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Effect of the Source Term on Steady Free Convection Boundary Layer Flows over an Vertical Plate in a Porous Medium. Part I

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Abstract. The Darcy free convection boundary layer flow over a vertical flat plate is considered in the presence of volumetric heat generation/absorption. In the present first part of the paper it is assumed that the heat generation/absorption takes place in a *self-consistent* way, the source term $q''' \equiv S$ of the energy equation being an analytical function of the local temperature difference $T - T_\infty$. In a forthcoming second part, the case of the *externally controlled* source terms $S = S(x, y)$ will be considered. It is shown that due to the presence of S, the physical equivalence of the up- and downflows gets in general broken, in the sense that the free convection flow over the upward projecting hot plate ("upflow") and over its downward projecting cold counterpart ("downflow") in general become physically distinct. The consequences of this circumstance are examined for different forms of S. Several analytical solutions are given. Some of them describe *algebraically decaying* boundary layers which can also be recovered as limiting cases of *exponentially decaying* ones. This asymptotic phenomenon is discussed in some detail.

Key words: volumetric heat generation, parallel flows, exact solutions, algebraic decay, exponential decay, Similar flows, Porous media.

Nomenclature

a	constant, Equations (25), (52)
b	constant, Equation (32)
c	constant, Equation (40b)
c_p	specific heat at constant pressure
f	similar stream function, Equation (45a)
g	magnitude of the acceleration due to gravity
Ge	Gebhart number, $= g\beta L/c_p$.
h	heat transfer coefficient, Equation (48)
k_m	effective thermal conductivity

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K	permeability of the porous medium
L	characteristic length
Nu	Nusselt number, Equation (24)
Ra	Darcy–Rayleigh number, $= g\beta K T_0 L/(\alpha \upsilon)$.
q_w	wall heat flux
Q_n	coefficients, Equation (9)
S	heat source/sink term, Equation (2c)
s_T, s_g	sign functions
T	fluid temperature
T_0	plate temperature scale
T_w	wall temperature
T_∞	ambient temperature
u, v	velocity components along x- and y-axes
x, y	Cartesian coordinates along the plate and normal to it, respectively
Y	dimensionless coordinate, Equations (26b), (40a)

Greek Symbols

α	effective thermal diffusivity
β	coefficient of thermal expansion
λ	power law exponent, Equation (1)
η	similarity independent variable, Equation (44b)
θ	dimensionless temperature, Equation (15)
ρ	fluid density
υ	kinematic viscosity
ψ	stream function

1. Introduction

Over recent several decades, fluid flow in porous media has intensively been studied and it has become a very productive field of research. The interest in the topic stems from its widespread practical applications in modern industries and in many environmental issues (as e.g. nuclear waste management, building thermal insulations, spread of pollutants, geothermal power plants, grain storage, packed-bed chemical reactors, oil recovery, ceramic processing, enhanced recovery of petroleum reservoirs, food science, medicine, etc.). This circumstance has resulted in a vast amount of both theoretical and experimental research work.

The mechanical and thermal characteristics of fluid flow in porous media are today well understood for a large number of surface geometries and (temperature and flux) boundary conditions. In this respect an enormous amount of scientific material has been collected and analyzed comprehensively in recent works by Bejan (1995), Ingham and Pop (1998, 2002, 2005), Nield and Bejan (1999), Vafai (2000, 2005), Pop and Ingham (2001), Ingham *et al.* (2004) and Bejan *et al.* (2004). Considerably less work has, however, been done for more complex situations, such as internal heat generation (other than viscous dissipation and pressure work) and/or absorption by sources and sinks distributed continuously in the bulk of porous medium. The aim of the present two-part paper is to add new results to this field of basic importance, by invoking, in order to be specific, the case of Darcy free convection boundary layer flow over a vertical plate.

According to the functional dependence which determines their volumetric distribution, the heat source/sink terms $q''' \equiv S\ [W/m^3]$ usually included in the energy equation (in addition to or instead of the viscous dissipation and pressure work terms), can be divided in two basic types, namely

$$\text{(I)}\ S = S(T - T_\infty) \quad \text{and} \quad \text{(II)}\ S = S(x, y, z)$$

The sources of type (I) depend only on the local temperature difference $T - T_\infty$, their space distribution being not externally controlled and thus, not known *a priori*. This circumstance results in a *self-consistent* mechanism of heat generation/absorption: the local intensity of sources is determined by the solution T of the energy equation, while this solution itself is governed by the activity of the sources. In other words, one is faced here with *feedback loops* connecting the sources to the temperature field. The actual space distribution of sources of type (I) is known only *a posteriori*, after the temperature problem has been resolved self consistently. This kind of sources occur in exothermic/endothermic chemical and biochemical reactions, where the reaction rate is controlled by the local temperature of the reactants.

The sources of type (II), in contrast, are *externally controlled*, their space distribution and intensity being either *prescribed*, or they result from some physical laws which do not depend on temperature directly, as e.g. the heat release by absorption of infrared radiation or of microwaves. In the latter cases the intensity of sources decay exponentially with the distance from the surface of incidence.

Concerning the published papers with internal heat generation S which fall into the first class (I), there are papers which describe the flow and heat transfer due to stretching surfaces and papers which describe free convection in cavities. Thus, Vajravelu and Nayfeh (1992, 1993), Vajravelu and Hadjinicalaou (1993, 1997), Chamkha (1999, 2003), Chamkha and Quadri (2002), and Abo-Eldahab and Aziz (2005) have studied the forced or convective heat transfer at a stretching sheet when S is a *linear* function of $T - T_\infty$, i.e. $S \sim (T - T_\infty)$. The papers by Hossain and Wilson (2002), Molla *et al.* (2004), and Hossain and Rees (2005) consider the free convection in a cavity with an internal heat generation term S also of the linear form $S \sim (T - T_0)$ where T_0 is a characteristic temperature of the fluid inside cavity. Furthermore, Abo-Eldahab and Aziz (2004) investigated the effect of blowing/suction on hydromagnetic heat transfer by mixed convection over a continuously stretching surface with power-law variation in the surface temperature or heat flux when the source term S is a combination of two functions of type (I) and (II),

$S = A(x) \cdot S_{\rm I}(T - T_\infty) + S_{\rm II}(x, y)$ where $S_{\rm I}$ is again a linear function of $T - T_\infty$. The same S has also been considered by Tashtoush and Duwairi (2003) in a transient mixed convection problem.

For the second class II of the previously studied problems, with $S(x, y) = A(x)\, e^{-\eta}$ where η is a similarity variable formed from x and y, the published papers are those due to Crepeau and Clarksean (1997) for a viscous and incompressible fluid (clear fluid) and by Postelnicu and Pop (1999), Postelnicu *et al.* (2000), Grosan and Pop (2001, 2002) and Bagai (2003) for the problem of free convection over a vertical flat plate or over a body of arbitrary shape embedded in a fluid-saturated porous medium by using the Darcy or non-Darcy flow models.

The present paper aims to add new results concerning the effect of sources on the free convection in porous media. In Part I some general features (as e.g. the breaking of the upflow/downflow equivalence) are considered which hold for both types of sources. Then possible parallel-flow as well as similarity solutions are investigated in the presence of self consistent heat generation/absorption mechanisms by different sources of type (I). In the forthcoming second part of the paper self similar flows in the presence of externally controlled sources of type (II) will be investigated in some detail.

2. Problem Formulation and Basic Equations

We consider the steady Darcy free convection in a fluid saturated porous medium adjacent to a heated or cooled semi-infinite vertical flat plate of power-law temperature distribution

$$T_{\rm w}(x) = T_\infty + s_T T_0 \left(\frac{x}{L}\right)^\lambda \tag{1}$$

Here T_∞ denotes the ambient temperature of the saturated porous medium, $T_0 > 0$ specifies the temperature scale of the plate, L is a reference length and the sign function $s_T = \mathrm{sgn}(T_{\rm w} - T_\infty)$ takes the value $s_T = +1$ for a "hot" plate, $T_{\rm w}(x) > T_\infty$ and the value $s_T = -1$ for a "cold" one, $T_{\rm w}(x) < T_\infty$. We further assume that in the porous medium continuously distributed heat sources are present. The rate S $[W/m^3]$ of the volumetric heat generation is considered first to be arbitrary, the only overall restriction being $S \to 0$ as $y \to +\infty$. Later in this paper, different specific forms of S as functions of the local temperature difference the dependence $T - T_\infty$ will be examined.

The flow domain and the choice of the coordinate system are sketched in Figure 1a, b where s_g denotes the projection of $\mathbf{g}/|\mathbf{g}|$ on the positive x-axis. Thus, $s_g = +1$ when the positive x-axis points in the direction of $\mathbf{g}$ (i.e. vertically downwards) and $s_g = -1$ when it points in the direction opposite to $\mathbf{g}$ (i.e. vertically upwards). In each of the two cases, depicted in

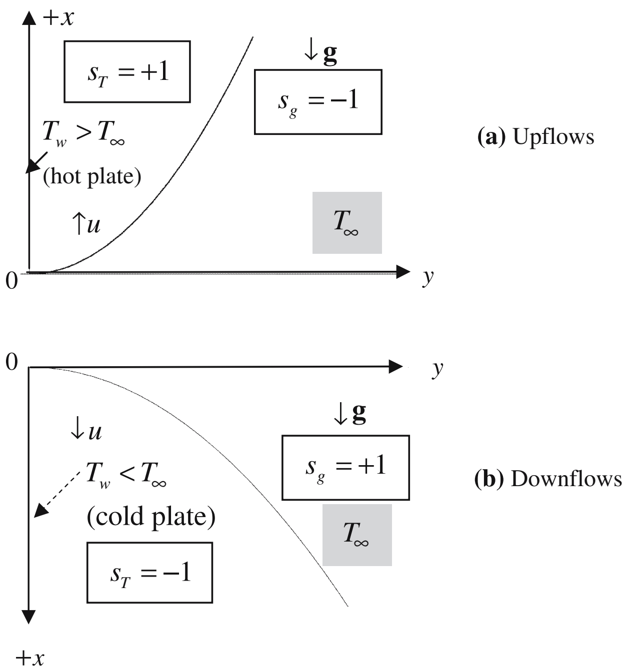

Figure 1. Representations of the free convection forward boundary layer up- and downflows over an upward projecting and downward projecting hot and cold plate, respectively. In the absence of the source term ($S = 0$) the two situations are physically equivalent. When, however $S \neq 0$, the up- and downflows in general become basically distinct.

Figure 1a, b, the "forward", i.e. the usual boundary layer flows are considered, where the definite edge of the plate, $x = 0$, represents its leading edge. The opposite situations, i.e. the cold plate in the configuration of Figure 1a and the hot one in the configuration of Figure 1b, correspond according to the nomenclature introduced by Goldstein (1965), to the "backward" boundary layer flows. The essential difference between the forward and backward boundary layer flows consist of the fact that in the latter case the leading edge recedes to an indefinite station infinitely far upstream, while the definite edge of the plate, $x = 0$, becomes a trailing edge. As a consequence, in the backward boundary layer flows the fluid has lost any memory of the perturbations introduced by the leading edge. This physical situation of the backward free convection boundary layer flows in saturated porous media has recently been investigated (for $S = 0$) by Magyari and Keller (2004), and will not be considered here. The forward boundary layer flows along the upward and downward projecting hot and cold plates shown in Figure 1a and b, will be referred to hereafter for short as "upflows" and "downflows", respectively.

Following Nield and Bejan (1999) we write the continuity, Darcy and energy equations corresponding to the situations of Figures 1a,b in the form

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{2a}$$

$$u=-s_g\frac{g\beta K}{\upsilon}\left(T-T_\infty\right), \tag{2b}$$

$$\rho c_p\left(u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}\right)=k_m\frac{\partial^2 T}{\partial y^2}+S, \tag{2c}$$

where one has assumed that the boundary-layer and the Boussinesq approximations hold. The plate is assumed impermeable, such that the boundary conditions of the problem associated with Equations (2) are

$$\begin{aligned} &v|_{y=0}=0, \qquad T|_{y=0}=T_{\mathrm{w}}(x),\\ &T|_{y\to\infty}\to T_\infty,\ S|_{y\to\infty}\to 0 \end{aligned} \tag{3a,b,c,d}$$

We notice that in both situations of Figure 1a and b the product of the sign functions s_T and s_g is the same,

$$s_T s_g=-1 \tag{4}$$

3. The Source Term can Break the Upflow/Downflow Equivalence

Since the flow is incompressible and two dimensional, it is convenient to introduce the stream function

$$(u,v)=\left(\frac{\partial\psi}{\partial y},-\frac{\partial\psi}{\partial x}\right) \tag{5}$$

and to express the energy equation (2c) in terms of ψ with the aid of the relationship

$$T=T_\infty+s_T\frac{\upsilon}{g\beta K}\frac{\partial\psi}{\partial y} \tag{6}$$

By doing so, we are left with the single partial differential equation

$$\rho c_p\left(\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial x\partial y}-\frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2}\right)=k_m\frac{\partial^3\psi}{\partial y^3}+s_T\frac{g\beta K}{\upsilon}S \tag{7}$$

accompanied by the boundary conditions

$$\left.\frac{\partial\psi}{\partial x}\right|_{y=0}=0,\quad \left.\frac{\partial\psi}{\partial y}\right|_{y=0}=\frac{\alpha}{L}Ra\left(\tfrac{x}{L}\right)^{\lambda},$$
$$\left.\frac{\partial\psi}{\partial y}\right|_{y\to\infty}\to 0,\ S|_{y\to\infty}\to 0 \tag{8a,b,c,d}$$

where $\alpha=k_m/(\rho c_p)$ and Ra denotes the Darcy Rayleigh number, $Ra=g\beta KT_0L/(\alpha\upsilon)$.

The boundary conditions (8) are always independent of the sign functions s_T and s_g while Equation (7) depends in general on s_T, as long as $S\neq 0$. If, however, the source term S is neglected, Equation (7) also becomes independent of s_T and thus we immediately recover the well known textbook result concerning the physical equivalence of the free convection flows over an upward projecting hot plate and over its downward projecting cold counterpart. If however in Equation (7) the source term S is present, then due to the sign $s_T=\pm 1$ in front of S this physical equivalence gets in general broken (as already anticipated in the title of the present Section as well as in the Caption of Figures 1). This means that the free convection flow over the upward projecting hot plate ("upflow", Figure 1a) and over its downward projecting cold counterpart ("downflow", Figure 1b) in general become physically distinct. The consequences of this circumstance will be examined in the following Sections for different forms of the source term S.

4. *S* is an Analytical Function of the Local Temperature Difference $T-T_\infty$

Except for the boundary condition (3d), no other restrictions on the source function S have been made until now. Hereafter an additional restriction will be adopted, namely we assume that S is an analytical function of the local temperature difference $T-T_\infty$, such that it can be expanded in a power series of $T-T_\infty$,

$$S=\sum_{n=0}^{\infty}Q_n\left(T-T_\infty\right)^n \tag{9}$$

We mention that, choosing T_∞ as origin of the temperature scale, $T-T_\infty$ represents precisely the local temperature of the fluid. Obviously, the above assumption of analyticity is a fairly weak restriction of generality.

As a consequence of the boundary conditions (3c) and (3d), the coefficient Q_0 of series (9) is always zero, while the other Q_n's may in general be non-vanishing. Thus, having in mind Equation (6), Equation (7) becomes

$$\rho c_p\left(\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial x\partial y}-\frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2}\right)=k_m\frac{\partial^3\psi}{\partial y^3}+\sum_{n=1}^{\infty}Q_n\left(\frac{s_T\upsilon}{g\beta K}\right)^{n-1}\left(\frac{\partial\psi}{\partial y}\right)^n \tag{10}$$

In this case the three relevant boundary conditions are (8a,b,c) since, owing to (8c), the condition (8d) is satisfied automatically.

A simple inspection of Equation (10) shows that it depends in general on the sign function s_T and the upflow/downflow equivalence gets broken. This conclusion also holds when S is an even function of $T - T_\infty$ (i.e. $Q_1 = Q_3 = Q_5 = \ldots = 0$). In this case Equations (9) and (6) yield

$$S \equiv S_{\text{even}} = \sum_{j=1}^{\infty} Q_{2j} \left(\frac{\upsilon u}{g\beta K} \right)^{2j} \tag{11}$$

When, however, S is an odd function of $T - T_\infty$ (i.e. $Q_2 = Q_4 = Q_6 = \ldots = 0$), then Equation (10) becomes independent of the sign function s_T, and S has the form

$$S \equiv S_{\text{odd}} = s_T \sum_{j=1}^{\infty} Q_{2j-1} \left(\frac{\upsilon u}{g\beta K} \right)^{2j-1} \tag{12}$$

Therefore, while for $S = S_{\text{even}}$ the upflow/downflow equivalence is broken, for $S = S_{\text{odd}}$ it holds, but with the important physical restriction that in one of the cases depicted in Figure 1, heat sources and in the other one heat sinks of the same intensity are present. From a practical point of view, this is an essential difference compared to the familiar upflow/downflow equivalence encountered in the case $S = 0$.

5. Parallel Flows Formed Over an Isothermal ($\lambda = 0$) Plate

5.1. General Considerations

We consider in this Section the important special case of constant plate temperature, $T_{\text{w}} = \text{constant} = T_\infty + s_T T_0$ corresponding in Equation (1) to $\lambda = 0$. In this case the problem does not possess for $\lambda = 0$ a natural length scale, i.e. L may be chosen arbitrarily. Our aim in the following is to examine the possible existence of a strictly parallel free convection flow over the vertical impermeable plate in the presence of a volumetric heat generation according to Equation (9) and to its special cases (11) and (12), respectively. Under "parallel" we mean (as usual) a plane boundary layer flow with identically vanishing transversal velocity component, $(u, v) = (u, 0)$, that is, with a stream function which depends only on the transversal coordinate, $\psi = \psi(y)$. In this case Equation (10) and the boundary conditions (8) reduce to

$$k_m \frac{\mathrm{d}^2 u}{\mathrm{d}y^2} + \sum_{n=1}^{\infty} Q_n \left(\frac{s_T \upsilon}{g\beta K} \right)^{n-1} u^n = 0 \tag{13}$$

$$u|_{y=0} = \frac{\alpha}{L} Ra,\ u|_{y\to\infty} \to 0 \tag{14a,b}$$

Since we are mainly interested in the heat transfer characteristics of the flow, it is convenient to transcribe the "velocity problem" (13), (14) with the aid of Equation (2b) into a "temperature problem" for the dimensionless temperature $\theta = \theta(y)$ defined by the relationship

$$T(y) = T_\infty + s_T T_0 \theta(y) \tag{15}$$

In terms of θ the expression (9) of S reads

$$S = \sum_{n=0}^{\infty} Q_n (s_T T_0)^n \theta^n \tag{16}$$

and Equations (2b) and (4) yield

$$u(y) = \frac{\alpha}{L} Ra\, \theta(y) \tag{17}$$

The equivalent temperature boundary value problem has the form

$$k_m \frac{\mathrm{d}^2\theta}{\mathrm{d}y^2} + \sum_{n=1}^{\infty} Q_n (s_T T_0)^{n-1} \theta^n = 0 \tag{18}$$

$$\theta(0) = 1, \quad \theta(\infty) = 0 \tag{19a,b}$$

Equation (18) admits the first integral

$$\frac{k_m}{2}\left(\frac{\mathrm{d}\theta}{\mathrm{d}y}\right)^2 + \sum_{n=1}^{\infty} \frac{Q_n}{n+1} (s_T T_0)^{n-1} \theta^{n+1} = \text{constant}. \tag{20}$$

Having in mind the boundary condition (19b) and assuming a sufficiently smooth decay of the temperature field as $y \to \infty$, we see that the integration constant in (20) is zero and thus we get

$$\frac{\mathrm{d}\theta}{\mathrm{d}y} = \pm\sqrt{-\frac{2}{k_m}\sum_{n=1}^{\infty} \frac{Q_n}{n+1} (s_T T_0)^{n-1} \theta^{n+1}} \tag{21}$$

This equation along with the boundary condition (19a) yields the solution in the implicit form $y = y(\theta)$,

$$y = \pm\int_1^{\theta} \left[-\frac{2}{k_m}\sum_{n=1}^{\infty} \frac{Q_n}{n+1} (s_T T_0)^{n-1} \theta^{n+1}\right]^{-1/2} \mathrm{d}\theta \tag{22}$$

In some simple but important special cases the implicit solution (22) can easily be inverted to the explicit form $\theta = \theta(y)$ (see below). Obviously, in order to be physical, the solution (22) of the problem (18), (19) must also be real and nonsingular. In addition, it must actually satisfy the boundary condition (19b) as being assumed above.

According to Equations (15), (21) and (19a), the wall heat flux corresponding to the solution (22) is

$$q_{\mathrm{w}} = -k_m \left.\frac{\partial T}{\partial y}\right|_{y=0} = \mp\sqrt{-2k_m \sum_{n=1}^{\infty} \frac{Q_n}{n+1} (s_T T_0)^{n+1}} \tag{23}$$

In account of Equations (1) and (23) the corresponding Nusselt number is

$$Nu = \frac{q_w L}{k_m |T_w - T_\infty|} = \mp s_T L \sqrt{-\frac{2}{k_m} \sum_{n=1}^{\infty} \frac{Q_n}{n+1} (s_T T_0)^{n-1}} \tag{24}$$

5.2. PARALLEL FLOW FOR $S = a(T - T_\infty)^2$

As a specific example for the upflow/downflow symmetry breaking described above, we consider the case when in Equation (9) the only non-vanishing coefficient is $Q_2 \equiv a > 0$. In this case we obtain for S the simplest even function of $T - T_\infty$,

$$S = a(T - T_\infty)^2, \, a > 0 \tag{25}$$

In this case Equation (22) yields the nonsingular solution

$$\theta(y) = \left(1 + \sqrt{-s_T}\, Y\right)^{-2}, \; Y = \sqrt{\frac{aT_0}{6k_m}}\, y \tag{26a,b}$$

The corresponding temperature and velocity fields (15) and (17) are

$$T(y) = T_\infty + s_T T_0 \left(1 + \sqrt{-s_T}\, Y\right)^{-2} \tag{27}$$

and

$$u(y) = \frac{\alpha}{L} Ra \left(1 + \sqrt{-s_T}\, Y\right)^{-2} \tag{28}$$

respectively. The Nusselt number (24) becomes in this case

$$Nu = s_T L \sqrt{-s_T \frac{2aT_0}{3k_m}} \tag{29}$$

We now immediately see that for $a > 0$ the above solution is only real for $s_T = -1$, while for $s_T = +1$ it becomes complex and thus non-physical. In other words, a source term of the form (25), which is an even function of $T - T_\infty$, actually breaks the usual upflow/downflow symmetry of the free convection flow over a vertical plate. The (algebraically decaying) parallel flow (28) can only be formed over the downward projecting cold plate (Figure 1b), but not over its upward projecting hot counterpart (Figure 1a). Obviously, for $a < 0$ (heat sinks) the opposite is true.

In the present context of the Darcy free convection over a vertical plate adjacent to a fluid saturated porous medium, a volumetric heat generation of type (23) is actually realized by the viscous dissipation. Indeed, in this case S is proportional to the dissipation function and has the expression (see Ene and Sanchez-Palencia, 1982; Bejan, 1995)

$$S = \frac{\rho \upsilon}{K} u^2 \tag{30}$$

Comparing Equations (30) to (25) and having in mind Equation (2b), we easily find that the constant aT_0/k_m occurring in Equations (26)–(29) can be put in the form $aT_0/k_m = RaGe/L^2$ where Ge is the Gebhart number, $Ge = g\beta L/c_p$. Thus, the Nusselt number (29), with $s_T = -1$, is given in this case by

$$Nu = -\sqrt{\frac{2}{3} RaGe} \tag{31}$$

The existence of the parallel flow solutions (27), (28) due to the presence of viscous dissipation, has first been pointed out by Magyari and Keller (2003). Later, Rees *et al.* (2003) have shown that these solutions represent precisely the asymptotic profiles toward which the classical Cheng–Minkowycz solution (Cheng and Minkowycz, 1977) evolves gradually with increasing distance x from the leading edge. On this reason, these solutions were named “Asymptotic Dissipation Profiles” (ADP’s).

5.3. PARALLEL FLOW FOR $S = b(T - T_\infty)$

We consider the case of the simplest odd function of $T - T_\infty$ which is obtained when in the general expression (9) of S the only non-vanishing the coefficient is $Q_1 \equiv b$ i.e., S is the linear function

$$S = b(T - T_\infty) \tag{32}$$

In this case Equation (22) yields the solution

$$\theta(y) = \exp\left(-\sqrt{-\frac{b}{k_m}}\, y\right) \tag{33}$$

The corresponding temperature and velocity fields (15) and (17) are

$$T\,(y) = T_\infty + s_T T_0 \exp\left(-\sqrt{-\frac{b}{k_m}}\,y\right) \tag{34}$$

and

$$u\,(y) = \frac{\alpha}{L} Ra \exp\left(-\sqrt{-\frac{b}{k_m}}\,y\right) \tag{35}$$

respectively.

One sees that this solution is only real for $b<0$, i.e. when the source term S is of the form

$$S = -s_T T_0\,|b| \exp\left(-\sqrt{\frac{|b|}{k_m}}\,y\right) \tag{36}$$

Therefore, the parallel flow solutions (34) and (35) can be realized over the cold plate ($s_T=-1$) if in the volume heat sources are present ($S>0$), and over the hot plate ($s_T=+1$) if heat sinks ($S<0$) of the same intensity $|S|$ are distributed in the volume of the saturated porous medium. This specific result is in full agreement with the general features described in Section 5.1. In contrast to the algebraically decaying parallel flow solutions (34) and (35) corresponding to the quadratic source term (25), the solutions (34), (35) associated with the linear law (32) show a rapid exponential decay.

The Nusselt number (24) with the lower sign becomes in this case

$$Nu = s_T L \sqrt{\frac{|b|}{k_m}} \tag{37}$$

5.4. PARALLEL FLOW FOR $S = a\,(T-T_\infty)^2 + b\,(T-T_\infty)$

After the pure power law cases (25) and (32), we consider the simplest "mixed case" in which in the expression (9) of S the coefficients are $Q_1 \equiv b$ and $Q_2 \equiv a > 0$ are simultaneously non-vanishing, but all the other Q_n's are zero, i.e. S is the quadratic function

$$S = a(T-T_\infty)^2 + b\,(T-T_\infty)\,,\ a>0 \tag{38}$$

It can be shown that in this case the boundary value problem (18), (19) does not admit solutions for $b>0$ (the condition $\theta\,(\infty)=0$ can not be satisfied for $b>0$). For $b<0$, however, there exist solutions for both signs $s_T = \pm 1$. They can again given in a explicit form, namely

$$\theta(y) = \frac{s_T}{c}\left[1 - \left(\frac{\sqrt{1 - s_T c} + \tanh Y}{1 + \sqrt{1 - s_T c}\tanh Y}\right)^2\right] \tag{39a}$$

or, equivalently,

$$\theta(y) = \frac{1}{\left(1 + \sqrt{1 - s_T c}\cdot\tanh Y\right)^2 \cosh^2 Y} \tag{39b}$$

where

$$Y = \sqrt{\frac{|b|}{k_m}}\frac{y}{2} \quad \text{and} \quad c = \frac{2aT_0}{3|b|} \tag{40a,b}$$

The corresponding Nusselt number (24) is

$$Nu = s_T L\sqrt{\frac{|b|}{k_m}}\sqrt{1 - s_T c} = Nu|_{a=0}\cdot\sqrt{1 - s_T c} \tag{41}$$

where $Nu|_{a=0}$ denotes the Nusselt number (37). One sees that for $s_T = +1$ the solution (39) is real only if

$$0 < c \leqslant 1 \qquad (s_T = +1) \tag{42}$$

In particular, for $c = 1$, Equations (39) and (41) reduce to

$$\theta(y) = \frac{1}{\cosh^2 Y}, Nu = 0, \qquad (s_T = +1, c = 1) \tag{43a,b}$$

For $s_T = -1$, on the other hand (40) the solution exists without further restriction on the positive parameter c.

As an illustration of the above results, in Figure 2 some of temperature profiles (39) are plotted as functions of the dimensionless transversal coordinate Y for $s_T = +1$, $s_T = -1$ and a couple of values of the parameter c. We see that the upflow ($s_T = +1$) to downflow ($s_T = -1$) equivalence is broken again. This becomes especially manifest for the value $c = 1$ where the corresponding Nusselt numbers are $Nu = 0$ and $Nu = -2\sqrt{2}$ for $s_T = +1$ and $s_T = -1$, respectively.

It is easy to show that in the limiting case $a \to 0$, which implies $c \to 0$, the exponentially decaying solution (39b) goes over, as it should be, in (33) which is valid for $b < 0$ and which is exponentially decaying, too. Of much more interest however, is the limiting case $b \to 0$ in which the *exponentially decaying* solution (39) should go over in the *algebraic decaying* asymptotic dissipation profile given by Equation (26) for $s_T = -1$. This is indeed the case since, according to Equation (40), for $b \to 0$ one has $c \to \infty$, $Y \to 0$ and thus Equation (39b) yields $\theta(y) \to \left(1 + \sqrt{c}\cdot Y\right)^{-2} = \left(1 + \sqrt{(aT_0/6k_m)}\cdot y\right)^{-2}$

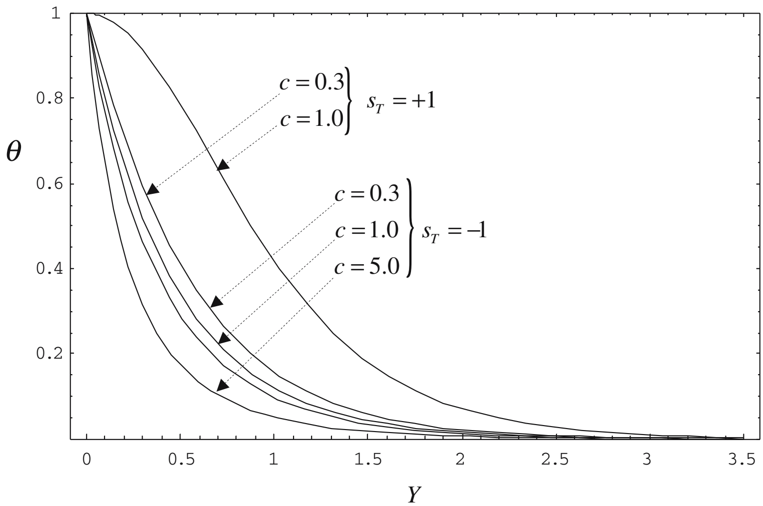

Figure 2. Plot of the temperature profiles (39) for $s_T = +1$ and $s_T = -1$ for specified values of the parameter c.

which is precisely Equation (26) for $s_T = -1$. This result is a further explicit example which bridges the historical disagreement concerning the feasibility of exponentially and algebraically decaying boundary layers (for historical details and further examples see Brown and Stewartson 1965; Merkin 1978; Kuiken 1981a,b, 1983; Kahn and Stewartson 1984; Magyari and Keller 2004; Magyari *et al.* 2005).

6. Self-similar Flows Formed over a Non-isothermal ($\lambda \neq 0$) Plate

The question whether the problem (7), (8) with a source term of the form (9) admits similarity solutions, is of an obvious basic interest. A suitable general similarity transformation which allows to find answer on this question is

$$\psi(x, y) = \alpha\sqrt{Ra}\left(\frac{x}{L}\right)^{\frac{\lambda+1}{2}} f(\eta)$$
$$\eta = \sqrt{Ra}\left(\frac{x}{L}\right)^{\frac{\lambda-1}{2}} \frac{y}{L} \tag{44a,b}$$

Inserting Equation (44a,b) in Equations (7) and (8) furnishes the boundary value problem for the similar stream function f,

$$f''' + \frac{\lambda+1}{2} ff'' - \lambda f'^2 + \frac{L^2}{k_m Ra}\sum_{n=1}^{\infty} Q_n (s_T T_0)^{n-1}\left(\frac{x}{L}\right)^{(n-2)\lambda+1} f'^n = 0 \tag{45a}$$

$$f(0) = 0, \quad f'(0) = 1, \quad f'(\infty) = 0 \tag{45b,c}$$

where the prime denotes differentiation with respect to η. The velocity and temperature field is given in terms of the solution of boundary value problem (45) as follows

$$u = \frac{\alpha}{L} Ra \left(\frac{x}{L}\right)^{\lambda} f'(\eta)$$
$$v = -\frac{\alpha}{L}\sqrt{Ra}\left(\frac{x}{L}\right)^{\frac{\lambda-1}{2}}\left[\frac{\lambda+1}{2} f(\eta) + \frac{\lambda-1}{2}\eta f'(\eta)\right] \tag{46a,b}$$

$$T = T_{\infty} + s_T T_0 \left(\frac{x}{L}\right)^{\lambda} f'(\eta) \tag{47}$$

The dimensionless similar wall heat transfer coefficient is obtained as

$$h = -s_T f''(0) \tag{48}$$

A simple inspection of Equation (45a) shows that the similarity reduction of the problem is only possible when in the expression (9) of S only a single coefficient Q_n is non-vanishing, namely that for which the relationship

$$(n-2)\lambda + 1 = 0 \tag{49}$$

holds. For $\lambda > 0$, Equation (49) admits (for $n =$ positive integer) a single solution which is $n = 1$, while for negative values of λ it admits an infinite number of solutions as e.g. $(\lambda = -1, n = 3)$, $(\lambda = -1/2, n = 4)$, $(\lambda = -1/3, n = 5)$ etc... Obviously, for a source term of the form $S = Q_0 (T - T_{\infty})^n$ with arbitrary n, the problem of similarity solutions makes sense (at least for $s_T = +1$) also for other values of the temperature exponent λ which satisfies Equation (49). In this case Equation (45a) becomes

$$f''' + \frac{\lambda+1}{2} ff'' - \lambda f'^2 + \frac{L^2 Q_0}{k_m Ra}(s_T T_0)^{\frac{\lambda-1}{\lambda}} f'^{\frac{2\lambda-1}{\lambda}} = 0 \tag{50}$$

In order to be specific, we restrict our considerations in the present Section to the case $\lambda = n = 1$. In this case Equation (50) reduces to

$$f''' + ff'' - f'^2 + af' = 0 \tag{51}$$

where

$$a = \frac{Q_0}{k_m}\frac{L^2}{Ra} \tag{52}$$

The problem admits the elementary solution

$$f(\eta) = \frac{1 - \exp\left(-\sqrt{1-a}\cdot\eta\right)}{\sqrt{1-a}} \tag{53}$$

The corresponding heat transfer coefficient (48) is

$$h = s_T\sqrt{1-a} \tag{54}$$

The explicit coordinate dependence of the source term is given by

$$S = Q_0\,(T - T_\infty) = s_T\,Q_0 T_0 \frac{x}{L}\exp\left(-\sqrt{1-a}\cdot\eta\right) \tag{55}$$

Obviously, the solution (52) is physical only for $a < 1$. We mention that the problem (51), (45b,c) admits physical solutions also for as well as $a \geqslant 1$ which, however can be obtained only numerically.

7. Summary and Conclusion

The steady Darcy free convection in a fluid saturated porous medium adjacent to a heated or cooled semi-infinite vertical flat plate of power-law temperature distribution was analyzed in this paper with focus on the effect of continuously distributed heat sources/sinks $q''' \equiv S[W/m^3]$ in the bulk of the porous medium. It has been assumed that these sources/sinks S are in general other than the viscous dissipation and pressure work terms of energy equation, but $S \to 0$ as the distance from the plate goes to infinity. The main results of the paper can be summarized as follows.

1. Due to the presence of S, the physical equivalence of the up- and downflows gets in general broken, in the sense that the free convection flow over the upward projecting hot plate ("upflow") and over its downward projecting cold counterpart ("downflow") in general become physically distinct.
2. When S is an analytical (but otherwise arbitrary) function of the local temperature difference $T - T_\infty$, the free convection problem for an isothermal plate admits in general a parallel flow solution which can be obtained by quadratures (see Equation (22)).
3. In the special case $S = a\,(T - T_\infty)^2$ of the quadratic dependence on $T - T_\infty$ with $a > 0$ (heat release), parallel flow can only be formed over the upward projecting hot (isothermal) plate, but not over its downward projecting cold counterpart. This result is one of the main consequences of the broken upflow/downflow equivalence in the presence of heat sources. The corresponding parallel free convection flow decay algebraically with increasing distance from the plate. Incidentally, the effect of sources of type $S = a\,(T - T_\infty)^2$ is equivalent in the Darcy free convection with the effect of viscous dissipation (see Section 5.2).
4. In the linear case $S = b\,(T - T_\infty)$ parallel flow solutions only exist for $b < 0$ which means volumetric heat absorption for the hot plate and heat

generation for the cold plate (see Equation (36). In contrast to the quadratic case $S = a\,(T - T_\infty)^2$, the solution decays now exponentially (see Equations (33–35)).

5. In the parabolic case $S = a\,(T - T_\infty)^2 + b\,(T - T_\infty)\,, a > 0$ examined in Section 5.4, parallel flow solutions exist both for the hot $(s_T = +1)$ and cold $(s_T = -1)$ plate but only for $b < 0$. These solutions decay exponentially with increasing distance from the plate for $b \neq 0$. However, the exponential decaying solution corresponding to $s_T = -1$ goes over (as it should) in the algebraic decaying solution of the above point 4 as $b \to 0$. This result furnishes an explicit example which bridges the historical disagreement concerning the feasibility of exponentially and algebraically decaying boundary.
6. The present free convection problem also admits plane boundary layer similarity solutions when the source term has the power law form $S \sim (T - T_\infty)^n$, regarding that n and the exponent λ of the plate temperature distribution satisfy the relationship $(n - 2)\,\lambda + 1 = 0$. For the special case $n = \lambda = 1$ (linearly rising plate temperature) an exponentially decaying (see Section 6) analytical solution could be found.

The forthcoming Part II of the paper (Magyari *et al.*, 2006, Submitted), will be concerned mainly with the similarity solutions of the present free convection problem in the presence of the externally controlled sources/sinks of type (II), $S = S(x, y)$.

References

Abo-Eldahab, E. M. and El Aziz, M. A.: 2004, Blowing/suction effect on hydromagnetic heat transfer by mixed convection from an inclined continuously stretching surface with internal heat generation/absorption, *Int. J. Thermal Sci.* **43**, 709–719.

Abo-Eldahab, E. M. and El Aziz, M. A.: 2005, Flow and heat transfer in a micropolar fluid past a stretching surface embedded in an non-darcian porous medium with uniform free stream. Appl, *Math. Comput.* **162**, 881–899.

Bagai, S.: 2003, Similarity solutions of free convection boundary layers over a body of arbitrary shape in a porous medium with internal heat generation, *Int. Comm. Heat Mass Transfer* **30**, 997–1003.

Bejan, A.: 1995, *Convective Heat Transfer*, 2nd ed., Wiley, New York.

Bejan, A., Dincer, I., Lorente, S., Miguel, A. F. and Reis, A. H.: 2004, *Porous and Complex Flow Structures in Modern Technologies*, Springer, New York.

Brown, S. N. and Stewartson, K.: 1965, On similarity solutions of the boundary-layer equations with algebraic decay, *J. Fluid. Mech.* **23**, 673–687.

Chamkha, Ali J.: 1999, Hydromagnetic three-dimensional free convection on a vertical stretching surface with heat generation or absorption, *Int. J. Heat Fluid Flow* **20**, 84–92.

Chamkha, Ali J.: 2003, MHD flow of a uniformly stretched vertical permeable surface in the presence of heat generation/absorption and a chemical reaction, *Int. Comm. Heat Mass Transfer* **26**, 1183–1191.

Chamkha, Ali J. and Quadri, M. M. A.: 2002, Combined heat and mass transfer by hydromagnetic natural convection over a cone embedded in a non-Darcian porous medium with heat generation/absorption effects. *Heat Mass Transfer* **38**, 487–495.

Cheng, P. and Minkowycz, W. J.: 1977, Free convection about a vertical flat plate embedded in a porous medium with application to heat transfer from a dike. *J. Geophys. Res.* **82**, 2040–2044.

Crepeau, J. C. and Clarksean, R.: 1997, Similarity solutions of natural convection with internal heat generation. *ASME J. Heat Transfer* **119**, 183–185.

Ene, H. I. and Sanchez-Palencia, E.: 1982, On thermal equations for flow in porous media. *Int. J. Engg. Sci.* **29**, 623–630.

Goldstein, S.: 1965, On backward boundary layers and flow in converging passages, *J. Fluid. Mech.* **21**, 33–45.

Grosan, T. and Pop, I.: 2001, Free convection over a vertical flat plate with variable wall temperature and internal heat generation in a porous medium saturated with a non-Newtonian fluid, *Technische Mechanik* **21**, 313–318.

Grosan, T. and Pop, I.: 2002, Free convection of non-Newtonian fluids over a vertical surface in a porous medium with internal heat generation, *Int. J. Appl. Mech. Engg.* **7**, 401–407.

Hossain, M. A. and Wilson, M.: 2002, Natural convection flow in a fluid-saturated porous medium enclosed by non-isothermal walls with heat generation. *Int. J. Thermal Sci.* **41**, 447–454.

Hossain, M. A. and Rees, D. A. S.: 2005, Natural convection flow of water near its density maximum in a rectangular enclosure having isothermal walls with heat generation. *Heat Mass Transfer* **41**, 367–374.

Ingham, D. B. and Pop, I. (eds.): 1998, vol. II 2002, *Transport Phenomena in Porous Media*. Pergamon, Oxford.

Ingham, D. B. and Pop, I. (eds.): 2005, *Transport Phenomena in Porous Media III*. Elsevier, Oxford.

Ingham, D. B., Bejan, A., Mamut, E. and Pop, I. (eds.): 2004, *Emerging Technologies and Techniques in Porous Media*. Kluwer, Dordrecht.

Kahn, M. A. and Stewartson, K.: 1984, On natural convection from a cooling vertical sheet, *Q. J. Mech. Appl. Math.* **37**, 325–338.

Kuiken, H. K.: 1981a, On boundary layers in fluid mechanics that decay algebraically along stretches of wall that are not vanishingly small, *IMA J. Appl. Math.* **27**, 387–405.

Kuiken, H. K.: 1981b, A backward free-convective boundary layers, *Q. J. Mech. Appl. Math.* **34**, 397–413.

Kuiken, H. K.: 1983, A class of backward free-convective boundary layer similarity solutions, *Int. J. Heat Mass Transfer* **26**, 655–661.

Magyari, E. and Keller, B.: 2003, The opposing effect of viscous dissipation allows for a parallel free convection boundary layer flow along a cold vertical flat plate, *Transport Porous Media* **51**, 227–230.

Magyari, E. and Keller, B.: 2004, Backward free convection boundary layers in porous media, *Transport Porous Media* **55**, 285–300.

Magyari, E. Rees, D. A. S. and Keller, B.: 2005, Effect of viscous dissipation on the flow in fluid saturated porous media, Chapter 9 in *Handbook of Porous Media (2nd edition)*, Vafai, K. (ed.), Taylor & Francis, New York.

Merkin, J. H.: 1978, On solutions of the boundary-layer equations with algebraic decay, *J. Fluid. Mech.* **88**, 309–321.

Molla, M. M., Hossain, M. A. and Yao, L. S.: 2004, Natural convection flow along a vertical surface with uniform surface temperature in presence of heat generation/absorption. *Int. J. Thermal Sci.* **43**, 157–163.

Nield, D. A. and Bejan, A.: 1999, *Convection in Porous Media*, 2nd ed., Springer, New York.

Pop, I. and Ingham, D. B.: 2001, *Convective Heat Transfer: Mathematical and Computational Modelling of Viscous Fluids and Porous Media*, Pergamon, Oxford.

Postelnicu, A., Pop, I.: 1999, Similarity solutions of free convection boundary layers with internal heat generation about a vertical and horizontal surfaces in porous media, *Int. Comm. Heat Mass Transfer* **26**, 1183–1191.

Postelnicu, A., Grosan, T. and Pop, I.: 2000, Free convection boundary layer over a vertical permeable flat plate in a porous medium with internal heat generation, *Int. Comm. Heat Mass Transfer* **27**, 729–738.

Rees, D. A. S., Magyari, E. and Keller, B.: 2003, The development of the asymptotic dissipation profile in a vertical free convective boundary layer flow in a porous medium, *Transport Porous Media* **53**, 347–355.

Tashtoush, B. and Duwairi, H. M.: 2003, Transient mixed convection with internal heat generation and oscillating plate temperature, *Acta Mechanica,* **174**, 185–199.

Vafai, K. (ed.): 2000, *Handbook of Porous Media*, Marcel Dekker, New York.

Vafai, K. (ed.): 2005, *Handbook of Porous Media*, 2nd ed., Taylor & Francis, New York.

Vajravelu, K. and Nayfeh, J.: 1992, Hydromagnetic convection at a cone and a wedge, *Int. Comm. Heat Mass Transfer* **19**, 701–710.

Vajravelu, K. and Hadjinicalaou, A.: 1993, Heat transfer in a viscous fluid over a stretching sheet with viscous dissipation and internal heat generation, *Int. Comm. Heat Mass Transfer* **20**, 417–430.

Vajravelu, K. and Hadjinicalaou, A.: 1997, Convective heat transfer in an electrically conducting fluid at a stretching surface with uniform free stream, *Int. J. Engg. Sci*. **35**, 1237–1244.