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Indecomposable Coverings with Concave Polygons

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Abstract We show that for any concave polygon that has no parallel sides and for any k, there is a k-fold covering of some point set by the translates of this polygon that cannot be decomposed into two coverings. Moreover, we give a complete classification of open polygons with this property. We also construct for any polytope (having dimension at least three) and for any k, a k-fold covering of the space by its translates that cannot be decomposed into two coverings.

Keywords Multiple coverings · Sensor networks

1 Introduction

A family of sets is a *k-fold* covering of a point set if every point is contained in at least k of the sets. A covering is *decomposable* if the sets can be partitioned into two (1-fold) coverings. We say that a planar set is *cover-decomposable* if there exists a k such that every k-fold covering of the plane by its translates is decomposable. Pach [7] conjectured that all convex planar sets are cover-decomposable, and this is still an open problem. The conjecture has been verified for open convex polygons in a series of papers. The main goal of this paper is to prove results about *non*-cover-decomposable polygons. We start the introduction by an overview and end it with a summary of this paper.

The problem of decomposing coverings is closely related to the so-called *sensor cover problem*. Here the problem is that if we are given an area that has to be monitored and a finite number of sensors, each of which can monitor a part of the given area and has a fixed amount of lifetime, then we have to come up with an optimal starting time for each sensor such that they together monitor the whole area for as

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long as possible (in this version, it is not allowed to switch on and off a sensor more than once). If every point is monitored by at least k sensors, each of which has a battery that is good for one hour, and we want to decide whether it is possible or not to monitor the whole area for two hours, then we get the decomposition problem. If we want to have longer coverage, we get a problem where we want to decompose into multiple coverings (not just two). It is another interesting problem when we allow one to switch sensors on and off an arbitrary number of times, about which little is known. Our paper deals with the simplest version of the problem when we are trying to decompose into two coverings. For more on decomposition to multiple coverings, see Aloupis et al. [1] and for results about the sensor cover problem, see Buchsbaum et al. [2]. A recent result of Gibson and Varadarajan [4] gives a possible generalization of Theorem C. about decomposition to multiple coverings. First we give a summary of the previous results about decomposition of coverings with the translates of a polygon.

Theorem A (Pach [7]) *Every centrally symmetric open convex polygon is cover-decomposable.*

Theorem B (Tardos and Tóth [10]) *Every open triangle is cover-decomposable.*

Theorem C (Pálvölgyi and Tóth [9]) *Every open convex polygon is cover-decomposable.*

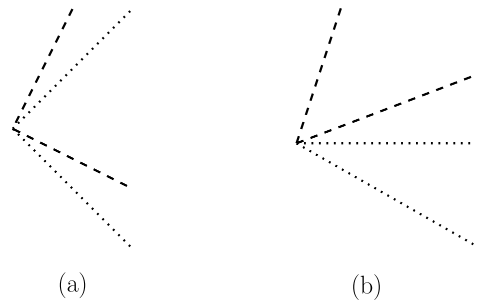
In fact, a stronger statement was proved in the last paper. To understand this, we need another definition. So far our definition only concerned coverings of the *whole* plane, but we could investigate coverings of any fixed planar point set. We say that a planar set is *totally-cover-decomposable* if there is k such that any k-fold covering of *any* planar point set by its translates is decomposable. In earlier papers this property was not defined, however, the proofs all work for this stronger version. This is the first paper that makes distinction of these definitions. To avoid confusion, in this paper we will call the cover-decomposable sets *plane-cover-decomposable*. By definition, if a set is totally-cover-decomposable, then it is also plane-cover-decomposable. On the other hand, we cannot rule out the possibility that there are sets, or even polygons, which are plane-cover-decomposable but not totally-cover-decomposable. From the negative direction very little is known.

Theorem D (Pach, Tardos, and Tóth [8]) *Concave quadrilaterals are not plane-cover-decomposable* (*thus neither totally-cover-decomposable*).

The main result of this paper is a generalization of Theorem D. We show that *almost* all (open or closed) concave polygons are *not* totally-cover-decomposable and even suspect that they are not plane-cover-decomposable.

To state our result precisely, we need to define *wedges*. Suppose we have two halflines, e and f, both with a common endpoint O. Then they divide the plane into two parts, W_1 W_2, which we call *wedges*. A *closed wedge* contains its boundary, and an *open wedge* does not. The point O where the two boundary lines meet is called

Fig. 1 Two possible pairs of special wedges



the *apex* of the wedges. The *angle* of a wedge is the angle between its two boundary halflines, measured inside the wedge. That is, the sum of the angles of W_1 and W_2 is 2π.

We say that a pair of wedges is *special* if

(i) There is a wedge of angle less than π that contains both of the wedges, and
(ii) None of the wedges contains the other wedge. (See Fig. 1)

For any polygon, the neighborhood of each vertex can be extended to a wedge. Those are called the *wedges of the polygon*. Note that a convex polygon cannot have a special pair of wedges. Pálvölgyi and Tóth proved the following:

Theorem E (Pálvölgyi and Tóth [9]) *Every open polygon that has no special pair of wedges is totally-cover-decomposable.*

Our main result is the following:

Theorem 1.1 *If a polygon has a special pair of wedges, then it is not totally-cover-decomposable.*

Together with the previous theorem, this gives a complete characterization of totally-cover-decomposable open polygons; an open polygon is totally-cover-decomposable if and only if it does not have a special pair of wedges.

We show that every concave polygon with no parallel sides has a pair of special wedges; therefore we have the following:

Theorem 1.2 *If a concave polygon has no parallel sides, then it is not totally-cover-decomposable.*

The problem of deciding plane-cover-decomposability for concave polygons is still open. However, in Sect. 3, we prove that a large class of concave polygons are not plane-cover-decomposable. We also show that any "interesting" covering of the plane uses only countably many translates. (However, we do not consider here the problem when every point is covered infinitely many times; the interested reader is referred to the recent paper of Elekes, Mátrai, and Soukup [3].)

Finally, in Sect. 4, we investigate the problem in three or more dimensions. The notion of totally-cover-decomposability extends naturally, and we can also introduce space-cover-decomposability. Previously, the following result was known.

Theorem F (Mani-Levitska and Pach [5]) *The unit ball is not space-cover-decomposable.*

Using our construction, we establish the first theorem for polytopes, which shows that the higher-dimensional case is quite different from the two-dimensional one.

Theorem 1.3 *No polytope is space-cover-decomposable.*

2 The Construction: Proof of Theorems 1.1 and 1.2

In this section, for any k and any polygon C that has a special pair of wedges, we present a (finite) point set and an indecomposable k-fold covering of it by (a finite number of) the translates of the polygon. We formulate (and solve) the problem in its dual form, like in [7] or [10]. Fix O, the center of gravity of C, as our origin in the plane. For a planar set S and a point p in the plane, we use $S(p)$ to denote the translate of S by the vector $\vec{Op}$. Let $\bar{C}$ be the reflection through O of C. For any point x, $x \in C(p_i)$ if and only if $p_i \in \bar{C}(x)$. To see this, apply a reflection through the midpoint of the segment xp_i. This switches $C(p_i)$ and $\bar{C}(x)$, and also switches p_i and x.

Consider any collection $\mathcal{C} = \{C(p_i) \mid i \in I\}$ of translates of C and a point set X. The collection $\mathcal{C}$ covers x at least k times if and only if $\bar{C}(x)$ contains at least k elements of the set $P = \{p_i \mid i \in I\}$. Therefore a k-fold covering of X transforms into a point set such that for every $x \in X$, the set $\bar{C}(x)$ contains at least k points of P. The required decomposition of $\mathcal{C}$ exists if and only if the set P can be colored with two colors so that every translate $\bar{C}(x)$ that contains at least k elements of P contains at least one element of each color. Thus constructing a finite system of translates of $\bar{C}$ and a point set where this latter property fails is equivalent to constructing an indecomposable covering using the translates of C.

If C has a special pair of wedges, then so does $\bar{C}$. We will use the following theorem to prove Theorem 1.1.

Theorem 2.1 *For any pair of special wedges V, W and for every k, l, there is a point set of cardinality $\binom{k+l}{k} - 1$ such that for every coloring of P with red and blue, either there is a translate of V containing k red points and no blue points or there is a translate of W containing l blue points and no red points.*

Proof Without loss of generality, suppose that the wedges are contained in the right halfplane.

For $k = 1$, the statement is trivial; just take l points such that any one is contained alone in a translate of W. Similarly k points will do for $l = 1$. Let us suppose that we already have a counterexample for all $k' + l' < k + l$, and let us denote these by $P(k', l')$. The construction for k and l is the following.

Place a point p in the plane and a suitable small scaled down copy of $P(k-1, l)$ left from p so that any translate of V with its apex in the neighborhood of $P(k-1, l)$ contains p, but none of the translates of W with its apex in the neighborhood of

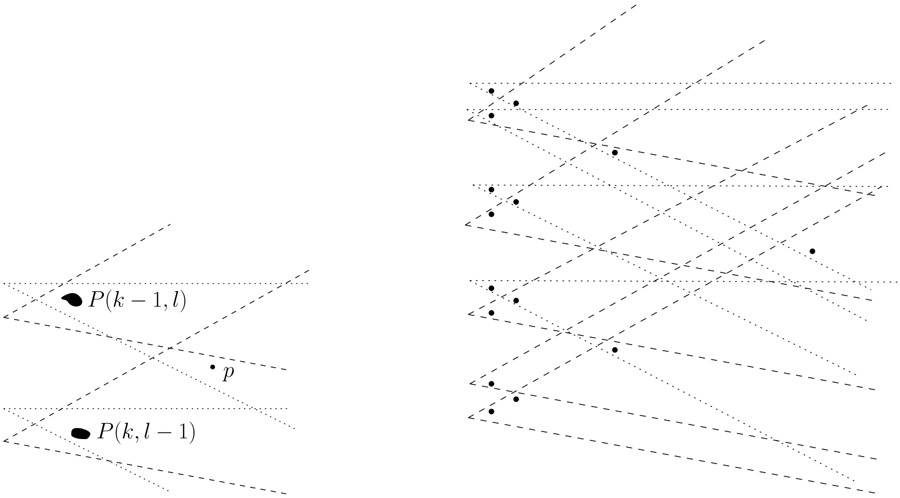


Fig. 2 Sketch of one step of the induction and the first few steps

$P(k-1,l)$ does. Similarly place $P(k,l-1)$ so that any translate of W with its apex in the neighborhood of $P(k,l-1)$ contains p, but none of the translates of V with its apex in the neighborhood of $P(k,l-1)$ does. (See Fig. 2.)

If p is colored red, then

- Either the $P(k-1,l)$ part already contains a translate of V that contains $k-1$ reds and no blues, and it contains p as well, which gives together k red points
- Or the $P(k-1,l)$ part contains a translate of W that contains l blues and no reds, and it does not contain p

The same reasoning works for the case where p is colored blue.

Now we can calculate the number of points in $P(k,l)$. For $l=1$ and $k=1$, we know that $|P(k,1)|=k$ and $|P(1,l)|=l$, while the induction gives $|P(k,l)|=1+|P(k-1,l)|+|P(k,l-1)|$. From this we have $|P(k,l)|=\binom{k+l}{k}-1$. □

It is easy to see that if we use this theorem for a pair of special wedges of $\bar{C}$ and $k=l$, then for every coloring of (a possibly scaled down copy of) the above point set with two colors, there is a translate of $\bar{C}$ that contains at least k points but contains only one of the colors. This is because "$\bar{C}$ can locally behave like any of its wedges." Therefore this construction completes the proof of Theorem 1.1.

Remark We note that for $k=l$, the cardinality of the point set is approximately $4^k/\sqrt{k}$, this significantly improves the previously known construction of Pach, Tardos, and Tóth [8], which used approximately k^k points and worked only for quadrilaterals, and in general, for "even more special" pairs of wedges (without giving the exact definition, see Fig. 1b). It can be proved that this exponential bound is close to optimal. Suppose that we have n points and $n<2^{k-2}$. Since there are two kinds of wedges, there are at most $2n$ essentially different translates that contain k points.

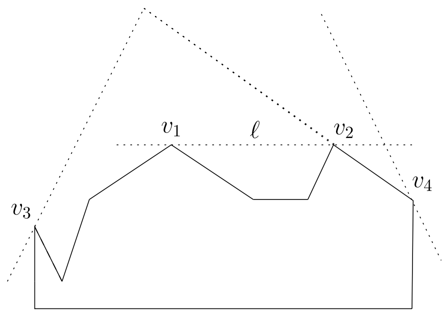


Fig. 3 How to find a special pair of wedges

There are 2^n different colorings of the point set, and each translate that contains k points is monochromatic for 2^{n-k+1} of the colorings. Therefore, there are at most $2n2^{n-k+1} < 2^n$ bad colorings, so there is a coloring with no monochromatic translates.

Theorem 1.2 follows directly from the next result.

Lemma 2.2 *Every concave polygon that has no parallel sides has a special pair of wedges.*

Proof Assume that the statement does not hold for a polygon C. There is a touching line ℓ to C such that the intersection of ℓ and C contains no segments and contains at least two vertices, v_1 and v_2. (Here we use that C has no parallel sides.) Denote the wedges at v_i by W_i. Is the pair W_1, W_2 special? They clearly fulfill property (i), and the only problem that can arise is that the translate of one of the wedges contains the other wedge. This means, without loss of generality, that the angle at v_1 contains the angle at v_2. Now let us take the two touching lines to C that are parallel to the sides of W_2. It is impossible that both of these lines touch v_2, because then the touching line ℓ would touch only v_2 as well. Take a vertex v_3 from the touching line (or from one of these two lines) that does not touch v_2. (See Fig. 3.) This cannot be v_1 because then the polygon would have two parallel sides. Is the pair W_2, W_3 special? They are contained in a halfplane (the one determined by the touching line). This means, again, that the angle at v_2 contains the angle at v_3. Now we can continue the reasoning with the touching lines to C parallel to the sides of W_3; if they would both touch v_3, then the touching line ℓ would touch only v_3. This way we obtain the new vertices $v_4, v_5, \dots$, which contradicts the fact the C may have only a finite number of vertices. $\square$

3 Versions of Cover-Decomposability

Here we consider different variants of cover-decomposability and prove relations between them.

3.1 Number of Sets: Finite, Infinite, or More

We say that a set is *finite/countable-cover-decomposable* if there exists k such that every k-fold covering of any point set by a finite/countable number of its translates is decomposable. So by definition we have: totally-cover-decomposable $\Rightarrow$ countable-cover-decomposable $\Rightarrow$ finite-cover-decomposable. But which of these implications can be reversed? We will prove that the first can be for "nice" sets.

It is well known that the plane is hereditary Lindelöf, i.e., if a point set is covered by open sets, then countably many of these sets also cover the point set. It is easy to see that the same also holds for k-fold coverings. This observation implies the following lemma.

Lemma 3.1 *An open set is totally-cover-decomposable if and only if it is countable-cover-decomposable.*

The same holds for "nice" closed sets, such as polygons or discs. We say that a closed set C is *nice* if there is t and a set $\mathcal{D}$ of countably many closed halfdiscs such that if t different translates of C cover a point p, then their union covers a halfdisc from $\mathcal{D}$ centered at p (meaning that p is halving the straight side of the halfdisc) and the union of their interiors covers the interior of the halfdisc. For a polygon, t can be the number of its vertices plus one, $\mathcal{D}$ can be the set of halfdiscs whose side is parallel to a side of the polygon and has rational length. For a disc, t can be 2, and $\mathcal{D}$ can be the set of halfdiscs whose side has a rational slope and a rational length. In fact every convex set is nice.

Claim 3.2 *Every closed convex set is nice.*

Proof Some parts of the boundary of the convex set C might be segments; we call them sides. Trivially, every convex set can have only countably many sides. Choose $t = 5$ and let the set of halfdiscs $\mathcal{D}$ be the ones whose side is either parallel to a side of C or its slope is rational and has rational length. Assume that five different translates of C cover a point p. Shifting these translates back to C, denote the points that covered p by $p_1, \dots, p_5$. If any of these points is not on the boundary of C, we are done. The $p_1 p_2 p_3 p_4 p_5$ pentagon has two neighboring angles the sum of whose degrees is strictly bigger than 2π, without loss of generality, p_1 and p_2. If $p_1 p_2$ is also the side of C, then the five translates cover a halfdisc whose side is parallel to $p_1 p_2$, else they cover one whose side has a rational slope. □

Taking a rectangle verifies that $t = 5$ is optimal in the previous proof.

Lemma 3.3 *A nice set is totally-cover-decomposable if and only if it is infinite-cover-decomposable.*

Proof We have to show that if we have a covering of some point set P by the translates of our nice, infinite-cover-decomposable set C, then we can suitably color the points of P. Denote by P^* the points that are covered by two copies of the same translate of the nice set C. Color one of these red, the other blue. Now we only have to deal with $P' = P \setminus P^*$, and we can suppose that there is only one copy of each translate. Now instead of coloring these translates, we rather show that we can choose countably many of them such that they still cover every point of P' many times. Using after this that the set is infinite-cover-decomposable finishes the proof. So now we show that if there is a set of translates of C that cover every point of P' at least kt times, then we can choose countably many of these translates that cover every point of P' at least k times. It is easy to see that it is enough if we show this for $k = 1$ (since we can repeat this procedure k times).

Denote the points that are contained in the interior of a translate by P_0. Because of the hereditary Lindelöf property, countably many translates cover P_0. If a point $p \in P'$ is covered t times, then because of the nice property of C, a halfdisc from $\mathcal{D}$ centered at p is covered by these translates. We say that this (one of these) halfdisc(s) belongs to p. Take a partition of $P' \setminus P_0$ into countably many sets $P_1 \cup P_2 \cup \cdots$ such that the ith halfdisc belongs to the points of P_i. Now it is enough to show that P_i can be covered by countably many translates. Denote the halfdisc belonging to the points of P_i by D_i. Using the hereditary Lindelöf property for P_i and open discs (not halfdiscs!) with the radius of D_i centered at the points of P_i, we obtain a countable covering of P_i. Now replacing the open discs with closed halfdiscs still gives a covering of P_i because otherwise we would have $p, q \in P_i$ such that p is in the interior of $q + D_i$, but interior of $q + D_i$ is covered by the interiors of translates of C, which would imply $p \in P_0$, contradiction. Finally we can replace each of the halfdiscs belonging to the points of P_i by t translates of C, and we are done. $\square$

Unfortunately, we did not manage to establish any connection among finite- and countable-cover-decomposability. We conjecture that they are equivalent for nice sets (with a possible slight modification of the definition of nice). If one manages to find such a statement, then it would imply that considering cover-decomposability, it does not matter whether the investigated geometric set is open or closed, as long as it is nice. For example, it is unknown whether closed triangles are cover-decomposable or not. We strongly believe that they are.

3.2 Covering the Whole Plane

Remember that by definition if a set is totally-cover-decomposable, then it is also plane-cover-decomposable. However, the other direction is not always true. For example, take the lower halfplane and "attach" to its top a pair of special wedges (see Fig. 4). Then the counterexample using the special wedges works for a special point set, and thus this set is not totally-cover-decomposable, but it is easy to see that a covering of the whole plane can always be decomposed.

For a given polygon C, our construction gives a set of points S and a nondecomposable k-fold covering of S by translates of C. It is not clear when we can extend this covering to a k-fold covering of the whole plane such that none of the new translates contain any point of S. This would be necessary to ensure that the covering remains nondecomposable.

We show that in certain cases it can be extended, but it remains an open problem to decide whether plane- and totally-cover-decomposability are equivalent or not for open polygons/bounded sets.

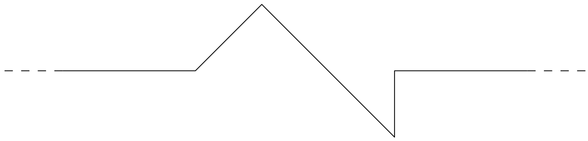


Fig. 4 The lower halfplane with a special pair of wedges at its top

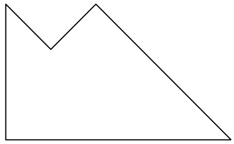


Fig. 5 A pentagon that is cover-decomposable but is not the union of a finite number of translates of the same convex polygon

Theorem 3.4 *If a concave polygon C has two special wedges that have a common locally touching line, and one of the two touching lines parallel to this line is touching C in only a finite number of points (i.e., does not contain a side), then it is not plane-cover-decomposable.*

Proof Assume, without loss of generality, that this locally touching line is vertical. We just have to extend our construction with the special wedges into a covering of the whole plane. Or, in the dual, we have to add more points to our construction such that every translate of C will contain at least k points. Of course, to preserve that the construction works, we cannot add more points into those translates that we used in the construction. Otherwise, our argument that the construction is correct does not work. From the proof of the construction we can see that the apices of the wedges all lay on the same vertical line. Therefore, the translates can all be obtained from each other via a vertical shift, because we had a vertical locally touching line to both wedges. Now we can simply add all points that are not contained in any of these original translates. Proving that every translate of C contains at least k points is equivalent to showing that the original translates do not cover any other translate of C. It is clear that they could only cover a translate that can be obtained from them via a vertical shift. On the touching vertical line each of the translates has only finitely many points. In the construction we have the freedom to perturbate the wedges a bit vertically, and this way we can ensure that the intersection of each other translate (obtainable via a vertical shift) with this vertical line is not contained in the union of the original translates. □

Corollary 3.5 *A pentagon is totally-cover-decomposable if and only if it is plane-cover-decomposable.*

Proof All totally-cover-decomposable sets are also plane-cover-decomposable. (See an example on Fig. 5.) If our pentagon is not totally-cover-decomposable, then it has a special pair of wedges and it must also have a touching line that touches it in these special wedges, and thus we can use the previous theorem. □

The same argument does not work for hexagons; for example, we do not know whether the hexagon depicted in Fig. 6c is plane-cover-decomposable or not.

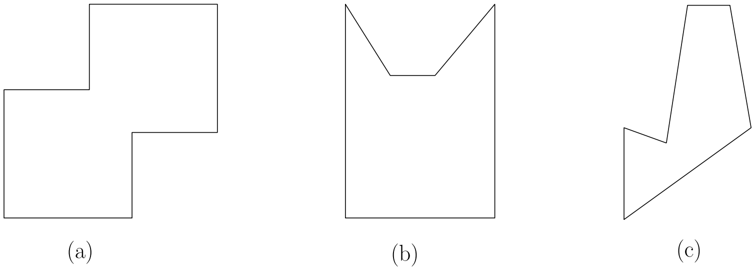


Fig. 6 Three different hexagons: (**a**) totally-cover-decomposable (hence, also plane-coverdecomposable), (**b**) not plane-cover-decomposable (hence neither totally-cover-decomposable), and (**c**) not totally-cover-decomposable but not known if plane-cover-decomposable

4 Higher Dimensions

The situation is different for the space. For any polytope and any k, one can construct a k-fold covering of the space that is not decomposable. First note that it is enough to prove this result for the three-dimensional space, since for higher dimensions, we can simply intersect our polytope with a three-dimensional space, use our construction for this three-dimensional polytope, and then extend it naturally. To prove the theorem for three-dimensional polytopes, first we need some observations about polygons. Given two polygons and one side of each of them that are parallel to each other, we say that these sides are *directedly parallel* if the polygons are on the “same side” of the sides (i.e., the halfplane which contains the first polygon and whose boundary contains this side of the first polygon can be shifted to contain the second polygon such that its boundary contains that side of the second polygon). We will slightly abuse this definition and say that a side is directedly parallel if it is directedly parallel to a side of the other polygon. We can similarly define *directedly parallel faces* for a single polytope. We say that a face is directedly parallel to another if they are parallel to each other and the polytope is on the “same side” of the faces (e.g., every face is directedly parallel to itself and, if the polytope is convex, to no other face).

Lemma 4.1 *Given two convex polygons, both of which have at most two sides that are directedly parallel, there is always a special pair among their wedges.*

Proof Take the smallest wedge of the two polygons, excluding the one whose sides are both directedly parallel if such exists. Without loss of generality, we can suppose that the right side of this minimal wedge is not directedly parallel and is going to the right (i.e., its direction is $(1, 0)$), while the left side goes upwards. Take a wedge of the other polygon both of whose sides go upwards (there always must be one since the right side of the first wedge was not directedly parallel). If this second wedge is not contained in the first, we found a special pair. If the second wedge is contained in the first wedge, then because of the minimality of the first wedge, the second wedge must have two directedly parallel sides. But then the second wedge must be a wedge of both polygons, while it cannot be in the same convex polygon as the first wedge, a contradiction. □

Theorem 3 *No polytope is space-cover-decomposable.*

Proof We will, as usual, work in the dual case. This means that to prove that our polytope C is not totally-cover-decomposable, we will exhibit a point set for any k such that we cannot color it with two colors so that any translate of C that contains at least k points contains both colors. These points will be all in one plane, and the important translates of C will intersect this plane either in a concave polygon or in one of two convex polygons. It is enough to show that this concave polygon is not cover-decomposable or that among the wedges of these convex polygons there is a special pair.

Take a plane π that is not parallel to any of the segments determined by the vertices of C. The touching planes of C parallel to π are touching C in one vertex each, A and B. Denote the planes parallel to π that are very close to A and B and intersect C by π_A and π_B. Denote $C \cap \pi_A$ by C_A and $C \cap \pi_B$ by C_B. Now we have two cases.

Case 1. C_A or C_B is concave.

Without loss of generality, assume that C_A is concave. Then since no two faces of C incident to A can be parallel to each other, with a perturbation of π_A we can achieve that the sides of C_A are not parallel. After this, using Theorem 1.2, we are done.

Case 2. Both C_A and C_B are convex.

Now by perturbing π we cannot necessarily achieve that C_A and C_B have no parallel sides, but we can achieve that they have at most two directedly parallel sides. This is true because there can be at most two pairs of faces that are directedly parallel to each other and one of them is incident to A, the other to B, since A is touched from above, and B from below by the plane parallel to π. Therefore C_A and C_B satisfy the conditions of Lemma 4.1, and this finishes the proof of totally-cover-decomposability.

To prove space-cover-decomposability, just as in the proof of Theorem 3.4, we have to add more points to the constructions such that every translate will contain at least k points, but we do not add any points to the original translates of our construction. This is the same as showing that these original translates do not cover any other translate. Note that there are two types of original translates (depending on which wedge of it we use) and translates of the same type can be obtained from each other via a shift that is parallel to the side of the halfplane in π that contains our special wedges. This means that the centers of all the original translates lay in one plane. With a little perturbation of the construction, we can achieve that this plane is in general position with respect to the polytope. But in this case it is clear that the translates used in our construction cannot cover any other translate, this proves space-cover-decomposability. □

5 Concluding Remarks

A lot of questions remain open. In three dimensions, neither polytopes nor unit balls are cover-decomposable. Is there any nice (e.g., open and bounded) set in three dimensions that is cover-decomposable? Maybe such nice sets only exist in the plane.

In the first part of Sect. 2, we have seen that interesting covers with translates of nice sets only use countably many translates. We could not prove, but conjecture, that every cover can be somehow reduced to a locally finite cover. Is it true that if a nice set is finite-cover-decomposable, then it is also countable-cover-decomposable? This would have implications about the cover-decomposability of closed sets. Are closed convex polygons cover-decomposable?

In the second part of Sect. 3, we have seen that our construction is not naturally extendable to give an indecomposable covering of the whole plane; maybe the reason for this is that it is impossible to find such a covering. Are there polygons that are not totally-cover-decomposable but plane-cover-decomposable?

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