ISSN 0081-5438, Proceedings of the Steklov Institute of Mathematics, 2008, Vol. 261, pp. 287–294. © Pleiades Publishing, Ltd., 2008.

Some Issues on the p-Laplace Equation in Cylindrical Domains

M. Chipot[a] **and Y. Xie**[b]

Received January 2007

Abstract—We investigate the asymptotic behavior of the solution to equations of the p-Laplacian type in cylindrical domains becoming unbounded and address some issues regarding the solution in unbounded domains.

DOI: 10.1134/S0081543808020235

1. INTRODUCTION

Many problems of mathematical physics are set in cylindrical domains. For instance, these are porous media flows in channels, plate theory, elasticity theory, etc. In this note we will address the problem of the p-Laplace equation. To be more precise, suppose that Ω_ℓ is a two-dimensional domain (for simplicity) pictured in the figure below, and let u_ℓ be the unique solution to the nonlinear problem

$$\begin{cases} -\Delta_p u_\ell = f(x_2) & \text{in } \Omega_\ell, \\ u_\ell = 0 & \text{on } \partial\Omega_\ell, \end{cases} \tag{1.1}$$

where Δ_p is the usual p-Laplace operator defined as $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, $p > 1$, and $\partial\Omega_\ell$ denotes the boundary of Ω_ℓ. Note that the p-Laplace operator reduces to the usual Laplacian when $p = 2$. In this note, we mainly consider the case $p \neq 2$. One refers to [2–7] for results in the linear case.

We notice that the data f of (1.1) depends only on the x_2-variable. Of course, due to the boundary conditions at the ends of the cylinder, u_ℓ is not a function independent of x_1. However, one expects that when $\ell \to +\infty$, u_ℓ is close to a function depending on x_2 only. To be more precise, let u_∞ be the solution to

$$\begin{cases} -\Delta_p u_\infty = f(x_2) & \text{in } (-1,1), \\ u_\infty(-1) = u_\infty(1) = 0. \end{cases} \tag{1.2}$$

Then a natural question is whether

$$u_\ell \to u_\infty$$

as $\ell \to +\infty$ in any bounded subdomain of $\mathbb{R} \times (-1,1)$.

We will investigate such convergence in Section 2. In the following Section 3, some property of the solution to the p-Laplace equation in an unbounded domain will be discussed.

Before we go into details, let us quote some useful inequalities:

[a] Institut für Mathematik, Abteilung für Angewandte Mathematik, Universität Zürich, Winterthurerstrasse 190, CH-8057 Zürich, Switzerland.

[b] Department of Mathematics, Ecole Polytechnique Fédérale de Lausanne, 1015 Lausanne, Switzerland.

E-mail addresses: m.m.chipot@math.unizh.ch (M. Chipot), yitian.xie@epfl.ch (Y. Xie).

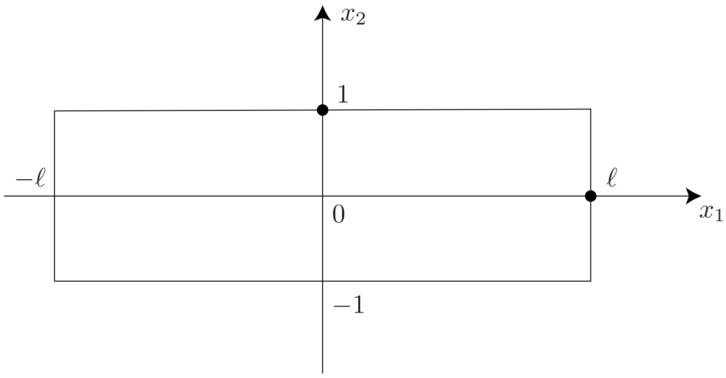

Figure.

Lemma 1.1 (see [1, 8] for a proof). *For all $p > 1$, $\delta \geq 0$, and $\xi, \eta \in \mathbb{R}^n$, it holds that for some constant c depending on p*

$$\big||\xi|^{p-2}\xi - |\eta|^{p-2}\eta\big| \leq c|\xi - \eta|^{1-\delta}(|\xi| + |\eta|)^{p-2+\delta}, \tag{1.3}$$

$$\big(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta, \xi - \eta\big) \geq c|\xi - \eta|^{2+\delta}(|\xi| + |\eta|)^{p-2-\delta}. \tag{1.4}$$

2. A CONVERGENCE RESULT

Let us consider the problem mentioned above in a more general setting. Denote by $X = (X_1, X_2) = (x_1, \dots, x_q, x_{q+1}, \dots, x_n)$ any point in $\mathbb{R}^n$, and let $\Omega_\ell = (-\ell, \ell)^q \times \omega$, where ω is an open bounded domain in $\mathbb{R}^{n-q}$.

Assume that u_ℓ and u_∞ are solutions to

$$\begin{cases} -\Delta_p u_\ell = f(X_2) & \text{in } \Omega_\ell, \\ u_\ell = 0 & \text{on } \partial\Omega_\ell \end{cases} \tag{2.1}$$

and

$$\begin{cases} -\Delta_p u_\infty = f(X_2) & \text{in } \omega, \\ u_\infty = 0 & \text{on } \partial\omega, \end{cases} \tag{2.2}$$

respectively. We consider here the weak solutions to (2.1) and (2.2) and assume $u_\ell \in W_0^{1,p}(\Omega_\ell)$ and $u_\infty \in W_0^{1,p}(\omega)$, where $W_0^{1,p}(\Omega_\ell)$ and $W_0^{1,p}(\omega)$ stand for the usual Sobolev spaces. We refer, for instance, to [2] for more information regarding these spaces.

Theorem 2.1. *Suppose that $f(X_2) \in L^{p'}(\omega)$ is nonnegative (or nonpositive for an analogous statement) and Ω is any bounded subset in $\mathbb{R}^q \times \omega$. Then u_ℓ is a nondecreasing sequence of nonnegative functions bounded above by u_∞, and it holds that*

$$u_\ell \to u_\infty \qquad \text{in } W^{1,p}(\Omega).$$

Proof. We notice that u_ℓ, the solution to (2.1), satisfies

$$\int_{\Omega_\ell} |\nabla u_\ell|^{p-2}\nabla u_\ell \nabla v \,\mathrm{d}x = \int_{\Omega_\ell} f v \,\mathrm{d}x \qquad \forall v \in W_0^{1,p}(\Omega_\ell). \tag{2.3}$$

Take $v = u_\ell^- \in W_0^{1,p}(\Omega_\ell)$, the negative part of u_ℓ, in (2.3). We obtain

$$\int_{\Omega_\ell} |\nabla u_\ell|^{p-2}\nabla u_\ell \nabla u_\ell^- \,\mathrm{d}x = \int_{\Omega_\ell} f u_\ell^- \,\mathrm{d}x \geq 0,$$

i.e.,

$$\int_{\Omega_\ell} |\nabla u_\ell^-|^p \, dx \le 0.$$

Hence we derive that u_ℓ is nonnegative. Following the same arguments, one can show that u_∞, the solution to (2.2), is nonnegative.

Letting $\ell < \ell'$, one has

$$\int_{\Omega_\ell} |\nabla u_\ell|^{p-2} \nabla u_\ell \nabla v \, dx = \int_{\Omega_\ell} f v \, dx \qquad \forall v \in W_0^{1,p}(\Omega_\ell), \tag{2.4}$$

$$\int_{\Omega_{\ell'}} |\nabla u_{\ell'}|^{p-2} \nabla u_{\ell'} \nabla v \, dx = \int_{\Omega_{\ell'}} f v \, dx \qquad \forall v \in W_0^{1,p}(\Omega_{\ell'}). \tag{2.5}$$

We remark that $u_{\ell'}$ is nonnegative in $\Omega_{\ell'}$. So

$$(u_\ell - u_{\ell'})^+ \in W_0^{1,p}(\Omega_\ell) \qquad ((u_\ell - u_{\ell'})^+ \text{ is the positive part of } u_\ell - u_{\ell'}).$$

Therefore, one derives from (2.4) and (2.5) that

$$\int_{\Omega_\ell} |\nabla u_\ell|^{p-2} \nabla u_\ell \nabla (u_\ell - u_{\ell'})^+ \, dx = \int_{\Omega_\ell} |\nabla u_{\ell'}|^{p-2} \nabla u_{\ell'} \nabla (u_\ell - u_{\ell'})^+ \, dx.$$

The above equation (see (1.4)) leads to

$$(u_\ell - u_{\ell'})^+ = 0,$$

which shows that $\{u_\ell\}$ is a nondecreasing sequence in ℓ. On the other hand, we have

$$\begin{cases} -\Delta_p u_\ell + \Delta_p u_\infty = 0 & \text{in } \Omega_\ell, \\ u_\ell - u_\infty = \begin{cases} -u_\infty \le 0 & \text{on } \partial(-\ell,\ell)^q \times \omega, \\ 0 & \text{on } (-\ell,\ell)^q \times \partial\omega. \end{cases} \end{cases} \tag{2.6}$$

By the weak maximum principle, this implies

$$0 \le u_\ell \le u_\infty \qquad \text{in } \Omega_\ell. \tag{2.7}$$

Consider now a smooth nonnegative function $\rho(x)$ such that

$$0 \le \rho \le 1, \qquad \rho \equiv 1 \quad \text{in } \Omega_{\ell_0}, \qquad \rho \equiv 0 \quad \text{outside } \Omega_{\ell_0+1}, \qquad |\nabla\rho| \text{ is bounded.} \tag{2.8}$$

Taking $u_\ell \rho^p$ in (2.1), we obtain

$$\int_{\Omega_\ell} |\nabla u_\ell|^{p-2} \nabla u_\ell \nabla \{u_\ell \rho^p\} \, dx = \int_{\Omega_\ell} f u_\ell \rho^p \, dx.$$

Therefore, for some constant c we have

$$\begin{aligned} \int_{\Omega_{\ell_0+1}} |\nabla u_\ell|^p \rho^p \, dx &= -p \int_{\Omega_{\ell_0+1}} |\nabla u_\ell|^{p-2} \nabla u_\ell \nabla \rho u_\ell \rho^{p-1} \, dx + \int_{\Omega_{\ell_0+1}} f u_\ell \rho^p \, dx \\ &\le c \int_{\Omega_{\ell_0+1}} |\nabla u_\ell|^{p-1} u_\ell \rho^{p-1} \, dx + \int_{\Omega_{\ell_0+1}} f u_\ell \rho^p \, dx. \end{aligned}$$

Applying the Young inequality on the right-hand side, we come to

$$\int\limits_{\Omega_{\ell_0+1}} |\nabla u_\ell|^p \rho^p \,\mathrm{d}x \le \frac{1}{2} \int\limits_{\Omega_{\ell_0+1}} |\nabla u_\ell|^p \rho^p \,\mathrm{d}x + c \int\limits_{\Omega_{\ell_0+1}} u_\ell^p \,\mathrm{d}x + \int\limits_{\Omega_{\ell_0+1}} f u_\ell \rho^p \,\mathrm{d}x \tag{2.9}$$

for some constant c. Since (2.7) holds, we derive from (2.9) that

$$\int\limits_{\Omega_{\ell_0}} |\nabla u_\ell|^p \,\mathrm{d}x \le 2c \int\limits_{\Omega_{\ell_0+1}} u_\ell^p \,\mathrm{d}x + 2 \int\limits_{\Omega_{\ell_0+1}} f u_\ell \,\mathrm{d}x \le 2c \int\limits_{\Omega_{\ell_0+1}} u_\infty^p \,\mathrm{d}x + 2 \int\limits_{\Omega_{\ell_0+1}} f u_\infty \,\mathrm{d}x \le C;$$

i.e., $\{u_\ell\}$ is uniformly bounded in Ω_{ℓ_0}. Up to a subsequence of u_ℓ, labeled still by ℓ, there exists a $u_0 \in W^{1,p}(\Omega_{\ell_0})$ vanishing on $(-\ell_0, \ell_0) \times \partial\omega$ such that

$$u_\ell \to u_0 \quad \text{in } L^p(\Omega_{\ell_0}), \qquad u_\ell \rightharpoonup u_0 \quad \text{in } W^{1,p}(\Omega_{\ell_0}).$$

We remark here that due to the monotonicity of $\{u_\ell\}$, the whole sequence converges to u_0 in L^p.

Fixing a positive constant h, for any integer $1 \le i \le q$ we consider $u_\ell(X_1 + he_i, X_2)$ and $u_{\ell+h}(X_1, X_2)$ in the domain $(-\ell,\ell)^{i-1} \times (-\ell-h, \ell-h) \times (-\ell,\ell)^{q-i} \times \omega$. We have

$$-\Delta_p u_\ell(X_1 + he_i, X_2) + \Delta_p u_{\ell+h}(X_1, X_2) = f(X_2) - f(X_2) = 0$$
$$\text{in} \quad (-\ell,\ell)^{i-1} \times (-\ell-h, \ell-h) \times (-\ell,\ell)^{q-i} \times \omega,$$

together with the boundary condition

$$u_\ell(X_1 + he_i, X_2) - u_{\ell+h}(X_1, X_2)$$
$$= \begin{cases} 0 & \text{on } (-\ell,\ell)^{i-1} \times (-\ell-h, \ell-h) \times (-\ell,\ell)^{q-i} \times \partial\omega, \\ 0 & \text{on } (-\ell,\ell)^{i-1} \times \{-\ell-h\} \times (-\ell,\ell)^{q-i} \times \omega, \\ -u_{\ell+h}(X_1, X_2) \le 0 & \text{on } (-\ell,\ell)^{i-1} \times \{\ell-h\} \times (-\ell,\ell)^{q-i} \times \omega. \end{cases}$$

By the maximum principle we have

$$u_\ell(X_1 + he_i, X_2) \le u_{\ell+h}(X_1, X_2) \qquad \text{in} \quad (-\ell,\ell)^{i-1} \times (-\ell-h, \ell-h) \times (-\ell,\ell)^{q-i} \times \omega.$$

Passing to the limit in Ω_{ℓ_0} leads to

$$u_0(X_1 + he_i, X_2) \le u_0(X_1, X_2).$$

By changing h to $-h$ and noticing that this holds for all i, one has

$$u_0(X_1, X_2) = u_0(X_2).$$

Now we denote by ρ a smooth nonnegative function such that (this is slightly different from (2.8))

$$0 \le \rho \le 1, \qquad \rho \equiv 1 \quad \text{in } \Omega_{\ell_0-1}, \qquad \rho \equiv 0 \quad \text{outside } \Omega_{\ell_0}, \qquad |\nabla\rho| \ \text{ is bounded.}$$

Taking $v = (u_\ell - u_0)\rho \in W_0^{1,p}(\Omega_{\ell_0})$ as a test function in (2.1), we obtain

$$\int\limits_{\Omega_{\ell_0}} |\nabla u_\ell|^{p-2} \nabla u_\ell \nabla\{(u_\ell - u_0)\rho\} \,\mathrm{d}x = \int\limits_{\Omega_{\ell_0}} f(u_\ell - u_0)\rho \,\mathrm{d}x,$$

i.e.,

$$\int_{\Omega_{\ell_0}} |\nabla u_\ell|^{p-2}\nabla u_\ell \nabla(u_\ell - u_0)\rho\,\mathrm{d}x = -\int_{\Omega_{\ell_0}} |\nabla u_\ell|^{p-2}\nabla u_\ell \nabla\rho(u_\ell - u_0)\,\mathrm{d}x + \int_{\Omega_{\ell_0}} f(u_\ell - u_0)\rho\,\mathrm{d}x. \quad (2.10)$$

Since $u_\ell \to u_0$ in $L^p(\Omega_{\ell_0})$, it follows that

$$\int_{\Omega_{\ell_0}} f(u_\ell - u_0)\rho\,\mathrm{d}x \to 0$$

and

$$\left|\int_{\Omega_{\ell_0}} |\nabla u_\ell|^{p-2}\nabla u_\ell \nabla\rho(u_\ell - u_0)\,\mathrm{d}x\right| \le c\int_{\Omega_{\ell_0}} |\nabla u_\ell|^{p-2}|\nabla u_\ell|\cdot|u_\ell - u_0|\,\mathrm{d}x$$

$$\le c\left\{\int_{\Omega_{\ell_0}} |\nabla u_\ell|^p\,\mathrm{d}x\right\}^{1/p'}\left\{\int_{\Omega_{\ell_0}} |u_\ell - u_0|^p\,\mathrm{d}x\right\}^{1/p} \le C\left\{\int_{\Omega_{\ell_0}} |u_\ell - u_0|^p\,\mathrm{d}x\right\}^{1/p} \to 0.$$

Then we can derive from (2.10)

$$\int_{\Omega_{\ell_0}} |\nabla u_\ell|^{p-2}\nabla u_\ell \nabla(u_\ell - u_0)\rho\,\mathrm{d}x \to 0.$$

Therefore, one has

$$\int_{\Omega_{\ell_0}} \{|\nabla u_\ell|^{p-2}\nabla u_\ell - |\nabla u_0|^{p-2}\nabla u_0\}\nabla(u_\ell - u_0)\rho\,\mathrm{d}x$$

$$= \int_{\Omega_{\ell_0}} |\nabla u_\ell|^{p-2}\nabla u_\ell \nabla(u_\ell - u_0)\rho\,\mathrm{d}x - \int_{\Omega_{\ell_0}} |\nabla u_0|^{p-2}\nabla u_0 \nabla(u_\ell - u_0)\rho\,\mathrm{d}x \to 0.$$

(Recall that $u_\ell \rightharpoonup u_0$ in $W^{1,p}(\Omega_{\ell_0})$.) We find that when $p \ge 2$ (see (1.4)),

$$\int_{\Omega_{\ell_0}} |\nabla(u_\ell - u_0)|^p\rho\,\mathrm{d}x \le c\int_{\Omega_{\ell_0}} \{|\nabla u_\ell|^{p-2}\nabla u_\ell - |\nabla u_0|^{p-2}\nabla u_0\}\nabla(u_\ell - u_0)\rho\,\mathrm{d}x \to 0. \quad (2.11)$$

When $1 < p < 2$, it holds that

$$\int_{\Omega_{\ell_0}} |\nabla(u_\ell - u_0)|^p\rho\,\mathrm{d}x$$

$$\le \left\{\int_{\Omega_{\ell_0}} \{|\nabla u_\ell| + |\nabla u_0|\}^{p-2}|\nabla(u_\ell - u_0)|^2\rho\,\mathrm{d}x\right\}^{p/2}\left\{\int_{\Omega_{\ell_0}} \{|\nabla u_\ell| + |\nabla u_0|\}^p\rho\,\mathrm{d}x\right\}^{(2-p)/2}$$

$$\le c\left\{\int_{\Omega_{\ell_0}} \{|\nabla u_\ell|^{p-2}\nabla u_\ell - |\nabla u_0|^{p-2}\nabla u_0\}\nabla(u_\ell - u_0)\rho\,\mathrm{d}x\right\}^{p/2}\left\{\int_{\Omega_{\ell_0}} \{|\nabla u_\ell| + |\nabla u_0|\}^p\rho\,\mathrm{d}x\right\}^{(2-p)/2}$$

$$\to 0. \quad (2.12)$$

From above, we derive that for $1 < p < +\infty$

$$\int_{\Omega_{\ell_0-1}} |\nabla(u_\ell - u_0)|^p \,\mathrm{d}x \le \int_{\Omega_{\ell_0}} |\nabla(u_\ell - u_0)|^p \rho \,\mathrm{d}x \to 0,$$

which shows that

$$u_\ell \to u_0 \qquad \text{in } W^{1,p}(\Omega_{\ell_0-1}).$$

Therefore, u_0 satisfies

$$-\Delta_p u_0 = f.$$

We finally conclude that $u_0 = u_\infty$ since u_0 is independent of X_1. This completes the proof of the theorem. □

Remark 2.2. Some other convergence results including some rate of convergence are available. We refer the reader to [9] for details.

3. A LIOUVILLE TYPE THEOREM

As we have seen at the end of the proof of Theorem 2.1, a key point in the convergence issue is the uniqueness of the solution to

$$\begin{cases} -\Delta_p u_0 = f(X_2) & \text{in } \mathbb{R}^q \times \omega, \\ u_0 = 0 & \text{on } \mathbb{R}^q \times \partial\omega, \end{cases} \tag{3.1}$$

when, eventually, some other assumptions on u_0, like boundedness, are imposed. This has a flavor of the Liouville theorem. In this direction let us prove

Theorem 3.1. *Let $u \in L^\infty(\mathbb{R}^q; L^p(\omega))$ be a weak solution to*

$$\begin{cases} -\Delta_p u = 0 & \textit{in } \mathbb{R}^q \times \omega, \\ u = 0 & \textit{on } \mathbb{R}^q \times \partial\omega. \end{cases} \tag{3.2}$$

Then $u \equiv 0$; i.e., problem (3.2) *does not admit any nontrivial solution.*

Before turning to the proof of the theorem, let us make our assumptions precise. $L^\infty(\mathbb{R}^q; L^p(\omega))$ denotes the space of functions from $\mathbb{R}^q$ with values in $L^p(\omega)$ that are essentially bounded. By a weak solution to (3.2) we mean a function u such that, for any domain $(-\ell, \ell)^q \times \omega$,

$$\begin{gathered} u \in W^{1,p}((-\ell,\ell)^q \times \omega) \cap L^\infty(\mathbb{R}^q; L^p(\omega)), \qquad u = 0 \quad \text{on } (-\ell,\ell)^q \times \partial\omega, \\ \int_{(-\ell,\ell)^q \times \omega} |\nabla u|^{p-2} \nabla u \nabla v \,\mathrm{d}x = 0 \qquad \forall v \in W_0^{1,p}((-\ell,\ell)^q \times \omega). \end{gathered} \tag{3.3}$$

Let us now prove the above theorem.

Proof of Theorem 3.1. Let ρ be a nonnegative smooth function such that

$$0 \le \rho \le 1, \qquad \rho \equiv 1 \quad \text{in } \left(-\frac{1}{2}, \frac{1}{2}\right), \qquad \rho \equiv 0 \quad \text{outside } (-1,1), \qquad |\rho'| \text{ is bounded},$$

and set $\rho_\ell(x) = \rho\left(\frac{x}{\ell}\right)$, $\Pi := \prod_{i=1}^q \rho_\ell(x_i)$. We have, if Π^p denotes the pth power of Π,

$$u\Pi^p \in W_0^{1,p}((-\ell,\ell)^q \times \omega).$$

Plugging the above function into (3.3) leads to

$$\int\limits_{(-\ell,\ell)^q\times\omega} |\nabla u|^{p-2}\nabla u\nabla(u\Pi^p)\,\mathrm{d}x = 0.$$

Therefore, we derive that

$$\begin{aligned}\int\limits_{(-\ell,\ell)^q\times\omega} |\nabla u|^{p-2}\nabla u\nabla u\Pi^p\,\mathrm{d}x &= -\int\limits_{(-\ell,\ell)^q\times\omega} |\nabla u|^{p-2}\nabla u\nabla\Pi^p u\,\mathrm{d}x\\ &= -p\int\limits_{(-\ell,\ell)^q\times\omega} |\nabla u|^{p-2}\nabla u\nabla\Pi u\Pi^{p-1}\,\mathrm{d}x\\ &\le \frac{c}{\ell}\int\limits_{(-\ell,\ell)^q\times\omega} |\nabla u|^{p-1}|u|\Pi^{p-1}\,\mathrm{d}x.\end{aligned}$$

Applying the Hölder inequality to the right-hand side of the above inequality, we obtain

$$\int\limits_{(-\ell,\ell)^q\times\omega} |\nabla u|^p\Pi^p\,\mathrm{d}x \le \frac{c}{\ell}\left\{\int\limits_{(-\ell,\ell)^q\times\omega} |\nabla u|^p\Pi^p\,\mathrm{d}x\right\}^{(p-1)/p}\left\{\int\limits_{(-\ell,\ell)^q\times\omega} |u|^p\,\mathrm{d}x\right\}^{1/p},$$

which is equivalent to

$$\int\limits_{(-\ell,\ell)^q\times\omega} |\nabla u|^p\Pi^p\,\mathrm{d}x \le \frac{c}{\ell^p}\int\limits_{(-\ell,\ell)^q\times\omega} |u|^p\,\mathrm{d}x. \tag{3.4}$$

We remark that u vanishes on $\mathbb{R}^q\times\partial\omega$, i.e., for a.e. X_1, $u(X_1,\cdot)\in W_0^{1,p}(\omega)$. Exploiting Poincaré's inequality in the X_2-direction (see [3]) yields

$$\int\limits_{(-\ell,\ell)^q\times\omega} |\nabla u|^p\Pi^p\,\mathrm{d}x \le \frac{c}{\ell^p}\int\limits_{(-\ell,\ell)^q\times\omega} |\nabla_{X_2}u|^p\,\mathrm{d}x$$

($\nabla_{X_2} = (\partial x_{q+1},\dots,\partial x_n)$). Due to the choice that we made for Π, we obtain

$$\int\limits_{(-\frac{\ell}{2},\frac{\ell}{2})^q\times\omega} |\nabla u|^p\,\mathrm{d}x \le \frac{c}{\ell^p}\int\limits_{(-\ell,\ell)^q\times\omega} |\nabla u|^p\,\mathrm{d}x.$$

After iterating $k-1$ times this inequality, it comes that

$$\int\limits_{(-\frac{\ell}{2},\frac{\ell}{2})^q\times\omega} |\nabla u|^p\,\mathrm{d}x \le \frac{c}{\ell^{p(k-1)}}\int\limits_{(-2^{k-1}\ell,\,2^{k-1}\ell)^q\times\omega} |\nabla u|^p\,\mathrm{d}x.$$

Hence we derive from (3.4) that

$$\int\limits_{(-\frac{\ell}{2},\frac{\ell}{2})^q\times\omega} |\nabla u|^p\,\mathrm{d}x \le \frac{c}{\ell^{pk}}\int\limits_{(-2^k\ell,\,2^k\ell)^q\times\omega} |u|^p\,\mathrm{d}x.$$

Since $u \in L^\infty(\mathbb{R}^q; L^p(\omega))$, we obtain

$$\int\limits_{(-\frac{\ell}{2},\frac{\ell}{2})^q\times\omega} |\nabla u|^p \, \mathrm{d}x \le \frac{c}{\ell^{pk}}(2^{k+1}\ell)^q \operatorname{ess\,sup} |u(X_1,\cdot)|^p_{L^p(\omega)}$$

(ess sup denotes the essential supremum in X_1). Choosing k so large that $pk > q$ and letting $\ell \to \infty$ lead to

$$u = 0.$$

This completes the proof of the theorem. □

Remark 3.2. One should notice that one cannot remove entirely the assumption

$$u \in L^\infty(\mathbb{R}^q; L^p(\omega)). \tag{3.5}$$

Indeed, suppose that

$$p = 2, \qquad q = 1, \qquad \omega = (-\pi, \pi). \tag{3.6}$$

Then the function

$$u = e^{x_1} \sin x_2 \tag{3.7}$$

is a nontrivial weak solution to

$$\begin{cases} -\Delta u = 0 & \text{in } \mathbb{R} \times \omega, \\ u = 0 & \text{on } \mathbb{R} \times \partial\omega. \end{cases} \tag{3.8}$$

ACKNOWLEDGMENTS

The authors have been supported by the Swiss National Science Foundation under the contracts #20-103300/1 and #20-111543/1.

REFERENCES

1. J. W. Barrett and W. B. Liu, "Finite Element Approximation of the Parabolic p-Laplacian," SIAM J. Numer. Anal. **31** (2), 413–428 (1994).
2. M. Chipot, *Elements of Nonlinear Analysis* (Birkhäuser, Basel, 2000).
3. M. Chipot, *ℓ Goes to Plus Infinity* (Birkhäuser, Basel, 2002).
4. M. Chipot and A. Rougirel, "On the Asymptotic Behaviour of the Solution of Elliptic Problems in Cylindrical Domains Becoming Unbounded," Commun. Contemp. Math. **4** (1), 15–44 (2002).
5. M. Chipot and A. Rougirel, "On the Asymptotic Behaviour of the Solution of Parabolic Problems in Cylindrical Domains of Large Size in Some Directions," Discrete Contin. Dyn. Syst. B **1** (3), 319–338 (2001).
6. M. Chipot and A. Rougirel, "Remarks on the Asymptotic Behaviour of the Solution to Parabolic Problems in Domains Becoming Unbounded," Nonlinear Anal., Theory Methods Appl. **47** (1), 3–11 (2001).
7. M. Chipot and Y. Xie, "On the Asymptotic Behaviour of the p-Laplace Equation in Cylinders Becoming Unbounded," in *Nonlinear Partial Differential Equations and Their Applications: Proc. Int. Conf., Shanghai, China, 2003*, Ed. by N. Kenmochi, M. Ôtani, and S. Zheng (Gakkotosho, Tokyo, 2004), pp. 16–27.
8. W. Liu and N. Yan, "Quasi-norm Local Error Estimators for p-Laplacian," SIAM J. Numer. Anal. **39** (1), 100–127 (2001).
9. Y. Xie, "On Asymptotic Problems in Cylinders and Other Mathematical Issues," Thes. (Univ. Zürich, May 2006).

This article was submitted by the authors in English