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Risk averse asymptotics in a Black–Scholes market on a finite time horizon

Peter Grandits · Stefan Thonhauser

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Abstract We consider the optimal investment and consumption problem in a Black–Scholes market, if the target functional is given by expected discounted utility of consumption plus expected discounted utility of terminal wealth. We investigate the behaviour of the optimal strategies, if the relative risk aversion tends to infinity. It turns out that the limiting strategies are: do not invest at all in the stock market and keep the rate of consumption constant!

Keywords Utility maximization · Risk aversion asymptotics · Black–Scholes market

1 Introduction

In this paper we consider a financial agent, who invests in a market described by geometric Bownian motion and a riskless bond. The agent has also the possibility to consume wealth according to some consumption rate. His aim is to maximize the sum of the total expected discounted utility from consumption over some finite time horizon $[0, T]$ plus the expected utility from terminal wealth. This is a classical problem and seminal papers dealing with this problem are Merton (1969, 1971) and Karatzas et al. (1987). In the papers by Merton the problem was solved for the so called HARA class of utility functions (basically the power functions and the logarithm) using methods

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P. Grandits
Institute for Mathematical Methods in Economics, TU Vienna, 1040 Vienna, Austria

S. Thonhauser (✉)
Institute of Actuarial Science, University of Lausanne, 1015 Lausanne, Switzerland
e-mail: stefan.thonhauser@unil.ch

from dynamic programming. In the paper by Karatzas et al. duality methods were used and they were able to give some representation formulas for the optimal investment process, as well as for the optimal consumption density. They were able to do this for a market model where the coefficients are given by bounded adapted processes. In their last chapter, where they give some explicit solutions, they restrict to the constant coefficient case again.

In our paper we want to investigate the behaviour of the optimal processes, if the Arrow-Pratt relative risk aversion $-\frac{xU''(x)}{U'(x)}$ of the used utility function U becomes large. Investigations in this direction are not too numerous in the literature. We cite just two of them: Carassus and Rásonyi (2006) and Grandits and Summer (2006) investigate the connection of increasing (absolute) risk aversion and superreplication in discrete models.

Our main result (Theorem 2.6) can be described in the following way. The relative amount of money, which the agent invests in the stock market vanishes for relative risk aversion tending to infinity, which is certainly what one would expect. For the consumption rate we show that it converges to some constant value of consumption, which is maybe at first sight not so obvious.

During the completion of this paper we got knowledge of a paper by Nutz (2011), who investigates basically the same problem but in a general semimartingale setting. If the price process is given by a continuous process and the underlying filtration is continuous his result (Theorem 3.1(ii) and Remark 2.4) coincides with ours. But note that in Nutz (2011) power utility is used, where the focus of our paper is a more general family of utility function with increasing relative risk aversion. To keep technicalities as little as possible we confine ourselves here to the case of one stock and one bond, as well as to the case of constant coefficients in a Black–Scholes setting.

1.1 Financial market model

We assume a Black–Scholes type financial market with a finite time horizon $[0, T]$. The available two assets are one riskless with price process P_0 and one risky with price process P_1. Their dynamics are given by

$$\begin{aligned} dP_0(t) &= r\, P_0(t)\, dt,\;\; P_0(0) = p_0, \\ dP_1(t) &= P_1(t)(\mu\, dt + \sigma\, dW_t),\;\; P_1(0) = p_1. \end{aligned}$$

The model parameters are constant interest rate and mean return $r,\ \mu \in \mathbb{R}$ and constant volatility $\sigma \in \mathbb{R}^+$. The driving stochastic process $W = (W_t)_{0\le t\le T}$ is a standard Brownian motion on some probability space $(\Omega, \mathcal{F}, P)$. The flow of information $(\mathcal{F}_t)_{0\le t\le T}$ is given by the augmented filtration generated by W, i.e. $\mathcal{F}_t = \sigma\{W_s,\ 0 \le s \le t\}$.

Consider an agent with wealth given by a process $X = (X_t)_{0\le t\le T}$, who is allowed to invest in the market and to consume from his wealth. The portfolio process $\pi = (\pi_t)_{0\le t\le T}$ describes the amount of money invested in the risky asset P_1 for $t \in [0, T]$. For being reasonable π needs to be adapted to $(\mathcal{F}_t)_{0\le t\le T}$ and

$$\int_0^T \pi_s^2 \, ds < \infty \quad \text{a.s.}$$

The consumption density process $C = (C_t)_{0 \leq t \leq T}$ describes the amount of money taken from the wealth, again it needs to be adapted, $C_t \in [0, \infty)$ and

$$\int_0^T C_s \, ds < \infty \quad \text{a.s.}$$

Because of the derivation of some PDEs linked to the utility of consumption and terminal wealth we have to introduce in the following wealth, admissible portfolio and consumption processes when starting at some time $t \in [0, T]$.

For a fixed pair (π, C) of portfolio and consumption processes and for $(t, x) \in [0, T] \times (0, \infty)$ the wealth of the agent $X = (X_s)_{t \leq s \leq T}$ is given by

$$X_s = x + \int_t^s (rX_u - C_u) du + \int_t^s (\mu - r)\pi_u \, du + \int_t^s \pi_u \sigma \, dW_u, \quad X_t = x. \tag{1}$$

Note, as common within this setting, the amount $X_t - \pi_t$ is invested in the riskless asset (or if negative borrowed at rate r).

Let $\mathcal{A}(t, x)$ denote the set of (π, C) such that X given through (1) stays non-negative. From Sect. 2 of Karatzas et al. (1987) we know that for given C there exists π such that $(\pi, C) \in \mathcal{A}(t, x)$ if and only if

$$\mathbb{E}\left(\int_t^T Z_s^t e^{-r(s-t)} C_s ds \right) \leq x, \tag{2}$$

where $Z^t = (Z_s^t)_{t \leq s \leq T}$ is the change of measure martingale from Girsanov's Theorem removing the drift without consumption in (1), $Z_s^t = \exp\{-\theta(W_s - W_t) - \frac{1}{2}|\theta|^2(s-t)\}$ $(\theta = \sigma^{-1}(\mu - r))$.

Under this new measure $\tilde{P}$, $\tilde{P}(A) = \mathbb{E}(Z_T^t I_A)$, discounted wealth plus consumption forms a local martingale and discounted X itself is a supermartingale, hence

$$\mathbb{E}\left(Z_T^t \, e^{-r(T-t)} X_T \right) \leq x. \tag{3}$$

The expectations in (2) and (3) are with respect to the measure $P_{(t,x)}$ which ensures that $P_{(t,x)}(X_t = x) = 1$.

1.2 Family of utility functions

Before stating an optimization problem we have to choose some utility functions as a basis for measuring the performance of a pair of investment and consumption strategies (π, C).

Let $\{U_m\}_{m\in\mathbb{N}}$ be a family of utility functions, such that for each m we have $U_m : (0,\infty) \to \mathbb{R}$ is strictly increasing, strictly concave, three times differentiable, with non-decreasing second derivative and $U_m(0) := \lim_{x\to 0} U_m(x) \geq -\infty$, $U_m'(\infty) := \lim_{x\to\infty} U_m'(x) = 0$ and $U_m'(0) = \lim_{x\to 0} U_m'(x) = \infty$. Furthermore

$$\lim_{m\to\infty} -\frac{xU_m''(x)}{U_m'(x)} = \infty, \quad \text{uniformly in } x \in (0,\infty). \tag{4}$$

This says that the relative risk aversion tends to infinity if $m \to \infty$.

Let $\alpha > 0$ be a fixed but arbitrary number, for later purposes we scale the family $\{U_m\}_{m\in\mathbb{N}}$ by $U_m'(\alpha) = 1$. One may notice that for fixed m the optimal strategies do not depend on such a normalization. Therefore their limit behaviour as m tends to infinity is not affected by this scaling.

When writing

$$\frac{xU_m''(x)}{U_m'(x)} =: -M_m(x), \tag{5}$$

we derive $\ln(U_m'(x)) = -\int^x \frac{M_m(z)}{z}\,dz + \tilde{C},\ \tilde{C} \in \mathbb{R}$ and finally by the scaling property we get

$$U_m'(x) = e^{-\int_\alpha^x \frac{M_m(z)}{z}\,dz}. \tag{6}$$

From this representation one obtains for $\alpha > y > 0$

$$\begin{aligned} &\lim_{m\to\infty} U_m'(x) = \infty \quad \text{uniformly on } (0, \alpha - y),\\ &\lim_{m\to\infty} U_m'(x) = 0 \quad \text{uniformly on } (\alpha + y, \infty). \end{aligned} \tag{7}$$

The properties of U_m yield that $U_m' : [0,\infty] \to [0,\infty]$ is strictly decreasing, therefore there exists a strictly decreasing inverse $I_m : [0,\infty] \to [0,\infty]$.

From scaling in α we have $I_m(y) \leq \alpha$ for $y \geq 1$ and $I_m(y) \geq \alpha$ for $0 \leq y \leq 1$.

Equations (4)–(6) give for $K > 1$ that (for a detailed proof we refer to the Appendix A.2)

$$\lim_{m\to\infty} I_m(y) = \alpha, \quad \text{uniformly for } y \in \left[\frac{1}{K}, K\right]. \tag{8}$$

Figure 1 illustrates the shapes of U_m' and I_m when scaling in $\alpha = 2$.

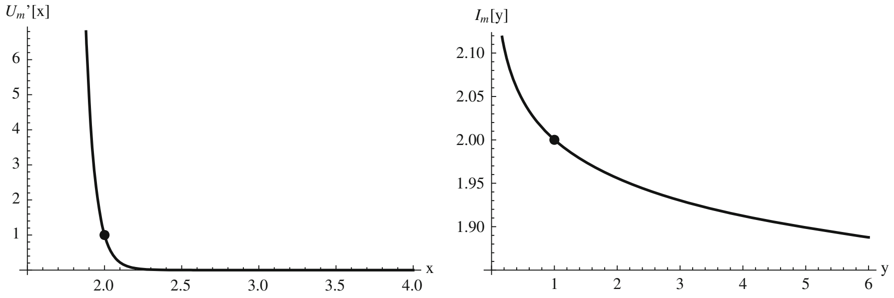

Fig. 1 Illustrations of behaviour of U'_m and I_m when $\alpha = 2$

Remark 1.1 The basic properties of $\{U_m\}_{m\in\mathbb{N}}$ are those stated in Sect. 3 of Karatzas et al. (1987) plus the $C^3(0,\infty)$-property of U_m as well as the condition $U'_m(0) = \infty$, but we do not impose the stronger conditions from Section 7 of the same paper.

For calculating the asymptotics of maximzing strategies if $m \to \infty$ we assume that for fixed m the function $M_m \in C^1[0,\infty)$, $M_m(x)$ is bounded from below and that

$$n = \inf_m \inf_{x\in\mathbb{R}^+} M_m(x) > 0. \tag{9}$$

This condition is certainly no restriction, since we are interested in the risk averse case. For example (9) holds true when $\inf_{x\in\mathbb{R}^+} M_1(x) > 0$ and $M_m(x)$ is increasing in m for fixed x. We get

$$U'_m(x) = e^{-\int_\alpha^x \frac{M_m(z)}{z}dz} \leq e^{-n(\ln x - \ln\alpha)} = \left(\frac{x}{\alpha}\right)^{-n}, \quad \text{for } x \geq \alpha.$$

Remember $I_m(y) \geq \alpha$ for $y \leq 1$ so we have

$$\begin{aligned} U'_m(I_m(y)) &\leq \left(\frac{I_m(y)}{\alpha}\right)^{-n}, \quad \text{for } y \leq 1 \\ \Leftrightarrow y &\leq \left(\frac{I_m(y)}{\alpha}\right)^{-n}, \quad \text{for } y \leq 1 \\ \Leftrightarrow \alpha y^{-(n)^{-1}} &\geq I_m(y), \quad \text{for } y \leq 1. \end{aligned} \tag{10}$$

When setting $\nu = (n)^{-1} > 0$ we can write (10) as

$$I_m(y) \leq \alpha y^{-\nu}, \quad \text{for } y \leq 1 \text{ and } \forall m.$$

Together with the previous discussions on I_m we have

$$I_m(y) \leq \alpha(y^{-\nu} \vee 1), \quad \text{for all } y \in \mathbb{R}^+. \tag{11}$$

We close this section with an usefull observation.

Lemma 1.1 *Condition* (4) *is equivalent to*

$$\lim_{m\to\infty} \frac{y I_m'(y)}{I_m(y)} = 0,$$

uniformly for $y \in \mathbb{R}^+$, *(i.e.* $\forall \varepsilon > 0\ \exists \hat{M}$ *such that* $|\, y I_m'(y)\,| \leq \varepsilon\, I_m(y)\ \forall\ m \geq \hat{M}$ *uniformly in* $y \in \mathbb{R}^+$).

Proof $-\frac{x U_m''(x)}{U_m''(x)} \xrightarrow{m\to\infty} \infty$ uniformly for $x \in \mathbb{R}^+$, if and only if

$$-\frac{I_m(y) U_m''(I_m(y))}{U_m'(I_m(y))} \xrightarrow{m\to\infty} \infty \quad \text{uniformly for } y \in \mathbb{R}^+$$
$$\Leftrightarrow \frac{y}{I_m(y) U_m''(I_m(y))} \xrightarrow{m\to\infty} 0 \quad \text{uniformly for } y \in \mathbb{R}^+.$$

Using $I_m'(y) U_m''(I_m(y)) = 1$ we get $\lim_{m\to\infty} \frac{y I_m'(y)}{I_m(y)} = 0$ uniformly for $y \in \mathbb{R}^+$. □

In addition to (4) we shall need a second condition describing high risk aversion, namely

$$\lim_{m\to\infty} \frac{y^2 I_m''(y)}{I_m(y)} = 0 \quad \text{uniformly for } y \in \mathbb{R}^+. \tag{12}$$

Remark 1.2 Similarly as in Lemma 1.1 one can show that (12) is equivalent to

$$\lim_{m\to\infty} \frac{U_m''(x)^3\, x}{U_m'(x)^2\, U_m'''(x)} = -\infty.$$

Note also that for power utility $U_m(x) = -\frac{1}{m} x^{-m}$ condition (12) transforms to $\lim_{m\to\infty} \frac{m+2}{(m+1)^2} = 0$.

In the Appendix A.1 sufficient conditions for the function $M_m(x)$ are derived such that the corresponding family of utility functions is in the scope of our assumptions. Furthermore some non-standard examples are given there.

1.3 Maximization problem and some representation results

After fixing the wealth process, the class of admissible controls $\mathcal{A}(t, x)$ and the family of utility functions we can state the following (classical) optimization problem

$$V_m(t, x) = \sup_{(\pi, C)\in\mathcal{A}(t,x)} \mathbb{E}\left(\int_t^T e^{-\beta s} U_m(C_s)\, ds + e^{-\beta T} U_m(X_T) \right), \tag{13}$$

where $\beta > 0$ is a constant discount factor. In the case $U_m(0) = -\infty$ we need for being an admissible consumption process and having a reasonable optimization problem in addition that

$$\mathbb{E}\left(\int_0^T e^{-\beta s} U_m^-(C_s)\,ds\right) < \infty, \quad \mathbb{E}\left(U_m^-(X_T)\right) < \infty.$$

In particular we are interested in the behaviour of the maximizing strategies if m tends to infinity, i.e. if the risk aversion tends to infinity.

As starting point we use the representation for these optimal controls obtained by Karatzas et al. (1987) by the martingale method. For that purpose define the process $\zeta^t = (\zeta_s^t)_{t\le s\le T}$ for $t \in [0, T]$ by

$$\zeta_s^t = e^{(\beta-r)(s-t)} Z_s^t.$$

Then the optimal consumption and terminal wealth will be of the form $I_m(y\zeta_s^t)$ and $I_m(y\zeta_T^t)$ for the right choice of $y \in (0, \infty)$. Having these general forms of the controls in mind one can define the following two functions for $(t, y) \in [0, T] \times (0, \infty)$ which are related to (13) and the sum of (2) and (3),

$$G^m(t, y) = \mathbb{E}\left(\int_t^T e^{-\beta(s-t)} U_m(I_m(y\zeta_s^t))\,ds + e^{-\beta(T-t)} U_m(I_m(y\zeta_T^t))\right),$$

$$S^m(t, y) = \mathbb{E}\left(\int_t^T e^{-\beta(s-t)} y\zeta_s^t\, I_m(y\zeta_s^t)\,ds + e^{-\beta(T-t)} y\zeta_T^t\, I_m(y\zeta_T^t)\right).$$

Furthermore define $\mathcal{H}^m(t, y) = \frac{S^m(t,y)}{y}$.

Notice $\mathcal{H}^m$ (see Karatzas et al. (1987, Section 7)) is continuous and strictly decreasing in $y \in (0, \infty)$ with $\lim_{y\to 0} \mathcal{H}^m(t, y) = \infty$ and $\lim_{y\to\infty} \mathcal{H}^m(t, y) = 0$. Therefore there exists an inverse of $\mathcal{H}^m$ in y denoted by $\mathcal{Y}^m(t, \cdot) : [0, \infty] \to [0, \infty]$ (i.e. $\mathcal{Y}^m(t, \mathcal{H}^m(t, y)) = y$). For later usage we want to point out that for $\mathcal{Y}^m(t, x)$ we have

$$\mathcal{Y}_x^m(t, \mathcal{H}^m(t, y)) = \frac{1}{\mathcal{H}_y^m(t, y)}, \quad \mathcal{Y}_t^m(t, \mathcal{H}^m(t, y)) = -\frac{\mathcal{H}_t^m(t, y)}{\mathcal{H}_y^m(t, y)}.$$

Splitting off $\mathcal{H}^m = \mathcal{H}^{(m,1)} + \mathcal{H}^{(m,2)}$ one observes that $\mathcal{H}^{(m,1)}$ corresponds to the maximization of pure consumption and $\mathcal{H}^{(m,2)}$ corresponds to pure maximization of terminal wealth. From Section 6 of Karatzas et al. (1987) we know when maximizing both, one virtually splits off the initial capital $x = x_1 + x_2$ such that x_1 is used for maximizing consumption only and x_2 is used for maximizing terminal wealth only. For getting both bounds (2) and (3) an equality (with respect to the virtual initial capital splitting x_1 and x_2) we need some y which fulfills:

$$\mathcal{H}^m(t,y) = \mathcal{H}^{(m,1)}(t,y) + \mathcal{H}^{(m,2)}(t,y) = x_1 + x_2 = x.$$

Therefore the special choice of y corresponds to $\mathcal{Y}^m(t,x)$. Finally the definitions

$$C_s^{(t,x,m)} = I_m(\mathcal{Y}^m(t,x)\zeta_s^t) \quad \text{and} \quad X_T^{(t,x,m)} = I_m(\mathcal{Y}^m(t,x)\zeta_T^t) \tag{14}$$

yield the optimal consumption process and the optimal terminal wealth.

For this special consumption process there exists a portfolio process $\pi^{(t,x,m)}$ delivering the terminal wealth $X_T^{(t,x,m)}$ and $(\pi^{(t,x,m)}, C^{(t,x,m)}) \in \mathcal{A}(t,x)$.

By Theorem 6.6 of Karatzas et al. (1987),

$$V_m(t,x) = \mathbb{E}\left(\int_t^T e^{-\beta s} U_m(C_s^{(t,x,m)})\,ds + e^{-\beta T} U_m(X_T^{(t,x,m)})\right)$$
$$V_m(t,x) = e^{-\beta t} G^m(t, \mathcal{Y}^m(t,x)).$$

Furthermore the definitions of $(\pi^{(t,x,m)}, C^{(t,x,m)})$ and $\mathcal{H}^m$ can be used for describing the optimal wealth process $X^{(t,x,m)}$ by

$$X_s^{(t,x,m)} = \mathcal{H}^m(s, \mathcal{Y}^m(t,x)\zeta_s^t). \tag{15}$$

Because of the Markovian structure of the optimization problem (13) for the constant coefficients setting there exists a feedback type form of $(\pi^{(t,x,m)}, C^{(t,x,m)})$ and we get finally:

Proposition 1.2 *The pair $(C^{(t,x,m)}, \pi^{(t,x,m)})$ constructed above is optimal for (13) and for $t \le s \le T$ can be written as*

$$C_s^{(t,x,m)} = I_m(\mathcal{Y}^m(s, X_s^{(t,x,m)})), \tag{16}$$

$$\pi_s^{(t,x,m)} = -\frac{(b-r)}{\sigma^2} \frac{\mathcal{Y}^m(s, X_s^{(t,x,m)})}{\mathcal{Y}_x^m(s, X_s^{(t,x,m)})}. \tag{17}$$

1.4 PDEs for S and $\mathcal{H}$

The starting point for the asymptotic calculations for $(\pi^{(t,x,m)}, C^{(t,x,m)})$ will be a PDE for $\mathcal{H}^m(t,y)$ going to be stated in (18).

Let L be the following operator linked to the discounted process ζ^t,

$$L\phi(t,y) := \gamma y^2 \frac{\partial^2 \phi(t,y)}{\partial y^2} + (\beta - r) y \frac{\partial \phi(t,y)}{\partial y} - \beta\phi(t,y),$$

with $\gamma = \frac{1}{2}|\theta|^2$.

Lemma 7.1 of Karatzas et al. (1987) applies in our situation, since $yI_m(y)$ does not depend on t, is Hölder continuous on compacts and fulfills a polynomial growth condition by the estimate (11). Therefore $S^m(t,y)$ is the unique solution to

$$\left(\frac{\partial}{\partial t}+L\right)S^m(t,y)+yI_m(y)=0,\quad (t,y)\in[0,T)\times(0,\infty),$$
$$S^m(T,y)=yI_m(y),\quad y\in(0,\infty).$$

The relation between $S^m(t,y)$ and $\mathcal{H}^m(t,y)$ yields

$$\mathcal{H}^m_t(t,y)+\gamma y^2\mathcal{H}^m_{yy}(t,y)+(\beta-r+2\gamma)y\mathcal{H}^m_y(t,y)-r\mathcal{H}^m(t,y)+I_m(y)=0,$$
$$(t,y)\in[0,T)\times(0,\infty),$$
$$\mathcal{H}^m(T,y)=I_m(y),\quad y\in(0,\infty). \tag{18}$$

2 Asymptotics

For simplifying notations we focus on the asymptotics of the optimal strategies maximizing $V_m(t,x)$ given in (13) when $t=0$. We denote the optimal consumption and portfolio processes by $C^{(m)}=C^{(0,x,m)}$ and by $\pi^{(m)}=\pi^{(0,x,m)}$, the corresponding wealth process by $X^{(m)}=X^{(0,x,m)}$ and use $\zeta=\zeta^0$.

2.1 Asymptotics of the consumption process

We start with transforming (18) into the backward heat equation. Using the following transformations

$$z=\ln y,\ z\in\mathbb{R},$$
$$\bar{I}_m(z)=I_m(e^z),$$
$$\bar{\mathcal{H}}^m(t,z)=\mathcal{H}^m(t,e^z)\,e^{-at-bz}$$

with $a=r+\frac{\kappa^2}{4\gamma}$ and $b=-\frac{\kappa}{2\gamma}$ where $\kappa=\beta-r+\gamma$ and finally when setting $\tau=T-t$ we arrive at

$$\bar{\mathcal{H}}^m_\tau=\gamma\bar{\mathcal{H}}^m_{zz}+I_m(z)e^{-a(T-\tau)-bz},\quad (\tau,z)\in(0,T]\times\mathbb{R},$$
$$\bar{\mathcal{H}}^m(\tau=0,z)=\bar{I}_m(z)e^{-aT-bz},\quad z\in\mathbb{R}. \tag{19}$$

Equation (19) has the explicit solution (see Ladyženskaja et al. 1967, IV 1.13)

$$\bar{\mathcal{H}}^m(\tau, z) = \int_0^\tau \int_{\mathbb{R}} \Gamma(z-w, \tau-s)\bar{I}_m(w)\, e^{-a(T-s)-bw}\, dw\, ds + \int_{\mathbb{R}} \Gamma(z-w, \tau)\bar{I}_m(w)\, e^{-aT-bw}\, dw, \tag{20}$$

where

$$\Gamma(x, \tau) = \frac{1}{\sqrt{4\gamma\pi\tau}} e^{-\frac{x^2}{4\gamma\tau}}.$$

Changing $z = \ln y$ in (11) we have

$$\bar{I}_m(z) \le \alpha(e^{-\nu z} \vee 1), \quad \forall m. \tag{21}$$

Now using the estimate obtained in (21) and dominated convergence we get from (20) and (8)

$$\lim_{m\to\infty} \bar{\mathcal{H}}^m(\tau, z) = \int_0^\tau \int_{\mathbb{R}} \Gamma(z-w, \tau-s)\alpha e^{-a(T-s)-bw}\, dw\, ds + \int_{\mathbb{R}} \Gamma(z-w, \tau)\alpha e^{-aT-bw}\, dw.$$

For $\tau = T$ we find

$$\lim_{m\to\infty} \bar{\mathcal{H}}^m(\tau = T, z) = \alpha\, e^{-bz} \frac{1 - e^{-aT+b^2\gamma T}}{a - b^2\gamma} + \alpha\, e^{-bz} e^{-aT+b^2\gamma T} =: \alpha\lambda e^{-bz}, \tag{22}$$

uniformly on compact sets for the z variable. Here λ is a positive constant depending only on the parameters of the problem.

From (22) we get by back substituting $\lim_{m\to\infty} \mathcal{H}^m(t = 0, z) = \alpha\lambda$ uniformly on compact sets for the z variable and furthermore

$$\lim_{m\to\infty} \mathcal{H}^m(t = 0, y) = \alpha\lambda \quad \text{uniformly on compact sets for } y. \tag{23}$$

Changing from $\mathcal{H}^m$ to $\mathcal{Y}^m$, (23) yields for all $m > M$, (m big enough) that

$$\mathcal{Y}^m(0, x) \ge 1 \quad \text{for } x \in \left(0, \frac{\alpha\lambda}{2}\right], \tag{24}$$

an interested reader can find the details in Sect. A.2 of the Appendix.

The following Lemma is the first out of three steps for proving that in the limit the consumptions process does not vary anymore.

Lemma 2.1 *We have*

$$\lim_{m\to\infty}\left\|\int_0^T |\, I_m'(\mathcal{Y}^m(0,x)\zeta_t)\mathcal{Y}^m(0,x)\zeta_t|\, dt\right\|_{L^2} = 0,$$

uniformly for all $x \in (0, \frac{\alpha\lambda}{2}]$.

Proof The basic property (11) of I_m yields

$$I_m(\mathcal{Y}^m(0,x)\zeta_t) \le \alpha\left((\mathcal{Y}^m(0,x)\zeta_t)^{-\nu} \vee 1\right),$$

using (24) we get

$$I_m(\mathcal{Y}^m(0,x)\zeta_t) \le \alpha\left(\zeta_t^{-\nu} \vee 1\right), \tag{25}$$

for $m > M$ and uniformly for $x \in (0, \frac{\alpha\lambda}{2}]$. Lemma 1.1 stated that $\forall \varepsilon > 0\ \exists \hat{M}$ such that $|\, y I_m'(y)\,| \le \varepsilon\, I_m(y)\ \ \forall\, m \ge \hat{M}$ uniformly in $y \in \mathbb{R}^+$. Therefore we get together with (25) that for every $\omega \in \Omega$ and for all $m > (\hat{M} \vee M) =: \overline{M}$ uniformly for $x \in (0, \frac{\alpha\lambda}{2}]$,

$$|\, I_m'(\mathcal{Y}^m(0,x)\zeta_t)\mathcal{Y}^m(0,x)\zeta_t| \le \varepsilon\,\alpha\left(\zeta_t^{-\nu} \vee 1\right).$$

Integration gives

$$\int_0^T |\, I_m'(\mathcal{Y}^m(0,x)\zeta_t)\mathcal{Y}^m(0,x)\zeta_t|\, dt \le \varepsilon\,\alpha \int_0^T \left(\zeta_t^{-\nu} \vee 1\right) dt$$

and finally

$$\left\|\left(\int_0^T |\, I_m'(\mathcal{Y}^m(0,x)\zeta_t)\mathcal{Y}^m(0,x)\zeta_t|\, dt\right)\right\|_{L^2} \le \left(\mathbb{E}\left[\varepsilon^2\alpha^2\left(\int_0^T \left(\zeta_t^{-\nu} \vee 1\right) dt\right)^2\right]\right)^{\frac{1}{2}} \le \varepsilon\, C,$$

for some constant $C > 0$, depending only on the model parameters, and $m > \overline{M}$, which proves the Lemma. □

Applying similar estimates together with assumption (12) prove the following Lemma.

Lemma 2.2 *We have*

$$\lim_{m\to\infty}\left\| \int_0^T | I_m''(\mathcal{Y}^m(0,x)\zeta_t)(\mathcal{Y}^m(0,x))^2\zeta_t^2| \, dt \right\|_{L^2} = 0,$$

$$\lim_{m\to\infty}\left\| \left(\int_0^T | I_m'(\mathcal{Y}^m(0,x)\zeta_t)\mathcal{Y}^m(0,x)\zeta_t|^2 \, dt \right)^{\frac{1}{2}} \right\|_{L^2} = 0,$$

uniformly for all $x \in (0, \frac{\alpha\lambda}{2}]$.

Proposition 2.3 *Let* $C^{(m)} = (C_t^{(m)})_{0\le t\le T}$ *denote the optimal consumption process for utility* U_m *and* $\|\cdot\|_{\mathcal{H}^2}$ *the* $\mathcal{H}^2$ *norm for special semimartingales, then*

$$\lim_{m\to\infty}\left\| C^{(m)} - C_0^{(m)} \right\|_{\mathcal{H}^2} = 0, \quad \textit{uniformly for initial wealth } x \in (0, \frac{\alpha\lambda}{2}].$$

Proof From (16) we have the representation $C_t^{(m)} = I_m(\mathcal{Y}^m(0,x)\zeta_t)$ for the optimal consumption process. Applying Itô's formula and using $d\zeta_t = (\beta - r)\zeta_t dt - \theta\zeta_t dW_t$ we derive

$$C_t^{(m)} = C_0^{(m)} + \int_0^t f_s^{(m)} ds + \int_0^t g_s^{(m)} dW_s,$$

where

$$f_t^{(m)} = I_m'(\mathcal{Y}^m(0,x)\zeta_t)\mathcal{Y}^m(0,x)\zeta_t(\beta - r) + \frac{1}{2} I_m''(\mathcal{Y}^m(0,x)\zeta_t)(\mathcal{Y}^m(0,x))^2\zeta_t^2\theta^2,$$
$$g_t^{(m)} = -I_m'(\mathcal{Y}^m(0,x)\zeta_t)\mathcal{Y}^m(0,x)\zeta_t\theta.$$

Looking at the definition of the $\mathcal{H}^2$ norm (see Protter (2004, p. 154), we have to replace the infinite time horizon there by our T) and application of Lemmas 2.1 and 2.2 shows the result of the Theorem. □

2.2 Asymptotics of the portfolio process

Intuitively if the risk aversion increases the proportion of capital invested in the risky asset should decrease. Via linking the optimal portfolio process given in (17) with the backward heat equation we are going to prove this intuitive result in this section.

The representation (17), comprehended as a process with $0 \leq s \leq T$, shows that we need to show

$$\lim_{m\to\infty} \left\| \frac{\mathcal{Y}^m(\cdot, X^{(m)}_\cdot)}{\mathcal{Y}^m_x\left(\cdot, X^{(m)}_\cdot\right) X^{(m)}_\cdot} \right\|_{\mathcal{S}^2} = 0.$$

For the definition of $\|\cdot\|_{\mathcal{S}^2}$, see Protter (2004, p. 244). Using (15) and the definition of $\mathcal{Y}^m$,

$$\begin{aligned} X^{(m)}_t &= \mathcal{H}^m(t, \mathcal{Y}^m(0,x)\zeta_t), \\ \mathcal{Y}^m_x(t, \mathcal{H}^m(t,y)) &= \frac{1}{\mathcal{H}^m_y(t,y)}, \\ \mathcal{Y}^m_x(t, \mathcal{H}^m(t, \mathcal{Y}^m(0,x)\zeta_t)) &= \frac{1}{\mathcal{H}^m_y(t, \mathcal{Y}^m(0,x)\zeta_t)}, \end{aligned}$$

this is equivalent to proving

$$\lim_{m\to\infty} \left\| \frac{\mathcal{Y}^m(0,x)\zeta_\cdot \mathcal{H}^m_y(\cdot, \mathcal{Y}^m(0,x)\zeta_\cdot)}{\mathcal{H}^m(\cdot, \mathcal{Y}^m(0,x)\zeta_\cdot)} \right\|_{\mathcal{S}^2} = 0. \tag{26}$$

Lemma 2.4 *The limit*

$$\lim_{m\to\infty} \frac{y\mathcal{H}^m_y(t,y)}{\mathcal{H}^m(t,y)} = 0$$

is attained uniformly for all $(t, y) \in [0, T] \times \mathbb{R}^+$.

If $\mathcal{H}^m(t, \mathcal{Y}^m(0,x)\zeta_t)$ is bounded in $\mathcal{S}^2$ then by Lemma 2.4 we have that (26) holds true.

Proof of Lemma 2.4 By applying $\frac{\partial}{\partial y}$ to (18) and multiplying it with y we derive a PDE for $y\mathcal{H}^m_y$,

$$\begin{aligned} &(y\mathcal{H}^m_y)_t + \gamma y^2 (y\mathcal{H}^m_y)_{yy} + (\beta - r + 2\gamma) y (y\mathcal{H}^m_y)_y - r(y\mathcal{H}^m_y) + yI'_m(y) = 0, \\ &y\mathcal{H}^m_y(T, y) = yI'_m(y). \end{aligned}$$

Introducing $\mathcal{K} = y\mathcal{H}^m_y$, $z = \ln y$ and $\tau = T - t$ we get (along the lines which led from (18) to (19) with explicit solution $\bar{\mathcal{H}}^m(\tau, z)$ given by (20))

$$\begin{aligned} \bar{\mathcal{K}}^m(\tau, z) = &\int_0^\tau \int_{\mathbb{R}} \Gamma(z - w, \tau - s) \bar{I}'_m(w)\, e^{-a(T-s)-bw}\, dw\, ds \\ &+ \int_{\mathbb{R}} \Gamma(z - w, \tau) \bar{I}'_m(w)\, e^{-aT-bw}\, dw. \end{aligned} \tag{27}$$

Note when switching from y to z we get $\bar{I}'_m(z)$ for $yI'_m(y)$ as inhomogeneity and boundary value.

From Lemma 1.1 we deduce that $\bar{I}'_m(z)/\bar{I}_m(z) \xrightarrow{m\to\infty} 0$ uniformly on $\mathbb{R}$ and get with (20) and (27),

$$\lim_{m\to\infty} \frac{\bar{\mathcal{K}}^m(\tau, z)}{\bar{\mathcal{H}}^m(\tau, z)} = 0 \quad \text{uniformly for } z \text{ and } \tau, \tag{28}$$

again details can be found in Sect. A.2 of the Appendix.

Switching back to y and t we have

$$\lim_{m\to\infty} \frac{y\mathcal{H}^m_y(t, y)}{\mathcal{H}^m(t, y)} = 0 \quad \text{uniformly for } y \text{ and } t,$$

which proves the Lemma. □

Proposition 2.5 *The proportion of wealth invested in the risky asset* $\hat{\pi}^{(m)} = \left(\frac{\pi_t^{(m)}}{X_t^{(m)}}\right)_{0\le t\le T}$ *converges to zero for* $m \to \infty$,

$$\lim_{m\to\infty} \left\| \hat{\pi}^{(m)} \right\|_{\mathcal{S}^2} = 0, \quad \textit{uniformly for initial wealth } x \in \left(0, \frac{\alpha\lambda}{2}\right].$$

Proof The only open question is the boundedness of $\mathcal{H}^m(t, \mathcal{Y}^m(0, x)\zeta_t)$. Because of $I_m(y) \le \alpha(y^{-\nu} \vee 1)$ and (20) we have $\mathcal{H}^m(t, y) \le C_1(y^{-\nu} \vee 1)$ for some constant C_1, the related calculations can be found in Sect. A.2 of the Appendix. From (24) we obtain

$$\mathcal{H}^m(t, \mathcal{Y}^m(0, x)\zeta_t) \le C_1((\mathcal{Y}^m(0, x)\zeta_t)^{-\nu} \vee 1) \le C_1((\zeta_t)^{-\nu} \vee 1)$$

for $m > M$ and $x \in (0, \frac{\alpha\lambda}{2}]$ and therefore for some constant C_2

$$\left\| \mathcal{H}^m(t, \mathcal{Y}^m(0, x)\zeta_t) \right\|_{\mathcal{S}^2} \le C_2,$$

for all $t \in [0, T]$ holds. □

Now we are able to determine $C_0^{(m)}$. From the representation $C_0^{(m)} = I_m(\mathcal{Y}^m(0, x))$ and (24) we have that $C_0^{(m)} \le \alpha$ for $m > M$ and $x \in (0, \frac{\alpha\lambda}{2}]$ with the consequence that there exists some converging subsequence $\{C_0^{(m_k)}\}_{k\in\mathbb{N}}$ with $\lim_{k\to\infty} C_0^{(m_k)} = \bar{C}$. Notice $C^{(m_k)} \xrightarrow{\mathcal{H}^2} \bar{C}$, fixing the corresponding optimal proportion invested in the risky asset $\hat{\pi}_t^{(m_k)} = \pi_t^{(m_k)}/X_t^{(m_k)}$ we get the wealth process,

$$dX_t^{(m_k)} = X_t^{(m_k)}[(r + (b - r)\hat{\pi}_t^{(m_k)})dt + \hat{\pi}_t^{(m_k)}\sigma dW_t] - C_t^{m_k} dt, \quad X_0^{m_k} = x.$$

The wealth process without investing and constant consumption $\bar{C}$ is given by

$$dX_t = X_t r dt - \bar{C} dt, \quad X_0 = x,$$

or in an explicit way

$$X_t = x\, e^{rt} - \bar{C} e^{rt} \left(\frac{1 - e^{-rt}}{r} \right). \tag{29}$$

We have (see Protter 2004, Theorem 15, p. 265) together with Propositions 2.3 and 2.5 that $X^{(m)} \xrightarrow{ucp} X$. From the representation of optimal terminal wealth and optimal consumption, i.e. (14), we have $C_T^{(m_k)} = X_T^{(m_k)}$, hence after $k \to \infty$, $\bar{C} = X_T$. Plugging into (29) and solving for $\bar{C}$ we get

$$\bar{C} = \frac{x\, r e^{rT}}{e^{rT} + r - 1}.$$

Assuming that there is another converging subsequence of $C_0^{(m)}$ with a limit $\tilde{C}$ all arguments from above are applicable. But this finally leads to $\tilde{C} = \bar{C}$ and therefore $\lim_{m\to\infty} C_0^{(m)} = \bar{C}$.The previous discussions, Propositions 2.3 and 2.5 and noting that α in $\frac{\alpha\lambda}{2}$ was an arbitrary positive constant, whereas λ was a positive constant, depending only on the model parameters, we get:

Theorem 2.6 *Letting the relative risk aversion tend to infinity, i.e. equation* (4) *holds, and assuming condition* (12) *we have*

$$\lim_{m\to\infty} \hat{\pi}^{(m)} \stackrel{\mathcal{S}^2}{=} 0,$$
$$\lim_{m\to\infty} C^{(m)} \stackrel{\mathcal{H}^2}{=} \bar{C},$$

uniformly on compact intervals for the initial capital x, where $\bar{C}$ is given by

$$\bar{C} = \frac{x\, r e^{rT}}{e^{rT} + r - 1}.$$

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A Appendix

A.1 Utility functions

In the first part of the Appendix we are going to examine utility functions which are in the scope of the stated conditions in Sect. 1.2. They are:

1.) $U_m' > 0$	2.) $U_m'' < 0$
3.) $U_m \in C^3(0,\infty)$	4.) $U_m''' \geq 0$
5.) $U_m(0) \geq -\infty$	6.) $U_m'(\infty) = 0$
7.) $U_m'(0) = \infty$	

The two conditions characterizing high relative risk aversion are:

$$8.)\ \lim_{m\to\infty} -\frac{xU_m''(x)}{U_m'(x)} = \infty \quad \text{uniformly in } x \in (0,\infty)$$

$$9.)\ \lim_{m\to\infty} \frac{U_m''(x)^3\, x}{U_m'(x)^2\, U_m'''(x)} = -\infty \quad \text{uniformly in } x \in (0,\infty)$$

In the sequel let $\alpha = 1$, any other choice would only lead to a different normalization of the family $\{U_m\}_{m\in\mathbb{N}}$ which would not affect further results since the aforementioned conditions are invariant with respect to those normalizations.

From formulas (5) and (6) we have the following representation

$$U_m'(x) = e^{-\int_1^x \frac{M_m(z)}{z}dz}, \tag{30}$$

for a function $M_m \in C^1(0,\infty)$ with $\lim_{m\to\infty} M_m(x) = \infty$ uniformly in $x \in [0,\infty)$. Therefore for getting appropriate utility functions we need to fix some additional sufficient conditions on $M_m(x)$.

From (30) and monotonicity, conditions 1.) and 5.) follow immediately. We have

$$U_m''(x) = e^{-\int_1^x \frac{M_m(z)}{z}dz}\left(-\frac{M_m(x)}{x}\right),$$

$$U_m'''(x) = e^{-\int_1^x \frac{M_m(z)}{z}dz}\,\frac{M_m(x)^2}{x^2} + e^{-\int_1^x \frac{M_m(z)}{z}dz}\,\frac{-xM_m'(x) + M_m(x)}{x^2},$$

which yields 3.) and 2.) because of $M_m \to \infty$ uniformly in x. Conditions 6.) and 7.) follow from (30) and this fact as well. The above calculations show that condition 4.) is equivalent to

$$M_m^2(x) + M_m(x) - xM_m'(x) \geq 0, \tag{31}$$

and that 9.) is equivalent to

$$\lim_{m\to\infty} \frac{-M_m(x)^3}{M_m(x)^2 + M_m(x) - xM_m'(x)} = -\infty \quad \text{uniformly in } x \in (0,\infty). \tag{32}$$

A sufficient condition for (31) and (32) would be the following special form of M_m,

$$M_m(x) = m + g(x), \quad \text{for } m \text{ large enough.} \tag{33}$$

where $g : [0, \infty) \to \mathbb{R}$ with $|xg'(x)|$ bounded.

We can conclude:

Lemma A.1 *A family of functions $\{M_m\}_{m\in\mathbb{N}}$ such that $M_m \in C^1[0,\infty)$, $\lim_{m\to\infty} M_m(x) = \infty$ uniformly in $x \in [0,\infty)$ and (33) hold, defines via representation (30) an appropriate family of utility functions $\{U_m\}_{m\in\mathbb{N}}$.*

Related to (33) on can for instance take $g(x) = \frac{1}{1+x}$ leading to

$$U_m'(x) = C\left(\frac{1+x}{x}\right)x^{-m}, \quad C \in \mathbb{R}^+.$$

Remark A.1 One gets a different family of proper relative risk aversions by setting

$$M_m(x) = m(1+x), \quad \text{resp.} \quad M_m(x) = m(1+x) + \frac{1}{1+x},$$

leading to

$$U_m'(x) = C\,e^{-mx}x^{-m}, \quad C \in \mathbb{R}^+, \quad \text{resp.} \quad U_m(x) = C\,e^{-mx}x^{-m}, \quad C \in \mathbb{R}^-.$$

A.2 Some detailed proofs

In the second part of the Appendix we are going to state some technical proofs of results used throughout the manuscript which may be not plausible at first glance.

Detailed proof of (8):

Equation (8) states that for $K > 1$ we have

$$\lim_{m\to\infty} I_m(y) = \alpha, \quad \text{uniformly for } y \in \left[\frac{1}{K}, K\right].$$

Since I_m is monotone it is enough to show

$$\lim_{m\to\infty} I_m\left(\frac{1}{K}\right) = \lim_{m\to\infty} I_m(K) = \alpha, \quad \text{for arbitrary } K > 1.$$

Therefore it is enough to show for arbitrary $L > 0$,

$$\lim_{m\to\infty} I_m(L) = \alpha.$$

Let x_m be the solution to equation $U_m'(x) = L$, hence we need to proof $\lim_{m\to\infty} x_m = \alpha$ for all $L > 0$.

From (6) we get

$$U_m'(x) = e^{-\int_\alpha^x \frac{M_m(z)}{z}\,dz} \Rightarrow \int_\alpha^x \frac{M_m(z)}{z}\,dz = -\ln(L) =: L'. \tag{34}$$

Assume now that there is some $\varepsilon_0 > 0$ such that $x_m - \alpha > \varepsilon_0$ for infinitely many m (the case $x_m - \alpha < -\varepsilon_0$ is similar). We obtain

$$\int_\alpha^x \frac{M_m(z)}{z}\,dz \geq \int_\alpha^{\alpha+\varepsilon_0} \frac{M_m(z)}{z}\,dz \geq \inf_z M_m(z)\,\ln\left(\frac{\alpha+\varepsilon_0}{\alpha}\right).$$

Since $M_m(x) \to \infty$ uniformly in x the last expression yields a contradiction to (34).

This argument also implies that the limit $\lim_{m\to\infty} \bar{I}_m(w) = \alpha$ is attained uniformly on compacts of the w variable and holds pointwise for all $w \in \mathbb{R}$.

Detailed proof of (24):

In the following we are going to demonstrate how one can use the asymptotic behaviour of $\mathcal{H}^m(t, y)$ as stated in formula (23) for deriving the estimate (24).

From Sect. 1.3 we know that the function $\mathcal{Y}^m(0, x)$ fulfills

$$\mathcal{H}^m(0, \mathcal{Y}^m(0, x)) = x.$$

From (23) and the fact that $\mathcal{H}^m(0, z)$ is a monotone decreasing function in z we have $\mathcal{H}^m(0, z) \geq \frac{3\alpha\lambda}{4}$ for given N with $z \in [0, N]$ and large enough $m \geq M(N)$.

Since $\lim_{z\to\infty} \mathcal{H}^m(0, z) = 0$, see Sect. 1.3, we get that

$$\mathcal{Y}^m(0, x) \geq N, \quad \forall x \in \left(0, \frac{\alpha\lambda}{2}\right] \quad \forall m \geq M(N).$$

Particularly (24) holds, i.e.

$$\mathcal{Y}^m(0, x) \geq 1 \quad \text{for } x \in \left(0, \frac{\alpha\lambda}{2}\right] \quad \text{and} \quad m \geq M(N).$$

Detailed proof of (28):

By Lemma 1.1 and a transformation to the z coordinate from Sect. 2.1 we have

$$\forall\, \varepsilon > 0 \quad \exists M \quad \text{such that} \quad \forall\, m \geq M: \quad \left|\frac{\bar{I}_m'(z)}{\bar{I}_m(z)}\right| \leq \varepsilon \quad \text{uniformly in } z. \tag{35}$$

Now we get using (20) and (27):

$$\left|\frac{\bar{\mathcal{K}}^m(\tau,z)}{\bar{\mathcal{H}}^m(\tau,z)}\right|$$
$$=\frac{\int_0^\tau\int_{\mathbb{R}}\Gamma(z-w,\tau-s)\left|\bar{I}'_m(w)\right|e^{-a(T-s)-bw}\,dw\,ds+\int_{\mathbb{R}}\Gamma(z-w,\tau)\left|\bar{I}'_m(w)\right|e^{-aT-bw}\,dw}{\int_0^\tau\int_{\mathbb{R}}\Gamma(z-w,\tau-s)\bar{I}_m(w)\,e^{-a(T-s)-bw}\,dw\,ds+\int_{\mathbb{R}}\Gamma(z-w,\tau)\bar{I}_m(w)\,e^{-aT-bw}\,dw}$$
$$=\frac{\int_0^\tau\int_{\mathbb{R}}\Gamma(z-w,\tau-s)\left|\frac{\bar{I}'_m(w)}{\bar{I}_m(w)}\right|\bar{I}_m(w)\,e^{-a(T-s)-bw}\,dw\,ds+\int_{\mathbb{R}}\Gamma(z-w,\tau)\left|\frac{\bar{I}'_m(w)}{\bar{I}_m(w)}\right|\bar{I}_m(w)\,e^{-aT-bw}\,dw}{\int_0^\tau\int_{\mathbb{R}}\Gamma(z-w,\tau-s)\bar{I}_m(w)\,e^{-a(T-s)-bw}\,dw\,ds+\int_{\mathbb{R}}\Gamma(z-w,\tau)\bar{I}_m(w)\,e^{-aT-bw}\,dw}$$
$$\leq\varepsilon.$$

The last inequality is due to (35) and holds uniformly in z and τ for $m\geq M$.

Detailed proof of $\mathcal{H}^m(t,y)\leq C_1(y^{-\nu}\vee 1)$, used in the proof of Proposition 2.5:

In the following let $C_1\in\mathbb{R}^+$ be a generic constant depending on the model parameters, on α and on ν. It may vary from step to step.

We start with examining some bound for the second integral in formula (20) which presents an explicit expression for the function $\bar{\mathcal{H}}^m(t,z)=\mathcal{H}^m(t,e^z)\,e^{-at-bz}$. From elementary calculations we obtain:

$$\int_{\mathbb{R}}\Gamma(z-w,\tau)\bar{I}_m(w)\,e^{-aT-bw}\,dw\leq\int_{\mathbb{R}}\Gamma(z-w,\tau)\alpha(e^{-\nu w}\vee 1)\,e^{-aT-bw}\,dw$$
$$=C_1\int_{\mathbb{R}}\frac{e^{-\frac{(z-w)^2}{4\gamma\tau}-bw}}{\sqrt{\tau}}(e^{-\nu w}\vee 1)\,dw=C_1\int_{\mathbb{R}^-}\frac{e^{-\frac{(z-w)^2}{4\gamma\tau}-bw-\nu w}}{\sqrt{\tau}}\,dw$$
$$+C_1\int_{\mathbb{R}^+}\frac{e^{-\frac{(z-w)^2}{4\gamma\tau}-bw}}{\sqrt{\tau}}\,dw$$
$$\leq C_1\int_{\mathbb{R}}\frac{e^{-\frac{(z-w)^2}{4\gamma\tau}-bw-\nu w}}{\sqrt{\tau}}\,dw+C_1\int_{\mathbb{R}}\frac{e^{-\frac{(z-w)^2}{4\gamma\tau}-bw}}{\sqrt{\tau}}\,dw$$
$$=C_1\left(2\sqrt{\gamma\pi}\,e^{-bz-\nu z+\gamma\tau(b+\nu)^2}+2\sqrt{\gamma\pi}\,e^{-bz+b^2\gamma\tau}\right)\leq C_1\left(e^{-bz-\nu z}+e^{-bz}\right).$$

Analogously and by exchanging τ by $\tau-s$ we get for the first integral appearing in (20):

$$\int_0^\tau C_1\left(2\sqrt{\gamma\pi}\,e^{-bz-\nu z+\gamma(\tau-s)(b+\nu)^2}+2\sqrt{\gamma\pi}\,e^{-bz+b^2\gamma(\tau-s)}\right)ds\leq C_1\left(e^{-bz-\nu z}+e^{-bz}\right).$$

Changing from $\bar{\mathcal{H}}^m$ to $\mathcal{H}^m$ we finally get,

$$\mathcal{H}^m(t,e^z)\leq C_1(e^{-\nu z}+1)\;\Rightarrow\;\mathcal{H}^m(t,y)\leq C_1(y^{-\nu}\vee 1).$$

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