightarrow \infty}\left(\mathcal{S}^{i}(P) \cap Q_{2}\right)=\tilde{P} \cap Q_{2}$. By Observation 2, $\lim _{i \rightarrow \infty}\left(\left(\mathcal{S}^{i}(P) \cap Q_{1}\right) \cup\left(\mathcal{S}^{i}(P) \cap Q_{2}\right)\right)=\left(\tilde{P} \cap Q_{1}\right) \cup$ $\left(\tilde{P} \cap Q_{2}\right)$. By Observation 3, $\lim _{i \rightarrow \infty}$ cl conv.hull $\left(\left(\mathcal{S}^{i}(P) \cap Q_{1}\right) \cup\left(\mathcal{S}^{i}(P) \cap Q_{2}\right)\right)=$ cl conv.hull $\left(\left(\tilde{P} \cap Q_{1}\right) \cup\left(\tilde{P} \cap Q_{2}\right)\right)$. Since moreover by Claim 3, $\lim _{i \rightarrow \infty} \mathcal{S}^{i+1}(P)=\tilde{P}$, it follows by Observation 4 that

$$
\tilde{P} \subseteq \operatorname{cl} \text { conv.hull }\left(\left(\tilde{P} \cap Q_{1}\right) \cup\left(\tilde{P} \cap Q_{2}\right)\right) .
$$

Since this holds for every $a \in \mathbb{Z}^{m}, \beta \in \mathbb{Z}$, we have that
$\tilde{P} \subseteq \bigcap_{(a, \beta) \in \mathbb{Z}^{m+1}} \operatorname{clconv\cdot hull(\{ (x,y)\in \tilde {P}:ax\leq \beta \} \cup \{ (x,y)\in \tilde {P}:ax\geq \beta +1\} ),~}$
which means that $\tilde{P} \subseteq \mathcal{S}(\tilde{P})$.

We now show that

$$
\tilde{P}=P_{I} .
$$

We prove this by contradiction, thus assume $\tilde{P} \neq P_{I}$. By Claim 3, $\tilde{P} \supseteq P_{I}$, thus $\tilde{P} \supset P_{I}$. Moreover $\tilde{P}$ is closed and convex. Using Rockafellar's notation [11, Sect. 18], the zero-dimensional faces of $\tilde{P}$ are called the extreme points of $\tilde{P}$. Moreover, an exposed point of $\tilde{P}$ is a point of $\tilde{P}$ through which there is a supporting hyperplane which contains no other point of $\tilde{P}$.

We now show that there exists an exposed point $\tilde{z}$ of $\tilde{P}$ with $\tilde{z} \notin P_{I}$. By Claim 3, we have that char.cone $\tilde{P}=$ char.cone $P_{I}=$ char.cone $P$. Since moreover lin.space $P=\{0\}$, it follows that lin.space $\tilde{P}=\{0\}$. Since $\tilde{P}$ is closed and convex, lin.space $\tilde{P}=\{0\}$, char.cone $\tilde{P}=$ char.cone $P_{I}$, and $P_{I} \subset \tilde{P}$, it follows that there exists an extreme point $\bar{z}$ of $\tilde{P}$ not in $P_{I}$ (see [11, Theorem 18.5]). It follows from Straszewicz's Theorem [11, Theorem 18.6] that $\bar{z}$ is the limit of some sequence of exposed points of $\tilde{P}$. Hence there exists an exposed point $\tilde{z}$ of $\tilde{P}$ arbitrarily close to $\bar{z}$. On the other hand, since $P_{I}$ is a polyhedron that does not contain $\bar{z}$, there cannot be points in $P_{I}$ arbitrarily close to $\bar{z}$, thus there exists an exposed point $\tilde{z}$ of $\tilde{P}$ not in $P_{I}$.

Since $\tilde{z}$ is exposed, let $c z \leq \gamma$ be an inequality valid for $\tilde{P}$, and such that $\tilde{z}$ is the only vector in $\tilde{P}$ contained in the corresponding hyperplane. Since $\tilde{z}=(\tilde{x}, \tilde{y}) \in \tilde{P} \backslash P_{I}$, it follows that $\tilde{x}$ is not integral, thus it contains a component not integer, say $\tilde{x}_{j}$. Let $Q_{1}:=\left\{(x, y) \in \mathbb{R}^{m+n}: x_{j} \leq\left\lfloor\tilde{x}_{j}\right\rfloor\right\}, Q_{2}:=\left\{(x, y) \in \mathbb{R}^{m+n}: x_{j} \geq\left\lceil\tilde{x}_{j}\right\rceil\right\}$. Since $\tilde{P}$ is closed and convex, so are $\tilde{P} \cap Q_{1}$ and $\tilde{P} \cap Q_{2}$. Since $\tilde{z}$ is the only point in $\tilde{P}$ that satisfies $c z=\gamma$, and $\tilde{z} \notin Q_{1} \cup Q_{2}$, it follows that there exists $\epsilon>0$ such that $c z \leq \gamma-\epsilon$ is valid for $\tilde{P} \cap Q_{1}$ and for $\tilde{P} \cap Q_{2}$. Hence $c z \leq \gamma-\epsilon$ is a split cut for $\tilde{P}$ which is not valid for $\tilde{z}$. It follows that $\mathcal{S}(\tilde{P}) \subset \tilde{P}$, which contradicts Claim 4.

## 3 Finite convergence

In this section we prove an analogue of Theorem 1 in mixed integer programming. Indeed, we show that the integral lattice-free closure of a polyhedron is again a polyhedron, and that repeatedly taking the integral lattice-free closure of $P$ gives $P_{I}$ after a finite number of iterations. Moreover, we show that in general it is not true that the lattice-free closure of $P$ equals $P_{I}$.

Given a full-dimensional polyhedron $L \in \mathbb{R}^{m}$ that contains integral points in every facet, the max-facet-width $w(L)$ of $L$ is the maximum value of $\beta-\min \{a x: x \in L\}$, where $a x \leq \beta$ defines a facet of $L$ with $a \in \mathbb{Z}^{m}$, and the greatest common divisor of the entries in $a$ is 1 . A result by Andersen et al. [1] shows that the closure of a polyhedron obtained from disjunctions associated with any family of maximal latticefree polyhedra with bounded max-facet width is a rational polyhedron. In such proof, the maximality assumption is only used to show that every facet of every lattice-free polyhedron in the family contains an integral point. Since such condition is clearly satisfied by any integral polyhedron, the result by Andersen et al. implies that the closure of a polyhedron obtained from disjunctions associated with any family of integral lattice-free polyhedra with bounded max-facet width is a rational polyhedron. In the following theorem we use such result. In the remainder of the paper, we call affine unimodular transformations the affine transformations that map $\mathbb{Z}^{m}$ onto $\mathbb{Z}^{m}$.

Theorem 3 For any rational polyhedron $P$, the set $\mathcal{L}(P)$ is again a rational polyhedron.
Proof Let $P$ be a polyhedron in $\mathbb{R}^{m+n}$. It follows by definition of integral lattice-free cut, that each integral lattice-free cut for $P$, which is not a valid inequality for $P$, corresponds to an integral lattice-free polyhedron of $\mathbb{R}^{m}$ of dimension $m$. Given two lattice-free polyhedra $L, L^{\prime}$ in $\mathbb{R}^{m}$ of dimension $m$ and with $L \subseteq L^{\prime}$, then clearly relint $L \subseteq$ relint $L^{\prime}$. It follows that each irredundant integral lattice-free cut for $P$ corresponds to an integral lattice-free polyhedron in $\mathbb{R}^{m}$ of dimension $m$ which is maximal with respect to inclusion. Thus let $Z$ be the family of such maximal latticefree polyhedra. A result by Averkov et al. [2] implies that the set $Z$ is finite up to affine unimodular transformations.

We now show that for every $L \in Z, w(L)$ is finite. By contradiction assume that $w(L)$ is not finite. Thus there exists a facet defining inequality $a x \leq \beta$ of $L$ such that $\beta-\min \{a x: x \in L\}$ is not finite. It follows that there exists a vector $v \in$ char.cone $L$ with $a v<0$. Let $L^{\prime}:=L+\operatorname{lin}$.hull $\{v\}$. Since $a x \leq \beta$ is valid for $L, v \notin \operatorname{lin}$.space $L$, thus $L \subset L^{\prime}$. Since $L$ is lattice-free and $v \in$ char.cone $L$, it can be checked that also $L^{\prime}$ is lattice-free. Moreover since $L$ is integral, also $L^{\prime}$ is integral. Hence $L^{\prime} \in Z$ and $L \subset L^{\prime}$, but this contradicts the maximality of $L$. Thus $w(L)$ is finite for every $L \in Z$.

It is well-known that affine unimodular transformations preserve the max-facet-width of polyhedra. Since moreover $Z$ is finite up to affine unimodular transformations, and every polyhedron in $Z$ has bounded max-facet-width, it follows that $Z$ is a family of integral lattice-free polyhedra with bounded max-facet-width. It follows by the proof of [1, Theorem 4.3] that the set $\mathcal{L}(P)$ is a rational polyhedron.

The main result of this paper is that, for each polyhedron $P$ there exists a number $k$ such that $\mathcal{L}^{k}(P)=P_{I}$. Our proof of this result is quite technical. It requires two

Lemmas that we state first. In order to streamline the presentation, we postpone the proof of Lemma 10 to Sect. 4. At this point, let us just mention that Lemma 10 applied to polytopes has already been proven by Jörg [8]. We adapt his proof technique to show the general result.

Lemma 10 Let $P$ be a polyhedron in $\mathbb{R}^{m+n}$, and let $c x+d y \leq \gamma$ be a valid inequality for $P_{I}$ such that

$$
\operatorname{proj}_{x}\left\{(x, y) \in P_{I}: c x+d y=\gamma\right\}
$$

is not lattice-free. Then $\exists k \in \mathbb{N}$ such that $c x+d y \leq \gamma$ is valid for $\mathcal{S}^{k}(P)$.
Lemma 11 Let $P$ be an integral lattice-free polyhedron in $\mathbb{R}^{m}$. Then there exists an integral lattice-free polyhedron $L \supseteq P$ of dimension $m$ such that relint $P \subseteq$ relint $L$.

Proof If $\operatorname{dim} P=m$ then $L=P$, so we now assume $d:=\operatorname{dim} P<m$. Since the lemma is invariant under affine unimodular transformations, we may assume that aff.hull $P=\mathbb{R}^{d} \times\{0\}^{m-d}$. It is then easy to verify that $L:=P+\left(\{0\}^{d} \times \mathbb{R}^{m-d}\right)$ is a lattice-free polyhedron of dimension $m$ with relint $P \subseteq$ relint $L$.

We now prove our main result.
Theorem 4 For each rational polyhedron $P$ there exists $k \in \mathbb{N}$ such that

$$
\mathcal{L}^{k}(P)=P_{I}
$$

Proof Let $P \subseteq \mathbb{R}^{m+n}$ be a polyhedron. If $P_{I}=\emptyset$, then by Theorem 2, $\lim _{i \rightarrow \infty} \mathcal{S}^{i}(P)=\emptyset$, which implies that there exists $k \in \mathbb{N}$ such that $\mathcal{S}^{k}(P)=\emptyset$. Since $\mathcal{L}^{k}(P) \subseteq \mathcal{S}^{k}(P)$, it follows that $\mathcal{L}^{k}(P)=\emptyset$. Thus, we now assume that $P_{I} \neq \emptyset$.

Since $P_{I} \neq \emptyset$, to prove the theorem we show that for every inequality $c x+d y \leq \gamma$ valid for $P_{I}$, there exists $k \in \mathbb{N}$ such that $c x+d y \leq \gamma$ is valid for $\mathcal{L}^{k}(P)$. Note that it suffices to show this because we only need to show it for some finitely many irredundant inequalities defining $P_{I}$. One can then take the maximum of the $k \in \mathbb{N}$ for each of these finitely many inequalities. We prove this by induction on $\operatorname{dim} F$, where $F:=\left\{(x, y) \in P_{I}: c x+d y=\gamma\right\}$.

We prove the first two base cases. If $F=\emptyset$, then there exists $\epsilon>0$ such that $c x+d y \leq \gamma-\epsilon$ is valid for $P_{I}$. By Theorem $2, \lim _{i \rightarrow \infty} \mathcal{S}^{i}(P)=P_{I}$, and by Observation 5, there exists $k \in \mathbb{N}$ such that $c x+d y \leq \gamma-\epsilon+\epsilon=\gamma$ is valid for $\mathcal{S}^{k}(P)$, and so for $\mathcal{L}^{k}(P)$. Now let $F$ be a minimal face of $P_{I}$. Then $F$ is an affine space and it contains $x$-integral vectors. It follows that $\operatorname{proj}_{x} F$ is an affine space too, and it contains integral vectors. Since $\operatorname{proj}_{x} F$ is an affine space, and the relative interior of every affine space is the same affine space, then $\operatorname{proj}_{x} F$ contains integral vectors in its relative interior, thus it is not lattice-free. Then by Lemma 10 , there exists $k \in \mathbb{N}$ such that $c x+d y \leq \gamma$ is valid for $\mathcal{S}^{k}(P)$, and so for $\mathcal{L}^{k}(P)$.

To prove the inductive step, assume that $F$ is a proper face of $P_{I}$ which is not minimal, and assume that the statement is true for every face of $P_{I}$ of dimension
strictly smaller than $\operatorname{dim} F$. If $\operatorname{proj}_{x} F$ is not lattice-free, then the statement follows from Lemma 10. Hence, we now assume that $\operatorname{proj}_{x} F$ is lattice-free.

Since $\operatorname{proj}_{x} F$ is an integral lattice-free polyhedron, it follows by Lemma 11 there exists an integral lattice-free polyhedron $L \subseteq \mathbb{R}^{m}$ of dimension $m$ such that proj${ }_{x} F \subseteq$ $L$ and relint $\left(\operatorname{proj}_{x} F\right) \subseteq$ relint $L$. Now let $a_{j} x \leq \beta_{j}, j \in J$, be a minimal system of inequalities defining $L$. The result follows if we show that there exists a $k \in \mathbb{N}$ such that $c x+d y \leq \gamma$ is an integral lattice-free cut for the polyhedron $\mathcal{L}^{k}(P)$. By definition of integral lattice-free cut, this happens if $c x+d y \leq \gamma$ is valid for every set

$$
\left\{(x, y) \in \mathcal{L}^{k}(P): a_{j} x \geq \beta_{j}\right\}, \quad j \in J .
$$

Claim 5 For every $j \in J$, there exists an inequality $c_{j} x+d y \leq \gamma_{j}$ such that:
(i) $c_{j} x+d y \leq \gamma_{j}$ is valid for $P_{I}$;
(ii) $F_{j}:=\left\{(x, y) \in P_{I}: c_{j} x+d y=\gamma_{j}\right\} \subset F$;
(iii) $c x+d y \leq \gamma$ is valid for $\left\{(x, y) \in \mathbb{R}^{m+n}: c_{j} x+d y \leq \gamma_{j}, a_{j} x \geq \beta_{j}\right\}$.

Proof of Claim Let $j \in J$. For every $\epsilon \geq 0$, consider the inequality $\left(\epsilon a_{j}+c\right) x+d y \leq$ $\epsilon \beta_{j}+\gamma$.
(i) Notice that, since $\max \left\{c x+d y:(x, y) \in P_{I}\right\}$ is attained in the face $F$, since $a_{j} x \leq \beta_{j}$ is valid for $F$, and since $\lim _{\epsilon \rightarrow 0} \epsilon a_{j}=0$, it follows that there exists $\bar{\epsilon}>0$ small enough such that $\max \left\{\left(\bar{\epsilon} a_{j}+c\right) x+d y:(x, y) \in P_{I}\right\}$ is attained in a face of $F$. Let $c_{j}:=\bar{\epsilon} a_{j}+c$, and $\gamma_{j}:=\bar{\epsilon} \beta_{j}+\gamma$. Since both inequalities $c x+d y \leq \gamma$, and $a_{j} x \leq \beta_{j}$ are valid for $F$, it follows that also their conic combination $c_{j} x+d y \leq \gamma_{j}$ is valid for $F$. Since $\max \left\{c_{j} x+d y:(x, y) \in P_{I}\right\}$ is attained in a face of $F$, and $c_{j} x+d y \leq \gamma_{j}$ is valid for $F$, then $c_{j} x+d y \leq \gamma_{j}$ is valid for $P_{I}$.
(ii) Since $c_{j} x+d y \leq \gamma_{j}$ is valid for $P_{I}$, and $\max \left\{c_{j} x+d y:(x, y) \in P_{I}\right\}$ is attained in a face of $F$, then $F_{j} \subseteq F$. To prove that the inclusion is proper, let $(\bar{x}, \bar{y}) \in F$ with $\bar{x} \in \operatorname{relint}\left(\operatorname{proj}_{x} F\right)$. Since $(\bar{x}, \bar{y}) \in F$, then $c \bar{x}+d \bar{y} \leq \gamma$. Moreover, since $\bar{x} \in \operatorname{relint}\left(\operatorname{proj}_{x} F\right)$, then $\bar{x} \in \operatorname{relint} L$, hence $a_{j} \bar{x}<\beta_{j}$. Since $\bar{\epsilon}>0$, it follows that $\left(\bar{\epsilon} a_{j}+c\right) \bar{x}+d \bar{y}<\bar{\epsilon} \beta_{j}+\gamma$. Hence $(\bar{x}, \bar{y}) \in F$ does not satisfy $c_{j} x+d y=\gamma_{j}$, implying $F_{j} \subset F$.
(iii) Follows by definition of the inequality $c_{j} x+d y \leq \gamma_{j}$, and the fact that $\bar{\epsilon}>0$.

For each $j \in J$, let $c_{j} x+d y \leq \gamma_{j}$ be an inequality as in Claim 5. We show next that there exists $k \in \mathbb{N}$ such that all the inequalities $c_{j} x+d y \leq \gamma_{j}, j \in J$, are valid for $\mathcal{L}^{k}(P)$. Let $j \in J$. By Claim 5(i) $c_{j} x+d y \leq \gamma_{j}$ is valid for $P_{I}$. It follows by Claim 5(ii) that the set $F_{j}$ is a face of $F$ different from $F$, which implies that $\operatorname{dim} F_{j}<\operatorname{dim} F$. Thus by hypothesis of the induction, there exists $k \in \mathbb{N}$ such that $c_{j} x+d y \leq \gamma_{j}$ is valid for $\mathcal{L}^{k}(P)$. Since $J$ is finite, it follows that there exists $k \in \mathbb{N}$ such that all the inequalities $c_{j} x+d y \leq \gamma_{j}, j \in J$, are valid for $\mathcal{L}^{k}(P)$.

Finally by Claim 5(iii), the inequality $c x+d y \leq \gamma$ is valid for every polyhedron $\left\{(x, y) \in \mathbb{R}^{m+n}: c_{j} x+d y \leq \gamma_{j}, a_{j} x \geq \beta_{j}\right\}, j \in J$. Since each inequality $c_{j} x+d y \leq \gamma_{j}, j \in J$, is valid for $\mathcal{L}^{k}(P)$, it follows that $c x+d y \leq \gamma$ is valid for every
polyhedron $\left\{(x, y) \in \mathcal{L}^{k}(P): a_{j} x \geq \beta_{j}\right\}, j \in J$. This implies that $c x+d y \leq \gamma$ is an integral lattice-free cut for the polyhedron $\mathcal{L}^{k}(P)$.

We conclude this section with an example that shows that generally it is not true that $\mathcal{L}(P)=P_{I}$. Let $P \subseteq \mathbb{R}^{2+1}$ be the convex hull of the vectors

$$
(-1 / 2,1 / 2,0),(1 / 2,-1 / 2,0),(1 / 2,3 / 2,0),(3 / 2,1 / 2,0),(1 / 2,1 / 2,1) .
$$

It follows that
$P_{I}=\operatorname{conv} \cdot h u l l\left(P \cap\left(\mathbb{Z}^{2} \times \mathbb{R}\right)\right)=\operatorname{conv} \cdot h u l l\{(0,0,0),(0,1,0),(1,0,0),(1,1,0)\}$.
To show that $\mathcal{L}(P) \neq P_{I}$ we show that the point $(1 / 2,1 / 2,1 / 4) \in P \backslash P_{I}$ satisfies every integral lattice-free cut for $P$.

Let $c x+d y \leq \gamma$ be an integral lattice-free cut for $P$, and let $L$ be an integral lattice-free polyhedron in $\mathbb{R}^{2}$ corresponding to $c x+d y \leq \gamma$. Clearly we can assume that $L$ is maximal among the integral lattice-free polyhedra in $\mathbb{R}^{2}$. It follows that $L$ is an affine unimodular transformation of either $S:=\left\{x \in \mathbb{R}^{2}: 0 \leq x_{1} \leq 1\right\}$, or $T:=\left\{x \in \mathbb{R}^{2}: x \geq 0, x_{1}+x_{2} \leq 2\right\}$.

It can be checked that there exists a point $\bar{x}$ among $(0,1 / 2),(1 / 2,0),(1 / 2,1)$, $(1,1 / 2)$ which is not contained in relint $L$. In fact, if $L$ is an affine unimodular transformation of $S$ this is trivial, while if $L$ is an affine unimodular transformation of $T$ it is easy to see that relint $L$ contains only three vectors $x$ such that $2 x \in \mathbb{Z}^{2}$.

Note that by definition of $P,(\bar{x}, 1 / 2) \in P$. It follows that $c x+d y \leq \gamma$ is valid for $(\bar{x}, 1 / 2)$. Since moreover the vectors $(0,0,0),(0,1,0),(1,0,0),(1,1,0)$ are in $P_{I}$, the inequality $c x+d y \leq \gamma$ is valid for all of them. It follows that $c x+d y \leq \gamma$ is also valid for conv.hull $\{(\bar{x}, 1 / 2),(0,0,0),(0,1,0),(1,0,0),(1,1,0)\} \ni(1 / 2,1 / 2,1 / 4)$. Hence $\mathcal{L}(P) \neq P_{I}$.

## 4 Proofs of technical lemmas

Lemma 8 Let $P$ be a polyhedron in $\mathbb{R}^{m+n}$, let $v \in \mathbb{Q}^{m+n}$ be a nonzero vector, and let $H$ be a subspace that is mixed integer invariant under projection along $v$. Then

$$
\operatorname{proj}_{v, H} \mathcal{S}^{i}(P) \subseteq \mathcal{S}^{i}(\underset{v, H}{\operatorname{proj}} P) \quad \text { for every } i \in \mathbb{N} .
$$

Proof The proof is by induction on $i \geq 0$, the case $i=0$ being trivial. We now show the base case $i=1$, i.e. that $\operatorname{proj}_{v, H} \mathcal{S}(P) \subseteq \mathcal{S}\left(\operatorname{proj}_{v, H} P\right)$. Let $\bar{P}:=\operatorname{proj}_{v, H} P$, and let $\bar{z} \notin \mathcal{S}(\bar{P})$. We want to show that $\bar{z} \notin \operatorname{proj}_{v, H} \mathcal{S}(P)$. If $\bar{z} \notin \bar{P}$, then clearly $\bar{z} \notin \operatorname{proj}_{v, H} \mathcal{S}(P)$, as $\mathcal{S}(P) \subseteq P$. So we now assume $\bar{z} \in \bar{P}$. Thus there exists a split cut $c z \leq \gamma$ for $\bar{P}$ such that $c \bar{z}>\gamma$. This implies that there exist $a \in \mathbb{Z}^{m}, \beta \in \mathbb{Z}$ such that $c z \leq \gamma$ is valid for both $\bar{P} \cap Q_{1}$ and $\bar{P} \cap Q_{2}$, where $Q_{1}=\left\{(x, y) \in \mathbb{R}^{m+n}\right.$ : $a x \leq \beta\}, Q_{2}=\left\{(x, y) \in \mathbb{R}^{m+n}: a x \geq \beta+1\right\}$.

Now let $c^{\prime} z \leq \gamma^{\prime}$ be the inequality defining the half-space

$$
\{z \in H: c z \leq \gamma\}+\operatorname{lin} \cdot \operatorname{hull}\{v\} .
$$

Notice that by construction $c^{\prime} v=0$, and $\{z \in H: c z \leq \gamma\}=\left\{z \in H: c^{\prime} z \leq\right.$ $\left.\gamma^{\prime}\right\}$. Moreover, let $Q_{j}^{\prime}:=\left(Q_{j} \cap H\right)+$ lin.hull $\{v\}$, for $j \in\{1,2\}$, and notice that $Q_{j}^{\prime} \cap H=Q_{j} \cap H$ for $j \in\{1,2\}$. Since every $x$-integral vector is contained in $Q_{1} \cup Q_{2}$, it follows that every $x$-integral vector in $H$ is in $Q_{1}^{\prime} \cup Q_{2}^{\prime}$. Since $H$ is mixed integer invariant under projection along $v$, it follows by definition of $Q_{1}^{\prime}, Q_{2}^{\prime}$ that every $x$-integral vector is contained in $Q_{1}^{\prime} \cup Q_{2}^{\prime}$. Note that by construction $Q_{1}^{\prime}$ and $Q_{2}^{\prime}$ are disjoint and defined by two parallel hyperplanes. Moreover $Q_{1}^{\prime} \cup Q_{2}^{\prime} \neq \mathbb{R}^{m+n}$ since $\bar{z} \notin Q_{1}^{\prime} \cup Q_{2}^{\prime}$. Hence it can be verified that there exist $a^{\prime} \in \mathbb{Z}^{m}$, and $\beta^{\prime} \in \mathbb{Z}$, such that $Q_{1}^{\prime} \supseteq\left(Q_{1}^{\prime}\right)_{I}=\left\{(x, y) \in \mathbb{R}^{m+n}: a^{\prime} x \leq \beta^{\prime}\right\}$, and $Q_{2}^{\prime} \supseteq\left(Q_{2}^{\prime}\right)_{I}=\left\{(x, y) \in \mathbb{R}^{m+n}:\right.$ $\left.a^{\prime} x \geq \beta^{\prime}+1\right\}$.

Since $c z \leq \gamma$ is valid for $\bar{P} \cap Q_{1}$, and $\bar{P} \subseteq H$, it follows that $c^{\prime} z \leq \gamma^{\prime}$ is valid for $\bar{P} \cap Q_{1}$, and so for $\bar{P} \cap\left(Q_{1}^{\prime}\right)_{I}$. Since $c^{\prime} v=0$, it follows that $c^{\prime} z \leq \gamma^{\prime}$ is valid for $\left(\bar{P} \cap\left(Q_{1}^{\prime}\right)_{I}\right)+$ lin.hull $\{v\} \supseteq P \cap\left(Q_{1}^{\prime}\right)_{I}$. Symmetrically $c^{\prime} z \leq \gamma^{\prime}$ is valid for $P \cap\left(Q_{2}^{\prime}\right)_{I}$. Hence $c^{\prime} z \leq \gamma^{\prime}$ is a split cut for $P$. Since $\bar{z} \in H$, and $c \bar{z}>\gamma$, it follows that $c^{\prime} \bar{z}>\gamma^{\prime}$. Since moreover $c^{\prime} v=0$, then $\bar{z} \notin \operatorname{proj}_{v, H}\left\{(x, y) \in \mathbb{R}^{m+n}: c^{\prime} z \leq \gamma^{\prime}\right\}$. Finally, since $c^{\prime} z \leq \gamma^{\prime}$ is a split cut for $P$, it follows that $\bar{z} \notin \operatorname{proj}_{v, H} \mathcal{S}(P)$. Thus we showed $\operatorname{proj}_{v, H} \mathcal{S}(P) \subseteq \mathcal{S}\left(\operatorname{proj}_{v, H} P\right)$.

To prove the inductive step, let $i \geq 2$, and assume that the statement is true for every $0 \leq j \leq i-1$. Then

$$
\begin{aligned}
\operatorname{proj}_{v, H} \mathcal{S}^{i}(P)= & \underset{v, H}{\operatorname{proj}} \mathcal{S}\left(\mathcal{S}^{i-1}(P)\right) \subseteq \mathcal{S}\left(\underset{v, H}{\operatorname{proj}} \mathcal{S}^{i-1}(P)\right) \\
& \subseteq \mathcal{S}\left(\mathcal{S}^{i-1}(\underset{v, H}{\operatorname{proj} P})\right)=\mathcal{S}^{i}\left(\operatorname{proj}_{v, H} P\right)
\end{aligned}
$$

Lemma 9 Let $P$ be an unbounded polyhedron in $\mathbb{R}^{m+n}$, let $v$ be a nonzero vector in char.cone $P$, and let $H$ be a subspace that is mixed integer invariant under projection along $v$. Then

$$
\underset{v, H}{\operatorname{proj}}\left(P_{I}\right)=(\underset{v, H}{(\operatorname{proj}} P)_{I} .
$$

Proof Let $\bar{P}=\operatorname{proj}_{v, H} P$. Clearly $\bar{P}_{I}$ is $x$-integral, and, since $P_{I}$ is $x$-integral and $H$ is mixed integer invariant under projection along $v$, it follows that also $\operatorname{proj}_{v, H}\left(P_{I}\right)$ is $x$-integral. Thus we only need to show that $\operatorname{proj}_{v, H}\left(P_{I}\right) \cap\left(\mathbb{Z}^{m} \times \mathbb{R}^{n}\right)=\bar{P}_{I} \cap\left(\mathbb{Z}^{m} \times \mathbb{R}^{n}\right)$.

Let $z \in \operatorname{proj}_{v, H}\left(P_{I}\right) \cap\left(\mathbb{Z}^{m} \times \mathbb{R}^{n}\right)$. Clearly $z \in \bar{P}$, and since $z$ is $x$-integral, it follows that $z \in \bar{P}_{I}$.

To prove the converse, let $z \in \bar{P}_{I} \cap\left(\mathbb{Z}^{m} \times \mathbb{R}^{n}\right)$. Since in particular $z \in \bar{P}$, it follows there exists a scalar $\lambda$ such that $z+\lambda v \in P$. As $z$ is $x$-integral, and $v$ is rational, it follows that there exists a scalar $\mu \geq 0$ such that $w=z+\lambda v+\mu v$ is $x$-integral. Since
$z+\lambda v \in P$, and $v$ is a nonzero vector in char.cone $P$, it follows that $w \in P$. This implies $w \in P_{I}$ and so $z=\operatorname{proj}_{v, H} w \in \operatorname{proj}_{v, H}\left(P_{I}\right)$.
Lemma 10 Let $P$ be a polyhedron in $\mathbb{R}^{m+n}$, and let $c x+d y \leq \gamma$ be a valid inequality for $P_{I}$ such that

$$
\operatorname{proj}_{x}\left\{(x, y) \in P_{I}: c x+d y=\gamma\right\}
$$

is not lattice-free. Then $\exists k \in \mathbb{N}$ such that $c x+d y \leq \gamma$ is valid for $\mathcal{S}^{k}(P)$.
Proof We define the sets $F:=\left\{(x, y) \in P_{I}: c x+d y=\gamma\right\}$ and $Q:=\{(x, y) \in$ $P: c x+d y>\gamma\}$. If $Q=\emptyset$ there is nothing to show, thus we assume $Q \neq \emptyset$. Let $\bar{Q}:=\{(x, y) \in P: c x+d y \geq \gamma\}$ be the topological closure of $Q$. Since $c x+d y \leq \gamma$ is valid for $P_{I}$, then $Q$ does not contain any $x$-integral vector. It follows that $\operatorname{proj}_{x} Q$ does not contain any integral vector, and that $\operatorname{proj}_{x} \bar{Q}$ is lattice-free, since it is the topological closure of $\operatorname{proj}_{x} Q$.

Now we show that there exists an inequality $a x \geq \beta$, with $a \in \mathbb{Z}^{m}, \beta \in \mathbb{Z}$, such that $a x=\beta$ is valid for $F$, and $a x>\beta$ is valid for $Q . F \subseteq \bar{Q}$ implies that $\operatorname{proj}_{x} F \subseteq \operatorname{proj}_{x} \bar{Q}$, and since $\operatorname{proj}_{x} \bar{Q}$ is lattice-free while by hypothesis $\operatorname{proj}_{x} F$ is not, $\operatorname{dim}\left(\operatorname{proj}_{x} F\right)<\operatorname{dim}\left(\operatorname{proj}_{x} \bar{Q}\right)$ and $\operatorname{proj}_{x} F$ is contained in a proper face of $\operatorname{proj}_{x} \bar{Q}$. Let $G$ be a minimal face of $\operatorname{proj}_{x} \bar{Q}$, with respect to inclusion, containing $\operatorname{proj}_{x} F$. It follows by the minimality assumption that $G$ is not lattice-free. Now let $a x \geq \beta$ be valid for $\operatorname{proj}_{x} \bar{Q}$ and such that $G=\left\{x \in \operatorname{proj}_{x} \bar{Q}: a x=\beta\right\}$. Then clearly $a x=\beta$ is valid for $\operatorname{proj}_{x} F$ and so for $F$. Moreover, since $\operatorname{proj}_{x} Q$ contains no integral point, and since $G$ is not lattice-free, then it is easy to verify that $a x>\beta$ is valid for $\operatorname{proj}_{x} Q$ and so for $Q$. Clearly, since $a$ is rational and $\operatorname{proj}_{x} F$ contains integral vectors, we can assume that $a \in \mathbb{Z}^{m}$ and $\beta \in \mathbb{Z}$.

We now complete the proof by showing that there exists $k \in \mathbb{N}$ such that $c x+d y \leq \gamma$ is a split cut for $\mathcal{S}^{k}(P)$. We introduce the sets $Q^{i}:=\left\{(x, y) \in \mathcal{S}^{i}(P): c x+d y>\gamma\right\}$ and $\bar{Q}^{i}:=\left\{(x, y) \in \mathcal{S}^{i}(P): c x+d y \geq \gamma\right\}$ for every $i \in \mathbb{N}$. By Theorem 2, $\lim _{i \rightarrow \infty} \mathcal{S}^{i}(P)=P_{I}$. By intersecting $P_{I}$ and $\mathcal{S}^{i}(P)$ for every $i \in \mathbb{N}$ with the halfspace corresponding to the inequality $c x+d y \geq \gamma$, it follows by Observation 6 that $\lim _{i \rightarrow \infty} \bar{Q}^{i}=F$. Since $a x=\beta$ is valid for $F$, it follows by Observation 5 that there exists $k \in \mathbb{N}$ such that $a x<\beta+1$ is valid for $\bar{Q}^{k}$. Since $Q^{i} \subseteq \bar{Q}^{i}$ for every $i \in \mathbb{N}$, it follows that $a x<\beta+1$ is valid for $Q^{k}$. Moreover, since $Q^{i} \subseteq Q$ for every $i \in \mathbb{N}$, and since $a x>\beta$ is valid for $Q$, it follows that $a x>\beta$ is valid for $Q^{k}$. Hence $\beta<a x<\beta+1$ is valid for $Q^{k}$. In other words, $c x+d y \leq \gamma$ is valid for both

$$
\left\{(x, y) \in \mathcal{S}^{k}(P): a x \leq \beta\right\} \quad \text { and } \quad\left\{(x, y) \in \mathcal{S}^{k}(P): a x \geq \beta+1\right\}
$$

implying that $c x+d y \leq \gamma$ is a split cut for $\mathcal{S}^{k}(P)$.

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